TRUNCATION OF FUNCTIONAL RELATIONS
IN THE XXZ MODEL

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Abstract. The integrable XXZ model with a special open boundary condition is considered. We study the Sklyanin transfer matrices after the quantum group reduction at roots of unity. In this case, the Sklyanin transfer matrices satisfy a closed system of truncated functional equations. The algebraic reason for the truncation is found. An important role in the proof of the result is played by the Zamolodchikov algebra introduced in the paper.

Key words and phrases. Quantum groups, quantum algebras, representation theory, integrable XXZ model.

1. Introduction

We consider the integrable XXZ model with a special open boundary condition. Its Hamiltonian is

\[ H_{XXZ} = \sum_{n=1}^{N-1} \left( \sigma_n^+ \sigma_{n+1}^- + \sigma_n^- \sigma_{n+1}^+ + \frac{\cosh(\eta)}{2} \sigma_n^z \sigma_{n+1}^z + \frac{\sinh(\eta)}{2} (\sigma_n^z - \sigma_{n+1}^z) \right). \]

This Hamiltonian is invariant under the quantum algebra \( \mathcal{U}_q(\mathfrak{sl}(2)) \), whose generators \( X, Y, \) and \( H \) satisfy the following commutation relations:

\[ [H, X] = X, \quad [H, Y] = -Y, \quad [X, Y] = [2H]_q, \]

where the function \([x]_q\) equals \( \frac{q^x - q^{-x}}{q - q^{-1}} \) and \( q \equiv e^{i\eta} \). We have the following representation for \( X, Y, \) and \( H \) in terms of the Pauli matrices:

\[ X = \sum_{n=1}^{N} q^{\frac{1}{2} \sum_{i=1}^{n-1} \sigma_i^+} \sigma_n^+ q^{-\frac{1}{2} \sum_{i=1}^{n-1} \sigma_i^+} q^{\frac{1}{2} \sum_{i=1}^{n-1} \sigma_i^+} \sigma_n^+ q^{-\frac{1}{2} \sum_{i=1}^{n-1} \sigma_i^+} \]

\[ Y = \sum_{n=1}^{N} q^{\frac{1}{2} \sum_{i=1}^{n-1} \sigma_i^+} \sigma_n^- q^{-\frac{1}{2} \sum_{i=1}^{n-1} \sigma_i^+} q^{\frac{1}{2} \sum_{i=1}^{n-1} \sigma_i^+} \sigma_n^- q^{-\frac{1}{2} \sum_{i=1}^{n-1} \sigma_i^+} \]

\[ H = \sum_{n=1}^{N} \frac{\sigma_n^z}{2}. \]

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We concentrate on the case in which $q^{p+1} = -1$. It is known \cite{1, 2} that if $q^{p+1} = -1$, then $X^{p+1} = 0$ and $Y^{p+1} = 0$; we can therefore consider the subquotient space $V_p = \text{Ker } X/ \text{Im } X^p$.

The result of restricting the XXZ model to $V_p$ is the Minimal Model of Integrable Lattice Theory $LM(p, p + 1)$, see \cite{2, 3}. The thermodynamic limit is the ordinary Minimal Model of CFT $M(p, p + 1)$ (see \cite{4}) with the Virasoro central charge $c = 1 - 6/p(p + 1)$, see \cite{2}.

The XXZ model is integrable \cite{5}, and, as shown by Sklyanin \cite{6}, there exists a family of transfer matrices that commute with the Hamiltonian and with one another. The Sklyanin transfer matrices are also $U_q(\hat{sl}(2))$ invariant \cite{7}. Therefore, they can be also restricted to $V_p$. As shown in \cite{3}, the result of this restriction is the truncation of the fusion functional relations \cite{8} for the transfer matrices. This statement was proved in \cite{3} using the $T$–$Q$ Baxter equation. In this work, we prove this fact more directly and algebraically. Namely, we prove that if $q^{p+1} = -1$, then the Sklyanin transfer matrix $t_{p/2}(u)$ with the spin $j = p/2$ in the auxiliary space vanishes after its restriction to $V_p$, because

$$t_{p/2}(u) = X^p M + N X,$$  \hfill (1)

where $M$ and $N$ are some operators in the quantum space.

In other words,

$$t_{p/2}(u) = 0 \quad \text{on } \text{Ker } X/ \text{Im } X^p.$$  

Then it follows from equation (1) that the fusion functional relations are truncated and transformed to a system of functional equations that can be used to obtain the eigenvalues of the transfer matrices.

This paper is organized as follows.

In Section 2, we recall some basic formulas that are necessary for introducing the Sklyanin transfer matrices in Section 3.

