DIMENSIONS OF QUANTIZED TILTING MODULES

VIKTOR OSTRIK

Abstract. Let $U$ be the quantum group with divided powers at $p$-th root of unity for prime $p$. To any two-sided cell $A$ in the corresponding affine Weyl group, one associates the tensor ideal in the category of tilting modules over $U$. In this note we show that for any cell $A$ there exists a tilting module $T$ from the corresponding tensor ideal such that the greatest power of $p$ which divides $\dim T$ is $p^{a(A)}$, where $a(A)$ is Lusztig’s $\alpha$-function. This result is motivated by a conjecture of J. Humphreys.


Key words and phrases. Quantum groups at roots of unity, tilting modules, special representations of Weyl groups.

1. Introduction

Let $G$ be a simply connected algebraic group. In [6] G. Lusztig proved the existense of a bijection between two finite sets: the set of two-sided cells in the affine Weyl group associated to $G$ (this set is defined combinatorially) and the set of unipotent $G$-orbits. The proof in [6] is quite involved and this bijection remains rather mysterious. In [3] J. Humphreys suggested a natural conjectural construction for Lusztig’s bijection using cohomology of tilting modules over algebraic groups in characteristic $p > 0$ or, similarly, over quantum groups at a root of unity. In [10, 11] the author proved some partial results towards the Humphreys Conjecture. Now this Conjecture is known to be true in the quantum group case thanks to the (unpublished) work of R. Bezrukavnikov.

For an element $w$ of a finite Weyl group $W_f$ one defines the number $a(w)$ as the Gelfand—Kirillov dimension of the highest weight simple module $L(w \cdot 0)$ over the corresponding semisimple Lie algebra. Generalizing this, G. Lusztig defined the $\alpha$-function on any Coxeter group, see [4]. The $\alpha$-function takes constant values on any two-sided cell and appears to be very useful for the theory of cells in Coxeter groups. J. Humphreys suggested that his construction of Lusztig’s bijection is compatible with the theory of $\alpha$-functions in the following way: the dimension of any tilting module corresponding to a two-sided cell $A$ is divisible by $p^{a(A)}$ and generically is not divisible by a higher power of $p$ (here $p$ is the characteristic of the field in the
case of an algebraic group and the order of the root of unity in the quantum group case). In [11] the author proved that the first statement (divisibility of dimensions by \( p^{\alpha(A)} \)) is a consequence of the Humphreys Conjecture, so it is a consequence of Bezrukavnikov’s work. The second statement (generic indivisibility by \( p^{\alpha(A)+1} \)) seems to be harder. The main result of this note is that in the quantum group case, for any cell \( A \) there exists a tilting module \( T \) corresponding to \( A \) whose dimension is not divisible by \( p^{\alpha(A)+1} \). So we determine the \( p \)-component of dimension of certain tilting modules, what seems to be of some interest independently of Humphreys’ Conjecture.

We will follow the notations of [9]. Let \((Y, X, \ldots)\) be a simply connected root datum of finite type. Let \( p \) be a prime number bigger than the Coxeter number \( h \). Let \( \zeta \) be a primitive \( p \)-th root of unity in \( \mathbb{C} \). Let \( U \) be the quantum group with divided powers associated to these data. Let \( T \) be the category of tilting modules over \( U \), see e.g. [1]. Recall that any tilting module is the sum of indecomposable ones, and indecomposable tilting modules are classified by their highest weights, see loc. cit. Let \( X_+ \) be the set of dominant weights, and for any \( \lambda \in X_+ \) let \( T(\lambda) \) denote the indecomposable tilting module with highest weight \( \lambda \). The tensor product of tilting modules is again a tilting module.

Let us introduce the following preorder relation \( \leq_T \) on \( X_+ \): \( \lambda \leq_T \mu \) if and only if \( T(\lambda) \) is a direct summand of \( T(\mu) \otimes \) (some tilting module). We say that \( \lambda \sim_T \mu \) if \( \lambda \leq_T \mu \) and \( \mu \leq_T \lambda \). Obviously, \( \sim_T \) is an equivalence relation on \( X_+ \). The equivalence classes are called weight cells. The set of weight cells has a natural order induced by \( \leq_T \). It was shown in [10] that the partially ordered set of weight cells is isomorphic to the partially ordered set of two-sided cells in the affine Weyl group \( W \) associated with \((Y, X, \ldots)\) \((W \) is a semidirect product of the finite Weyl group \( W_f \) with the dilated coroot lattice \( pY \)).

