CONTEMPORARY MATHEMATICS

702

Topological Complexity and Related Topics

Mini-Workshop
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February 28–March 5, 2016
Mathematisches Forschungsinstitut Oberwolfach,
Oberwolfach, Germany

Mark Grant
Gregory Lupton
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AMERICAN MATHEMATICAL SOCIETY

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Preface

This is the proceedings volume of a conference on Topological Complexity and Related Topics, held at the Mathematisches Forschungsinstitut Oberwolfach (MFO) from February 28 to March 5, 2016, under the auspices of their Mini-Workshop program. There were 16 participants. The talks were a mixture of presentations of research results and surveys, as reflected in the contents of this volume. Details about the conference, including a list of participants as well as short abstracts of the talks presented, are available from the corresponding Oberwolfach Report [Report No. 15/2016, *Mini-Workshop: Topological Complexity and Related Topics*, Oberwolfach Reports, Vol. 13, No. 1 (2016), pp. 705–740].

The notion of topological complexity (TC) was introduced by Farber in 2003. This is a numerical homotopy invariant of a space, of Lusternik–Schnirelmann type, which provides a topological approach to the study of the complexity of the motion planning problem in robotics. If X is the *configuration space* of a mechanical system, that is, the space of all the possible states of the system, then a *motion planner* in X is a (not necessarily continuous) function that assigns to each pair $(x, y) \in X \times X$ a path in X from x to y prescribing the motion of the system from the initial state x to the final state y. The point of departure for this notion as a topic of interest is a basic result that says a global continuous motion planner is possible only when the configuration space X is contractible. Roughly speaking, then, $\mathrm{TC}(X)$ corresponds to the minimum number of local continuous motion planners needed to determine a complete motion planner in X. It soon becomes apparent that $\mathrm{TC}(X)$ is determined by the topology of the space X in ways that are often not well-understood, making it a delicate invariant to compute. By specifying some intermediate states in the motion, the concept was generalized to the notion of higher topological complexity by Rudyak in 2010.

These invariants have been intensively studied in the last decade, and their values have been determined for numerous interesting spaces. Much work has been done in particular on the case of *classical configuration spaces*, whose points consist of ordered n-tuples of distinct points in a given space, and which model the collision-free motion planning problem with n agents. Some of theses spaces are *Eilenberg-Mac Lane spaces* $K(G, 1)$. It remains a challenge to understand the topological complexity of such spaces in terms of the properties of the group G. The survey of Cohen presents the methods of determination of the topological complexity of the configuration spaces of various classical spaces (such as Euclidean spaces and compact orientable surfaces). The discussion is also extended to some related spaces such as orbits of configuration spaces and some Eilenberg-Mac Lane spaces $K(G, 1)$. In the same direction, the article of González and Gutiérrez determines the higher topological complexity of the configuration spaces of ordered distinct points in an orientable surface. The article of Grant and Recio-Mitter studies the (higher) topological complexity of Eilenberg-Mac Lane spaces $K(G,1)$ where G is a certain type of subgroup of Artin's full braid group, while the article by Fieldsteel is dedicated to the (higher) topological complexity of the complement in $\mathbb{C}^n$ of some arrangements of hyperplanes associated to a graph. In most of these works, an important tool is the cohomological lower bound of TC given by the zero-divisors cup-length. As a potential useful step towards the computation of this cohomological lower bound, the article of Davis analyses the top cohomology class of the space of isometry classes of polygons whose side lengths satisfy some condition.

As mentioned above, TC is related to the *Lusternik–Schnirelmann category* (LS-category, or cat), the theory of which progressed rapidly in the last decade of the twentieth century. Many tools and techniques of rational homotopy theory and of classical homotopy theory have been developed in order to study the properties of this invariant, and to solve outstanding problems such as the Ganea conjecture on $\mathrm{cat}(X \times S^p)$. Topological complexity (and its higher versions) and LS-category are special cases of the *Schwarz genus*, or *sectional category*, of a fibration. Several studies have been dedicated to this unifying notion in the recent past, with efforts being made to generalize the ideas, techniques, and approximations developed in the context of LS-category to sectional category, in order to make them available for the study of TC. The survey of Carrasquel presents the algebraic tools and results which have been developed in rational homotopy theory to study approximations of sectional category and TC using Sullivan models. In a more topological approach, the article by Doeraene, El Haouari and Ribeiro develops a notion of sectional category of a class of maps which generalizes the classical notion of sectional category, while the article of Fernández-Suárez and Vandembroucq studies a notion of Q-topological complexity inspired by the notion of Q-category introduced by Scheerer, Stanley and Tanré. In the last decade, some work has also been done to understand the (close) relationship between $\mathrm{TC}(X)$ and the LS-category of the cofibre of the diagonal map $\Delta : X \to X \times X$. The article by González, Grant and Vandembroucq pursues this investigation in the special case of two-cell complexes, making use of the Berstein-Hilton-Hopf invariants, which played an important role in Iwase's resolution of the Ganea conjecture for LS-category.

Since the introduction of TC, many extensions and variations of the concept have been conceived that take into account some additional aspects of the motion planning problem. For instance, various authors have developed versions of *equivariant TC* in order to study the complexity of the motion planning problem when the configuration space is given with the action of a group. The survey of Ángel and Colman reviews these different approaches, and discusses the properties and advantages of each invariant. In another direction, the notion of topological complexity of a map has recently been introduced, in order to take into account the information encoded by the so-called *kinematic map* associated to a mechanical system. The survey of Pavešić reviews various aspects of robotics which are relevant in the study of the motion planning problem, discussing in particular the properties of kinematic maps and the study of this new extension of topological complexity.

Finally, we note that the articles of Ángel–Colman, Cohen, Davis and Pavešić use Farber's original definition of topological complexity for which the topological complexity of a contractible space is 1, whereas the other articles have adopted the normalized version which assigns 0 to a contractible space (as is usual in more homotopical approaches to invariants of Lusternik–Schnirelmann type).

The AMS publications department has been very encouraging and supportive throughout the preparation of this volume. We would like to thank Christine Thivierge, especially, for her guidance at each stage. The editors, who were also the conference organisers, thank the MFO for support with overall organization of the conference, and for providing an ideal location with outstanding facilities for our activity. This activity was partially supported by a grant from the Simons Foundation (#209575 to Gregory Lupton), and by funds from the (Portuguese) Fundação para a Ciência e a Tecnologia, through the Project UID/MAT/0013/2013. The MFO and the editors thank the National Science Foundation for supporting the participation of junior researchers in the workshop under the grant DMS-1049268, "US Junior Oberwolfach Fellows."

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Topological Complexity • Grant et al., Editors

This volume contains the proceedings of the mini-workshop on Topological Complexity and Related Topics, held from February 28–March 5, 2016, at the Mathematisches Forschungsinstitut Oberwolfach.

Topological complexity is a numerical homotopy invariant, defined by Farber in the early twenty-first century as part of a topological approach to the motion planning problem in robotics. It continues to be the subject of intensive research by homotopy theorists, partly due to its potential applicability, and partly due to its close relationship to more classical invariants, such as the Lusternik–Schnirelmann category and the Schwarz genus.

This volume contains survey articles and original research papers on topological complexity and its many generalizations and variants, to give a snapshot of contemporary research on this exciting topic at the interface of pure mathematics and engineering.

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