PERRON—FROBENIUS SPECTRUM FOR RANDOM MAPS
AND ITS APPROXIMATION

MICHAEL BLANK

Dedicated to Robert Minlos on the occasion of his 70th birthday

ABSTRACT. To study the convergence to equilibrium in random maps, we develop the spectral theory of the corresponding transfer (Perron—Frobenius) operators acting in a certain Banach space of generalized functions (distributions). The random maps under study in a sense fill the gap between expanding and hyperbolic systems, since among their (deterministic) components there are both expanding and contracting ones. We prove the stochastic stability of the Perron—Frobenius spectrum and develop its finite rank operator approximations by means of a ‘stochastically smoothed’ Ulam approximation scheme. A counterexample to the original Ulam conjecture about the approximation of the SBR measure and the discussion of the instability of spectral approximations by means of the original Ulam scheme are presented as well.


Key words and phrases. Perron—Frobenius operator, invariant measure, spectrum, random map, mixing, finite rank approximation.

1. Introduction

Let \{T_i\} be a collection of nonsingular maps from a d-dimensional smooth manifold \(X\) into itself with a metrics \(\rho\) on it, and let \(\{p_i\}\) be a collection of nonnegative real numbers, such that \(\sum_i p_i = 1\). A random dynamical system on the space \(X\) is a stationary stochastic process \(T_1, T_2, \ldots : X \rightarrow X\) (i.e., with values in a space of maps), see for instance [17]. There are several approaches to further detalization of this object, and in the sequel we shall use the following Markov one.

By a random map \(T\) we shall mean the Markov random process on \(X\) given by the following family of transition probabilities \(P(x, A) := \sum_i p_i 1_{T_i^{-1}(A)}(x)\) from a point \(x \in X\) to a subset \(A \subseteq X\). In other words, at every time step the map \(T_i\) is chosen from the collection \(\{T_i\}\) independently from the previous choices with the probability given by the distribution \(\{p_i\}\).

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In the literature (especially physical) the above defined random map is often called the \textit{iterated function system} (with probabilities). Naturally, a purely deterministic setup was studied as well. This can be done as follows. According to the given collection of maps \{T_i\}, one can define a new multivalued map \( \tilde{T} : X \to X \) as \( \tilde{T}x := \bigcup_i T_ix \). Assuming that the maps \( T_i \) are continuous and strictly contracting, one can show [13] that the multivalued map \( \tilde{T} \) possesses a global attractor \( X_{\tilde{T}} \subset X \), namely the Hausdorff distance between the sets \( \tilde{T}^nY \) and \( X_{\tilde{T}} \) decreases exponentially fast for any closed nonempty set \( Y \subset X \).

It is worth noting that our aim in this paper is not to study the most general setup (for example, one can consider an infinite (continual) collection of maps \( \{T_i\} \), while the distribution \( \{p_i\} \) might be place dependent (on the space variable)), but rather to give a deeper analysis of dynamical and statistical properties of these systems related to the convergence to equilibrium, i.e. on spectral problems of the corresponding transfer operators practically not studied in the literature (except [1]). Therefore we shall not consider well studied problems of the dimensional analysis of invariant sets and measures. The reader can find results and further references of this type e.g. in [9].

One can always realize the random map under study as a deterministic one on the extended phase space. Denote by \( \Omega \) the space of one-side sequences \( \omega := \{\omega_1, \omega_2, \ldots\} \in \Omega \), where each \( \omega_i \) belongs to the set of indices of the collection \( T_i \).

On the space \( \Omega \) one defines the left-shift map \( \sigma : \Omega \to \Omega \) according to the rule \( (\sigma\omega)_i := \omega_{i+1} \). The topology in the space \( \Omega \) is defined as the direct product of the discrete topologies on the set of indices, and a Borel measure \( \mu_p \) on \( \Omega \) is defined as the product of distributions \( \{p_i\} \) on the set of indices. Naturally, there exists one-to-one correspondence between the sequences of maps \( T_i \) and the elements of the space \( \Omega \). On the extended space \( \Omega \times X \) one can define a new map, the skew product on the left-shift map: \( \tilde{T}(\omega, x) := (\sigma\omega, T_{\omega_i}x) \).

