DYNAMICS OF TRAFFIC JAMS: ORDER AND CHAOS

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ABSTRACT. By means of a novel variational approach, we study the ergodic properties of a model of a multi lane traffic flow, considered as a (deterministic) wandering of interacting particles on an infinite lattice. For a class of initial configurations of particles (roughly speaking satisfying the Law of Large Numbers) the complete description of their limit behaviour (in time) is obtained, as well as estimates of the transient period. In this period the main object of interest is the dynamics of ‘traffic jams’, which is rigorously defined and studied. It is shown that the dynamical system under consideration is chaotic in the sense that its topological entropy (calculated explicitly) is positive. Statistical quantities describing limit configurations are obtained as well.

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1. Introduction

Despite the self evident practical importance of the analysis of traffic flows and a relatively long history of attempts of their scientific treatment (going back to the fifties) only recently (at the end of the nineties) reasonable mathematical models of traffic flows and methods to study them were introduced. Previous attempts were based on ideas borrowed from such classical fields of physics as mechanics and hydrodynamics. Not going into the details of the qualitative and quantitative comparison of the hydrodynamic type models with practice (which one can find, for example, in a recent survey [19] and references therein), we note the following practical observation. It turns out that going by foot in a slowly moving crowd it is faster to go against the “flow” than in the same direction as other people go. A mathematical model describing this effect in the case of one lane traffic was introduced in [2]. A standard probabilistic model of the diffusion of particles against/along the flow clearly contradicts this observation, which very likely points to a very special (nonrandom) intrinsic structure of the flow in this case. The main
aim of the present paper is to study how this structure emerges from arbitrary (random) initial configurations in a simple model of multi lane traffic flow.

Recent progress in the analysis of traffic flows was due to the introduction of discrete (in time and in space) cellular automata models of one lane traffic flow in \cite{14,13}, later studied by many authors (see \cite{4} for a survey and further references). Various approaches starting from the mean-field approximation \cite{11} to combinatorial techniques and statistical mechanics methods \cite{10} were used in their analysis. All these models were based on the idea of describing the dynamics in terms of deterministic or random cellular automata (see results about stochastic models in \cite{6,7,8,9}) and to a large extent were studied by means of numerical simulation (especially because of the low computational cost of numerical realization of cellular automata rules, which made it possible to realize large-scale real-time simulations of urban traffic \cite{20}).

Roughly speaking, the one lane road in these models is associated to a finite one-dimensional integer lattice of size $N$ with periodic boundary conditions, each position on the lattice being either occupied by a particle (representing a vehicle), or empty. At the next time step each particle remains in its place if the next position is occupied, and moves forward by one place otherwise. In \cite{14,10,4} it was shown (mainly numerically and by some physical argument) that the limit (as time goes to infinity) behaviour of the dynamical system under consideration depends only on the density of particles in the initial configuration. This result was generalized to the case of the dynamics on the infinite lattice and proved mathematically in \cite{2}, where a novel variational approach was introduced.

Despite various generalizations, the one lane restriction of these models was crucial, for example, in an attempt to study a multi lane model satisfying standard traffic rules \cite{2} no mathematically interesting phenomena were found. Only two years ago \cite{15,16}, a first nontrivial multi lane generalization was introduced for the case of motion on a finite lattice with periodic boundary conditions, based on the so-called ultradiscrete limit of the well-known Burgers equation.

In the present paper we study ergodic properties of this model. As we shall show, this analysis boils down to the study of the dynamics of ‘traffic jams’ (see the rigorous definition in Section 4), which mainly depends of the density of particles in the initial configuration.

One of the main quantities of interest in traffic models — the average velocity of cars and its dependence on the density of cars $\rho$ (called the fundamental diagram) is typically studied in the steady state. From our results it follows that in the multi lane model that we consider, the average velocity in the steady state is equal to $\max\{1, K/\rho - 1\}$, which immediately brings to mind similar result known in the one lane case.

The paper is organized as follows. In Section 2 we describe the model in detail and introduce the basic notation, including the important notions of dual configurations and maps. In Section 3 we introduce the space of regular (statistically defined) configurations, show that this space is invariant with respect to the dynamics and formulate the main result of the paper — Theorem 3.1. Qualitatively this result means that in a steady state any configuration either consists of free (moving independently) particles, or this property holds for all empty places on the
lattice. In terms of the variational principle mentioned above, this can be formulated to say that the total number of free particles between two fixed ones can only grow in time. The proof of this result in the next Section 4 is based on the detailed analysis of the dynamics of traffic jams. It is worth noticing that as distinct from the one lane case, the formal description of a traffic jam is rather nontrivial and some individual particles in it can move, but still represent obstacles to the motion of other particles. Section 5 is dedicated to the proof of the chaoticity of this model: we explicitly calculate its topological entropy and show that it is strictly positive. In the last Section 6 we derive statistical quantities describing typical limit configurations.

We tried to define rigorously all the important objects that we consider in the text; but of course we were unable to introduce all the standard mathematical definitions. The reader can find exact definitions and further references related to dynamical systems (especially of those acting on discrete phase spaces), for example, in [1, 12].

2. Multi lane traffic flow models: dynamics in space of configurations

The model corresponds to the highway traffic flow on a road with $K$ lanes. Let $X^K_0 := \{0, 1, \ldots, K\}^Z_1$ be an infinite lattice, positions on which we call (lattice) sites. For a sequence $X \in X^K_0$ and $x \in Z_1$, by $X(x)$ we denote the $x$-th element of this sequence. Consider a map $T: X^K_0 \rightarrow X^K_0$, defined by the relation

$$TX(x) = X(x) + \min\{X(x - 1), K - X(x)\} - \min\{X(x), K - X(x + 1)\}. \quad (2.1)$$

Remark. In [15, 16] the above map was introduced for the case of the finite lattice with periodic boundary conditions. Note that a finite lattice of arbitrary size $N < \infty$ with periodic boundary conditions is a particular case of the $Z_1$ lattice considered in our paper and restricted only to $N$-periodic configurations. The paper [16] claims an estimate of the transient period in the $2N$-periodic case as $N$. Since the construction in that paper substantially depends on the length of the period, it cannot be extended to the case of unbounded lattices with the general nonperiodic initial configurations that we consider. It is of interest that $2N + 1$ periodic initial configurations lead (as we shall show) to much worse estimates of the transient period $2N$.

The above map can be described in a different way in terms of configurations of particles. Let us introduce several definitions.

A collection of particles $\Xi$ on the lattice $Z_1$ will be called ordered if there is a function (called the index function) $I: \Xi \rightarrow Z_1$ such that for any two particles $\xi, \xi' \in \Xi, \xi \neq \xi'$, the corresponding indices satisfy the inequality $I(\xi) \neq I(\xi')$ and if these particles are located at sites $|\xi| < |\xi'|$ on the lattice, then $I(\xi) < I(\xi')$, where $|\xi|$ stands for the location of the particle $\xi$ on the lattice.

To a configuration $X \in X^K_0$ we associate an ordered collection (finite or infinite) of particles on the lattice $Z_1$ containing not more than $K$ particles at each site so that $X(x)$ for $x \in Z_1$ stands for the number of particles located at the site $x$. Then the set of all possible ordered configurations of particles containing not more
than $K$ particles at each site forms the phase space $X^K_0 := \{0, 1, \ldots, K\}^{\mathbb{Z}_1}$ of the system under consideration.

For a given positive integer $K$ and a given configuration $X \in X^K_0$, the action of the map $T$ can be described as follows. For each site $x \in \mathbb{Z}_1$ (independently of other sites) we move $\min\{X(x), K - X(x + 1)\}$ particles with the largest indices from the site $x$ to the site $x + 1$.

**Lemma 2.1.** The order function $I$ is preserved under the action of the map $T$.

**Proof.** Straightforward. \hfill $\square$

For a given configuration $X$ associated to the collection of particles $\Xi$, for each particle $\xi \in \Xi$ we introduce the notion of velocity $v(\xi)$ which is equal to 1 if the particle moves after the application of the map $T$ or 0 otherwise. Accordingly, we shall say that the particle $\xi$ is free if $v(\xi) > 0$ and jammed otherwise. Summing the velocities of all individual particles, we obtain the moments (the total velocity of particles at a given site) of lattice sites in the configuration $X$:

$$v(X, x) := \sum_{|\xi| = x} v(\xi).$$

An immediate calculation yields the following equality.

**Lemma 2.2.** We have $v(X, x) = \min\{X(x), X^*(x + 1)\}$. \hfill $\square$

From the point of view of the description in terms of individual particles, we introduce the notion of dual configuration $X^*(x) := K - X(x)$ for any $x \in \mathbb{Z}_1$, which describes empty places in the original configuration of particles $X$. Therefore, in order to describe the dynamics of empty places, we consider the dual map $T^*: X^K_0 \to X^K_0 = (X^K_0)^*$ whose action is defined by the relation $T^*X = (TX^*)^*$ and can be written as follows.