Let \( G \) and \( g \) be the simply connected algebraic group and the Lie algebra (both over \( \mathbb{C} \)) associated to \((Y, X, \ldots)\), and let \( N \) be the nilpotent cone in \( g \), i.e., the variety of \( \text{ad} \)-nilpotent elements. It is well known that \( N \) is the union of finitely many \( G \)-orbits, called nilpotent orbits. Using the theory of support varieties one defines Humphreys’ map

\[ \mathcal{H}: \{\text{the set of weight cells}\} \rightarrow \{\text{the set of closed } G\text{-invariant subsets of } N\}, \]

see [11]. The construction is as follows: it is known that the cohomology ring \( H^*(u) \) of a small quantum group \( u \subset U \) is isomorphic to the ring of regular functions on \( N \) (this is a theorem due to V. Ginzburg and S. Kumar, see [2]); now let \( A \) be a weight cell and \( \lambda \in A \) any weight, then \( \text{Ext}^*(T(\lambda), T(\lambda)) \) is naturally a module over \( H^*(u) \), so it can be considered as a coherent sheaf on \( N \) and finally the Humphreys map \( \mathcal{H}(A) \) is just the support of this sheaf. The conjecture due to Humphreys says that the image of the map \( \mathcal{H} \) consists of irreducible varieties, i.e., the closures of nilpotent orbits; moreover, Humphreys conjectured that this map coincides with Lusztig’s bijection between the set of two-sided cells in the affine Weyl group and the set of nilpotent orbits, see [3] and [11]. In particular, the Humphreys map must preserve Lusztig’s \( a \)-function; this function is equal to half of the codimension in \( N \) of the nilpotent orbit and is defined purely combinatorially on the set of two-sided cells, see [6]. The aim of this note is to show that the Humphreys map does not
decrease the $a$-function: for a weight cell $A$ corresponding to a two-sided cell $\mathcal{A}$ in $W$, we have the inequality
\[
\text{codim}_{\mathcal{W}} \mathcal{H}(A) \geq a(A).
\]
This inequality follows easily from the definition of $\mathcal{H}$, Theorem 4.1 in [11], and the Main Theorem below:

**Main Theorem.** Let $A$ be a weight cell corresponding to a two-sided cell $\mathcal{A}$ in the affine Weyl group. Then there exists a weight cell $B \leq T$ and a regular weight $\lambda \in B$ such that $\dim T(\lambda)$ is not divisible by $p^{a(\mathcal{A})+1}$ provided $p$ is sufficiently large.

**Remark.** It follows from Humphreys’ Conjecture proved by Bezrukavnikov that in the Main Theorem $B = A$.

The proof of this theorem is based on formulas for characters of indecomposable tilting modules obtained by W. Soergel in [12, 13]. In what follows we will freely use notation and results from [12] and [11].

**Warning.** The character formulas for tilting modules use certain Kazhdan—Lusztig elements in the Iwahori—Hecke algebra of $W$, and in modules thereof. The Iwahori—Hecke algebra is an algebra over $\mathbb{Z}[v, v^{-1}]$, and we will only need its specialization at $v = 1$. So all the notions related to it (e.g. Kazhdan—Lusztig bases) will be understood in the specialization $v = 1$.

2. Proof of the Main Theorem

Let $\leq$ denote the Bruhat order on the affine Weyl group $W$. For any $x \in W$ let $(-1)^x$ denote the sign of $x$, which is $(-1)^{l(x)}$, where $l(x)$ is the length of $x$.

2.1. We may and will suppose that our root system $R$ is irreducible. Let $S$ be the set of simple reflections in the affine Weyl group $W$. For any $s \in S$ let $W_s$ be the parabolic subgroup generated by $S - \{s\}$. The subgroup $W_s$ is finite. There exists a unique point $\mu_s \in X \otimes \mathbb{Q}$ invariant under the $W_s$-action. In general, $\mu_s \not\in X$, but the denominators of its coordinates contain only bad primes for $R$.

In particular, let $s_a \in S$ be the unique affine reflection. Then $W_{s_a} = W_f$ is the finite Weyl group. There exists a natural projection $W \rightarrow W_f$, $x \mapsto \bar{x}$. This projection embeds all the subgroups $W_s$ into $W_f$.

Recall from [12] that the set $W_f \setminus W$ is identified with the set of dominant alcoves.