Let us give a few simple examples of random maps, demonstrating that even in the simplest cases the dynamics of random maps can be rather nontrivial. For the sake of simplicity, in all our examples both here and in the sequel, the collection of maps will consist of only two maps \( T_1, T_2 \) from the unit interval into itself, and thus the corresponding distribution is described by one parameter \( p := p_1 \).

**Example 1.1.** Pure contractive maps: \( T_1(x) := x/2, T_2(x) := x/2 + 1/2. \)

According to [13] (see also the next section), for each value of the parameter \( 0 < p < 1 \) the corresponding random map \( T \) has the unique invariant measure, whose support typically (when \( p \neq 1/2 \)) is a Cantor set.

**Example 1.2.** A mixed case: \( T_1(x) := 1 - |2x - 1|, T_2(x) := x/2. \)

In contrast to the previous example, depending on the choice of the parameter \( p \) the properties of the random map \( T \) differ qualitatively. Namely, for \( 0 \leq p < \frac{1}{2} \) the random map possesses the unique invariant measure concentrated at 0. For \( p > \frac{1}{2} \) the invariant measure becomes non-unique but there exists a unique absolutely continuous invariant one, whose density \( h_T \) is such that \( h_T|I_k = \gamma_k \) for any \( k = 1, 2, \ldots \) and the intervals \( I_k := (2^{-(k-1)}, 2^{-k}] \). However, the constants \( \gamma_k \) are bounded on \( k \) while \( 2/3 < p < 1 \) and go to infinity with \( k \) for \( 1/2 < p < 2/3 \).
The above examples are typical for statistical problems related to random maps, and in the next section we shall separately discuss properties of random maps expanding on average and contracting on average; and thus we postpone the proof of the above claims till all needed technicalities will be introduced. It is worth noting that our results related to the expanding on average random maps are very similar to those known for deterministic piecewise expanding maps [1] (see also the results about the convergence to the Sinai—Bowen—Ruelle (SBR) measure for multidimensional expanding on average random maps in [16, 6]). Therefore in the corresponding sections we mainly demonstrate the similarities between these two types of systems. On the other hand, the spectral analysis of the contracting on average random maps is completely new. In fact, our interest in this type of systems is due to the analysis of spectral properties of Anosov maps in [4], where the presence of stable foliations relates the situation to our case. Some of the methods and ideas used in this paper originated from the construction in [4] needed to study the behavior of the transfer operators ‘along the stable foliation’ and are based technically on the establishing of the so called Lasota—Yorke type inequalities. The nature of the system under study gives some advantages compared to the Anosov maps, in particular, the functional Banach spaces in our case do not depend on the finite structure of the map, the spectrum stability results are proven for a much broader class of random perturbations, and a more direct approximation scheme is elaborated.

The paper is organized as follows. In the following two sections (2 and 3) we introduce the notions of contractive and expanding on average random maps and discuss some of their basic properties, slightly generalizing known results about them. The main results of the paper are described in Section 4, where we construct and study the Perron—Frobenius spectrum for random maps. We also prove stochastic stability of isolated eigenvalues in this spectrum and construct two schemes of their finite rank operator approximation by means of a special random smoothing. To some extent, these schemes generalize the well known Ulam approximation scheme [22] (see also discussion of its realization in [1, 7, 11, 18]). We discuss in detail why the spectrum approximation by means of the original Ulam scheme is not stable in the case of random maps (see also [1, 4]). Since the original Ulam conjecture about the approximation of the SBR measure (the leading element of the spectrum) still holds in our setting, in Lemma 4.12 we construct the first (to the best of our knowledge) counterexample to this conjecture. Note however that the map in this counterexample is not only discontinuous, but the discontinuity occurs in a periodic turning point (compare to the instability results for general random perturbations in [1]).