**Lemma 2.3.** For $X \in X^K_0$ we have

$$T^*X(x) = X(x) - \min\{X(x), X^*(x - 1)\} + \min\{X(x + 1), X^*(x)\}. \hfill \square$$

Observe that the dynamics of empty places is exactly the same as the dynamics of particles, but occurs in the opposite direction. Obviously, both the above formula and relation (2.1) describe the mass conservation rule: the number of particles at a given site in the new configuration is equal to the number of particles at the same site in the original configuration minus the number of particles leaving it and plus the number of those coming to this site.

By a jammed cluster (of particles) we shall mean a locally maximal group of consecutive sites on the lattice containing at least one jammed particle at each site. Respectively, the jammed cluster in the dual configuration defines a cluster of free empty places in the original configuration. The locally maximal property means that any enlarging of the considered group contradicts the definition, i.e., both immediate neighboring sites to the cluster do not contain jammed particles.

Consider two subspaces of the space of configurations $X^K_0$; they shall play an important role in our analysis. The first of them is the space of configurations of free particles:

$$\text{Free}(K) := \{X \in X^K_0 : v(\xi) = 1 \ \forall \xi \in X\},$$
and the second one is the space of (spacial) $n$-periodic configurations:

$$\text{Per}(n, K) := \{X \in X^n_K : X(x) = X(x + n) \forall x \in \mathbb{Z}^1\}.$$ 

A trivial calculation shows that both of these spaces are invariant with respect to the dynamics.

**Lemma 2.4.** We have $T : \text{Free}(K) \to \text{Free}(K), \ T : \text{Per}(n, K) \to \text{Per}(n, K)$ for any $n$ and $K \in \mathbb{Z}_+$. 

**Proof.** Observe that the restriction of the map $T$ to the space of configurations of free particles is equivalent to the shift operator in this space, from which the first statement follows immediately. To prove the second statement, note that according to formula (2.1),

$$TX(x) = X(x) + \min\{X(x - 1), K - X(x)\} - \min\{X(x), K - X(x + 1)\}$$

$$= X(x + n) + \min\{X(x - 1 + n), K - X(x + n)\}$$

$$- \min\{X(x + n), K - X(x + 1 + n)\}$$

$$= TX(x + n)$$

due to the $n$-periodicity of the configuration $X \in \text{Per}(n, K)$. 

Denote by $X = \langle \alpha \rangle$ the $n$-periodic configuration $X \in \text{Per}(n, K)$ consisting of a periodically repeated word $\alpha = a_1a_2\ldots a_n$ with $a_i \in \{0, 1, \ldots, K\}$ and such that $X(1) = a_1$, for example $X = \langle 1234 \rangle = \ldots 123412341234 \ldots$.

It is worth noting that despite the statement of the previous Lemma, the minimal period of the configuration may not be preserved under the dynamics. Indeed, consider the 4-periodic configuration $\langle 1100 \rangle$ and observe that for $K = 1$ we have $T(\langle 1100 \rangle) = \langle 1010 \rangle \in \text{Per}(2, 1)$.

To deal with more general and still statistically homogeneous configurations, in the next section we introduce a more interesting subset of configurations — regular configurations, for which as we shall show the statistical description makes sense.

**3. Space of regular configurations**

For a configuration $X \in X^n_K$, we define the notion of subconfiguration, namely $X^n_k := \{X(k), X(k + 1), \ldots, X(n)\}$, i.e., a collection of entries of $X$ between the pair of given indices $k < n$, and introduce the corresponding density and the average velocity:

$$\rho(X^n_k) := \frac{m(X^n_k)}{n - k + 1}, \quad V(X^n_k) := \frac{1}{m(X^n_{k-1})} \sum_{x=k}^{n-1} v(X, x),$$

where $m(X^n_k) := \sum_{x=k}^{n} X(x)$ denotes the number of particles in the subconfiguration $X^n_k$. 

1Observe that in the definition of the average velocity we consider only particles from sites up to $n - 1$. This is related to the fact that the velocities of particles at the site $n$ are not defined by the subconfiguration $X^n_k$. 
By the density and the average velocity (of particles) of an entire configuration $X \in X^K_0$ we mean the following limits (if they exist):

$$\rho(X) := \lim_{n \to \infty} \rho(X_n), \quad V(X) := \lim_{n \to \infty} V(X_n),$$

otherwise we can consider the corresponding upper and lower limits, which we denote by $\rho_{\pm}(X)$ and $V_{\pm}(X)$.

Note that both these quantities are well defined in the case of space periodic configurations (belonging to $\text{Per}(K)$) in contrast even to the simplest case, in which a configuration $X$ consists of free particles (i.e., belongs to $\text{Free}(K)$). Thus, in the general case the important statistical quantities $\rho(X), V(X)$ may be not well defined. To be able to deal with the space of configurations satisfying a reasonable statistical description, we introduce the following space of configurations.

We shall say that a configuration $X$ satisfies the regularity assumption (or simply is regular) if there exists a number $\rho \in [0, K]$ and a strictly monotone function $\varphi(n) \to 0$ as $n \to \infty$ (which we call the rate function) such that for any $n \in \mathbb{Z}^1$, $N \in \mathbb{Z}^1^+$ and any subconfiguration $X_{n+1}^{n+N}$ of length $N$ the number of particles in this subconfiguration $m(X_{n+1}^{n+N})$ satisfies the inequality

$$m(X_{n+1}^{n+N}) - \rho \leq \varphi(N). \quad (3.1)$$

It is clear that for a configuration $X$ satisfying the regularity assumption, the density of particles $\rho(X)$ is well defined and is equal to the value $\rho$ in the assumption. The space of configurations from $X^K_0$ satisfying the regularity assumption with density $\rho$ and the rate function $\varphi$ is denoted by $\text{Reg}(\rho, \varphi, K)$.

The main result of the paper formulated below describes the restriction of the dynamics to the space of regular configurations and will be proved in the rest of this section and the next one.

**Theorem 3.1.** Let the initial configuration $X \in \text{Reg}(\rho, \varphi)$ with $\rho \neq K/2$. Then after a finite number of iterations

$$t \leq t_c = t_c(\rho, \varphi) := \frac{1}{4} \left( \frac{K}{2} - \rho \right)^2$$

for the configuration $T^4 X$, the average velocity of particles becomes well defined and is equal to $\min\{1, K/\rho - 1\}$. Moreover, for any $t \geq t_c$ we have

$$T^4 X \in \text{Free}(K) \quad \text{if } \rho < K/2,$$

$$T^4 X^* \in \text{Free}(K) \quad \text{if } \rho > K/2.$$

To analyze the properties of regular configurations, we introduce the binary relation domination, which we denote by $\vdash$, on the set of configurations $X^K_0$ as follows: $X \vdash Y$ if and only if for any $n \in \mathbb{Z}$, $N \in \mathbb{Z}^+$ there exists a pair $n_-, n_+ \in \mathbb{Z}$ such that

$$m(X_{n-1}^{n+N}) \leq m(Y_{n+1}^{n+N}) \leq m(X_{n+1}^{n+N}).$$

**Lemma 3.1.** The relation $\vdash$ is an order relation, i.e., it is reflexive and transitive, but not symmetric.
Proof. The proof of reflexivity, i.e., of the relation \( X \vdash X \) for any \( X \in X^K_0 \), is straightforward. To prove the second statement, consider a triple of configurations \( X \vdash Y \vdash Z \). By definition, for all \( n, k \in \mathbb{Z} \), \( N \in \mathbb{Z}^+ \), we have

\[
\begin{align*}
m(X_{n-1}^{n+N}) &\leq m(Y_{n+1}^{n+N}) \leq m(X_{n+1}^{n+N}), \\
m(Y_{n-1}^{n+N}) &\leq m(Z_{n+1}^{n+N}) \leq m(Y_{n+1}^{n+N}).
\end{align*}
\]

Thus for any \( k, N \) there exist \( n_-, n_+, n'_-, n'_+ \in \mathbb{Z} \) such that

\[
m(X_{n-1}^{n+N}) \leq m(Y_{n-1}^{n+N}) \leq m(Z_{n+1}^{n+N}) \leq m(Y_{n+1}^{n+N}).
\]

Therefore \( X \vdash Z \). It remains to prove the absence of symmetry, i.e., that there exists a pair of configurations \( X \vdash Y \) such that the relation \( Y \vdash X \) does not hold. Let \( X(1) = 1 \), while \( X(x) = 0 \) for all \( x \neq 1 \), and let \( Y(x) = 0 \) for all \( x \in \mathbb{Z} \). Then clearly \( X \vdash Y \) but the opposite relation does not hold. \( \square \)

Lemma 3.2. Let \( X \vdash Y \) and \( X \in \text{Reg}(\rho, \varphi, K) \). Then \( Y \in \text{Reg}(\rho, \varphi, K) \) as well.

Proof. According to the definition for any \( n \) and \( N \) there exists a pair \( n_-, n_+ \) such that

\[
m(X_{n-1}^{n+N}) \leq m(Y_{n-1}^{n+N}) \leq m(X_{n+1}^{n+N}).
\]

Thus

\[
\varphi(N) \leq \frac{m(X_{n-1}^{n+N})}{N} - \rho \leq \frac{m(Y_{n-1}^{n+N})}{N} - \rho \leq \frac{m(X_{n+1}^{n+N})}{N} - \rho \leq \varphi(N),
\]

which yields the desired statement. \( \square \)

Now we are ready to prove that the set of regular configurations is invariant under the dynamics.