2.2. Recall that any two-sided cell of $W$ intersects some $W_s$ nontrivially, see [6]. In the group algebra of $W_s$, there are two remarkable bases: the Kazhdan—Lusztig base $\mathcal{H}_s$, $w \in W_s$, and the dual Kazhdan—Lusztig base $\mathcal{H}_s^\dagger$, $w \in W_s$ (notation from [12]). Recall that
\[
\mathcal{H}_s = \sum_{x \leq w} p_{x,w} x, \quad \mathcal{H}_s^\dagger = \sum_{x \leq w} p_{x,w} (-1)^{xw} x,
\]
where $p_{x,w}$ are the values at 1 of Kazhdan—Lusztig polynomials.

Let $V = X \otimes \mathbb{R}$ be the reflection representation of $W_f$. For any $s \in S$ the restriction of $V$ to $W_s$ is isomorphic to the reflection representation $V_s$ of $W_s$. 

We refer the reader to [4] for the definition and properties of Lusztig’s a-function. This function is defined on the set of elements of a Coxeter group and takes values in $\mathbb{N} \cup \infty$. We will use the following properties of a-function:

(i) a-function is constant on any two-sided cell, see [4] 5.4.

(ii) Suppose that $w \in W_0 \subset W$, where $W_0$ is a parabolic subgroup of $W$. Then the values of the a-function of $w$ calculated with respect to the Coxeter groups $W_0$ and $W$ coincide, see [5] 1.9 (d).

(iii) Let $w \in W_s$. The element $\overline{H}_w$ acts trivially on $S^i(V_s)$ for $i < a(w)$, see [7]. The space $S^{a(w)}(V_s)$ contains exactly one irreducible component (the special representation) such that elements $\overline{H}_w$, $w' \sim_{LR} w$, act nontrivially on it, see loc. cit. Moreover, these elements generate an action of the full matrix algebra on this component, see [8], Chapter 5. We will say that this special representation corresponds to $w$.

Convention. The equivalence relation $\sim_{LR}$ depends on the ambient group, e.g. if $w_1, w_2 \in W_s$, then $w_1 \sim_{LR} w_2$ in $W$ does not imply $w_1 \sim_{LR} w_2$ in $W_s$. In what follows, the equivalence relation $\sim_{LR}$ is considered with respect to $W_s$. In spite of this we apply the notation $\leq_{LR}$ with respect to $W$. We hope that this does not cause ambiguity in what follows.

2.3. Let

$$\Delta(\lambda) = \prod_{\alpha \in R_+} \left( \frac{\langle \lambda, \alpha^\vee \rangle}{\langle \rho, \alpha^\vee \rangle} \right)$$

be the Weyl polynomial. For any $w \in W_s$ and $y \in W_f$ let us consider the following polynomial in $\lambda$ and $\mu$:

$$\Delta(y, W_s, w, \mu, \lambda) = \sum_{x \leq w} p_{x,w} \Delta(\mu + yx^{-1} \lambda).$$

Lemma 1. The lowest degree term of $\Delta(y, W_s, w, \mu, \lambda)$ in $\mu$ has degree $\geq a(w)$.

Proof. It is well known that the polynomial $\Delta(\lambda)$ is skew-symmetric with respect to the $W_f$-action: $\Delta(y \lambda) = (-1)^y \Delta(\lambda)$. Hence

$$\Delta(y, W_s, w, \mu, \lambda) = \Delta(1, W_s, w, y^{-1} \mu, y^{-1} \lambda).$$

So for the proof of the Lemma it suffices to consider the case $y = 1$. Using the skew-symmetry of $\Delta(\lambda)$ with respect to the $W_f$-action once again, we obtain

$$\Delta(1, W_s, w, \mu, \lambda) = \sum_{x \leq w} p_{x,w} (-1)^x \Delta(x^{-1} \mu + \lambda). \quad (*)$$

The element $\sum_{x \leq w} p_{x,w} (-1)^x x^{-1}$ equals $\overline{H}_{w^{-1}}$, see e.g. the proof of Theorem 2.7 in [12]. Since $a(w) = a(w^{-1})$ (see [4]), the lemma follows from 2.2.

2.4. Now we consider $\Delta(\mu + \lambda)$ as a polynomial in two variables $\mu, \lambda \in V$. The action of the Weyl group $W_f$ on the space $S^*(V \oplus V)$ of all polynomials in two variables $\mu$ and $\lambda$ via the variable $\mu$ is well defined and preserves the degrees of polynomials with respect to both $\mu$ and $\lambda$. 
Lemma 2. Let the group $W_s$ act on the polynomial $\Delta(\mu + \lambda)$ according to the rule $x\Delta(\mu + \lambda) = \Delta(x\mu + \lambda)$. Then the representation generated by the summand of degree $a(w)$ in $\mu$ contains the special representation corresponding to $w^{-1}$.

