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Preface

0.1. This volume constitutes a reworking of the main parts of Chapters VI and VII in Hartshorne's *Residues and Duality* [**7**], in greater generality, and by a local, rather than global, approach.

"Greater generality" signifies that we work throughout with arbitrary (quasi-coherent, torsion) Cousin complexes on (noetherian) formal schemes, not just with residual complexes on ordinary schemes. And what emerges at the end is a duality pseudofunctor (alias 2-functor) on the category of composites of compactifiable maps between those formal schemes which admit dualizing complexes.[1]

"Local approach" signifies that the compatibilities between certain pseudofunctors associated to smooth maps on the one hand and to closed immersions on the other (base-change and residue isomorphisms...), compatibilities underlying the basic process of pasting together these two pseudofunctors, are treated by means of explicitly-defined—through formulas involving generalized fractions—maps between local cohomology modules over commutative rings, and in particular, residue maps. This way of dealing with compatibilities seems to us to have advantages over the classical one. In regard to relative complexity, one might for instance compare Chapter 6 of [**8**], where the compatibilities we need are taken care of, with [**2**, Chap. 2, §7], where the compatibilities needed in the global approach of [**7**, Chap. VI, §2] are discussed. (To pursue the global approach here, one would have to redo everything for formal schemes, with the added complication introduced by the necessary presence of the derived torsion functor.)[2] Moreover, the connection between local and global behaviors is made transparent, the latter being defined entirely in terms of the former.

Indeed, one motivation behind this work has been to gain a better understanding of the close relation between local properties of residues and global properties of the dualizing pseudofunctor.

0.2. Classical Grothendieck Duality theory [**7**], [**10**], [**6**], [**2**] concerns itself with a contravariant pseudofunctor $(-)^!$ on the category (say) of finite-type maps of noetherian separated schemes X, taking values in derived categories $\mathbf{D}^+_{\mathrm{qc}}(X)$ whose objects are the $\mathcal{O}_X$-complexes $M^\bullet$ with quasi-coherent homology modules $H^n(M^\bullet)$ which vanish for $n \ll 0$. To each such scheme map $f : X \to Y$, $(-)^!$ assigns a functor $f^! : \mathbf{D}^+_{\mathrm{qc}}(Y) \to \mathbf{D}^+_{\mathrm{qc}}(X)$, and to each composition $X \xrightarrow{f} Y \xrightarrow{g} Z$ a functorial isomorphism $C^!_{f,g} : f^!g^! \xrightarrow{\sim} (gf)^!$. Using Nagata's compactifications and the formal arguments of [**4**, p. 318, Prop. 3.3.4], one finds that this pseudofunctor is characterized *up to isomorphism* by the following data a) and b), which exist and satisfy c):

[1]Nagata showed that every separated finite-type map of (noetherian) schemes is compactifiable, that is, factors as an open immersion followed by a proper map, see [**9**], [**3**]; this is not known to be so for formal schemes (where "proper" becomes "pseudo-proper," see below). In [**7**], "pseudofunctor" = "theory of variance." The definition of the duality pseudofunctor will be self-contained with respect to this volume; but the proof given here of its duality properties needs the existence of a right adjoint for the direct image functor on derived torsion categories, see [**1**, p. 59].

[2]A novel and very efficient way of handling compatibilities for finite tor-dimension maps of schemes over a regular base has recently been developed by Yekutieli and Zhang [**12**].

(a) An isomorphism between the restriction of $(\_)^!$ to the subcategory of open immersions (or more generally, étale maps) and the pseudofunctor associating the inverse image functor f^* to $f : X \to Y$.

(b) (Proper duality) A bifunctorial isomorphism, for proper $f : X \to Y$,

$$\mathrm{Hom}(\mathbf{R}f_*F, G) \xrightarrow{\sim} \mathrm{Hom}(F, f^!G) \qquad (F \in \mathbf{D}^+_{\mathrm{qc}}(X),\ G \in \mathbf{D}^+_{\mathrm{qc}}(Y)),$$

compatible, in the natural sense, with the pseudofunctorial structures—respectively covariant and contravariant—on $\mathbf{R}f_*$ and $f^!$. Thus $f^!$ is right-adjoint to $\mathbf{R}f_*$, and there is a functorial *trace map* $\mathbf{R}f_*f^! \to 1$, "transitive" vis-à-vis $X \xrightarrow{f} Y \xrightarrow{g} Z$.