Lemma 3.3. We have \( T: \text{Reg}(\rho, \varphi, K) \rightarrow \text{Reg}(\rho, \varphi, K) \) for any triple \( (\rho, \varphi, K) \).

Proof. Consider a configuration \( X \in \text{Reg}(\rho, \varphi, K) \). We need to show that the configuration \( TX \) also satisfies the same assumption. To do this we shall prove that \( X \vdash TX \), from which, by Lemma 3.2, we shall obtain the desired statement. Fix arbitrary integers \( n \in \mathbb{Z} \) and \( N \in \mathbb{Z}^+ \) and consider the subconfiguration \((TX)_{n+1}^{n+N}\). The number of particles in this subconfiguration differs from the number of particles in the subconfiguration \(X_{n+1}^{n+N}\) by the number of particles \( P_- \) coming from the site \( n \) to the site \( n + 1 \) and the number of particles \( P_+ \) coming from the site \( n + N \) to the site \( n + N + 1 \), i.e.,

\[
m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + P_- - P_+.
\]

There may be four possible situations:

(a) \( X(n) + X(n+1) \leq K \) and \( X(n+N) + X(n+N+1) \leq K \). Then \( P_- = X(n) \), \( P_+ = X(n+N) \), and thus

\[
m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + X(n) = X(n+N) = m(X_{n}^{n+N-1}).
\]
proximating (by the number of particles) those in $TX$ the statement of the lemma. Therefore by Per

$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + X(n) + K - X(n + N)$ $= m(X_{n}^{n+N-1}) + X(n + N) + X(n+N+1) - K > m(X_{n}^{n+N-1}).$

On the other hand,

$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + X(n) + K - X(n + N)$

(c) $X(n) + X(n + 1) > K$ and $X(n + N) + X(n + N + 1) \leq K$. Then we have $P_{-} = K - X(n + 1), P_{+} = X(n + N), and$

$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + K - X(n + 1) - X(n + N)$ $= m(X_{n}^{n+N-1}) - X(n) + X(n + N) + K - X(n + 1) - X(n + N)$ $> m(X_{n}^{n+N-1}).$

On the other hand,

$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + K - X(n + 1) - X(n + N)$

(d) $X(n) + X(n + 1) > K$ and $X(n + N) + X(n + N + 1) > K$. Then we have $P_{-} = K - X(n + 1), P_{+} = K - X(n + N + 1)$ and

$m((TX)_{n+1}^{n+N}) = m(X_{n+1}^{n+N}) + K - X(n + 1) - K + X(n + N + 1) = m(X_{n+1}^{n+N}).$

Therefore in all four possible cases we have found subconfigurations in $X$ approximating (by the number of particles) those in $TX$ from both sides, which yields the statement of the lemma. 

**Lemma 3.4.** We have $(\text{Reg}(\rho, \varphi, K))^* = \text{Reg}(K - \rho, \varphi, K)$.

**Proof.** Let $X \in \text{Reg}(\rho, \varphi, K)$. Then for any $n \in \mathbb{Z}^1$, $N \in \mathbb{Z}_1^1$, we have $m((X^*)_{n+1}^{n+N}) = m((\langle K \rangle - X)_{n+1}^{n+N}) = K \cdot N - m(X_{n+1}^{n+N}).$ Therefore

$$
\rho((X^*)_{n+1}^{n+N}) - (K - \rho) = \frac{m((X^*)_{n+1}^{n+N})}{N} - (K - \rho) = \frac{m(X_{n+1}^{n+N})}{N} - \rho \leq \varphi(N).
$$

Now consider the connection between spaces of periodic configurations and those of regular ones. Clearly, for any configuration $X \in \text{Per}(n, K)$ the notion of density $\rho(X)$ is well defined and $\rho(X) = m(X_{n}^{n})/n$. To specify the density, we denote by $\text{Per}_{\rho}(n, K)$ the set of configurations from $\text{Per}(n, K)$ having the same density $\rho$. 


Lemma 3.5. For any $\rho$, $n$ and $K$, we have

$$T : \text{Per}_\rho(n, K) \rightarrow \text{Per}_\rho(n, K), \quad \text{Per}_\rho(n, K) \subset \text{Reg} \left( \rho, \rho \left(1 - \frac{\rho}{K}\right) \frac{n}{N}, K \right).$$

Proof. For a given configuration $X \in \text{Per}(n, K)$ let $\rho : = \rho(X) = n(X^i)$. The first statement immediately follows from Lemma 2.4 and the fact that the number of particles on the period of the configuration cannot change under the dynamics. Now each positive integer $N$ can be represented as $N = kn + l$, where $k \in \{0, 1, \ldots\}$, $l \in \{0, 1, \ldots, n - 1\}$. For any $l \leq n - \frac{n}{K}$ the number of particles in the subconfiguration $X_{x+1}$ can be estimated from below

$$m(X_{x+1}^N) \geq \rho kn.$$

Therefore

$$\rho - \frac{m(X_{x+1}^N)}{N} \leq \rho - \frac{\rho kn}{kn + l} = \rho l = \left(1 - \frac{\rho}{K}\right) \frac{n}{N} =: \varphi(N).$$

Otherwise, if $l > n - \frac{n}{K}$, we have

$$m(X_{x+1}^N) \geq \rho kn + K(l - n + \frac{\rho}{K}n) = \rho kn + Kl - Kn + \rho n.$$

Thus

$$\rho - \frac{m(X_{x+1}^N)}{N} \leq \rho - \frac{\rho kn + Kl - Kn + \rho n}{kn + l} = \frac{1}{N}(n - l)(K - \rho) < \frac{1}{N} \frac{mn}{K}(K - \rho) = \varphi(N).$$

Now we shall use estimates for the number of particles in the subconfiguration $X_{x+1}^N$ from above. If $l \leq \frac{n}{K}$, then

$$m(X_{x+1}^N) \leq \rho kn + Kl,$$

$$\frac{m(X_{x+1}^N)}{N} - \rho \leq \frac{\rho kn + Kl}{kn + l} - \rho = \frac{1}{N}(K - \rho)l \leq \frac{1}{N} (K - \rho) \frac{n}{K} = \varphi(N).$$

Otherwise, if $l > \frac{n}{K}$, then

$$m(X_{x+1}^N) \leq \rho kn + \rho n,$$

$$\frac{m(X_{x+1}^N)}{N} - \rho \leq \frac{\rho kn + \rho n}{kn + l} - \rho = \frac{1}{N} \rho(n - l) < \frac{1}{N} \rho n = \frac{1}{N} \rho (n - \frac{n}{K}) = \frac{1}{N} mn(1 - \frac{\rho}{K}) = \varphi(N).$$

One can easily check that for any (spatial) periodic configuration the notion of average velocity is well defined. It is of interest that for more general classes of regular configurations this is not the case even for $K = 1$. Denote $a = 1100$ and $b = 1010$ and consider the configuration $X$ constructed as follows:

$$\ldots bbbbaaa bbaa ba ab aabb aaaa bbbb \ldots,$$

i.e., $X^a_8 = ab$, $X^0_7 = ba$, $X^2_4 = aabb$, $X^{-2}_3 = bbaa$, etc. Note that in each subsequent series the number of consecutive elements $aa \ldots a$ and $bb \ldots b$ doubles.
Lemma 3.6. The configuration $X$ defined by (3.2) is regular, i.e., we have $X \in \text{Reg}(1/2, 1/N, K)$, but the average velocity is not well defined.

Proof. Observe that $m(X_{i+j}^{i+4j}) = 2k$ for any $i, k$, while $2k \leq m(X_{i+j}^{i+4j}) \leq 2k + 2$ for any $j \in \{1, 2, 3\}$. Therefore the configuration (3.2) is regular with density $1/2$ and rate function $\varphi(N) = 1/N$.

Now let us calculate the average velocity on various subconfigurations. First consider the subconfiguration starting from the first element and containing the full series $aa...bb$, i.e., $X_1^{2(k+1)-1}$. This subconfiguration for any $k$ contains the same number of elements $a$ and $b$, and hence

$$V(X_1^{2(k+1)-1}) = \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 1 = \frac{3}{4}.$$  

Similarly, due to the symmetry of the configuration $X$, we have $V(X_{-2}^{2(k+1)-1}+1) = 3/4$ and therefore $|V(X_{-2}^{2(k+1)+1}) - 3/4| < 5 \cdot 2^{-(k+3)}$. Thus

$$V(X_{-2(k+1)} ightarrow 3/4 \text{ as } k \rightarrow \infty.$$  

Another type of subconfiguration that we consider differs from the previous one by the fact that it contains an additional (full) series of elements $a...a$ at the end, i.e., $X_1^{3(2k+1)-1} = X_1^{3(2k+1)-2}$. We have

$$V(X_1^{3(2k+1)-2}) = \frac{1}{2} \cdot \frac{2k+1 + 3}{2(k+1) + 2(2k+1) - 1} = \frac{2 \cdot 2k+1 - \frac{3}{2}}{3 \cdot 2k+1 - 2} \rightarrow \frac{2}{3}$$  

as $k \rightarrow \infty$. Therefore using the symmetry of the configuration $X$ again, we see that

$$V(X_{-3}^{2(k+1)} \rightarrow 2/3 \text{ as } k \rightarrow \infty,$$

and thus different subsequences of $k$ lead to different average velocities. □

4. Traffic jams and simple properties of the dynamics

Recall that in a configuration $X$, the sites between $x'$ and $x''$ belong to a jammed cluster if for any integer $x \in [x', x'']$ the inequality $X(x) + X(x + 1) > K$ holds. Similarly, consecutive sites for which this inequality does not hold belong to a free cluster. The site $x$ is called free if $X(x) + X(x + 1) \leq K$.