Proof. Let $E_1 \subset S^a(w)(V)$ be the special representation of $W_s$ corresponding to the element $w^{-1}$. According to Section 3 in [7], the $W_f$-representation $E$ generated by $E_1$ is irreducible, occurs with multiplicity 1 in the space of polynomials of degree $a(E_1) = a(w)$, and does not occur in the spaces of polynomials of lower degree. Moreover, $E$ lies in the space of harmonic polynomials, which is identified with the cohomology of the flag variety $H^2_n(w)(G/B)$. Hence, Lemma 2 is reduced to the following statement:

Lemma 3 (R. Bezrukavnikov). Let $W_f$ act on the polynomial $\Delta(\mu + \lambda)$ by the rule $x\Delta(\mu + \lambda) = \Delta(x\mu + \lambda)$. Then the representation generated by the summand of degree $i$ in $\mu$ contains any irreducible constituent of $H^2_n(G/B)$.

Proof. We identify the cohomology space $H^•(G/B \times G/B)$ with the space of harmonic polynomials in two variables $\mu$ and $\lambda$ (with respect to the group $W \times W$). It is well known that the diagonal class is represented by $\Delta(\mu + \lambda)$. Using Poincaré duality, we identify $H^•(G/B \times G/B)$ with $\text{End}(H^•(G/B))$ (this identification is not $(W \times W)$-equivariant since the fundamental class is $W$-antiinvariant, but it is $W$-equivariant with respect to the action of the first copy of $W$). Now any vector $v \in H^•(G/B)$ defines a $W$-equivariant map

$$\text{End}(H^•(G/B)) \to H^•(G/B), \quad x \mapsto xv,$$

where $W$ acts on $\text{End}(H^•(G/B)) = H^•(G/B) \otimes (H^•(G/B))^*$ via the first factor and under this map we have $1 \mapsto v$. The diagonal class $\Delta(\mu + \lambda)$ corresponds to $1 \in \text{End}(H^•(G/B))$ and the summand of degree $i$ in $\mu$ corresponds to $1 \in \text{End}(H^{2i}(G/B))$. □

2.5. Let $N$ denote the degree of the polynomial $\Delta(\lambda)$.

Lemma 4. Let us choose $\mu$ so that $\langle \mu, \alpha \rangle \neq 0$ for any $\alpha \in R$. Then there exists an element $w' \in W_s$, $w' \sim_{LR} w$, such that the summand of $\Delta(y, W_s, w', \mu, \lambda)$ of degree $N - a(w)$ in $\lambda$ is nontrivial.

Proof. We may and will assume that $y = 1$. By formula $(\ast)$, we have

$$\Delta(1, W_s, w_1, \mu, \lambda) = \tilde{H}_{w_1^{-1}} \Delta(\mu + \lambda),$$

where $W_s$ acts on $\Delta(\mu + \lambda)$ via the variable $\mu$. Since the elements $\tilde{H}_{w_1^{-1}}$, with $w_1 \sim_{LR} w$, generate an action of the full matrix algebra on the special representation corresponding to $w^{-1}$ by 2.2, Lemma 2.4 shows that the set of summands of degree $a(w)$ in $\mu$ of $\tilde{H}_{w_1^{-1}} \Delta(\mu + \lambda)$, where $w_1$ runs over all $w_1 \sim_{LR} w$, contains a basis over the field of rational functions in $\lambda$ of special representation corresponding to $w^{-1}$. Evidently, these summands are exactly the summands of $\Delta(1, W_s, w_1, \mu, \lambda)$ of degree $N - a(w)$ in the variable $\lambda$.

Our Lemma claims that this set contains at least one nonzero element when we specialize $\mu$ to any weight satisfying the conditions of Lemma 4. Consider the ideal generated by this set in the ring of polynomials in $\mu$ with coefficients which
are rational functions in \( \lambda \). Evidently, Lemma 4 is a consequence of the following statement:

**Lemma 5.** Let \( U \) be an irreducible \( W_f \)-submodule of \( S^*(V) \) not contained in \( (S^*(V))^{W_f} \). In other words, suppose that the module \( U \) projects nontrivially to \( S^*(V)/(S^*(V))^{W_f} = H^2(G/B) \). Then the zero set of the ideal of \( S^*(V) \) generated by \( U \) is contained in the union of hyperplanes \( \langle \mu, \alpha^\vee \rangle = 0, \alpha \in R \).