In Section 4, we show how to obtain the operator $X$ from the Sklyanin monodromy matrix.

This will allow us then to obtain the commutation relations between the generators of the quantum algebra $U_q(\hat{sl}(2))$ and the Sklyanin monodromy matrices in Section 5.

In Sections 6, 7, 8, we express the $L$ operators in terms of the generators of the Zamolodchikov algebra and obtain the commutation relations between these new operators and the generators of the quantum algebra $U_q(\hat{sl}(2))$.

In Section 9, we rewrite the Sklyanin transfer matrix in the new variables.

In Section 10, we introduce the operator $\text{ad } X$.

In Section 11, we write down some useful properties of $\text{ad } X$.

In Sections 12 and 13, we prove (1), which is our main statement (the $\text{ad } X$ theorem).

2. The Yang—Baxter equation

The integrable structure of XXZ is contained in the well-known equation

$$R_{um}^{bn}(u - v) L^b_c(u) L^m_k(v) = L^b_m(v) L^m_u(u) R_{ck}^{mn}(u - v),$$  \hfill (2)
where $L$ is a monodromy matrix whose entries $L_n^m$ ($n, m = 1, \ldots, 2j + 1$) are operators acting in the so-called quantum space, which is the tensor product of two two-dimensional linear spaces in our case. The dimension $2j + 1$ of the so-called auxiliary space can vary; strictly speaking, we should write something like $(L_j)_n^m$ instead of $L_n^m$ and $(R_{j_1j_2})^m_n$ instead of $R_{bn}^m$, but sometimes for simplicity we do not do that and assume that everything is clear from the context. The $R_{j_1j_2}$-matrix acts in the tensor product of two auxiliary spaces and satisfies the Yang--Baxter equation

$$R_{bn}^m(u - v) R_{ck}^{mp}(u) R_{ks}^{mq}(v) = R_{mk}^{np}(v) R_{bs}^{qr}(u) R_{ck}^{jm}(u - v),$$

and the unitarity relation

$$(R_{j_1j_2})^m_n(u) (R_{j_1j_2})^n_m(-u) = \phi_{j_1j_2}(u) \phi_{j_1j_2}(-u) \delta_{um}^m.$$

The monodromy matrix $L$ can be realized through an $N$-fold product of $R$-matrices,

$$L(u) = R_N(u) \ldots R_1(u),$$

here the comultiplier $R_n$ stands for the $R$-matrix acting in the tensor product of the auxiliary space and of the $n$-th comultiplier of the quantum space. We need the related matrix $\hat{L}^i(u)$,

$$\hat{L}^i(u) = R_1(u) \ldots R_N(u) \sim L^{-1}(-u).$$

It is easy to show that

$$\hat{L}^i_j(u) = \xi^N_j(-u) L^{-1}_j(-u),$$

where $\xi_j(u) \equiv \phi_{1/2j}(u) \phi_{1/2j}(-u)$ and $L^{-1}_j(u)$ is defined by the equation

$$(L_j)^b_u(u) (L^{-1}_j)^b_u(u) = \delta^c_u \text{Id}.$$}

Using this equality, it is possible to obtain the formulas

$$R_{bn}^m(u - v) L^m_n(v) L^b_c(u) = L^m_n(u) L^n_m(v) R_{ck}^m(u - v),$$

$$L^c_n(u) L^m_k(v) R_{ck}^n(u + v) R_{cm}^k(u + v) L^b_n(v) L^m_k(u).$$

In our case, the $R$-matrix is

$$R(u) = \begin{pmatrix} \sinh(u + (\frac{1}{2} + \hat{H}) \eta) & \sinh(\eta) \hat{F} \\ \sinh(\eta) \hat{E} & \sinh(u + (\frac{1}{2} - \hat{H}) \eta) \end{pmatrix}. $$

The $\hat{E}$, $\hat{F}$, $\hat{H}$ are the generators of the quantum algebra $U_q(sl(2))$ in the auxiliary representation: $[\hat{H}, \hat{E}] = \hat{E}, [\hat{H}, \hat{F}] = -\hat{F}, [\hat{E}, \hat{F}] = [2\hat{H}]_q$.

In the representation with the spin $j$, we have

$$\pi_j(\hat{H})_{mn} = (j + 1 - n) \delta_{m,n}, \quad m, n = 1, 2, \ldots, 2j + 1,$$

$$\pi_j(\hat{E})_{mn} = \omega_m \delta_{m,n-1},$$

$$\pi_j(\hat{F})_{mn} = \omega_n \delta_{m-1,n},$$

where

$$\omega_n \equiv \sqrt{[n]_q [2j + 1 - n]_q}.$$

The spin $j$ takes values from the set $(0, \frac{1}{2}, 1, \ldots)$. 