Lemma 4.1. For each configuration $X \in X_0^K$ and for each site $x \in \mathbb{Z}^1$, we have

$$\min\{X(x-1), X(x), X(x+1)\} \leq (TX)(x) \leq \max\{X(x-1), X(x), X(x+1)\}$$  

and thus for any $t \in \mathbb{Z}^1$,

$$\min_{x} \{X(x)\} \leq \min_{x} \{(TX)(x)\} \leq \max_{x} \{(TX)(x)\} = \max_{x} \{X(x)\}.$$  

Moreover, the upper and lower limits may be not preserved under the dynamics:

- $\exists X \in X_0^K$ such that $\max_e \{(TX)(x)\} < \max_e \{X(x)\}$;
- $\exists X \in X_0^K$ such that $\min_e \{(TX)(x)\} > \min_e \{X(x)\}$;
Proof. Fix a configuration $X \in \mathbf{X}_0^K$ and a site $x \in \mathbb{Z}_1^1$. Let
\[ P(x) := \min\{X(x-), K - X(x)\} \]
be the number of particles moving to the site $x$ (from the site $x-1$) under the action of the map $T$; then we have
\[ (TX)(x) = X(x) + P(x) - P(x+1). \]
Consider all four possibilities:
(a) $X(x-1) \leq K - X(x)$ and $X(x) \leq K - X(x+1)$. Then
\[ (TX)(x) = X(x) + X(x-1) - X(x) = X(x-1). \]
(b) $X(x-1) > K - X(x)$ and $X(x) \leq K - X(x+1)$. Then
\[ (TX)(x) = X(x) + (K - X(x)) - X(x) = K - X(x) < X(x-1). \]
On the other hand, in this case
\[ (TX)(x) = K - X(x) \geq X(x+1). \]
(c) $X(x-1) \leq K - X(x)$ and $X(x) > K - X(x+1)$. Then
\[ (TX)(x) = X(x) + X(x-1) - (K - X(x+1)) = X(x-1) - (K - X(x)) + X(x+1) \leq X(x+1). \]
On the other hand, in this case
\[ (TX)(x) = X(x-1) + X(x) - (K - X(x+1)) > X(x-1) + X(x) - X(x) = X(x-1). \]
(d) $X(x-1) > K - X(x)$ and $X(x) > K - X(x+1)$. Then
\[ (TX)(x) = X(x) + (K - X(x)) - (K - X(x+1)) = X(x+1). \]
So the first statement of the lemma holds in all cases.

It remains to construct examples of configurations satisfying the last two statements of the lemma. Let $K = 2$. Then we have
\[ T: (221022) \rightarrow (211122), \quad T: (002100) \rightarrow (001110). \]
In the first example, the minimal value 0 becomes 1, while in the second example the maximal value 2 becomes 1 under the action of the dynamics. \qed

Introduce a map marking global maxima of a configuration $M: \mathbf{X}_0^K \rightarrow \mathbf{X}_1^K$ as follows: $MX(x) := 1$ if $X(x) = \max_y \{X(y)\}$ and $MX(x) := 0$ otherwise. We also define arithmetic operations with configurations $X, Y \in \mathbf{X}_0^K$:
\[ (X + Y)(x) := \min\{X(x) + Y(x), K\}, \quad (X - Y)(x) := \max\{X(x) - Y(x), 0\}. \]
Using this notation, we can formulate the following decomposition result.

Lemma 4.2. If for a given $X \in \mathbf{X}_1^K$ we have $X(x-1) + X(x) \leq K$ and $X(x) + X(x+1) \leq K$ for all $x \in \mathbb{Z}_1^1$ such that $MX(x) = 1$, then
\[ TX = T(X - MX) + T(MX), \]
otherwise
\[ TX = T|_{K-1}(X - MX) + T]_1 MX, \]
where \( T|_{K-1} \) stands for the restriction of the map \( T \) to \( X_0^{K-1} \). On the other hand, there exists a configuration \( X \in X^K_0 \) such that 
\[
(T^t X)(x) \neq (T|_{K-1}^t X)(x)
\]
even if \( \max_x\{X(x)\} < K \).

**Proof.** The statement about the decomposition follows immediately from the definition of the dynamics, while the following example proves the second statement: 
\[
T|_3: (1221) \rightarrow (1212), \quad T|_2: (1221) \rightarrow (2211).
\]

These results demonstrate some rather counterintuitive properties of the model of traffic flow under consideration. For example, from Lemma 4.1 it follows that if for a given initial configuration one traffic lane is not occupied (along the entire lattice), then this property holds for any moment of time. So it looks as if the dynamics will not change if the road will be made narrower by one lane. However this is completely wrong, which was shown by the second statement of Lemma 4.2.

Another example gives the following seemingly evident (but wrong) decomposition, which one would expect instead of the more complex decomposition described in Lemma 4.2. Assume that for a configuration \( X \) we have \( X(x) > 1 \) for all \( x \in \mathbb{Z}^1 \). Then it looks reasonable that the dynamics of the configuration, restricted to the lanes \( 2, 3, \ldots, K \) should be the same as in the original one, i.e.,
\[
T|_K X = T|_{K-1}(X - (1)) + T|_K((1)).
\]
The following example of a periodic configuration shows that this is not the case:
\[
T|_2(1221) = (2211), \quad T|_1(0110) + T|_2(1111) = (0101) + (1111) = (1212).
\]

**Lemma 4.3.** Let \( X_{k+n}^{k+1} \) be a jammed cluster of length \( n \) in the configuration \( X \). Then 
\[
(TX)(x) = X(x + 1) \quad \forall x \in \{k + 2, \ldots, k + n\},
\]
\[
(TX)(k + n + 1) = K - X(k + n + 1),
\]
\[
(TX)(k + 1) = X(k) + X(k + 1) + X(k + 2) - K
\]
and if the site \( k - 1 \) does not belong to another jammed cluster, then
\[
(TX)(k - 1) = X(k - 2), \quad (TX)(k) = X(k - 1),
\]
otherwise
\[
(TX)(k - 1) = X(k), \quad (TX)(k) = K - X(k).
\]

**Proof.** First let us show that \( TX(x) = X(x + 1) \) for all \( x \in \{k + 2, \ldots, k + n\} \). Observe that by the definition of a jammed cluster we have 
\[
X(x - 1) + X(x) > K, \quad X(x) + X(x + 1) > K.
\]
Thus 
\[
X(x - 1) > K - X(x), \quad X(x) > K - X(x + 1).
\]
Therefore, after the application of the map \( T \) exactly \( K - X(x) \) particles come to the site \( x \) from the site \( x - 1 \), while \( K - X(x + 1) \) particles leave it. Therefore 
\[
TX(x) = K - X(x) + X(x) - (K - X(x + 1)) = X(x + 1),
\]
which proves equality (4.1). Observe that this equality makes sense only if \( n \geq 2 \).

The site \( k + n + 1 \) is the first free site after the jammed cluster. Therefore all particles from it move to the site \( k + n + 2 \) under the action of the map \( T \), while exactly \( K - X(k + n + 1) \) particles move to the site \( k + n + 1 \) from the last site of the considered jammed cluster. This gives equality (4.2). Notice that this inequality implies

\[
(TX)(k + n) + (TX)(k + n + 1) = K,
\]

i.e., the site \( k + n \) does not belong to a jammed cluster in the configuration \( TX \).

Clearly the site \( k \) cannot belong to another jammed cluster, otherwise the site \( k + 1 \) would not be the first site of the considered jammed cluster.

By definition, under the action of the map \( T \) all particles from the site \( k \) move to the site \( k + 1 \), so that exactly \( X(k + 2) - K \) particles move to the site \( k + 2 \). Thus we get equality (4.3).

Now consider the case in which the site \( k - 1 \) does not belong to another jammed cluster, i.e., \( X(k - 1) + X(k) \leq K \). This immediately gives formulas (4.4) for the number of particles at the sites \( k - 1 \) and \( k \).

If \( n = 1 \), then

\[
(TX)(k + 1) + (TX)(k + 2) = X(k) + X(k + 1) + X(k + 2) - K + K - X(k + 2) = X(k) + X(k + 1) \leq K.
\]

If \( n \geq 2 \), then

\[
(TX)(k + i) + (TX)(k + i + 1) = X(k + i - 1) + X(k + i)
\]

for all \( i \in \{1, \ldots, n - 2\} \). Thus

\[
(TX)(k + n - 1) + (TX)(k + n) = X(k + n) + X(k + n + 1) > K,
\]

\[
(TX)(k + n) + (TX)(k + n + 1) = X(k + n + 1) + K - X(k + n + 1) = K.
\]

Therefore in both cases the site which was the last one in the jammed cluster \( X_{k+1}^{k+n+1} \) becomes free, and if \( n \geq 2 \) the site \( k + n - 1 \) turns out to be the last site in the jammed cluster in \( TX \).