**Proof.** Evidently, the ideal generated by \( U \) is \( W_f \)-invariant. By Poincaré duality, for any \( 0 \neq v \in H^i(G/B) \) there exists a \( v' \in H^{2N-i}(G/B) \) such that \( vv' \) represents the fundamental class of \( G/B \). Hence the ideal generated by \( U \) contains an element \( \omega \in S^N(V) \) which projects nontrivially onto \( H^2N(G/B) \). The alternation \( \omega' = \frac{1}{|W_f|} \sum_{w \in W_f} (-1)^w w(\omega) \) is also contained in our ideal and projects nontrivially onto \( H^{2N}(G/B) \). But \( \omega' \) must be a nonzero multiple of Weyl polynomial \( \Delta(\bar{\lambda}) \) since the Weyl polynomial is unique up to a scalar \( W \)-antiinvariant in \( S^N(V) \). \( \square \)

2.6. Let \( \mathcal{A} \subset W \) be a two-sided cell. Choose \( W_s \) so that \( W_s \cap \mathcal{A} \neq \emptyset \) (this is possible by Theorem 4.8(d) in [6]). Let us fix \( w_1 \in W_s \cap \mathcal{A} \). We choose a minimal \( y \in W_f \) with the property:

For some \( w \in W_s \) such that \( w \sim_{LR} w_1 \), the summand of \( \Delta(y, W_s, w, y\mu_s, \tilde{y}\lambda) \) of degree \( N - a(w) \) in \( \lambda \) is nonzero. \( (***) \)

By Lemma 4 such a \( y \) exists since there exists a \( y \in W_f \) such that \( y\mu_s \) lies strictly inside the dominant Weyl chamber.

In the following lemma we use the notation from [12].

**Lemma 6.** Let \( y \in W_f \) and \( w \in W_s \) be as above. Then the element \( N_y H_w \in \mathcal{N} \) is the sum of elements \( N_x H_w \), \( x \leq_{LR} \mathcal{A} \), with positive integral coefficients, and hence can be considered as a character of a tilting module in a regular block.

**Proof.** By the formulas at the end of Proposition 3.4 of [12], we have:

\[
N_y H_w = \begin{cases} 
  N_x & \text{if } x \in W_f, \\
  0 & \text{if } x \notin W_f.
\end{cases}
\]

So, \( N_y H_w = N_x H_w \) and the Lemma follows from the definition of cells, together with the positivity properties of multiplication in the Iwahori—Hecke algebra, see e.g. §3 of [4]. \( \square \)

2.7. **Proof of the Main Theorem.** We can rewrite the element \( N_y H_w \) as

\[
N_y H_w = N_1 \sum_{y_1 \in W_f} n_{y_1, y} \sum_{x \leq w} p_{x, w} H_{y_1 x}.
\]

Let \( \lambda_1 \) be a regular weight from the fundamental alcove. The dimension of the tilting module \( T \) in the linkage class of \( \lambda_1 \) with character given by \( N_y H_w \) is equal
to
\[ \sum_{y_1 \in W^f} \sum_{y_1 \leq y} p_{x,w} \Delta(y_1 x \cdot \lambda + \rho). \]

Now let us write \( \lambda_1 = -\rho + p\mu_s + \lambda \).

We have
\[ \dim T = \sum_{y_1 \leq y} n_{y_1,y} \sum_{x \leq w} p_{x,w} \Delta(y_1 p\mu_s + y_1 x \lambda) = \sum_{y_1 \leq y} n_{y_1,y} \Delta(y_1, W_s, w, y_1 p\mu_s, y_1 \lambda). \]

According to (**), for some \( w \sim_{LR} w_1 \) the polynomial \( \dim T \) has a nonvanishing summand of degree \( N - a(A) \) in \( \lambda \). Hence, for \( p \gg 0 \) it is possible to choose \( \lambda \) so that this summand is not divisible by \( p \), and \( \lambda_1 \) lies in the lowest alcove. The Main Theorem is proved.

Acknowledgements. This note is the result of conversations with many mathematicians. Especially I wish to thank R. Bezrukavnikov, M. Finkelberg, J. Humphreys, J. C. Jantzen, and G. Rybnikov for their generous help and extremely useful discussions. I am grateful to the referee for a careful reading of the paper and useful comments.

References


Department of Mathematics, Massachusetts Institute of Technology, Cambridge MA, 02139

E-mail address: ostrik@math.mit.edu