3. The Sklyanin transfer matrix

It was shown by Sklyanin in [6] that if the transfer matrices are defined as

$$ t_j(u) = \text{tr}_\pi (e^{-2(u+\eta)} \hat{H} L_j(u) e^{2\hat{H}u} \hat{L}_j(u)) $$

$$ \equiv \sum_{n,k=1}^{2j+1} e^{2(n-k)u} q^{-2(j+1-n)} (L_j)_n^k(u) (\hat{L}_j)_n^k(u), $$

where, as above,

$$ (L_n^m)_{i_j\ldots i_1} = (L_n^m)_{i_j\ldots i_1}, $$

then the transfer matrices commute with one another,

$$ [t_{j_1}(u_1), t_{j_2}(u_2)] = 0, $$

and with the Hamiltonian,

$$ [t_j(u), H_{XXZ}] = 0. $$

The basic transfer matrix $t_{1/2}(u)$ with the spin of the auxiliary space $j = 1/2$ is related to the $XXZ$ Hamiltonian by

$$ H_{XXZ} = \frac{\sinh(\eta)}{2} \frac{d \log t_{1/2}(u)}{du} \bigg|_{u=0} = \frac{\sinh^2(\eta)}{2 \cosh(\eta)} - \frac{N}{2} \cosh(\eta). $$

4. Behavior of $L$ as $u \to \pm \infty$

It is well known that the generators of the quantum affine algebra $U_q(\hat{sl}(2))$ can be easily obtained in the limit of the $L$ operator. To do that, we consider the monodromy matrix in the representation with the spin $1/2$. It is a $2 \times 2$ matrix whose entries are some operators. Using the expression of the monodromy matrix as an $N$-fold product of $R$-matrices with the same auxiliary and different quantum spaces, it is easy to deduce its behavior as $u \to \pm \infty$:

$$ \lim_{u \to \pm \infty} L(u) = 2^{-N} e^{(u+\eta/2)(N-1)} \begin{pmatrix} e^{u+\eta/2} q^H & (q-q^{-1})X_0 \\ (q-q^{-1}) X & e^{u+\eta/2} q^{-H} \end{pmatrix}, $$

$$ \lim_{u \to \pm \infty} L(u) = (-1)^N 2^{-N} e^{-\eta/2} \begin{pmatrix} e^{-(u+\eta/2)} q^H & -(q-q^{-1})Y_0 \\ -(q-q^{-1}) Y & e^{-(u+\eta/2)} q^{-H} \end{pmatrix}. $$

The explicit expressions for $X_0$ and $Y_0$ in terms of the Pauli matrices can be derived from those for $X$ and $Y$ by changing $q \to q^{-1}$ and $\sigma^\pm \to \sigma^\mp$. The $X$, $Y$, $X_0$, $Y_0$, and $H$ are the generators of the quantum affine algebra $U_q(\hat{sl}(2))$. Namely, if we denote: $x_1 \equiv X$, $y_0 \equiv Y_0$, $x_0 \equiv X_0$, $y_1 \equiv Y$, $h_0 = -h_1 \equiv H$, then the relations $[x_i, y_j] = [2H]_q$, $[x_i, y_j] = 0$, $[H, x_0] = -x_0$, $[H, y_1] = -y_1$, $[H, x_1] = x_1$ and $[H, y_0] = y_0$ are satisfied, as well as the Serre relations

$$ (\text{ad}_q x_i)^3 x_j = 0, \quad (\text{ad}_q y_i)^3 y_j = 0, $$

where

$$ (\text{ad}_q x_i)^3 x_j \equiv (x_i)^3 x_j - [3]_q (x_i)^2 x_j x_i + [3]_q x_i x_j (x_i)^2 - x_j (x_i)^3. $$
5. Transformation of $L_n^m$ and $\hat{L}_n^m$ under the quantum group action

