

# CONTEMPORARY MATHEMATICS

517

## Gems in Experimental Mathematics

AMS Special Session  
Experimental Mathematics  
January 5, 2009  
Washington, DC

Tewodros Amdeberhan  
Luis A. Medina  
Victor H. Moll  
Editors



American Mathematical Society

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**American Mathematical Society**  
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## Preface

The editors of these proceedings organized a special session on Experimental Mathematics at the Joint Meetings of the American Mathematical Society that was held in Washington, D. C., January 2009.

The point of view of *Experimental Mathematics* has been recently formalized by a small number of research groups. The basic philosophical principles of this branch of mathematics have appeared in books initiated by the groups around David Bailey, Jonathan Borwein, Doron Zeilberger among others.

The goal of the editors is to bring to this volume a collection of papers reflecting the experimental nature of many mathematical problems. In a real sense, this collection is a continuation of **Tapas in Experimental Mathematics**, volume 457 of this series.

The volume contains most of the lectures presented at the Washington meeting, as well as some papers specially requested by the editors. An effort was made to include authors not traditionally included in the Experimental Mathematics world.

The editors wish to thank all participants at the special session and all contributors to this volume. We also wish to acknowledge the invaluable help of the referees and the American Mathematical Society, in particular the editors of the Contemporary Mathematics series, that have made this volume a reality. It is hoped that the papers appearing here will inspire many researchers to join the growing Experimental Mathematics community.

*The editors*

New Orleans and San Juan  
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These proceedings reflect the special session on Experimental Mathematics held January 5, 2009, at the Joint Mathematics Meetings in Washington, DC as well as some papers specially solicited for this volume.

Experimental Mathematics is a recently structured field of Mathematics that uses the computer and advanced computing technology as a tool to perform experiments. These include the analysis of examples, testing of new ideas, and the search of patterns to suggest results and to complement existing analytical rigor.

The development of a broad spectrum of mathematical software products, such as Mathematica® and Maple™, has allowed mathematicians of diverse backgrounds and interests to use the computer as an essential tool as part of their daily work environment.

This volume reflects a wide range of topics related to the young field of Experimental Mathematics. The use of computation varies from aiming to exclude human input in the solution of a problem to traditional mathematical questions for which computation is a prominent tool.

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