Summarizing for the case in which the site \( k - 1 \) is free, we obtain the following representation for \( TX \) in the neighborhood of the considered jammed cluster:

\[
X = \ldots X(k-1) X(k) [ X(k+1) X(k+2) X(k+3) \ldots \]
\[
TX = \ldots X(k-2) X(k-1) (TX)(k+1) X(k+3) X(k+4) \ldots \]

\[
X = \ldots \ldots \ldots \ldots X(k+n-1) X(k+n) [ X(k+n+1) \ldots \]
\[
TX = \ldots \ldots \ldots \ldots X(k+n) [ X(k+n+1) K - X(k+n+1) \ldots
\]

By square brackets we denote the boundaries of jammed clusters. Observe that the first site of the new cluster might be either \( k \) or \( k + 1 \).
In the alternative case, when the site \( k - 1 \) is the last site of the previous jammed cluster, the representation differs only at the sites \( k - 1 \) and \( k \):

\[
\begin{array}{cccccc}
\ldots & k-1 & k & k+1 & k+2 & k+3 \ldots \\
X &=& \ldots X(k-1) & X(k) & X(k+1) & X(k+2) & X(k+3) & \ldots \\
TX &=& \ldots X(k) & [K - X(k)] & (TX)(k+1) & X(k+3) & X(k+4) & \ldots
\end{array}
\]

Clearly, in this case the site \( k \) does not belong to another jammed cluster (i.e., \( k \) is the last site of the jammed cluster). Therefore, in the worst case the new jammed cluster has the same length and is located at the sites from \( k \) to \( k + n - 1 \):

\[
X = \ldots X(k-1) \quad X(k) \quad X(k+1) \quad X(k+2) \quad X(k+3) \quad \ldots
\]

\[
TX = \ldots X(k) \quad [K - X(k)] \quad (TX)(k+1) \quad X(k+3) \quad X(k+4) \quad \ldots
\]

Indeed, applying the first statement of the lemma (which has been already proved) to the previous cluster, we obtain \((TX)(k-1) = X(k)\). On the other hand, the number of particles moving from the site \( k - 1 \) to the site \( k \) is equal to \( K - X(k) \), while all the particles that were at the site \( k \) move to the site \( k + 1 \) (since the site \( k \) does not belong to a jammed cluster). Therefore,

\[
(TX)(k) + (TX)(k + 1) = K - X(k) + X(k) + X(k + 1) + X(k + 2) - K
\]

\[
= X(k + 1) + X(k + 2) > K;
\]

hence the new jammed cluster has the same length and is located at the sites from \( k \) to \( k + n - 1 \).

**Corollary 4.4.** For each jammed cluster \((TX)^{k+n+1}_{k+1}\) of length \( n > 1 \) we have \((TX)(k + n + 2) + (TX)(k + n + 3) = K\).

**Proof.** Immediately follows from (4.2).

This implies that the distance between two consecutive jammed clusters is at least 2.

**Lemma 4.5.** Let \( X^k_{k+1} \) be a jammed cluster of length \( n \) in the configuration \( X \). Then neither its length, nor the number of particles in it, can increase under the dynamics. Moreover, if \( X(k - 1) + X(k) < K \), then the number of particles in the jammed cluster decreases at least by \( K - (X(k - 1) + X(k)) \) after the application of the map.

**Proof.** First consider the case in which the site \( k - 1 \) does not belong to another jammed cluster, i.e.,

\[
X(k - 1) + X(k) \leq K, \quad X(k) + X(k + 1) \leq K.
\]

Clearly, in this case the site \( k - 1 \) cannot be the first site of the jammed cluster in the configuration \( TX \). Therefore, in the worst case the jammed cluster is located at sites from \( k \) to \( k + n - 1 \), i.e., its length is not larger than that of the considered one. Applying Lemma 4.3, we can estimate the difference between the number of particles in the new cluster and the old one from above as follows:

\[
[(TX)(k) + (TX)(k + 1)] - [X(k + 1) + X(k + 2)]
\]

\[
= X(k - 1) + X(k) + X(k + 1) + X(k + 2) - K - X(k + 1) - X(k + 2)
\]

\[
= X(k - 1) + X(k) - K \leq 0.
\]
Hence the number of particles in this case cannot increase, moreover, it decreases if $X(k - 1) + X(k) < K$.

It remains to consider the case when the jammed cluster under consideration is immediately preceded by another jammed cluster. Again by Lemma 4.3,

$$(TX)(k - 1) + (TX)(k) = X(k) + K - X(k) = K.$$ 

Hence the site $k - 1$ does not belong to the jammed cluster. On the other hand,

$$(TX)(k) + (TX)(k + 1) = K - X(k) + X(k + 1) + X(k + 2) - K = X(k + 1) + X(k + 2) > K,$$

since the site $k + 1$ does belong to the jammed cluster. Thus the site $k + 1$ is the first site of the jammed cluster in the configuration $TX$ that lies from $k$ to $k + n$, i.e., its length is exactly the same as of the old one. Applying the same trick as that above to calculate the difference between the number of particles in the new cluster and the old one, we obtain:

$$(TX)(k) + (TX)(k + 1) - (X(k + 1) + X(k + 2)) = K - X(k) + X(k + 1) + X(k + 2) - K - X(k + 1) - X(k + 2) = 0.$$ 

Therefore, even the number of particles in the jammed cluster is preserved in this case. □

**Lemma 4.6.** Let $n \in \mathbb{Z}^+_\nu$, $\rho(X_{k+1}^{k+2n+1}) \leq K/2$, and suppose that in the subconfiguration $X_{k+1}^{k+2n+1}$ there exists at least one jammed site. Then there is an integer $i \in \{1, \ldots, 2n - 1\}$ such that $X(k - i) + X(k + i + 1) < K$.

**Proof.** Assume that this statement does not hold. Then for any two consecutive sites $x$ and $x + 1$ in this subconfiguration, we have $X(x) + X(x + 1) \geq K$. On the other hand, for the jammed site $y$ we have $X(x) + X(x + 1) \geq K + 1$. Thus

$$\sum_{x=1}^{2n} X(k + x) \geq n + 1,$$

which contradicts the fact that the density is less than or equal to 1/2. □

These results yield the following property: For any given subconfiguration the number of particles in any jammed cluster completely contained in this subconfiguration is a nonincreasing function of time and achieves its lowest possible level under the dynamics.

**Lemma 4.7.** Let $X \in \text{Reg}(\rho, \varphi, K)$ with density $\rho < K/2$. Then after at most $t_c = t_c(\rho, \varphi) = \frac{1}{4} \left( \frac{K}{\varphi} - \rho \right)^2$ iterations all particles in $T^t X$ for $t \geq t_c$ will become free.
\begin{center}
\begin{tabular}{|c|c|c|c|}
\hline
\(t\) & \(X\) & \(V(X)\) & \(X\) & \(V(X)\) \\
\hline
0 & \(\{21200\}\) & \(3/5\) & \(\{22300\}\) & \(4/7\) \\
1 & \(\{12020\}\) & \(4/5\) & \(\{13030\}\) & \(6/7\) \\
2 & \(\{10202\}\) & \(4/5\) & \(\{10303\}\) & \(6/7\) \\
3 & \(\{11021\}\) & \(4/5\) & \(\{21031\}\) & \(6/7\) \\
4 & \(\{11111\}\) & 1 & \(\{12112\}\) & 1 \\
\hline
\end{tabular}
\end{center}

\textbf{Figure 1.} Two examples of the dynamics of \(n\)-periodic configurations: (a) \(K = 2, n = 5, \rho = 1 = K/2\), (b) \(K = 3, n = 5, \rho = 7/5 < K/2 = 3/2\).

\textbf{Proof.} According to the definition of a regular configuration, the maximal number \(M(n)\) of particles in length \(n\) subconfigurations of the configuration \(X\) for each \(n \in \mathbb{Z}_+\) satisfies the inequality

\[
\frac{M(n)}{n} \leq \rho + \varphi(n).
\]

Thus, for any \(n > N_c := \varphi^{-1}(K/2 - \rho)\) it follows that \(M(n) < n/(2K)\). By Lemma 4.6, in each subconfiguration of length \(n\) there is a pair of consecutive sites whose total number of particles \(Q\) is strictly less than \(K\). Consider the dynamics of this pair of sites. According to our previous results, while the site ahead of them is free, these two sites will simply move forward by one position. In the opposite case, when the next site is the first site of some jammed cluster, Lemma 4.5 implies that the number of particles in this cluster will decrease by \(K - Q > 0\). On the other hand, free particles and jammed clusters move in opposite directions, each with velocity 1. Thus, the maximum time between the consecutive meetings of a jammed cluster and a pair of consequent sites with total number of particles less than \(K\) does not exceed \(n/2\). Let \(n\) be the smallest integer greater than or equal to \(N_c\). Since after each such meeting the number of particles in the corresponding jammed cluster decreases at least by 1 and since the number of particles in this cluster is less than or equal to \(M(n)\), we obtain the following upper estimate of the transient period:

\[
t_c \leq \frac{M(n) \cdot n}{2} \leq \frac{n^2}{4K} \leq \frac{1}{4} \left(\varphi^{-1}(\frac{K}{2} - \rho) + 1\right)^2.
\]

\textbf{Proof of Theorem 3.1.} After the preparation made in Lemmas 3.2–4.7 we are able to finish the proof of our main result. Indeed, in the case of a regular configuration \(X \in \text{Reg}(\rho, \varphi, K)\) with density \(\rho < K/2\), Lemma 4.7 implies that for any integer \(t \geq t_c\) the configuration \(T^tX\) consists of free particles only. In the opposite case, when \(\rho > K/2\), we consider the dual configuration \(X^* \in \text{Reg}(K - \rho, \varphi, K)\) (by Lemma 3.4) and since the action of the dual map is equivalent to the main one but proceeds in the opposite direction, we obtain \(T^tX^* \in \text{Free}(K)\).