We consider equations (2) and (3) in the limit as $u \to \pm \infty$. We take the monodromy matrix $L(u)$ in the representation with the spin $j = 1/2$. We take the second monodromy matrix $L(v)$ in the representation with an arbitrary spin $j$. In the limit, the $L(u)$ matrix yields the generators of the algebra $U_q(\hat{sl}(2))$ with is given by the formulas in the previous section. We thus obtain the following commutation relations between $L_n^m$, $\hat{L}_n^m$, and the generators of $U_q(\hat{sl}(2))$. Here we write only those containing $X$ and $q^{\pm H}$:

$$q^H L_k^n = q^{n-k} L_k^n q^H,$$
$$q^H \hat{L}_k^n = q^{k-n} \hat{L}_k^n q^H,$$  \hspace{1cm} (4)

$$X L_k^n - q^{2(j+1)-k-n} L_k^n X = \omega_k - 1 \lambda q^{j+1-n} L_{k-1}^n q^{-H} - \omega_n \lambda q^{2-k} L_{k+1}^n q^H,$$
$$X \hat{L}_k^n - q^{k+n-2(j+1)} \hat{L}_k^n X = \omega_n - 1 \lambda q^{-j-1} L_{k-1}^n q^{-H} - \omega_{k-1} \lambda q^{-n-2} L_{k+1}^n q^H,$$  \hspace{1cm} (5)

where $\lambda \equiv e^\nu$.

These commutation relations can be used to prove (1). However, this way is rather difficult, and we choose another one. Equations (4) and (5) show that elements of the monodromy matrix $L$ are transformed under the quantum group action like a composition of two representations of it. We therefore introduce a new algebra (which is just the Zamolodchikov algebra) such that each element $L_n^m$ or $\hat{L}_n^m$ is a product of two generators, and equations (2), (4), and (5) follow from the defining relations of this new algebra. Rewriting the Sklyanin transfer matrix $t_{p/2}(u)$ in terms of the generators of the new algebra, we can easily prove our main statement (1).

6. $L_n^m$ and $\hat{L}_n^m$ as a composition of elements of the Zamolodchikov algebra

We introduce the Zamolodchikov algebra generated by the operators $\theta^a_n(u)$, $\theta^1_j(u)$, $\theta^0_n(u)$ and $\hat{\theta}^a_n(u)$, where $j$ is any positive half integer and $n = 1, \ldots, 2j + 1$, with the following defining relations:

$$(R_{j1,j2})^{\theta^a_n}_{h_n}(u - v) \theta^a_{j1}(u) \theta^a_{j2}(v) = \phi_{j1,j2}(u - v) \theta^a_{j2}(v) \theta^a_{j1}(u),$$
$$(R_{j1,j2})^{\theta^1_n}_{h_n}(u - v) \theta^1_m(v) \theta^1_n(u) = \phi_{j1,j2}(u - v) \theta^a_{j2}(v) \theta^1_n(u),$$
$$\theta^a_{j1}(u) \theta^a_{j2}(v) = \tau_{j1,j2}(u - v) \theta^a_{j2}(v) \theta^a_{j1}(u),$$
$$(R_{j1,j2})^{\theta^1_n}_{\hat{\theta}^a_n}(u - v) \hat{\theta}^a_{j1}(u) \hat{\theta}^a_{j2}(v) = \phi_{j1,j2}(u - v) \theta^a_{j2}(v) \hat{\theta}^a_{j1}(u),$$
$$\hat{\theta}^a_{j1}(u) \hat{\theta}^a_{j2}(v) = \tau_{j1,j2}(u - v) \hat{\theta}^a_{j2}(v) \hat{\theta}^a_{j1}(u),$$
$$(R_{j1,j2})^{\theta^0_n}_{\hat{\theta}^a_n}(u + v) \theta^1_m(v) \theta^0_n(u) = \chi_{j1,j2}(u + v) \theta^1_m(v) \theta^0_n(u),$$
$$(R_{j1,j2})^{\theta^0_n}_{h_n}(u + v) \theta^1_m(v) \theta^0_n(u) = \chi_{j1,j2}(u + v) \theta^1_m(v) \theta^0_n(u),$$
$$\theta^1_m(v) \theta^0_n(u) = \rho_{j1,j2}(u + v) \theta^0_n(u) \theta^1_m(v),$$
$$\theta^a_{j1}(u) \hat{\theta}^a_{j2}(v) = \rho_{j1,j2}(u + v) \hat{\theta}^a_{j2}(v) \theta^a_{j1}(u),$$
$$\hat{\theta}^a_{j1}(u) \theta^a_{j2}(v) = \rho_{j1,j2}(u + v) \theta^a_{j2}(v) \hat{\theta}^a_{j1}(u),$$
$$\theta^1_m(v) \theta^0_n(u) = \rho_{j1,j2}(u + v) \theta^0_n(u) \theta^1_m(v),$$
$$\theta^a_{j1}(u) \hat{\theta}^a_{j2}(v) = \rho_{j1,j2}(u + v) \hat{\theta}^a_{j2}(v) \theta^a_{j1}(u),$$
$$\hat{\theta}^a_{j1}(u) \theta^a_{j2}(v) = \rho_{j1,j2}(u + v) \theta^a_{j2}(v) \hat{\theta}^a_{j1}(u).$$  \hspace{1cm} (9)
where \( \phi_{j_1 j_2}, \tau_{j_1 j_2}, \rho_{j_1 j_2} \) and \( \chi_{j_1 j_2} \) are some functions. These functions depend on the specific form of \( R_{j_1 j_2} \). For example, \( \phi_{j_1 j_2} \) is defined from the unitarity equation and is equal to the multiplier in the right-hand side of this equation. We don’t know how to obtain the form of the three other functions in the general case, but we can find the needed combination of them from the self-consistency requirement.