(c) Given a cartesian square

$$\begin{array}{ccc} X' & \xrightarrow{\ v\ } & X \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\ u\ ]{} & Y \end{array}$$

with u (hence v) an open immersion and f (hence g) proper, the following natural diagram of functorial maps, with unlabeled arrows arising from $(\_)^!$, commutes:

$$\begin{array}{ccccc} (fv)^! & \longrightarrow & v^!f^! & \xrightarrow{\text{(a)}} & v^*f^! \\ \Big\| & & & & \big\downarrow{\scriptstyle \beta} \\ (ug)^! & \longrightarrow & g^!u^! & \xrightarrow[\text{(a)}]{} & g^!u^* \end{array}$$

where the *base-change map* β is defined to be the adjoint—via (b)—of the natural composition

$$\mathbf{R}g_*v^*f^! \xrightarrow{\sim} u^*\mathbf{R}f_*f^! \xrightarrow{\text{trace}} u^*,$$

and is in fact an *isomorphism* for any flat u.

The richness of the theory lies to a large extent in concrete representations of the basic components. For example, the restriction of $(-)^!$ to the category of *finite* maps is isomorphic to the pseudofunctor which assigns to f the functor $\bar{f}^*\mathbf{R}\mathcal{H}om(f_*\mathcal{O}_X, -)$, where $\bar{f}$ is the (flat) ringed space map $(X, \mathcal{O}_X) \to (Y, f_*\mathcal{O}_X)$ induced by f. Or, when $f : X \to Y$ is *Cohen-Macaulay* (i.e., flat, with Cohen-Macaulay fibers), of relative dimension d, then $f^!\mathcal{O}_Y$ has a single non-vanishing homology sheaf ω_f, the *relative dualizing sheaf,* which is flat over Y; and there is a functorial isomorphism

$$f^!C^\bullet \cong f^*C^\bullet \otimes_{\mathcal{O}_X} \omega_f[d] \qquad (C^\bullet \in \mathbf{D}^+_{\mathrm{qc}}(Y)).$$

which can be made *pseudofunctorial* over the category of Cohen-Macaulay maps. Under further mild restrictions, the relative dualizing sheaf, determined a priori only up to isomorphism, has a canonical representative, namely the sheaf of *regular d-forms,* coinciding over the smooth locus of f with the sheaf Ω^d_f of holomorphic d-forms (see [**5**]); and explication of the functorial isomorphism $C^!_{f,g}$ when gf and f are both finite and g is smooth is intimately tied to the local theory of residues. Thus differential forms, their traces and, more generally, their residues, play a vital role in the development of the concrete aspects of duality.

0.3. The papers in this volume continue efforts, begun in [**1**], to generalize all of Grothendieck duality theory to formal schemes (always understood to be noetherian). Why formal schemes, aside from their just being there? For one thing, the category F of formal schemes contains the category of ordinary schemes, that is, formal schemes whose structure sheaf of topological rings has the discrete topology. Also, F contains the category opposite to that of local homomorphisms of complete noetherian local rings. Thus the category of formal schemes offers, potentially, a framework for treating local and global duality as aspects of a single theory.

For example, suppose $f\colon X \to Y$ is a proper map of noetherian schemes, and that $x \in X$ and $y := f(x) \in Y$ are closed points. Set $R := \mathcal{O}_{X,x}$, $S := \mathcal{O}_{Y,y}$; and let $f_x\colon \mathrm{Spec}(R) \to \mathrm{Spec}(S)$ be the map induced by f. One can imagine that many properties of the global map f could be approached through simpler properties of local maps like f_x. But in the passage from f to f_x, properness—which is clearly important in duality theory because of (b) above—is lost.

