CONTEMPORARY MATHEMATICS

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Differential Geometry and Integrable Systems

A Conference on
Integrable Systems in Differential Geometry
University of Tokyo, Japan
July 17–21, 2000

Martin Guest
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Preface

Geometry is perhaps the oldest branch of mathematics, and the concept of integrability goes back to the early days of the calculus of several variables. It is not surprising that the tides of interaction have ebbed and flowed throughout this long period of coexistence. What may be surprising is the lack of a systematic theory relating the two subjects: each has developed in its own way, with its own problems, techniques and priorities. There is good reason to believe that this state of affairs is changing, however, and the main reason for this change — and consequent optimism for a more systematic approach — is the development of new mathematical tools over the last century and especially over the last thirty years.

These new tools include infinite dimensional Lie theory, the theory of moduli spaces of geometric structures, algebraic topology, symplectic geometry — and computers. The computer experiments of the 1950's, which led to the discovery of soliton theory, were the catalyst for rapid theoretical developments, employing a dazzling array of techniques across the whole spectrum of mathematics. The renewal of strong links between mathematics and quantum physics since the 1970's came at a fortuitous time, because many "integrable systems" have their origins in physics, and differential geometry plays a fundamental role here as well. Another phenomenon which has proved favourable to the subject is the return to mathematical fashion of concrete examples and explicit calculations, following several decades of preoccupation with abstract principles.

In view of these developments, an international conference was held in Japan, at the University of Tokyo, during the period 17-21 July 2000, with the title "Integrable Systems in Differential Geometry". Choosing a suitable title was not an easy task. The words "geometry" and "integrable systems" were essential, but the precise combination implies a deliberate choice of emphasis, and we were reluctant to impose our own artificial constraints on such a rapidly emerging subject. In the end we decided that differential geometry should be the underlying theme, partly because of the large number of differential geometers in Japan who were likely to attend, and partly because of the research areas of those who had already expressed interest in taking part in such a conference. But we intended to cover relations between integrable systems and geometry in the widest sense, and we were gratified that many speakers and participants represented areas such as physics, algebraic geometry, differential equations, and topology. We believe that a successful aspect of the conference was the juxtaposition of speakers and topics from a variety of subjects, some very much at the heart of recent connections between geometry and integrable systems, others in areas where there is potential for interaction in the future. We hope that the conference, and in particular these proceedings, will encourage differential geometers to embrace new ideas and techniques from integrable systems, and researchers in integrable systems to recognise the role of geometry as a source of intuition and concrete examples.

For practical purposes the proceedings are divided into three volumes. "Surveys on Geometry and Integrable Systems", published by the Mathematical Society of Japan in the series *Advanced Studies in Pure Mathematics*, contains mainly expository articles which relate geometry and integrable systems. "Differential Geometry and Integrable Systems", published by the American Mathematical Society in the series *Contemporary Mathematics*, contains both research and expository articles whose main focus is differential geometry. "Integrable Systems, Topology, and Physics", in the same series, contains articles related to geometry in a wider sense: topology, algebraic geometry, symplectic geometry, and mathematical physics. We have provided brief topical introductions for each of the three volumes, but there are a number of excellent general articles on geometry and integrable systems where the reader can find more details, such as the survey article [**4**] and the introductions to [**1**] and [**5**]. The article by Hitchin in [**5**] gives an overview of some major links between differential geometry, algebraic geometry and integrable systems, and the collections [**2**], [**3**] contain many more examples. References are listed at the end of the Introduction.

The conference was organised under the auspices of the Mathematical Society of Japan, as its 9-th International Research Institute, and financial support was provided by the Mathematical Society of Japan, the Japan Association for Mathematical Sciences, and from individual research grants of the Japan Society for Promotion of Science. The members of the Organising Committee were Takushiro Ochiai, Chairman (University of Tokyo), Martin Guest (Tokyo Metropolitan University), Mitsuhiro Itoh (University of Tsukuba), Reiko Miyaoka (Sophia University), Tohru Morimoto (Nara Women's University), Yoshihiro Ohnita (Tokyo Metropolitan University), Simon Salamon (Oxford University), Takeshi Sasaki (Kobe University), Chuu-Lian Terng (Northeastern University), and Gudlaugur Thorbergsson (Cologne University). On behalf of the entire Organising Committee we would like to thank several institutions and many individuals whose cooperation was essential for the success of the conference. In addition to the above funding agencies we thank the University of Tokyo for providing facilities at the Graduate School of Mathematical Sciences building in Komaba. Our conference secretaries Junko Furihata, Shiho Imai, and Yukari Takigawa, as well as the student assistants, all worked hard to ensure the smooth running of day to day activities and the associated administrative procedures. All articles were refereed, and we thank those mathematicians who served as anonymous referees for their important contributions. Carolyn Dong provided invaluable assistance with the preparation of TeX files. Finally we thank all speakers and participants for making the conference possible.

Martin Guest, Reiko Miyaoka and Yoshihiro Ohnita
9-th MSJ-IRI Proceedings Editors
Tokyo, June 2002

Introduction

The articles in this volume have a differential geometric emphasis. Many examples in the theory of integrable systems have their origins in differential geometry, where, conversely, questions of integrability have always been fundamental.