The Zamolodchikov algebra is associative for arbitrary \( \phi_{j_1 j_2}, \tau_{j_1 j_2}, \rho_{j_1 j_2} \) and \( \chi_{j_1 j_2} \), provided only that the \( R \)-matrix satisfies the triangle equation. A similar algebra was considered in [9].

Using (7)–(9), it is easy to check that if

\[
(L_j)_{0}^n(u) = \theta_j^n(u) \bar{\theta}_j^n(u), \quad (\bar{L}_j)_{0}^n(u) = \bar{\theta}_j^n(u) \theta_j^n(u), \tag{10}
\]

then \( L \) and \( \bar{L} \) satisfy the Yang—Baxter equations (2) and (3).

7. Transformation of \( \theta_n, \bar{\theta}_n \) and \( \bar{\theta}_n \) under the quantum group action

The following commutation relations between \( L_n^b, \bar{L}_n^b \) and the generators of the Zamolodchikov algebra, which follow from (7)–(10), are sufficient for our purposes:

\[
R_{00}^n(u - v) L_n^b(u) \theta_j^n(v) = \theta_j^n(v) L_n^b(u) \phi(u - v), \tag{11}
\]

\[
R_{00}^n(u - v) \theta_j^n(v) L_n^b(u) = \phi(u - v) L_n^b(u) \theta_j^n(v), \tag{12}
\]

\[
R_{00}^n(u - v) \bar{\theta}_j^n(v) \bar{L}_n^b(u) = \bar{L}_n^b(u) \bar{\theta}_j^n(v) \phi(u - v), \tag{13}
\]

\[
R_{00}^n(u - v) \bar{L}_n^b(u) \bar{\theta}_j^n(v) = \phi(u - v) \bar{\theta}_j^n(v) \bar{L}_n^b(u). \tag{14}
\]

Here we write \( L_{j \bar{j}} \) as just \( L \), \( R_{j \bar{j}} \) as \( R \), and set \( \phi(u) = \phi_{j \bar{j}}(u) \tau_{j \bar{j}}(u) \). We already know how to obtain the operator \( X \) from \( L(u) \). Having taken the limit \( u \to \pm \infty \), we can obtain the commutation relation between \( X \) and \( \theta \). To do that, we must know what the function \( \phi(u) \) is equal to. But the function \( \phi(u) \) depends on the function \( \tau_{j \bar{j}}(u) \), which is unknown. Actually, we only need to know the behavior of \( \phi(u) \) in the limit \( u \to +\infty \). Taking this limit, we obtain

\[
q^H \bar{\theta}_n^j(v) = \varepsilon(v) q^{a-j-1} \bar{\theta}_n^j(v) q^{-H},
\]

\[
q^{-H} \theta_n^j(v) = \varepsilon(v) q^{a+1-n} \theta_n^j(v) q^{-H},
\]

\[
q^H \theta_n^j(v) = \varepsilon(v)^{-1} q^{a+1-n} \theta_n^j(v) q^{-H},
\]

\[
q^{-H} \bar{\theta}_n^j(v) = \varepsilon(v)^{-1} q^{a-j-1} \bar{\theta}_n^j(v) q^{-H},
\]

where

\[
\varepsilon(v) \equiv \lim_{u \to +\infty} (2\phi(u - v) e^{-(u-v+n/2)}).
\]

It is obvious that these equations are self-consistent only if

\[
\varepsilon(v) = 1.
\]
Supposing that this is true, we also obtain the commutation relations between $X$ and $\theta$, thus,

$$
\begin{align*}
q^H \theta_j^n &= q^{n-j-1} \theta^n_{j+1} q^H, \\
q^H \theta_j^1 &= q^{j-n+1} \theta^n_{j-1} q^H, \\
q^H \theta^n_1 &= q^{n-j+1} \theta^n_{j+1} q^H, \\
X \theta_j^n &= -q^{j+1-n} \theta^n_1 X = -\omega_n \lambda q^{n+1} \theta^n_1 q^H, \\
X \theta_j^1 &= -q^{j+1-n} \theta^n_1 X = \omega_n \lambda -1 \theta^n_{j+1} q^H, \\
X \theta^n_1 &= -q^{n-j-1} \theta^n_1 X = \omega_n \lambda -1 \theta^n_{j+1} q^H, \\
X \theta^n_1 &= -q^{n-j-1} \theta^n_1 X = -\omega_n \lambda \bar{q}^{-1} \theta^n_{j+1} q^H.
\end{align*}
$$

(15)

It is easy to verify that these formulas are consistent with (4) and (5) if (10) is taken into account.