2. Contracting on average random maps

An important and well known case of random maps corresponds to the situation when all the maps $T_i$ are continuous and their contracting constants

$$A_{T_i} := \sup_{x,y \in X} \frac{|T(x) - T(y)|}{\rho(x, y)} \leq \Lambda < 1.$$
Only the list of literature devoted to various theoretical and applied aspects of the theory of this class of random maps would fill several pages, therefore we only give a reference to one of the first (and giving a very essential contribution) works in this field—[13], to the monograph [9], and to one of the recent publications [10], where the reader can find more references to recent results. Note that these papers are devoted mainly to questions related to dimensional and multifractal properties of invariant sets and measures which we shall not touch in this work.

We start the analysis of ergodic properties of these systems using a weaker assumption, namely that the random map $T$ is contracting on average.

**Definition 2.1.** We shall say that the random map $T$ is contracting on average if its contracting constant

$$\Lambda_T := \sum_i p_i \Lambda_{T_i} < 1.$$ 

Let $\mathcal{M}$ be the space of probability measures on $X$. Then the Markov operator associated with the random map $T$ for each probabilistic measure $\mu \in \mathcal{M}$ can be written as

$$T\mu := \sum_i p_i \mu \circ T_i^{-1}.$$ 

Following the standard scenario, we introduce the special metrics in the space $\mathcal{M}$ (often called the Hutchinson metrics) [13]:

$$\rho_H(\mu, \nu) := \sup \left\{ \int h \, d\mu - \int h \, d\nu, \ h \in C^0(X, \mathbb{R}), \ |h(x) - h(y)| \leq \rho(x - y), \ \forall x, y \in X \right\},$$

where $\rho$ is the metrics on our original phase space $X$. It is well known (see, for example, [13]) that the pair $(\mathcal{M}, \rho_H)$ defines a compact metric space.

**Lemma 2.1.** Let $h: X \to \mathbb{R}$ be a continuous function and let $\mu \in \mathcal{M}$. Then

$$\int h \, d(T\mu) = \sum_i p_i \int h \circ T_i \, d\mu.$$

**Proof.** For each continuous function $h: X \to \mathbb{R}$ there is a sequence of piecewise constant approximating functions $h_n$ converging to $h$ uniformly on $X$. Therefore

$$\int h_n \, d(T\mu) = \sum_i p_i \int h_n \, d(\mu \circ T_i^{-1})$$

$$= \sum_i p_i \int_{T_i X} h_n \, d(\mu \circ T_i^{-1}) = \sum_i p_i \int h_n \circ T_i \, d\mu.$$ 

On the other hand, $\int h_n \, d(T\mu) \to \int h \, d(T\mu)$ as $n \to \infty$, while for each pair $i, n$ the function $h_n \circ T_i$ is piecewise constant. Thus the sequence of functions $\{h_n \circ T_i\}$ converges uniformly on $n$ to the function $h \circ T_i$, which yields the convergence

$$\sum_i p_i \int h_n \circ T_i \, d\mu \to \sum_i p_i \int h \circ T_i \, d\mu.$$

□
The following result is a simple generalization of the well known Hutchinson Theorem [13].

**Lemma 2.2.** We have $\rho_H(T\mu, T\nu) \leq \Lambda_T \cdot \rho_H(\mu, \nu)$ for any measures $\mu, \nu \in \mathcal{M}$. In particular, contraction on average yields the strict ergodicity of the random map $T$.

**Proof.** Introduce the notation

$$H := \{ h \in C^0(X, \mathbb{R}), |h(x) - h(y)| \leq \rho(x, y), \forall x, y \in X \}.$$ 

Then

$$\rho(T\mu, T\nu) = \sup \left\{ \int h \, d(T\mu) - \int h \, d(T\nu), h \in H \right\}$$

$$= \sup \left\{ \sum_i p_i \int h \circ T_i \, d\mu - \sum_i p_i \int h \circ T_i \, d\nu, h \in H \right\}.$$ 