Variational principles give rise to natural geometric objects such as geodesics, minimal surfaces, harmonic maps, and Yang-Mills connections, so these have the same foundations as the various Euler-Lagrange equations of mathematical physics. Such equations often have a Hamiltonian formulation via the Legendre transformation, and this is the starting point of symplectic geometry and the study of symmetries of differential equations, a major theme of the theory of integrable systems. While some yield directly to integration through the Liouville-Arnold theorem, most partial differential equations remain very hard to solve explicitly by analytical methods, and the opportunity to exploit symmetries of naturally-occurring equations is one of the attractions of the integrable systems approach. Sometimes this approach is spectacularly successful, the KdV equation being a well known case, and this has certainly attracted the attention of geometers.

These symmetry groups are in general infinite dimensional, and they are usually not immediately visible in the initial geometrical formulation of the problem. For this reason, new techniques are required beyond the familiar methods of (finite dimensional) Lie group theory in differential geometry. Since infinite dimensional Lie theory has been developed most effectively at the infinitesimal level — that is, for Lie algebras — the subject has at times an algebraic character. But all these symmetries tend to be related, and the hidden, infinite dimensional groups can usually be traced back to more evident, finite dimensional ones. Thus, classical Lie theory and the theory of symmetric spaces in differential geometry feature in many of the articles in this volume.

The greatest impact of the theory of integrable systems on differential geometry has so far been in surface theory, especially in the study of surfaces with special properties in Euclidean space, and several articles deal with this subject. Other articles are concerned with integrability in the geometry of higher dimensional manifolds, and this topic seems likely to receive a lot more attention as the results from surface theory become more widely known and understood. The theory of harmonic maps (and their generalizations such as p-harmonic maps and harmonic morphisms) into symmetric spaces combines aspects of both these directions, as does classical submanifold theory.

Infinite dimensional symmetry groups arise naturally when these kinds of problems can be reformulated as zero curvature equations, i.e. equations which express the flatness of a certain connection, for then the gauge group comes into play. An alternative point of view regards the zero curvature equation as a family of Lax equations, or a single Lax equation on a function space. The resulting infinite dimensional "symmetry groups" have a strong influence on the geometry of the original problem, and provide a variety of approaches to revealing new information. For example one can study orbits of the action, or try to characterize solutions of a certain type in terms of properties of their orbits, or one can regard the group action simply as a way of producing new solutions from old ones.

Differential geometers may have been reluctant to venture into this field in the past, perhaps because it seems to demand a background in analysis or algebra or even physics. But this much is certain: the methods of integrable systems theory have already been so productive in geometry that they cannot in future be ignored. The discovery of Wente tori, the work of Hitchin and of Uhlenbeck on harmonic maps, the new insights into submanifold theory — all of which are related to the presence of previously unimaginable symmetry groups — have established a bond between integrable systems and differential geometry which is likely to endure.

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[5] C.-L. Terng and K. Uhlenbeck (eds.), *Surveys in Differential Geometry: Integrable Systems*, volume IV, International Press, 1998.

List of speakers and titles

All speakers are listed below in alphabetical order. The locations of articles published in the conference proceedings are indicated as follows:

SGIS: "Surveys on Geometry and Integrable Systems", *Advanced Studies in Pure Mathematics*, Mathematical Society of Japan

DGIS: "Differential Geometry and Integrable Systems", *Contemporary Mathematics*, American Mathematical Society

ISTP: "Integrable Systems, Topology, and Physics", *Contemporary Mathematics*, American Mathematical Society

The title refers to the published article when this differs from the title of the talk.

N. Ando [DGIS]
The Index of an Isolated Umbilical Point on a Surface
J. Berndt [—]
Ricci Curvature and Staeckel Systems
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M.A. Guest* [ISTP]
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F. Hélein and P. Romon [SGIS]
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M. Igarashi [—]
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On Submersive p-Harmonic Morphisms and their Stability
D. Joyce [SGIS]
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K. Kiyohara [DGIS]
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*invited speaker who either did not attend the conference or did not give a talk

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Ideas and techniques from the theory of integrable systems are playing an increasingly important role in geometry. Thanks to the development of tools from Lie theory, algebraic geometry, symplectic geometry, and topology, classical problems are investigated more systematically. New problems are also arising in mathematical physics. A major international conference was held at the University of Tokyo in July 2000. It brought together scientists in all of the areas influenced by integrable systems. This book is the first of three collections of expository and research articles.

This volume focuses on differential geometry. It is remarkable that many classical objects in surface theory and submanifold theory are described as integrable systems. Having such a description generally reveals previously unnoticed symmetries and can lead to surprisingly explicit solutions. Surfaces of constant curvature in Euclidean space, harmonic maps from surfaces to symmetric spaces, and analogous structures on higher-dimensional manifolds are some of the examples that have broadened the horizons of differential geometry, bringing a rich supply of concrete examples into the theory of integrable systems.

Many of the articles in this volume are written by prominent researchers and will serve as introductions to the topics. It is intended for graduate students and researchers interested in integrable systems and their relations to differential geometry, topology, algebraic geometry, and physics.

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