CONTEMPORARY MATHEMATICS

604

Recent Advances in Real Complexity and Computation

UIMP-RSME Lluís A. Santaló Summer School
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July 16–20, 2012
Universidad Internacional Menéndez Pelayo,
Santander, Spain

José Luis Montaña
Luis M. Pardo
Editors

American Mathematical Society
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Dedicated to our beloved friend Jean-Pierre Dedieu.

Contents

Editors' preface

This volume is composed of six contributions derived from the lectures given during the UIMP–RSME Lluís Santaló Summer School on "Recent Advances in Real Complexity and Computation". The goal of this Summer School was to present some of the recent advances on *Smale's 17th Problem*. This Problem was stated by Steve Smale as follows:

PROBLEM 1 (Smale's 17th Problem). *Can a zero of n complex polynomial equations in n unknowns be found approximately, on the average, in polynomial time with a uniform algorithm?*

These contributions cover several aspects around this problem: from numerical to symbolic methods in polynomial equation solving, computational complexity aspects (both worse and average cases, both upper and lower complexity bounds) and even aspects of the underlying geometry of the problem. Some of the contributions also deal with either real or multiple solutions solving.

The School was oriented to graduate mathematicians, as to Master or Ph. D. students in Mathematics and to senior researchers interested on this topic.

The School was promoted and supported by the Spanish Royal Mathematical Society (RSME) and hosted by the Universidad Internacional Menéndez Pelayo (UIMP), from July 16th to July 20th of 2012, in El Palacio de la Magdalena, Santander. Partial financial support was also granted by the University of Cantabria and the Spanish Ministry of Science Grant MTM2010-16051. We thank these institutions and grants for their financial support.

The speakers (in alphabetical order) and their courses in this Summer School were the following ones:

- Carlos Beltrán, *"Stability, precision and complexity in some numerical problems"*.
- Marc Giusti, *"Polar, co–polar and bipolar varieties: real solving of algebraic varieties with intrinsic complexity"*.
- Joos Heintz, *"On the intrinsic complexity of elimination problems in effective algebraic geometry"*.
- Gregorio Malajovich, *"From the quadratic convergence of Newton's method to problems of counting of the number of solutions"*.
- Klaus Meer, *"Real Number Complexity Theory and Probabilistically Checkable Proofs (PCPs)"*.
- Michael Shub, *"The geometry of condition and the analysis of algorithms"*.
- Jean-Claude Yakoubsohn, *" Tracking multiplicities"*.

The present volume extends the Summer School by expository articles presenting the state of art of each of the topics. The reader will find the following contributions in forthcoming pages:

(1) Martijn Baartse and Klaus Meer, *"Topics in real and complex number complexity theory"*
The contribution intends to introduce into topics relevant in real and complex number complexity theory. This is done in a survey style. Taking as starting point the computational model introduced by Blum, Shub, and Smale the following issues are addressed: Basic results concerning decidability and NP-completeness, transfer results of open questions between different models of computation, structural complexity inside $\mathrm{NP}_{\mathbb{R}}$, computational universality, and probabilistically checkable proofs over the real and complex numbers.

(2) Bernd Bank, Marc Giusti and Joos Heintz, *"Polar, bipolar and copolar varieties: Real solving of algebraic varieties with intrinsic complexity"*.
This survey covers a decade and a half of joint work with L. Lehmann, G. M. Mbakop, and L. M. Pardo. The authors address the problem of finding a smooth algebraic sample point for each connected component of a real algebraic variety, being only interested in components which are generically smooth locally complete intersections. The complexity of their algorithms is essentially polynomial in the degree of suitably defined generalized polar varieties and is therefore intrinsic to the problem under consideration.

(3) Carlos Beltrán and Michael Shub, *"The complexity and geometry of numerical solving polynomial equations"*.
This contribution contains a short overview on the state of the art of efficient numerical analysis methods that solve systems of multivariate polynomial equations. The authors focus on the work of Steve Smale who initiated this research framework, and on the collaboration between Stephen Smale and Michael Shub, which set the foundations of this approach to polynomial system–solving, culminating in the more recent advances of Carlos Beltrán, Luis Miguel Pardo, Peter Bürgisser and Felipe Cucker.

(4) Marc Giusti and Jean–Claude Yakoubsohn, *"Multiplicity hunting and approximating multiple roots of polynomials systems"*.
The computation of the multiplicity and the approximation of isolated multiple roots of polynomial systems is a difficult problem. In recent years, there has been an increase of activity in this area. Our goal is to translate the theoretical background developed in the last century on the theory of singularities in terms of computation and complexity. This paper presents several different views that are relevant to address the following issues : predict the multiplicity of a root and/or determine the number of roots in a ball, approximate fast a multiple root and give complexity results for such problems. Finally, we propose a new method to determine a regular system, called equivalent but deflated, i.e., admitting the same root as the initial singular one.

(5) Joos Heintz, Bart Kuijpers and Andrés Rojas Paredes, *"On the intrinsic complexity of elimination problems in effective algebraic geometry"*.
The representation of polynomials by arithmetic circuits evaluating them is an alternative data structure which allowed considerable progress in polynomial equation solving in the last fifteen years. The authors present in this contribution a circuit based computation model which captures the core of all known symbolic elimination algorithms that avoid unnecessary branchings in effective algebraic geometry and show the intrinsically exponential complexity character of elimination in this complexity model.

(6) Gregorio Malajovich, *"Newton iteration, conditioning and zero counting"*.
This contribution deals with the problem of counting the number of real solutions of a system of multivariate polynomial equations with real coefficients. You can also find in this contribution a crash-course in Newton iteration. We will state and analyze a Newton iteration based 'inclusion-exclusion' algorithm to count (and find) roots of real polynomials.

In recent months, two members of our scientific community left us: our colleague *Mario Wschebor* and our beloved friend *Jean-Pierre Dedieu*. Jean-Pierre was invited to the Summer School and his talk was scheduled as the closing talk of the School. Unfortunately, a long illness prevented him from being with us at the School and, sadly, he left us on 15 June 2012. Let this volume serve as a remembrance of both of them.

The editors wish to thank the RSME for giving us the opportunity to organize this event. It is also a pleasure to thank the patronage of the UIMP. Their help in the organization and the experience in Las Caballerizas del Palacio de la Magdalena are not to be easily forgotten. Our deepest gratitude goes to the speakers, who did an excellent job, and also to the students, whose interest and dedication created a great atmosphere. We finally wish to thank the authors for their excellent contributions to this volume.

José Luis Montaña & Luis M. Pardo

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