Consider the function $\tilde{h} := \frac{1}{\Lambda_T} \sum_i p_i h \circ T_i$. For each pair of points $x, y \in X$ we have

$$|\tilde{h}(x) - \tilde{h}(y)| \leq \frac{1}{\Lambda_T} \sum_i p_i |h \circ T_i(x) - h \circ T_i(y)|$$

$$\leq \frac{1}{\Lambda_T} \sum_i p_i \rho(T_i(x), T_i(y)) \leq \frac{1}{\Lambda_T} \sum_i p_i \Lambda_{T_i} \cdot \rho(x, y) = \rho(x, y).$$

Hence $\tilde{h} \in H$. Introducing another set of functions

$$\tilde{H} := \left\{ \tilde{h} \in H: \text{ there exists } h \in H: \quad \tilde{h} = \frac{1}{\Lambda_T} \sum_i p_i h \circ T_i \right\},$$

we can rewrite the distance between the images of the measures as follows:

$$\rho_H(T\mu, T\nu) = \sup \left\{ \frac{1}{\Lambda_T} \int \tilde{h} \, d\mu - \frac{1}{\Lambda_T} \int \tilde{h} \, d\nu: \quad \tilde{h} \in \tilde{H} \right\}.$$ 

Now, since $\tilde{H} \subset H$, we come to the desired estimate

$$\rho_H(T\mu, T\nu) \leq \Lambda_T \cdot \rho_H(\mu, \nu),$$

and thus the contraction on average yields the uniform contraction in the space of measures. \qed

Note that Example 1.1 satisfies the conditions of Lemma 2.2, while for Example 1.2 the conditions of Lemma 2.2 hold only when $0 < p < 1/3$. Indeed,

$$\sum_i p_i \Lambda_i = 2p + \frac{1}{2}(1 - p) = \frac{3}{2}p + \frac{1}{2} < 1.$$ 

At first sight, it seems that the continuity of the maps $T_i$ was not used in the proof, however it plays a very important role in it. One can easily construct an example when the absence of this property leads to the nonuniqueness of the invariant measure.

**Example 2.1.** $T_1(x) := \frac{x}{2} 1_{[0,1/2]}(x) + \frac{x+1}{2} 1_{(1/2,1]}(x), \quad T_2(x) := x.$
For each $p \in (0, 1)$ the random map corresponding to Example 2.1 possesses exactly two ergodic invariant measures concentrated at points 0 and 1 respectively.

The contraction on average, that we assume in this section, does not prevent some of the maps $T_i$ to be expanding. Therefore, despite the fact that some of the maps $T_i$ may possess several (not necessarily a finite number) ergodic invariant measures, the random map $\mathcal{T}$ under the assumptions of Lemma 2.2 is strictly ergodic.

3. Expanding on average random maps

The results obtained in the previous section are based technically on the contraction on average property. Now we are going to show that the opposite assumption about the expansion on average leads to an ideologically close result — the existence of an absolutely continuous invariant measure.

For the sake of simplicity, we shall restrict ourselves here to the analysis of piecewise $C^2$-smooth maps of the unit interval $[0, 1]$ into itself with nondegenerate expanding constants

$$
\lambda_{T_i} := \inf_x |T_i'(x)| \geq \lambda > 0.
$$

It is straightforward to show that the transfer operator corresponding to the random map $\mathcal{T}$ in the space $L^1$ can be written as

$$
P_{\mathcal{T}} := \sum_i p_i P_i,
$$

where $P_i$ is the Perron—Frobenius operator corresponding to the map $T_i$ and describing the dynamics of densities of measures under its action (see a detailed discussion of the properties of these operators, for example, in [1]).

Denoting by $\Omega$ the set of values of the index $i$, for a given number $\Lambda$ we consider the sequence of sets

$$
\Omega(n)(\Lambda) := \{\omega \in \Omega: |(T^n_\omega x)'| > \Lambda^n \text{ for a. a. } x \in X\},
$$

where

$$
T^n_\omega x := T_{\omega_n} \circ T_{\omega_{n-1}} \circ \cdots \circ T_{\omega_1} x
$$

is the $n$-th point of a realization of a trajectory of the random map $\mathcal{T}$ starting from the point $x$, and introduce the following regularity assumption: there exist two constants $\Lambda > 1$ and $C < \infty$ such that

$$
P(\Omega^{(n)}(\Lambda)) \geq 1 - Ce^{-\sqrt{n}}
$$

(3.1)

for each positive integer $n$.