Consider, however, X and Y completed along closed subsets $V \subset X$ and $W \subset Y$ such that $x \in V$, $y \in W$, and $f(V) \subset W$, so that f induces a formal-scheme map $\widehat{f}\colon \widehat{X} \to \widehat{Y}$. Further, let $\hat{R}$ and $\hat{S}$ be the completions of R and S at their respective maximal ideals, and let $\widehat{f_x}\colon \mathrm{Spf}(R) \to \mathrm{Spf}(S)$ be the resulting formal-scheme map. Then there is a natural commutative diagram

$$\begin{array}{ccc} \mathrm{Spf}(R) & \longrightarrow & \widehat{X} \\ {\scriptstyle \widehat{f_x}}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{f}} \\ \mathrm{Spf}(S) & \longrightarrow & \widehat{Y} \end{array}$$

with *both the maps* $\widehat{f}$ *and* $\widehat{f_x}$ *pseudo-proper.* ("Pseudo-proper" means that one of—hence each of—the ordinary-scheme maps obtained by factoring out ideals of definition in the source and target is *proper.*)

This primitive example suggests that the relation between local and global duality properties might well become more apparent in the context of formal schemes. In practice, it does! (This holds in the present volume, and will, it is planned, be supported in depth in a paper, in preparation, on the Residue Theorem for formal schemes, consolidating a number of results in the literature on the relation between local residues and canonical global realizations of duality for formal schemes.)

0.4. In [**1**], items (b) and (c) in 0.1 are extended, with certain restrictions, to where f and g are *pseudo-proper* formal-scheme maps. It turns out that on a formal scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ we can deal with coherent sheaves; but quasi-coherence does not have enough manageable properties unless we restrict to *torsion sheaves*—those modules over the sheaf of topological rings $\mathcal{O}_{\mathcal{X}}$ each of whose sections over any open $\mathcal{U} \subset \mathcal{X}$ is annihilated by some open ideal of $\mathcal{O}_{\mathcal{X}}|\mathcal{U}$. (On ordinary schemes, where (0) is an open ideal, *all* modules are torsion sheaves.) So, for instance, with $\mathbf{D}^+_{\mathrm{qct}}(\mathcal{X})$ denoting the derived category of homologically bounded-below $\mathcal{O}_{\mathcal{X}}$-complexes with quasi-coherent torsion homology, and f pseudo-proper, Theorems 6.1 and 7.4 of [**1**] provide a right adjoint $f^!$ for $\mathbf{R}f_*\colon \mathbf{D}^+_{\mathrm{qct}}(\mathcal{X}) \to \mathbf{D}^+_{\mathrm{qct}}(\mathcal{Y})$, satisfying items (b) and (c) (with "pseudo-" in front of "proper"). The variant for coherent sheaves is covered by *ibid,* p. 89, Corollary 8.3.3 and Theorem 8.4.

The most notable obstruction to dealing with more general separated pseudo-finite-type maps is that, as mentioned before, we know of no theorem to the effect that such a map is *compactifiable.*[3] Nevertheless, we can still work with those pseudo-finite separated formal-scheme maps which can be built up from pseudo-proper maps and open immersions, i.e., work within the subcategory F^0 of F having the same objects, but only those maps which are composites of compactifiable ones. The category F^0 includes all separated finite-type maps of ordinary noetherian schemes, since, by the above-mentioned theorem of Nagata, they are compactifiable. And indeed, the main theorem in [**7**] can be extended to F^0, as follows. (Here, and below, we want only to convey a preliminary idea of what is done in this volume, not precise statements. The introductions to the individual papers further explain the terminology, methods, and results.)

A basic problem, suggested by (0.2), is how to paste together the above pseudofunctor $(-)^!$ for pseudo-proper maps and the inverse image pseudofunctor $(-)^*$ on the category of open immersions to form a $\mathbf{D}^+_{\mathrm{qct}}$-valued pseudofunctor on all of F^0. One would like to have an abstract pasting procedure in the spirit of 3.3.4 in [**4**, p. 318], a Proposition which, as indicated before, applies to ordinary schemes, but which cannot be applied to formal schemes because we don't know that the composition of two compactifiable maps is still compactifiable.

Nayak's paper "Pasting pseudofunctors" in this volume *provides such an abstract procedure,* whose applicability to the preceding problem for F^0 is shown to result from certain formal properties of a base-change isomorphism established in [**1**]. (See Theorems 7.13 and 7.14 in Nayak's paper.) The resulting F^0-pseudofunctor is still denoted by $(-)^!$.

0.5. Sastry's paper "Duality for Cousin complexes" gives, in many situations (see below), a *concrete, canonical realization of the* F^0*-pseudofunctor* $(-)^!$.

The approach taken overlaps—and was inspired by—that in [**7**, Chap. 7], but it is both more concrete and more general.

