

# CONTEMPORARY MATHEMATICS

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Centre de Recherches Mathématiques Proceedings

## Geometric and Computational Spectral Theory

Séminaire de Mathématiques Supérieures  
Geometric and Computational Spectral Theory  
June 15–26, 2015

Centre de Recherches Mathématiques,  
Université de Montréal, Montréal, Québec

Alexandre Girouard  
Dmitry Jakobson  
Michael Levitin  
Nilima Nigam  
Iosif Polterovich  
Frédéric Rochon  
Editors



American Mathematical Society  
Providence, Rhode Island

Centre de Recherches Mathématiques  
Montréal, Québec, Canada

# Geometric and Computational Spectral Theory



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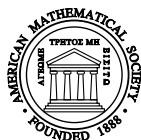
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## Preface

The 2015 edition of the *Séminaire de Mathématiques Supérieures* (SMS) took place on June 15-26, 2015, at the Université de Montréal. The topic of the meeting was *Geometric and computational spectral theory*, and it was perhaps the largest school in spectral geometry since the late nineties. The event was sponsored by the Centre de recherche mathématiques, as well as by the Fields Institute (Toronto), the Pacific Institute for the Mathematical Sciences (Vancouver), the Mathematical Sciences Research Institute (Berkeley), the Institut des sciences mathématiques (Montréal) and the Canadian Mathematical Society.

The summer school brought together students and internationally renowned experts in the geometric and computational aspects of spectral theory. The area of spectral theory has fascinated mathematicians and physicists for centuries, and recent years have seen remarkable progress in several branches of the field. The scientific program consisted of twelve minicourses in four main themes: geometry of eigenvalues, geometry of eigenfunctions, computational spectral theory, and spectral theory on singular spaces. A particular emphasis was made on the interplay between these topics, notably between the computational and the geometric part — which was one of the most novel aspects of the school. The minicourses were complemented by exercise sessions, computer labs and short presentations by selected junior participants who have already made important contributions to the subject. The school featured about 90 participants from 13 countries spanning five continents. The current volume contains the lecture notes and survey papers based on nine minicourses delivered at the summer school.

The lecture notes by Bruno Colbois (Neuchâtel) begin with an overview of some basic facts on the spectrum of the Laplace operator on Euclidean domains and Riemannian manifolds. The main part of the notes is concerned with the geometric eigenvalue inequalities, notably the Cheeger lower bound on the first eigenvalue and the Korevaar's method for estimating from above the eigenvalues of the Laplacian on a Riemannian manifold in a fixed conformal class. The last two sections cover some related recent results obtained by the author and his collaborators. In particular, bounds on the eigenvalues of hypersurfaces are discussed, as well as the notions of the topological and conformal spectra.

The lecture notes by Gregory Berkolaiko (Texas A & M) provide an introduction to spectral theory of Schrödinger operators on metric graphs, which in this setting are known as quantum graphs. A number of illuminating examples are treated in detail. A particular emphasis is made on the count of zeros of the eigenfunctions of quantum graphs.

A survey paper by Dorin Bucur (Chambéry) and Pedro Freitas (Lisbon) focusses on the celebrated Faber-Krahn inequality for the fundamental tone of the



Dirichlet Laplacian from the viewpoint of free boundary problems. The proof of the Faber-Krahn inequality which is presented is purely variational and, surprisingly, does not use rearrangement arguments. The advantage of this approach is its adaptability to extremal problems for higher eigenvalues and to more general Robin boundary conditions.

The paper by Pierre Bérard (Institut Fourier) and Bernard Helffer (Nantes and Orsay) is concerned with the nodal properties of the quantum harmonic oscillator and other Schrödinger operators on a Euclidean plane. In particular, the authors show that for the two-dimensional isotropic quantum harmonic oscillator there exists an infinite sequence of eigenfunctions with exactly two nodal domains. For the spherical Laplacian and the Dirichlet Laplacian on a square a similar result has been proved long time ago by A. Stern and H. Lewy. Other questions in nodal geometry are also discussed, such as bounds on the length of the zero set of a Schrödinger eigenfunction in the classically permitted region.

The next three contributions to the volume deal with various computational aspects of spectral theory. The paper by Jessica Bosch and Chen Greif (UBC) provides a review of the numerical methods for computing eigenvalues of matrices. A number of methods are discussed in detail, including the power method, the divide-and-conquer algorithms for tridiagonal matrices, as well as several approaches to compute the eigenvalues of large and sparse matrices, such as Lanczos, Arnoldi and Jacobi-Davidson methods.

The notes by Guido Kanschat (Heidelberg) give an introduction to the finite element methods for variational eigenvalue problems. In particular, Galerkin approximation is discussed in detail. Several eigenvalue problems are considered, including the standard boundary value problems for the Laplacian, as well as problems arising in the study of Maxwell and Stokes equations. In the appendix, some computer programs for experiments are presented.

The lecture notes by Alexander Strohmaier (Leeds) provide a link between the “numerical” and the “geometric” themes of this volume. The notes focus on the method of particular solutions and its applications to the computation of eigenvalues of the Laplace operator on Riemannian manifolds, including hyperbolic surfaces. Several Mathematica programs are included for illustration, which is most helpful. Computations of various spectral quantities, such as the spectral zeta function and the zeta-regularized determinant of the Laplace operator, are also discussed.

The remaining two contributions are based on minicourses requiring more advanced analytic background. The lecture notes by Daniel Grieser (Oldenburg) give an introduction to analysis on manifolds with corners in the spirit of R. Melrose. Its tools are used to construct quasimodes (i.e. approximate eigenfunctions) for degenerating families of domains, including adiabatic limit families and families exhibiting certain types of scaling behaviour.

The notes by Colin Guillarmou (ENS) are concerned with geometric inverse problems on surfaces with boundary, such as the lens rigidity problem and its special case, the boundary rigidity problem. These problems naturally arise when one needs to recover the Riemannian metric on a surface from the boundary measurements. A closely related notion of the X-ray transform is also discussed.

The Proceedings of the *2015 Séminaire de Mathématiques Supérieures on Geometric and Computational Spectral Theory* cover a large variety of topics and methods, combining geometric, analytic and numerical ideas. We hope that this volume

will serve both as a useful reference for experts and as an inspiring reading for young mathematicians who would like to learn more about this fascinating and rapidly developing area of mathematics. The editors would like to express their gratitude to all the contributing authors, as well as to all the speakers at the summer school. Last but not least, it is our pleasure to thank the SMS coordinator, Sakina Benhima, for all her hard work and help with the organization of this event.



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Geometric and Computational Spectral Theory • Girouard et al., Editors

The book is a collection of lecture notes and survey papers based on the mini-courses given by leading experts at the 2015 Séminaire de Mathématiques Supérieures on Geometric and Computational Spectral Theory, held from June 15–26, 2015, at the Centre de Recherches Mathématiques, Université de Montréal, Montréal, Quebec, Canada.

The volume covers a broad variety of topics in spectral theory, highlighting its connections to differential geometry, mathematical physics and numerical analysis, bringing together the theoretical and computational approaches to spectral theory, and emphasizing the interplay between the two.

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