It remains to prove the statement about the average velocity of the configuration \(T^tX\) for each \(t \geq t_c\). Again we start with the case of low density \(\rho < K/2\). Since
the configuration $T^t X$ consists of free particles, the velocity of each particle is equal to 1. Thus

$$V((T^t X)^n_{-n}) \equiv 1 \quad \forall n \in \mathbb{Z}_+^1,$$

which shows both that the average velocity is well defined and that $V(T^t X) = 1$. Now if $\rho > K/2$, the configuration dual to the configuration $Y := T^t X$ again consists of free particles. Hence

$$V(Y^n_n) = \frac{1}{m(Y^n_{-n})} \sum_{x=-n}^{n-1} v(Y, x) = \frac{m((Y^*)^n_{-n})}{2nK} \rightarrow \frac{K}{\rho} - 1$$

as $n \to \infty$, since the density of the configuration $Y^*$ is equal to $K - \rho$. □

Observe that in the proof of Theorem 3.1 we actually derived an estimate of the length of the transient period:

$$t \leq t_c = t_c(\rho, \varphi) := \frac{1}{4} \left( \varphi^{-1} \left( \frac{K}{2} - \rho \right) + 1 \right)^2,$$

which goes to infinity as $\rho \to 1/2$. This is the reason why Theorem 3.1 does not cover the boundary case $\rho = 1/2$, which we discuss below.

**Theorem 4.1.** Let the initial configuration $X \in \text{Reg}(K/2, \varphi, K)$ and let $x'(t) < x''(t)$ be positions of two fixed arbitrary particles at an arbitrary moment $t$. Then the average velocity of the subconfiguration $X^{x''(t)}_{x'(t)}$ converges to 1 as $t \to \infty$.

**Proof.** Denoting $\rho := K/2$ and choosing a positive integer $M$, we consider the configuration $-M X$ obtained from the configuration $X \in \text{Reg}(\rho, \varphi, K)$ by the following operation: for each integer $k$ we remove from the configuration $X$ the closest particle from behind the position $kM$. For a given positive integer $M$, any integer $N$ can be written as

$$N = kM + l \quad \text{with} \quad l \in \{-M, -M + 1, \ldots, M - 1, M\} \quad \text{and} \quad k \in \mathbb{Z}^1.$$

Then

$$m(-M X_{n+1}^{n+kM+l}) = m(X_{n+1}^{n+kM+l}) - k$$

and thus

$$\left| \frac{m(-M X_{n+1}^{n+N})}{N} - \left( \rho - \frac{1}{M} \right) \right| = \left| \frac{m(X_{n+1}^{n+N})}{N} - \left( \rho - \frac{k}{N} \right) \right|. $$

On the other hand,

$$\left| \frac{k}{N} - \frac{1}{M} \right| = \left| \frac{k}{kM + l} - \frac{1}{M} \right| = \frac{l}{(kM + l)M} < \frac{1}{N}. $$

Therefore, $-M X \in \text{Reg}(\rho - 1/M, \varphi + 1/N, K)$, and according to Theorem 3.1 after a finite number of iterations $t_c$ the average velocity of the configuration $T^{t_c}(-M X)$ becomes equal to 1 (since all the particles in this configurations are free).

Performing the opposite operation, namely inserting a particle in the configuration $X$ into the empty position closest from behind $kM$ for each integer $k$, we obtain another regular configuration $+M X \in \text{Reg}(\rho + 1/M, \varphi + 1/N, K)$. Again,
by Theorem 3.1, after a finite number of iterations the average velocity of this configuration becomes equal to

\[
\frac{K}{\rho - \frac{1}{\pi}} - 1 = 1 + \frac{4}{KM - 2} \to 1 \quad \text{as} \quad M \to \infty.
\]

Thus both (arbitrary close as \( M \to \infty \)) approximations \( \pm M X \) to the configuration \( X \) have average velocities deviating from 1 by \( O(1/M) \) after a finite number of iterations (depending on \( M \)). It remains to show that the average velocity of a subconfiguration of the configuration \( X \) can be estimated from above and from below by those from the above approximations. Let \( X \) and \( Y \) be two configurations such that \( X(x) \leq Y(x) \) for all \( x \) and let \( x'(t) < x''(t) \) be the positions of two fixed particles in the configuration \( X \) at the arbitrary moment \( t \). Denote by \( y'(t) < y''(t) \) the positions of the same particles in the configuration \( Y \). Then

\[
V(X_{x''(t)}) \geq V(X_{x'(t)})
\]

for any moment of time \( t \). Indeed, the additional particles in the configuration \( Y \) present only obstacles to the motion of other particles, thus making the average velocity slower (or at least not faster). \( \Box \)

In the case of (space) \( n \)-periodic configurations, the numerical examples below demonstrate a much better estimate of the transient period: \( t_c = n - 1 \), but it is rather unclear if it is possible to generalize this result to more general regular configurations.

\[
\begin{array}{cccc}
\hline
 t & X & V(X) & X & V(X) \\
0 & (0414232) & 9/14 & (1204440) & 7/15 \\
1 & (0142313) & 10/14 & (0124404) & 9/15 \\
2 & (3123131) & 13/14 & (4034040) & 12/15 \\
3 & (3213131) & 13/14 & (0430404) & 12/15 \\
4 & (3222131) & 13/14 & (4313040) & 12/15 \\
5 & (2222213) & 13/14 & (3131304) & 12/15 \\
6 & (2222222) & 1 & (1313133) & 13/15 \\
7 & (2222222) & 1 & (3131331) & 13/15 \\
8 & (2222222) & 1 & (1313313) & 13/15 \\
9 & (2222222) & 1 & (3133131) & 13/15 \\
10 & (2222222) & 1 & (1331313) & 13/15 \\
11 & (2222222) & 1 & (3313131) & 13/15 \\
12 & (2222222) & 1 & (3131313) & 13/15 \\
\hline
\end{array}
\]

\( \text{(a)} \quad \text{(b)} \)

Figure 2. Long transient periods \( (t_c = n - 1) \) of \( n \)-periodic configurations with \( K = 4 \) and \( n = 7 \): (a) \( \rho = 2 = K/2 \),
(b) \( \rho = 15/7 > K/2 \).

Moreover, it turns out that even the upper (attainable) estimate of the length of the transient period for an initial configuration from \( \text{Per}_\rho(n, K) \) is not monotonous on the length of the period \( n \) and heavily depends on its parity. The above example
demonstrates the estimate $t_c \leq n - 1$ for odd values of $n$. Now, mainly following the ideas proposed in [15], we shall show that this estimate is rather different for even values of the period.

**Lemma 4.8.** Let $X \in \text{Per}_\rho(2n, K)$ for some $n \geq 1$. Then the length of the transient period $t_c \leq n$.

**Proof.** We introduce an operator $G$ mapping the space of configurations $X_0^K$ into the space of two-sided sequences of real numbers defined as follows:

$$GX(x) := \sum_{i=0}^{x-1} X(i) - (x - 1)K/2,$$

where we set $\sum_{i=0}^{x} = \sum_{i=-x}^{0}$ for any positive integer $j$.

One can easily show that for any $x \in \mathbb{Z}$,

$$X(x) = GX(x + 1) - GX(x) + K/2,$$

and if additionally $X \in \text{Per}_\rho(2n, K)$, then

$$GX(x + 2n) - GX(x) = \sum_{i=x}^{x+2n-1} (X(i) - K/2) = 2nK(\rho - 1/2) \quad (4.6)$$

and for any $x \in \mathbb{Z}$

$$G(TX)(x) = \max\{GX(x - 1), GX(x) - K/2, GX(x + 1)\}. \quad (4.7)$$

This yields

$$G(T^1X)(x) = \max\{\max\{GX(x - t), GX(x - t + 2), \ldots, GX(x + t)\},\max\{GX(x - t + 1), GX(x - t + 3), \ldots, GX(x + t + 1)\} - K/2\}.$$

Consider all three possibilities.