8. New variables $\psi$

We can make our formulas more convenient if we introduce the new variables $\psi$ by

$$
\begin{align*}
\theta_j^n &= \psi^n_j q^{-nH}, \\
\theta_j^1 &= \psi^n_j q^nH, \\
\theta^n_1 &= \psi^n_j q^{-nH}, \\
\theta^n_1 &= \bar{\psi}^n_j q^nH,
\end{align*}
$$

(17)

We rewrite the Sklyanin transfer matrices in the new variables $\psi$. The result suggests that the transfer matrices consist of two other objects. Replacing the old variables by the new ones,

$$
(L_j)_{Q}^Q (\bar{L}_j)_{K}^Q = \theta_j^n \theta_j^1 \theta_j^n \theta_j^n = q^{k(j+1-k)} q^{-(n+1)(j+1-n)} \psi^n_j \psi^n_j \psi^n_j \psi^n_j,
$$

we obtain

$$
t_j(u) = \sum_{n=1}^{2j+1} \lambda^{2n} q^{-(n+1)(j+1-n)} \psi^n_j \left( \sum_{k=1}^{2j+1} \lambda^{-2k} q^{k(j+1-k)} \psi^n_k \psi^n_k \psi^n_k \psi^n_k \right) \bar{\psi}^n_j,
$$

(19)

(Here and below $\lambda = e^u$). We see that the transfer matrix consists of two independent structures. If we set

$$
\begin{align*}
g_j^-(u) &= \sum_{n=1}^{2j+1} \lambda^{-2k} q^{k(j+1-k)} \psi^n_k \psi^n_k, \\
g_j^+(u) &= \sum_{n=1}^{2j+1} \lambda^{2n} q^{-(n+1)(j+1-n)} \psi^n_j \psi^n_j.
\end{align*}
$$

(20)
then the transfer matrix has the form
\[ t_j(u) = \sum_{n=1}^{2j+1} \lambda^{2n} q^{-(n+1)(j+1-n)} \psi_j^n g_j^- (u) \bar{\psi}_j^n. \]

The new objects \( g_j^- (u) \) and \( g_j^+ (u) \) are remarkable, because they are separately invariant with respect to the quantum algebra \( U_q(\mathfrak{sl}(2)) \),
\[ [g_j^\pm (u), U_q(\mathfrak{sl}(2))] = 0. \]

We prove this formula after introducing the ad X operator.

10. The ad X operator

It is convenient to introduce a linear operator (we call it the ad X operator) such that the following properties hold:
\[ \text{ad} \ X (\Psi + \Psi') = \text{ad} \ X (\Psi) + \text{ad} \ X (\Psi'), \quad (21) \]
and also if \( \Psi \) and \( \Psi' \) have definite degrees,
\[ \text{ad} \ X (\Psi \Psi') = \text{ad} \ X (\Psi) \Psi' + q^{\deg(\Psi)} \Psi \text{ad} \ X (\Psi'), \quad \text{deg} (\Psi'') = \deg (\Psi) + \deg (\Psi'). \quad (22) \]

By definition, we set
\[ \text{ad} \ X (\psi_j^n) = \psi_j^n - q^{j+1-2n} \bar{\psi}_j^n \psi_j^n X = -\omega_n \lambda q \psi_j^{n+1}, \]
\[ \text{ad} \ X (\bar{\psi}_j^n) = \psi_j^n - q^{2n-j-1} \bar{\psi}_j^n \psi_j^n X = \omega_{n-1} \lambda^{-1} \bar{\psi}_j^{n-1}, \]
\[ \text{ad} \ X (\bar{\psi}_j^n) = \psi_j^n - q^{j+1-2n} \bar{\psi}_j^n X = \omega_{n-1} \lambda \psi_j^{n-1}, \]
\[ \text{ad} \ X (\psi_j^n) = \psi_j^n - q^{2n-j-1} \bar{\psi}_j^n \psi_j^n X = -\omega_n \lambda^{-1} q^{-1} \bar{\psi}_j^{n+1} \]
(compare with (18)). We define the degrees of these operators by
\[ \deg (\psi_j^n) = j + 1 - 2n, \]
\[ \deg (\bar{\psi}_j^n) = 2n - j - 1, \]
\[ \deg (\bar{\psi}_j^n) = j + 1 - 2n, \]
\[ \deg (\bar{\psi}_j^n) = 2n - j - 1, \]
and also
\[ \deg (X^N \psi_j^n) = \deg (\psi_j^n) - 2N, \]
\[ \deg (X^N \bar{\psi}_j^n) = \deg (\bar{\psi}_j^n) - 2N, \]
\[ \deg (X^N \bar{\psi}_j^n) = \deg (\psi_j^n) + 2N, \]
\[ \deg (X^N \bar{\psi}_j^n) = \deg (\bar{\psi}_j^n) + 2N. \quad (24) \]

