CONTEMPORARY MATHEMATICS

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Algebraic and Geometric Aspects of Integrable Systems and Random Matrices

AMS Special Session
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January 6–7, 2012
Boston, MA

Anton Dzhamay
Kenichi Maruno
Virgil U. Pierce
Editors

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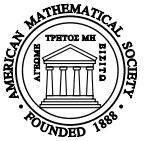

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Preface

The articles in this volume result from the AMS Special Session on *Algebraic and Geometric Aspects of Integrable Systems and Random Matrices* at the Joint Mathematics Meetings 2012 in Boston, MA.

In planning this special session we aimed at not only discussing important new results in the theory of integrable systems and soliton equations, but also at emphasizing the relationship between the theory of integrable systems and other branches of mathematics, such as Algebraic Geometry, Probability Theory, and Combinatorics. Consequently, the papers in this volume can be loosely grouped as follows.

Nonlinear integrable equations and soliton-type solutions. The paper by Adler, Cafasso, and van Moerbeke describes a connection between the integrable kernels that appear in the Random Matrix theory and the wave functions of the KP-hierarchy and then uses the KP integrable structure to show how to derive nonlinear PDEs for the Fredholm determinants of these kernels. With concise overview of the KP theory and careful treatment of some important examples, this paper provides a very nice introduction into this circle of ideas. The paper by DiFranco and Miller is one in the series of papers by the authors on the semi-classical analysis of the modified nonlinear Schrödinger equation. Using a variety of techniques and methods, some of them new, and all carefully explained, the authors study the semi-classical limit of solutions for carefully chosen multi-parameter family of Cauchy initial-value problems. One very interesting feature of this paper is that the authors found a way to work with the exact, and not approximate, solutions corresponding to this initial data. Very clear and detailed, this paper is an important contribution to the field of semi-classical analysis of the NLS-type equations. The paper by Szmigielski and Zhou studies the interaction of singular soliton-type solutions, called the peakons, of the Degasperis-Procesi equation. Such multipeakon solutions provide models for wave breaking for this class of wave equations, and the study of peakon-type solutions to non-linear wave equations has recently been attracting a lot of attention. The authors set up the general formalism to study peakons and antipeakons interactions and then apply it to the detailed study of possible interactions of three peakons. The paper by Kasman starts with some remarks about the action-angle duality in the theory of integrable systems and its relation to bispectrality. The author then proceeds to consider a somewhat different notion of duality for the Calogero systems constrained by a harmonic potential and shows how this duality can be realized as an algebraic map between two manifolds of matrix pairs satisfying an important rank-one condition, identifying the eigenvalue dynamic of one system to the eigenvalue dynamic of its dual.

Painlevé equations and their higher order analogues. Painlevé-type equations, that have already appeared in this volume before in the paper of Adler et. al., now take the central stage. The paper of Suzuki briefly reviews how Painlevé equations can be obtained from integrable hierarches by a reduction procedure. The author then uses similarity reduction to derive higher-order Painlevé systems from the Drinfield-Sokolov hierarchies and proceeds to study their Hamiltonian structures, affine Weyl group symmetries, and Lax-pair representations. The paper of Kawakami, Nakamura, and Sakai describes the classification of the four-dimensional Painlevé-type equations from the point of view of the theory of isomonodromic deformations by using the classification of the corresponding Fuchsian systems. They also list the Hamiltonians for these equations and describe their degeneration scheme. The paper by Ohyama and Okumura considers the question that goes back to the work of R. Fuchs over a hundred years ago. Among all possible solutions to Painlevé equations there is a distinguished class of algebraic solutions. Such solutions, that are completely classified, only occur for special values of parameters in Painlevé equations. Consider now a linear equation whose isomonodromic transformations give a particular Painlevé equation. R. Fuch's problem is to show that, if this Painlevé equation admits an algebraic solutions, then its associated linear equation can be transformed to a confluent hypergeometric equation or its degeneration. For Painlevé-VI, whose associated linear equation has only regular singular points, this problem was solved previously. In this paper the authors review their work on this problem in the case when associated linear equation has irregular singularities, which corresponds to Painlevé I–V, and explain how to solve it by explicitly constructing the required covering transformations.