(a) Assume first that $\rho < K/2$. Then from equality (4.6) we obtain the inequality $GX(x + 2n) < GX(x)$ for each $x$. Then for $t \geq n$ we get

$$G(T^tX)(x) = \max\{\max\{GX(x - t), GX(x - t + 2), \ldots\},\max\{GX(x - t + 1), GX(x - t + 3), \ldots\} - K/2\}.$$

Thus $G(T^{t+1}X)(x) = G(T^tX)(x - 1)$ and $T^{t+1}X(x) = T^t(x - 1)$. Substituting these equalities into (4.7), we obtain

$$0 = \max\{0, G(T^tX)(x) - G(T^tX)(x - 1) - K/2, G(T^tX)(x + 1) - G(T^tX)(x - 1)\} = \max\{0, T^tX(x - 1) - K, T^tX(x - 1) + T^tX(x) - K\},$$

which implies $T^tX(x) \leq K - T^tX(x + 1)$ for any $t$ and $x$. Therefore, for $t \geq n$ all particles in the configuration $T^tX$ are free.

(b) $\rho = K/2$. Since in this case $GX(x + 2n) = GX(x)$ for all $x$, it follows that for $t \geq n$

$$G(T^tX)(x) = \max\{\max\{GX(2), GX(4), \ldots, GX(2n)\},\max\{GX(1), GX(3), \ldots, GX(2n - 1)\} - K/2\}.$$
if \( x - t \) is even, and

\[
G(T^iX)(x) = \max\{\max\{GX(1), GX(3), \ldots, GX(2n-1)\},
\max\{GX(2), GX(4), \ldots, GX(2n)\} - K/2\}
\]

otherwise. Hence \( G(T^{t+1}X)(x) = G(T^tX)(x) = T^tX(x \pm 1) \) for all \( x \in \mathbb{Z}^1 \), which yields \( T^tX \in \text{Free}(K) \).

(c) \( \rho > K/2 \). This case follows from the same argument as in case (a), since we can consider the dual configuration \( X^* \), for which the density of particles is less than \( K/2 \).

**Lemma 4.9.** Let \( X \in \text{Per}_\rho(2n + 1, K) \) for some \( n \geq 1 \). Then the length of the transient period satisfies the inequality \( t_c \leq 2n + 1 \).

**Proof.** Any configuration \( X \in \text{Per}_\rho(2n + 1, K) \) also belongs to \( \text{Per}_\rho(4n + 2, K) \). On the other hand, the number \( 4n + 2 \) is even and thus by Lemma 4.8 we obtain the desired estimate of the transient period. \( \square \)

It is interesting that in the spacial periodic case we can give more detailed information about the dynamics in time.

**Proposition 4.10.** For each configuration \( X \in \text{Per}(n, K) \) and any integer \( t \geq t_c = n \) the sequence \( \{T^tX(x)\} \) is \( n \)-periodic in \( t \) for each \( x \in \mathbb{Z}^1 \).

**Proof.** This result follows from the fact that for each \( t \) the configuration \( T^tX \) is \( n \)-periodic in space and by Lemmas 4.8 and 4.9 the length of the transient period \( t_c \leq n \). Thus, for \( t \geq t_c \) the configuration \( T^tX \) either consists of free particles (if \( \rho(X) \leq K/2 \)), or its dual satisfies this property. Therefore \( T^{t+t_c}X(x) = T^tX(x) \) for any \( x \in \mathbb{Z}^1 \). Note that this period in time might be not minimal (the latter can be as small as 2). \( \square \)

Observe that this construction heavily depends on the periodic space structure of the configurations, which means that this cannot be carried over to a more general situation.

5. **On the chaoticity of the dynamics**

In the previous sections it was shown that for sufficiently large time the dynamics occurs either in \( \text{Free}(K) \) or in \( (\text{Free}(K))^* \), i.e., the corresponding dual configurations belong to this space. Therefore to study asymptotic (as time goes to \( \infty \)) properties of the dynamics, we consider the restriction to the space of configurations of free particles (which contains the union of all attractors of the map \( T \) restricted to the set of regular configurations with density less than \( K/2 \)). The following result shows that this map is chaotic in the sense that its topological entropy (see the definitions in [12]) is positive.

**Theorem 5.1.** For any \( K \in \mathbb{Z}^1_+ \) we have

\[
h_{\text{top}}(T, \text{Free}(K)) = \ln \left( \frac{2(K + 1)}{\pi} + \frac{1}{\pi} + \frac{R(K)}{(K + 1)^2} \right) > 0
\]

where the remainder \( R(K) \) satisfies the inequality \( |R(K)| \leq 2 \).
Proof. All the particles in configurations on the largest (i.e., containing all others) attractor are free, i.e., \( X(x) + X(x + 1) \leq K \). Thus the action of the map is equivalent to the right shift map with upper triangular transition matrix (i.e., all elements in the first line are 1, all but the last one are 1’s in the 2nd line, etc.). It is well known (see, for example, [12]) that the logarithm of the largest eigenvalue of this matrix gives the topological entropy of the right shift map and thus the topological entropy \( h_{\text{top}}(T) \) of the traffic flow. Therefore the representation of the largest eigenvalue of the transition matrix which we shall give below finishes the proof. □

To simplify the notation, we write \( N := K + 1 \). Let \( A(N) = (a_{ij}) \) be the \( N \times N \) left triangular matrix, i.e., \( a_{ij} = 1 \) for all \( i + j \leq N + 1 \) and \( a_{ij} = 0 \) otherwise. This is a nonnegative symmetric matrix, therefore its spectrum belongs to the real line and its largest eigenvalue \( \lambda_{\text{max}}(A(N)) \) is positive.

\[ \lambda_{\text{max}}(A(N)) = \frac{2\pi}{N} + \frac{1}{\pi} + \frac{R}{N^2} \]

where the remainder satisfies the inequality \( |R| = |R(N)| \leq 2 \).

Proof. For an integrable function \( f \in L^2 \), consider the operator
\[ Lf(x) := \int_0^{1-x} f(s) \, ds. \]
According to [3], the eigenvalues of the operator \( L \) ordered by their absolute values are equal to
\[ \lambda_k := \frac{(-1)^{k+1}}{k - 1/2}, \]
while \( e(x) := \cos \frac{x\pi}{2} \) is the eigenfunction corresponding to the leading eigenvalue \( \lambda_1 \). For simplicity we shall use the notation \( A := A(N) \), \( \lambda := \lambda_1 = 2/\pi \), \( e_k := e(k/N) \), and \( \varepsilon = 1/N \). Now, since the function \( e(x) \) is analytical, decreases monotonically, and its second derivative satisfies the inequality \( |\frac{d^2e(x)}{dx^2}| < \frac{\pi^2}{4} \), it follows that for each \( k = 1, 2, \ldots, N \) we have
\begin{align*}
(Ae)_k &= \int_0^{1-k/N} e(s) \, ds + \frac{1}{2N} \sum_{i=0}^{N-k} \left( e\left( \frac{i}{N} \right) - e\left( \frac{i+1}{N} \right) \right) + \frac{R_1}{N^2} \quad (5.1)
\end{align*}
where the remainder satisfies \( |R_1| = |R_1(N)| \leq \frac{\pi^2}{16} < 1 \). Thus, introducing the operator \( Gf(x) := f(0) - f(1 - x) \), we rewrite the last equality as
\[ (Ae)_k = Le(k/N) + \frac{\varepsilon}{2} Ge(k/N) + R_1\varepsilon^2. \]

At this step our aim is to show that there exists a function \( g \in C^1[0, 1] \) orthogonal to \( e \) (meaning \( \int g \cdot e = 0 \)) such that
\begin{align*}
(L + \frac{\varepsilon}{2} G) \left( e + \frac{\varepsilon}{2} g \right) &= \left( \frac{2}{\pi} + \frac{\varepsilon}{\pi} \right) \left( e + \frac{\varepsilon}{2} g \right) + R_3\varepsilon^2, \quad (5.2)
\end{align*}
where again \( |R_3| \leq 1 \).
Since the operator $L$ is symmetric, the orthogonal complement to the function $e$ is invariant with respect to $L$. Thus, for some constant $\alpha$ and a bounded function $h \in C^1$ independent of $\varepsilon$ and orthogonal to $e$, we have

$$Ge = \alpha e + h.$$  

Let us calculate the constant $\alpha$. Since $h$ is orthogonal to $e$, multiplying both sides of the previous equality by $e$ and integrating (observe that $\int_0^1 e \cdot h = 0$) we obtain

$$\int_0^1 e(x) \cdot (e(0) - e(1 - x)) \, dx = \alpha \int_0^1 e^2(x) \, dx.$$  

On the other hand,

$$\int_0^1 \cos \frac{\pi x}{2} \, dx = \frac{1}{2}, \quad \int_0^1 \cos \frac{\pi x}{2} \, dx = \frac{2}{\pi},$$

$$\int_0^1 \cos \frac{\pi x}{2} \cos \frac{\pi(1 - x)}{2} \, dx = \frac{1}{2} \int_0^1 \sin \frac{\pi x}{x} = \frac{1}{\pi}.$$  

Thus

$$\frac{2}{\pi} - \frac{1}{\pi} = \frac{1}{2} \alpha,$$

wherefrom $\alpha = 2/\pi$.

