CONTEMPORARY MATHEMATICS

676

Noncommutative Geometry and Optimal Transport

Workshop on
Noncommutative Geometry and Optimal Transport
November 27, 2014
Besançon, France

Pierre Martinetti
Jean-Christophe Wallet
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Preface

The distance formula in noncommutative geometry was introduced by Connes at the end of the 1980s. Given a so-called *spectral triple* $(\mathcal{A}, \mathcal{H}, D)$, that is, an involutive algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ via a representation π, and an operator D on $\mathcal{H}$ such that the commutator $[D, \pi(a)]$ is bounded for any a in $\mathcal{A}$, one defines on the space $\mathcal{S}(\mathcal{A})$ of states of $\mathcal{A}$ the (possibly infinite) distance

$$(1) \qquad d(\varphi, \varphi') = \sup_{a \in \mathcal{A}} \{ |\varphi(a) - \varphi'(a)|, \ ||[D, \pi(a)]|| \leq 1 \} \quad \forall \varphi, \varphi' \in \mathcal{S}(\mathcal{A}).$$

For $\mathcal{A} = C_0^\infty(\mathcal{M})$ the commutative algebra of smooth functions vanishing at infinity on a locally compact and complete manifold $\mathcal{M}$, acting on the Hilbert space $\mathcal{H}$ of square integrable differential forms and D the signature operator, this distance computed between pure states gives back the geodesic distance on $\mathcal{M}$.

In this sense, eq. (1) is a generalization of Riemannian geodesic distance that makes sense in a noncommutative setting, and provides an original tool to study the geometry of the space of states on an algebra.

Besides its mathematical interest, Connes distance also has an intriguing echo in physics, for it yields a metric interpretation for the Higgs field, the missing piece of the *Standard Model of Fundamental Interactions* recently discovered by the Large Hadronic Collider at CERN.

In the 1990s, Rieffel noticed that (1) was a noncommutative version of the Wasserstein distance of order 1 in the theory of optimal transport (the modern version of Monge *déblais et remblais* problem). More exactly, this is a noncommutative generalization of Kantorovich dual formula of the Wasserstein distance. Formula (1) thus offers an unexpected connection between an ancient mathematical problem and the most recent discovery in high energy physics.

The meaning of this connection is far from clear. Yet, Rieffel's observation suggests that Connes distance may provide an interesting starting point for a theory of optimal transport in noncommutative geometry, as well as a possible interpretation of the Higgs field as a cost function on spacetime.

More specifically, one may wonder

- What remains of the duality Wasserstein (minimizing a cost)/Kantorovich (maximizing a profit) in the noncommutative setting? Is there some "noncommutative cost" that one is minimizing while computing the supremum in the distance formula?
- May the noncommutative geometry point of view on the Wasserstein distance help to solve some problems in optimal transport? Vice versa, can one use results of optimal transport to address relevant issues in noncommutative geometry?

- Is such a generalization of the Riemannian distance truly interesting for physics?
- How useful can this distance be to study the space of states of an algebra, in particular does it yield interesting topologies?
- What are the properties of this distance? For instance how does it behave under the product of spectral triples?

These questions were at the origin of the mini-workshop *Optimal transport and noncommutative geometry* held in Besançon in November 2014, and whose proceedings are presented in this volume. Both optimal transport and noncommutative geometry are active areas of research, but with little intersection. In addition, the metric aspect of noncommutative geometry is a part of the theory that has been relatively little studied so far. Nevertheless several results—including explicit computations—have been obtained in recent years, links with other areas of geometry (like sub-Riemannian geometry) have been discovered, and general properties (e.g. topological) have been worked out.

This volume opens with a survey, which aims at providing an account of the metric aspect of noncommutative geometry readable by nonexperts. The contribution of Latrémolière then provides an extensive state of the art on the topological aspects of Connes distance (and suitable generalizations of it). The Dubois-Violette lecture is a presentation of his noncommutative generalization of the classical moment problem. Franco and Wallet recall how the Moyal plane is an example of a locally compact noncommutative space, and test on it some ideas of causal structure in noncommutative geometry. D'Andrea's text is an account on the properties of Connes distance under the product of spectral triples. Finally Guillemard's contribution presents some structures in dynamical systems that may have application in optimal transport.

P. Martinetti & J.-C Wallet

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NCG & Optimal Transport • Martinetti and Wallet, Editors

This volume contains the proceedings of the Workshop on Noncommutative Geometry and Optimal Transport, held on November 27, 2014, in Besançon, France.

The distance formula in noncommutative geometry was introduced by Connes at the end of the 1980s. It is a generalization of Riemannian geodesic distance that makes sense in a noncommutative setting, and provides an original tool to study the geometry of the space of states on an algebra. It also has an intriguing echo in physics, for it yields a metric interpretation for the Higgs field. In the 1990s, Rieffel noticed that this distance is a noncommutative version of the Wasserstein distance of order 1 in the theory of optimal transport. More exactly, this is a noncommutative generalization of Kantorovich dual formula of the Wasserstein distance. Connes distance thus offers an unexpected connection between an ancient mathematical problem and the most recent discovery in high energy physics. The meaning of this connection is far from clear. Yet, Rieffel's observation suggests that Connes distance may provide an interesting starting point for a theory of optimal transport in noncommutative geometry.

This volume contains several review papers that will give the reader an extensive introduction to the metric aspect of noncommutative geometry and its possible interpretation as a Wasserstein distance on a quantum space, as well as several topic papers.

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