Denote by $\text{var}(\cdot)$ the standard one-dimensional variation of a function, and by $\text{BV}$ the space of functions of bounded variation equipped with the norm $\|\cdot\|_{\text{BV}} := \text{var}(\cdot) + \|\cdot\|_{L^1}$.

The following result gives the decomposition for the transfer operator under the considered assumptions.

**Theorem 3.1** [19, 1]. Let the regularity assumption (3.1) hold. Then for each pair of positive integers $n$, $k$ the following decomposition takes place for the random
map $\mathcal{T}$:

\[
P_n^T = P_{n,k} + Q_{n,k},
\]

\[
\var(P_{n,k}h) \leq \text{Const}(\alpha^n \var(h) + \beta^k \|h\|),
\]

\[
\|Q_{n,k}h\| < \text{Const} \sqrt{k} e^{-\sqrt{k}} \|h\|,
\]

for each function $h \in BV$ and $0 < \alpha < 1 < \beta < \infty$. All the constants above depend on the choice of the maps $T_i$, but do not depend on $n$ and $k$.

In the literature devoted to the question of the existence of absolutely continuous invariant measures in our setting, one can find two types of sufficient conditions for this existence. The first of these conditions, obtained in [21], corresponds to the strong expansion on average

\[
\sum_i p_i \lambda_{T_i} < 1,
\]

while the second, described in [19], is a weaker condition:

\[
\prod_i \lambda_{T_i}^p > 1.
\]

One can easily show that the first of these assumptions yields the second one. On the other hand, as we shall show, there is an important difference between the properties of invariant measures and respectively random maps under these assumptions. To explain the difference, let us return to our regularity assumption and show that it is even more general than inequality (3.5). For this purpose, we shall need the following simple technical estimate.

**Lemma 3.1.** Let $\{\xi_i\}_i$ be a sequence of independent identically distributed (i.i.d.) random variables having exponential moments up to some positive order $s_0$, i.e. $\mathbb{E}[e^{s\xi}] < \infty$ for all $0 < s < s_0$. Then for each number $R > \mathbb{E}[\xi_i]$ there are constants $a < 1$, $A < \infty$ such that

\[
\mathcal{P}\left\{\frac{1}{n} \sum_{i=1}^n \xi_i > R\right\} < Aa^n
\]

for each positive $n$.

**Proof.** By the exponential Chebyshev inequality

\[
\mathcal{P}\left\{\frac{1}{n} \sum_{i=1}^n \xi_i > R\right\} \leq e^{-sR} \mathbb{E}[e^{s\xi}]
\]

for each positive number $s$. For our purpose it is enough to show that the right-hand side of this inequality decreases exponentially fast. Note that for each number $x$ the following inequality holds:

\[
|e^x - 1 - x| \leq e^{|x|} - 1 - |x|.
\]

Indeed, this is trivial for $x \geq 0$ (since $e^x \geq 1 + x$ and $e^{-x} \leq e^x$), while for $x < 0$ we have $e^x \leq 1 + x$. Thus the inequality can be reduced to