Note that applying the operator X to \( \psi \) and \( \bar{\psi} \) with the upper index \( n \) decreases their degrees and, conversely, applying to the ones with the lower index \( n \) increases their degrees. It is now easy to show that \( \text{ad} \ X (g_j^\pm (u)) = 0. \)
Indeed,
\[
\text{ad } g^\pm_j(u) = \sum_{k=1}^{2j+1} \lambda^{-2k} q^{k(1-j)} \omega_k \psi_k^j \bar{\psi}_k^j + q^{j+1-2k} \omega_k \lambda^{-1} q^{-1} \psi_k^j \bar{\psi}_k^j \]
= 0.

Since \( \text{deg}(g^\pm_j(u)) = 0 \), we have \( Xg^\pm_j(u) = g^\pm_j(u)X \). In fact, even the more general statement
\[
[g_j^\pm(u), U_q(\mathfrak{sl}(2))] = 0
\]
is true.

11. Properties of the \( \text{ad } X \) operator

Using properties (21) and (22) of the operator \( \text{ad } X \), it is easy to prove the identities
\[
(\text{ad } X)^N (\psi^A \bar{\psi}^B) = \sum_{n=0}^{N} q^n (\text{deg}(\psi^A) + n - N) C_n^N (\text{ad } X)^{N-n} (\psi^A) (\text{ad } X)^n (\bar{\psi}^B),
\]
\[
(\text{ad } X)^N (\psi^A \bar{\psi}^B) = \sum_{n=0}^{N} (-1)^n q^n (\text{deg}(\psi^A) - 1 + N) C_n^N \psi^A X^n,
\]
\[
(\text{ad } X)^N (\psi^A) = \sum_{n=0}^{N} (-1)^n q^n (\text{deg}(\psi^A) - 1 + N) C_n^N \psi^A X^n,
\]
where
\[
C_n^N = \frac{[N]_q!}{[n]_q! [N-n]_q!}
\]
is the \( q \)-binomial coefficient. We can observe a slight difference in these formulas for \( \psi \) and \( \bar{\psi} \) with the upper and lower indices.

12. Proof of the \( \text{ad } X \) theorem in terms of \( \theta \) operators

Using the definition of the operator \( \text{ad } X \), we can obtain
\[
\psi_n^j = a_n (\text{ad } X)^{n-1} (\psi_j^1), \quad \bar{\psi}_n^j = b_n (\text{ad } X)^{j+1-n} (\psi_{2j+1}^j),
\]
\[
\bar{\psi}_n^j = c_n (\text{ad } X)^{j+1-n} (\bar{\psi}_{2j+1}^j), \quad \bar{\psi}_n^j = d_n (\text{ad } X)^{n-1} (\bar{\psi}_1^j),
\]
It is easy to check that
\[ a_n = \prod_{k=1}^{n-1} (-\lambda^{-1} q^{-1} \omega_n^{-1}), \quad b_n = \prod_{k=1}^{2j+1-n} (\lambda^{-1} \omega_{2j+1-k}^{-1}), \]
\[ c_n = \prod_{k=1}^{2j+1-n} (\lambda \omega_{2j+1-k}^{-1}), \quad d_n = \prod_{k=1}^{n-1} (-\lambda q \omega_n^{-1}). \]

Therefore,
\[ \psi_n \bar{\psi}_n = (-1)^{n-1} \omega^{-1} q^{1-n} \lambda^{2(n-j-1)} (\text{ad} X)^{2j+1-n} (\psi_{2j+1})^2 (\text{ad} X)^{n-1} (\bar{\psi}_{2j+1}^2), \]
\[ \psi_j \bar{\psi}_j = (-1)^{n-1} \omega^{-1} q^{1-n} \lambda^{2(j+1-n)} (\text{ad} X)^n (\psi_{2j+1}^2) (\text{ad} X)^{j+1-n} (\bar{\psi}_{2j+1}^2). \]