It begins with the canonical pseudofunctor $(-)^\sharp$ to whose construction the joint paper "Pseudofunctorial behavior of Cousin complexes on formal schemes" of Lipman, Nayak and Sastry is devoted. Roughly speaking, $(-)^\sharp$ is defined over a suitable category $\mathbb{F}_c$ of formal schemes $\mathcal{X}$ with codimension functions Δ, assigning to each object $(\mathcal{X}, \Delta)$ the category $\mathrm{Coz}_\Delta(\mathcal{X})$ of quasi-coherent torsion Δ-Cousin $\mathcal{O}_\mathcal{X}$-complexes.

Briefly, having in mind that $(-)^\sharp$ is meant to be a concrete approximation to $(-)^!$, one first describes the functor $f^\sharp$ for f a closed immersion or a smooth map, by "Cousinifying" the concrete examples given above toward the end of §0.2. Then, noting that every $\mathbb{F}_c$-map factors locally as (smooth) $\circ$ (closed immersion), one defines $(-)^\sharp$ for such factorizable maps by composing the functors associated to the factors. This construction turns out to be independent of the factorization used, so finally it is possible to define $(-)^\sharp$ globally by gluing the local definitions. Carrying this all out involves careful attention to a great many details, a good portion of which have already been dealt with by Huang in [**8**], where he constructed, in essence, the restriction of $(-)^\sharp$ to Cousin complexes with vanishing differentials.

In [**7**, Chap. 6, §3], there is constructed a canonical pseudofunctor $(-)^\Delta$ on noetherian schemes, with values in categories of residual complexes (i.e., those Cousin

[3]Nor do we know a counterexample. But there is an example of a closed subscheme $\mathcal{Z}$ of an open subscheme of the completion of $\mathbb{C}^4$ along a line, whose inclusion map does not have an obvious compactification, i.e., $\mathcal{Z}$ is not an open subscheme of a closed subscheme, see [**1**, Correction].

complexes which are "pointwise dualizing").[4] See also [**2**, §3.2]. Our pseudofunctor $(-)^\sharp$ is more general, because while each $f^\sharp$ does take residual complexes to residual complexes, it operates on a larger class of Cousin complexes, and over formal schemes. It should be said, however, that the basic elements of the strategy for constructing $(-)^\sharp$, as outlined in the preceding paragraph, can all be found in [**7**].

Let us return to Sastry's paper. The proof of the Duality Theorem in [**7**, Chapter 7] begins with a trace map $f_*f^\Delta K \to K$ of graded modules, defined when $f\colon X \to Y$ is a finite-type map of noetherian schemes and K is a residual $\mathcal{O}_Y$-complex. What is called there the Residue Theorem states that when the map f is proper, "trace" is a map of complexes. Using local residues, Sastry defines, for every $\mathbb{F}_c$-map $f\colon (\mathfrak{X}, \Delta_1) \to (\mathfrak{Y}, \Delta)$ and Δ-Cousin $\mathcal{O}_\mathfrak{Y}$-complex $\mathcal{F}$, a functorial trace

$$\mathrm{Tr}_f(\mathcal{F})\colon f_*f^\sharp\mathcal{F} \to \mathcal{F};$$

and proves: *for pseudo-proper f, $\mathrm{Tr}_f(\mathcal{F})$ is a map of complexes* (Trace Theorem).

Via basic properties of the above F^0-pseudofunctor $(-)^!$, the Trace Theorem enables the construction of a canonical pseudofunctorial derived-category map

$$\gamma^!_f(\mathcal{F}) : f^\sharp\mathcal{F} \to f^!\mathcal{F} \qquad (\mathcal{F} \in \mathrm{Coz}_\Delta(\mathfrak{Y})).$$

Applying the usual Cousin functor E makes this an *isomorphism* $f^\sharp\mathcal{F} \xrightarrow{\sim} E(f^!\mathcal{F})$. Moreover, $\gamma^!_f$ itself is an isomorphism whenever f is flat or $\mathcal{F}$ is an injective complex. One finds then, with Q the canonical functor from the category of complexes to the derived category, that if one restricts to flat maps and Cohen-Macaulay complexes (the derived-category complexes isomorphic to $Q(C)$ for some Cousin complex C), or to Gorenstein complexes (the derived-category complexes isomorphic to $Q(C)$ for some *injective* Cousin complex C), then, *$Qf^\sharp E$ is a pseudofunctor satisfying the conditions* (a), (b) and (c) in §0.2.