Therefore, for any function $g \in C^0$ we have

$$\left(L + \frac{\varepsilon}{2} G\right) e + \frac{\varepsilon}{2} g = Le + \frac{\varepsilon}{2} Lg + \frac{\varepsilon}{2} Ge + \frac{\varepsilon^2}{4} Gg = \left(\frac{2}{\pi} + \frac{\varepsilon}{2} \frac{2}{\pi}\right) e + \frac{\varepsilon}{2} (h + Lg) + \frac{\varepsilon^2}{4} Gg.$$  

Comparing this relation with (5.2), we come to the conclusion that

$$h + Lg = \frac{2}{\pi} g \quad \text{i.e.,} \quad g = \left(L - \frac{2}{\pi}\right) h.$$  

Note that the right-hand side of the last expression makes sense, $h$ being orthogonal to $e$. Thus

$$g(x) = \left(L - \frac{2}{\pi}\right)^{-1} \left(e(0) - e(1 - x) - \frac{1}{\pi} e(x)\right).$$

From the first two leading eigenvalues, we see that the norm of the operator $L - 2/\pi$ restricted to the orthogonal complement to the leading eigenfunction can be estimated from above by $2/\pi - 2/3\pi = 2/3\pi$. Therefore

$$|g| \leq \frac{4}{3\pi} \left(1 - \frac{1}{\pi}\right), \quad \frac{1}{4} |Gg| \leq \frac{3\pi (1 + 1/\pi)}{4 \cdot 4} < 1.$$  

Combining the above estimates, we deduce that there exist two vectors $v, \xi \in \mathbb{R}^N$ such that

$$Av = \left(\frac{2}{\pi} N + \frac{1}{\pi}\right) v + \xi, \quad |\xi| \leq \frac{2}{N^2} |v|,$$

which yields the statement of the theorem by the following a posteriori matrix perturbation argument [17]: Let the equality $Av = \mu v + \xi$ be satisfied for a symmetric matrix $A$, two vectors $v, \xi \in \mathbb{R}^n$ with $||\xi|| \leq \varepsilon |v|$, and a scalar $\mu$. Then, the eigenvalue $\lambda$ of the matrix $A$ closest to $\mu$ satisfies the inequality $|\lambda - \mu| \leq \varepsilon$.  

Indeed, for \( \mu = \lambda \) the inequality becomes trivial, while otherwise
\[
\|v\| \leq \|(A - \mu I)^{-1}\| \cdot \|(A - \mu I)v\| = \frac{1}{|\mu - \lambda|} \cdot \|\xi\|.
\]
Thus \( |\mu - \lambda| \leq \|\xi\|/\|v\| \leq \varepsilon \). \( \square \)

As we have already mentioned, this result corresponds to the steady states of our model for the case of initial regular configurations with ‘low’ traffic. Note that in the opposite case of ‘high’ traffic \( \rho > K/2 \) each jammed configuration is in one-to-one correspondence with its dual one, for which \( \rho < K/2 \). Therefore a statement similar to Theorem 5.1 holds in this case as well, moreover
\[
h_{\text{top}}(T, \text{Free}(K) \cup (\text{Free}(K))^*) = h_{\text{top}}(T, \text{Free}(K)).
\]
Observe that for any positive integer \( K \) the topological entropy for the map \( T \) is strictly positive, which yields the chaoticity of the map.

6. Statistics of typical configurations

In this section we shall derive statistical information about typical configurations of particles. For configurations \( X \in \text{Free}(K) \) let us denote by \( S(n, K) \) the total number of different subconfigurations \( X^n \) of length \( n \in \mathbb{Z}_+ \).

**Lemma 6.1.** We have \( S(n, K) = \lambda_{\text{max}}^n(A(K + 1)) + o(\lambda_{\text{max}}^n(A(K + 1))) \), where \( A(N) \) is the \( N \times N \) left triangular matrix.

**Proof.** Denote by \( S_i(n, K) \) the number of subconfigurations of length \( n \) consisting of only free particles and starting with the symbol \( i \in \{0, 1, \ldots, K\} \). Then we have the following recurrence relation:
\[
S_i(n + 1, K) = \sum_{j=0}^{K-i} S_j(n, K).
\]
The number of particles in each site of a configuration \( X \) may vary from 0 to \( K \), i.e., it may take \( N := K + 1 \) different values, and the only additional relation that should be satisfied is
\[
X(x) + X(x + 1) \leq K \quad \forall x \in \{1, 2, \ldots, n\}.
\]
Therefore, these configurations are completely described by \( N \times N \) left triangular transition matrices \( A = A(N) \). Thus we get \( S(n, K) = \sum_{i=0}^{K} S_i(n, K) \), which together with Theorem 5.2 yields the statement of the lemma. \( \square \)

We shall say that a subconfiguration is **blocking** (non-blocking) if it contains (does not contain) the symbol \( K \). Then the number of non-blocking subconfigurations of length \( n \) is equal to \( S(n, K - 1) \). The fraction of blocking subconfigurations of length \( n \) is equal to
\[
\frac{S(n, N) - S(n, N - 1)}{S(n, N)} = 1 - \frac{S(n, N - 1)}{S(n, N)}.
\]
Applying the asymptotic representation of the leading eigenvalue of the matrix $A(N)$, we obtain the following estimate:

$$S(n, N) = \left( \frac{2}{\pi} N + \frac{1}{\pi} + o\left(\frac{1}{N}\right) \right)^n.$$ 

Therefore

$$\frac{S(n, N-1)}{S(n, N)} = \left( \frac{2}{\pi} (N-1) + \frac{1}{\pi} + o\left(\frac{1}{N}\right) \right)^n = \left( 1 - \frac{1}{N} + o\left(\frac{1}{N}\right) \right)^n.$$ 

To derive deeper information about the statistics, and to be able to deal with periodic configurations, one can use an approach based on Markov chain approximations.

For a given integer $i \in \{0, 1, \ldots, K\}$ and a configuration $X \in \text{Free}(K)$, denote by $\bar{\pi}_i(X^n)$ the fraction of sites $x \in \{1, 2, \ldots, n\}$ at which $X(x) = i$, i.e.,

$$\bar{\pi}_i(X^n) := \frac{1}{n} \#\{ x \in \{1, \ldots, n\} : X(x) = i \},$$

while $\bar{\pi}_i^{(n)}$ denotes the average of these fractions over all possible different subconfigurations of free particles of length $n$:

$$\bar{\pi}_i^{(n)} := \frac{\sum_{X \in \text{Free}(X)} \bar{\pi}_i(X^n)}{S(n, K)}.$$ 

Lemma 6.2. We have

$$\bar{\pi}_i^{(n)} \to \frac{K + 1 - i}{K + 1} \frac{2}{K + 2} \quad \text{as } n \to \infty.$$ 

Proof. Continuing the same argument as in the proof of Lemma 6.1, we see that each configuration can be regarded as the realization of a Markov chain with $N = K + 1$ states numbered as 0, 2, $\ldots$, $K$ and the following transition probabilities:

$$p_{ij} := \begin{cases} 
\frac{1}{K - i + 1} & \text{if } j \leq K - i, \\
0 & \text{otherwise}.
\end{cases}$$

Clearly the $N$-th power $P^N$ of the transition matrix $P = (p_{ij})$ is strictly positive and thus the Markov chain is ergodic. Denote by $\pi_i, i \in \{0, \ldots, K\}$, its stationary probabilities, i.e., the probability to have $X(x) = i$. Then these quantities must satisfy the following system of equalities:

$$\pi_i = \sum_{j=0}^{K-i} \frac{\pi_j}{K - j + 1},$$

which implies

$$\pi_i - \pi_{i+1} = \frac{\pi(K-i)}{i + 1}$$

for all $i = 0, 1, \ldots, K$. Solving the last system of difference equations, we find

$$\pi_i = \frac{K + 1 - i}{K + 1} \pi_0,$$
and eventually (since they sum up to 1) we come to
\[ \pi_i = \frac{K + 1 - i}{K + 1} \frac{2}{K + 2}. \]

To study the statistics of \( n \)-periodic configurations of free particles, one should take into account the fact that there is an additional constraint: \( X(1) + X(n) \leq N \). Denote by \( S(n, N) \) the total number of subconfigurations of length \( n \) consisting of free particles and satisfying this constraint. Then the fraction of nonadmissible \( n \)-periodic configurations (which do not satisfy the above constraint)

\[
\sum_{i=0}^{K} \pi_i \sum_{j=K-i}^{K} \pi_j = \sum_{i=0}^{K} \pi_i \left( 1 - \sum_{j=0}^{K-i} \pi_j \right)
\]

is asymptotically (for large \( K \)) equal to

\[
\int_0^1 \pi(x) \int_{1-x}^1 \pi(y) \, dy \, dx = \frac{1}{6}
\]

where \( \pi(x) := 2(1 - x) \) corresponds to the limit (as \( K \to \infty \)) distribution.

Using these statistics, one can easily obtain all correlation functions and study large deviations. For example, we see that the fraction of blocking configurations in \( \text{Free}(K) \) is equal to \( \pi_K = 2/((K + 1)(K + 2)) \), which in terms of traffic estimates shows how often the road with \( K \) lanes is completely blocked by moving cars.

References


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