Inserting this into the expression for \( g_j^- (u) \), we obtain
\[ g_j^- (u) = \sum_{k=1}^{2j+1} \lambda^{-2k} q^k \psi_k^2 \bar{\psi}_k \]
\[ = \omega^{-1} \lambda^{-2(j+1)} q^j \sum_{n=0}^{2j} (-1)^n q^{n(j-n)} (\text{ad} X)^{2j-n} (\psi_{2j+1})^2 (\text{ad} X)^n (\bar{\psi}_{2j+1}^2). \]

It makes sense to compare this with the similar formula
\[ (\text{ad} X)^{2j} (\psi_{2j+1} \bar{\psi}_{2j+1}^2) = \sum_{n=0}^{2j} q^{-2(j+1)} q^{n(j-n)} C_{2j}^n (\text{ad} X)^{2j-n} (\psi_{2j+1}^2) (\text{ad} X)^n (\bar{\psi}_{2j+1}^2). \]

Taking into account that if \( q^{p+1} = -1 \), then \( C_{p+1}^n = 1 \) and \( q^{(p+1)n} = (-1)^n \), we obtain the conclusion
\[ g_{p/2}^- (u) = \omega^{-1} \lambda^{-2(p/2+1)} q^{p/2} (\text{ad} X)^p (\psi_{p+1} \bar{\psi}_{p+1}^2). \]  \( \text{(28)} \)

Similarly, we have
\[ g_j^+ (u) = \sum_{n=1}^{2j+1} \lambda^{2n} q^{-(n+1)(j+1-n)} \psi_j^n \bar{\psi}_j^n \]
\[ = (-1)^{2j} \lambda^{2(j+1)} q^{i(2j+1)} \]
\[ \times \sum_{n=0}^{2j} (-1)^{j-n} q^{n(j+n)} (\text{ad} X)^{2j-n} (\psi_j^1) \bar{\psi}_j^{2j+1} \]  \( \text{(29)} \)

for \( g_j^+ (u) \); as above, if \( q^{p+1} = -1 \), we obtain
\[ g_{p/2}^+ (u) = (-1)^p \omega^{-1} \lambda^{2(p/2+1)} q^{p(p+1)/2} (\text{ad} X)^p (\psi_{p/2}^1 \bar{\psi}_{p/2}^{p+1}). \]
If $q^{p+1} = -1$, we can use (29) and $\text{ad}X(g_f^-(u)) = 0$ to rewrite formula (19) for the transfer matrix $t_{p/2}(u)$ as

$$t_{p/2}(u) = (\text{ad}X)^p(G_{p/2}(u)),$$

(30)

where we introduce the notation

$$G_{p/2}(u) \equiv (-1)^p \omega^{-1} \lambda^{2(p/2+1)} q^{p(p+1)/2} \left( \psi_{p/2}^1 g_{p/2}^-(u) \tilde{\psi}_{p/2}^{p+1} \right).$$

(31)

Because of property (25) of $\text{ad}X$ and since $q^{p+1} = -1$ implies $C^p_n = 1$, we can also rewrite (30) as

$$t_{p/2}(u) = \sum_{n=0}^{p} X^{p-n} G_{p/2}(u) X^n.$$

(32)

We have thus proved, in fact, our main result (the ad $X$ theorem). In the next section we return from the “virtual” variables $\psi$ and $\tilde{\psi}$ to the variables $L$ and $\tilde{L}$.

13. Return to $L$ and $\tilde{L}$

By returning from the variables $\psi$ and $\tilde{\psi}$ to the variables $L$ and $\tilde{L}$ in (31), we can easily show that

$$G_{p/2}(u) \equiv (-1)^p \omega^{-1} \lambda^{2(p/2+1)} \sum_{k=1}^{p+1} (\lambda^{2(p/2+1-k)} (L_{p/2})^1_k (\tilde{L}_{p/2})^{p+1}_k) q^{-pH}.$$

Although we have proved the main theorem using intermediate calculations with the “virtual” operators $\theta$, which somehow do not exist, this proof is nevertheless valid. Indeed, to prove equation (32), we could transpose the operator $X$ with the entire $L$ and $\tilde{L}$ to the right without splitting $L$ and $\tilde{L}$ into the operators $\theta$. Doing this, we would not meet these operators at all, but the result of this permutation must be the same (and equation (32) must be satisfied) because of the associativity of the Zamolodchikov algebra and the absence of any linear relations between the quartic products of its generators of the form $\theta^a_j \theta^b_k \theta^c_m \theta^d_n$. This completes the proof of the ad $X$ theorem.

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