Using $\gamma^!_f$, Sastry also proves a *canonical Duality Theorem* for pseudo-proper maps $f\colon (\mathfrak{X}, \Delta') \to (\mathfrak{Y}, \Delta)$ and Δ-Cousin $\mathcal{O}_\mathfrak{Y}$-complexes $\mathcal{F}$: *the pair* $(f^\sharp\mathcal{F}, \mathrm{Tr}_f(\mathcal{F}))$ *represents the functor* $\mathrm{Hom}_\mathfrak{Y}(f_*C, \mathcal{F})$ *of Δ'-Cousin $\mathcal{O}_\mathfrak{X}$-complexes C.*

In summary, *$f^\sharp$ is a canonical concrete approximation to the duality functor $f^!$.*

0.6. Finally, the canonicity of $\gamma^!_f$ and uniqueness properties of residual complexes enable one to draw closer to the holy grail of defining canonically a duality pseudofunctor $(-)^!$ for all pseudo-finite-type maps $f\colon \mathfrak{X} \to \mathfrak{Y}$, at least in the presence of bounded residual complexes (or equivalently, dualizing complexes), and under suitable coherence hypotheses. The idea, taken from [**7**], is to define $f^!$ as being dualization on $\mathfrak{Y}$ with respect to a fixed residual complex $\mathcal{R}$ (i.e., application of the functor $\mathcal{H}om^\bullet_\mathfrak{Y}(-, \mathcal{R})$), followed by $\mathbf{L}f^*$, followed by dualization on $\mathfrak{X}$ with respect to the residual complex $f^\sharp\mathcal{R}$. More details appear in the last section of Sastry's paper.

* * *

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[4]This Δ is not to be confused with the Δ used throughout to denote a codimension function.

References

[1] L. Alonso Tarrío, A. Jeremías López, and J. Lipman, *Duality and flat base change on formal schemes,* in: Studies in Duality on Noetherian formal schemes and non-Noetherian ordinary schemes, Contemporary Mathematics, **244**, Amer. Math. Soc., Providence, RI, 1999, 1–87. See also *Correction...,* Proc. Amer. Math. Soc. **131** (2003), no. 2, 351–357.

[2] B. Conrad, *Grothendieck duality and base change,* Lecture Notes in Mathematics, **1750**, Springer-Verlag, Berlin, 2000.

[3] ———, *Deligne's notes on Nagata compactifications,* item **5** at < `http://www-math.mit.edu/~dejong/#brian` >

[4] P. Deligne, *Cohomologie à supports propres,* in: SGA 4 Tome 3, Lecture Notes in Mathematics bf 305, Springer-Verlag, Berlin, 1973, pp. 250–461.

[5] R. Hübl and P. Sastry, *Regular differential forms and relative duality,* Amer. J. Math. **115** (1993), no. 4, 749–787.

[6] A. Neeman, *The Grothendieck duality theorem via Bousfield's techniques and Brown representability,* J. Amer. Math. Soc. **9** (1996), 205–236.

[7] R. Hartshorne, *Residues and duality,* Lecture Notes in Math., **20**, Springer-Verlag, Berlin, 1966.

[8] I. C. Huang, *Pseudofunctors on modules with zero-dimensional support,* Memoirs Amer. Math. Soc. **114** (1995), no. **548**.

[9] W. Lütkebohmert, *On compactification of schemes,* Manuscr. Math. **80** (1993), no. 1, 95–111.

[10] J. L. Verdier, *Base change for twisted inverse image of coherent sheaves,* Algebraic Geometry, Oxford Univ. Press, London, 1969, pp. 393–408.

[11] A. Yekutieli, *Smooth formal embeddings and the residue complex,* Canadian J. Math. **50** (1998), no. 4, 863–896.

[12] ———, J. J. Zhang *Rigid dualizing complexes and perverse sheaves on schemes,* preprint, math.AG/0405570.

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Robert Hartshorne's 1966 book, *Residues and Duality*, introduced the notion of residual complexes and developed a duality theory (Grothendieck duality) on the category of maps of noetherian schemes.

The three articles in this volume constitute a reworking of the main parts of the corresponding chapters in Hartshorne's 1966 book in greater generality using a somewhat different approach. Additionally, the authors' motivation is to help readers gain a better understanding of the relation between local properties of residues and global properties of the dualizing pseudofunctor.

The book is suitable for graduate students and researchers working in algebraic geometry.

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