The paper by Chakravarty presents a unified approach to systematically deriving a six-parameter family of third order nonlinear differential equations whose general solutions are given in terms of Schwarz triangle functions. This family includes third order nonlinear differential equations studied in the classical works of Jacobi, Halphen, Chazy, and Ramanujan, and many equations in this family are of Chazy-type, i.e., they are third-order ODEs that possess the Painlevé property. Each equation in this family admits, as its group of automorphisms, a special subgroup of $PSL_2(\mathbb{C})$ called the Fuchsian triangle group. Special solutions of such equations include, for example, the Eisenstein series E_2, E_4, and E_6, introduced by Ramanujan, that play an important role in number theory. The author also shows that each equation in this family can be obtained from the generalized Darboux-Halphen system by reduction.

Integrable systems and geometry. Papers in this group concern the relationship between integrable systems and algebraic geometry. The paper of Carstea uses the birational geometry of the space of initial conditions, in the spirit of Sakai's approach to the classification of discrete and continuous Painlevé equations, to study a traveling wave reduction from the Q_4 lattice equation in the Adler-Bobenko-Suris classification. One interesting feature of this example is that it lifts to the automorphism of a generalized Halphen surface of index one, whereas the Painlevé case corresponds to the Halphen surface of index zero. The paper of Doliwa studies the integrability of the non-commutative analogue of the Hirota bilinear equation using some geometric ideas, and in particular, the incidence geometry. The author introduces the notion of the Desargues map, which is based on a certain collinearity condition, and then interprets it as a linear problem for the Hirota system. He then

presents the theory of the Darboux-type transformations for the Desargue maps and describes the symmetries of the Hirota system in terms of affine Weyl group actions. Finally, using the observation that one can use the Desargue maps to construct solutions of functional pentagonal equations, the author specializes some of these constructions from the general non-commutative setting to the quantum case. The paper by Korotkin and Zograf concerns an application of the tau-function formalism to some questions of algebraic geometry. The notion of a tau-function is central in the modern theory of integrable systems and often such tau-functions encode some interesting geometric information. In this paper the authors use the Bergman tau functions to study the geometry of the moduli space of holomorphic quadratic differentials on Riemann surfaces and also the relationship between this moduli space and the moduli space of abelian differentials under the double-cover pullback map. The main result of the paper is the explicit decomposition, in the Picard group, of the Prym and Hodge classes in terms of the tautological and boundary divisor classes.

Applications to enumerative problems. The volume concludes with two papers that consider applications of some tools and ideas from the theory of integrable systems to the problems of enumerative combinatorics. The paper of Dmitriescu, Mulase, Safnuk, and Sorkin considers very recent and not yet fully understood Eynard-Orantin theory for topological recursion and counting. This theory, similar in spirit to mirror symmetry, establishes a correspondence between some counting problem (the A-side of the theory) and the Eynard-Orantin topological recursion formalism (the B-side). An important ingredient of the B-side of the theory is the choice of the spectral curve and the recursion kernel (in some sense, the seed of the recursion process). The authors propose that, rather than doing a heuristic search for these ingredients, one should just take the Laplace transform of some geometric data on the A-side. Thus, this Laplace transform plays a role of mirror symmetry in the theory. While not establishing rigorously this conjecture, the authors illustrate it with four interesting and important counting examples: dessins d'enfants (or the higher-genus analogues of the Catalan numbers), which is a new result, intersection theory of the moduli space $\overline{\mathcal{M}}_{g,n}$ of pointed stable curves, single Hurwitz numbers, and the stationary Gromov-Witten invariants of $\mathbb{P}^1$. In addition to formulating this new and original conjecture, the paper also gives a concise introduction into the Eynard-Orantin theory and contains numerous bibliographics references, which makes it also a very nice introductory paper for this new and active research area. The Catalan numbers (and their higher and fractional generalizations) also appear, via their generating functions that govern the asymptotic expansions of the partition functions of certain random matrix ensembles, in the paper by Pierce. The author uses the connections between these partition functions, orthogonal polynomials, and the Toda lattice hierarchy, to study the asymptotic expansions of the partition functions. He shows that the coefficients in this expansion can be expressed in terms of the auxiliary variable that is implicitly given as a solution of the algebraic equation defining the Catalan-type generating function. This paper gives a comprehensive survey of some known results and then discusses the possibilities and challenges of extending it to more general potential functions.

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Integrable Systems and Random Matrices • Dzhamay et al., Editors

This volume contains the proceedings of the AMS Special Session on Algebraic and Geometric Aspects of Integrable Systems and Random Matrices, held from January 6–7, 2012, in Boston, MA.

The very wide range of topics represented in this volume illustrates the importance of methods and ideas originating in the theory of integrable systems to such diverse areas of mathematics as algebraic geometry, combinatorics, and probability theory. The volume offers a balanced combination of survey articles and research papers with important new results.

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