

# CONTEMPORARY MATHEMATICS

745

## Motivic Homotopy Theory and Refined Enumerative Geometry

Workshop on  
Motivic Homotopy Theory  
and Refined Enumerative Geometry  
May 14–18, 2018  
Universität Duisburg-Essen, Essen, Germany

Federico Binda  
Marc Levine  
Manh Toan Nguyen  
Oliver Röndigs  
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## Preface

Refined enumerative geometry, an emerging area of study, enriches our understanding of solutions to geometric problems. Instead of a mere integer counting such solutions, refined enumerative geometry produces a class in the Grothendieck-Witt ring of quadratic forms over a field, whose dimension is the integer that classical enumerative geometry provides. Such a perspective already occurs in real algebraic geometry, where signs are introduced in order to capture certain counting phenomena over real closed fields, corresponding to the signature of quadratic forms.

Motivic or  $\mathbf{A}^1$ -homotopy theory supplies a convenient foundational framework for this endeavour. A standard target for Euler characteristics is the endomorphism ring of the unit in a suitable tensor triangulated category, such as the  $\mathbf{P}^1$ -stable homotopy category over a field. Morel's theorem identifies this endomorphism ring with the Grothendieck-Witt ring of the field. The  $\mathbf{P}^1$ -stable homotopy category over a field also contains Voevodsky's motivic Eilenberg-MacLane spectrum representing motivic cohomology, and in particular Chow groups of smooth varieties. The latter invariant constitutes a classical tool for enumerative geometry. Between the theories of  $\mathbf{P}^1$ -stable homotopy and motivic cohomology sits the theory of Chow-Witt groups, represented by algebraic cycles with additional orientation data. Chow-Witt groups are rich enough to capture quadratic form information yet manageable enough to allow for successful computations.

This new viewpoint on enumerative geometry allows beautiful refinements of very classical results (such as counting lines on a cubic surface) but also opens up new possibilities in various other subjects (such as Gromov-Witten theory). In order to promote this viewpoint, the workshop “Motivic Homotopy Theory and Refined Enumerative Geometry” took place in Essen (NRW, Germany) from the 14th to the 18th of May 2018. Central to the workshop were three series of lectures held by as many experts:

- Denis-Charles Cisinski: Intersection theory and mixed motives with rational coefficients
- Jean Fasel: Introduction to Chow-Witt groups
- Marc Levine: Enumerative geometry via quadratic forms

In addition to these lectures, nine research talks on refined enumerative geometry, motivic homotopy theory, and algebraic  $K$ -theory were given by Alexey Ananyevskiy, Frédéric Déglise, Jens Hornbostel, Oliver Röndigs, Shuji Saito, Marco Schlichting, Matthias Wendt, Kirsten Wickelgren, and Paul Arne Østvær. The workshop was supported by the DFG Priority Programme “Homotopy theory and algebraic geometry”. We would like to express our sincere gratitude to all lecturers, speakers, participants, and administrative personnel for contributing to the success

of this workshop. Moreover, we would like to thank all authors for their contributions to these proceedings, as well as the expert referees and the staff at AMS (in particular Chris Thivierge) for their valuable efforts.

Federico Binda  
Marc Levine  
Manh Toan Nguyen  
Oliver Röndigs



FIGURE 1. Workshop picture



## List of talks

### Lecture series

**Denis-Charles Cisinski:** Intersection theory and mixed motives with rational coefficients

**Jean Fasel:** Introduction to Chow-Witt groups

**Marc Levine:** Enumerative geometry via quadratic forms

### Research talks

**Alexey Ananyevskiy:**  $SL$ -oriented cohomology theories

**Frédéric Déglise:** Characteristic classes in non oriented theories

**Jens Hornbostel:** Chow-Witt groups of classifying spaces

**Oliver Röndigs:** On very effective hermitian  $K$ -theory

**Shuji Saito:** Rigid analytic  $K$ -theory

**Marco Schlichting:** Nisnevich descent for Grothendieck-Witt groups

**Matthias Wendt:** Chow-Witt rings of Grassmannians and oriented Schubert calculus

**Kirsten Wickelgren:** An arithmetic count of lines through 4 general lines in  $\mathbb{P}^3$

**Paul Arne Østvær:** A motivic Segal conjecture for the group of order two





## List of participants

Alexey Ananyevskiy  
St. Petersburg State University

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Université Paul Sabatier (Toulouse III)

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Universität Düsseldorf

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This volume contains the proceedings of the Workshop on Motivic Homotopy Theory and Refined Enumerative Geometry, held from May 14–18, 2018, at the Universität Duisburg-Essen, Essen, Germany.

It constitutes an accessible yet swift introduction to a new and active area within algebraic geometry, which connects well with classical intersection theory. Combining both lecture notes aimed at the graduate student level and research articles pointing towards the manifold promising applications of this refined approach, it broadly covers refined enumerative algebraic geometry.



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