\[-e^x + 1 + x \leq e^{-x} - 1 + x \quad \text{or} \quad e^{-x} + e^x \geq 2,
\]
which is evidently correct. Therefore
\[ |e^{s\xi} - 1 - s\xi| \leq e^{|s\xi|} - 1 - |s\xi|. \]
First assume that the values $\xi_i$ are bounded from below. Then
\[ E[e^{s\xi}] - 1 - E[|s\xi|] < \infty \]
for $|s| < s_0$, which yields the negativity of the left-hand side of the previous inequality. Therefore there are positive constants $s \in (0, \tilde{s}_0)$ and $C$ such that
\[ E[e^{s\xi}] \leq 1 + E[\xi] + Cs^2 \leq e^{sE[\xi] + Cs^2}. \]
Thus, setting
\[ s = \frac{(R - E[\xi])/(2C)}{e^{-(R - E[\xi]}/(4C)}, \quad a = e^{-(R - E[\xi]}/(4C)}, \]
we get the desired estimate. Observe that $s < \tilde{s}_0$ by construction. Therefore our inequalities make sense only if
\[ R \leq R_0 := E[\xi] + 2Cs_0. \]
This means that larger values of $R$ should be changed to $R_0$. To finish the proof, note that if the random values $\xi_i$ are not bounded from below, it is enough to ‘cut’ them from below by means of some constant and to apply the above argument to the result.

Lemma 3.2. The inequality (3.5) implies the regularity assumption (3.1).

Proof. It is enough to apply Lemma 3.1 to the sequence of i.i.d. random values $\xi_i := \ln \lambda_T$.

On the other hand, the following result shows that the opposite statement does not hold.

Example 3.1.
\[
T_1(x) := \begin{cases} 
\frac{3}{4} - 2x, & \text{if } 0 \leq x < \frac{1}{4}, \\
\frac{1}{4} - \frac{2}{3}x, & \text{if } \frac{1}{4} \leq x < \frac{1}{2}, \\
2 - 2x (\text{mod 1}), & \text{otherwise},
\end{cases}
\]
while $T_2(x) := T_1(x) + 1/12$ (mod 1). Graphs of these maps are shown on Fig. 1.

Lemma 3.3. The random map $T$ in Example 3.1 for any $0 < p < 1$ satisfies the regularity assumption, while condition (3.5) fails. It is interesting that in this example the stronger assumption (3.4) holds for the second iterate $T^2$.

Proof. Both maps $T_i$ are piecewise linear and the absolute values of their derivatives take only two values: 2 (on the intervals $(0, \frac{1}{4})$ and $(\frac{1}{4}, 1)$) and $\frac{2}{3}$ (on the interval $(\frac{1}{2}, 1)$). Since both these maps transform the interval $(\frac{1}{4}, \frac{1}{2})$ into $(0, \frac{1}{4})$ and on the remaining interval $(\frac{1}{2}, 1)$ the derivatives of both maps are equal to 2, it follows that for any $i, j \in \{1, 2\}$ the inequality
\[ |(T_iT_j)x)| \geq 2 \cdot \frac{2}{3} = \frac{4}{3} > 1 \]
Figure 1. Example when the regularity condition holds while condition (3.5) fails

holds. Thus we have checked the regularity assumption. Now observe that the
derivatives of both maps are strictly less than 1 on the interval \(x \in (\frac{1}{4}, \frac{1}{2})\), which
contradicts condition (3.5).

It remains to check condition (3.4) for the second iterate \(T^2\), which turns out to
be a consequence of the fact that the expanding constants for the maps \((T_i,T_j,x)\)
are not less than \(\frac{4}{3} > 1\) (according to the inequality above).

Note that since the right-hand side of the inequality (3.2) contains the term \(\beta^k\)
with \(\beta > 1\), Theorem 3.1 guarantees only estimates of the type

\[
\text{var}(P_{n,k}h) \leq \text{Const} \beta^k \frac{1}{1 - \alpha},
\]

which means that despite the fact that the density of the invariant measure is
integrable, it might be not a function of bounded variation. In fact, Example 1.2
demonstrates this phenomenon when the parameter \(p\) belongs to \((\frac{1}{2}, \frac{2}{3})\). Moreover,
in this example the density is not only a function of bounded variation, but it goes
to infinity at the vicinity of the origin.

It turns out that under a stronger assumption (3.4) the standard Lasota—Yorke
inequality is valid for the random map, and thus the invariant density is a function
of bounded variation.

**Theorem 3.2** [21]. For a. a. \(x \in [0, 1]\) let the inequality

\[
\sum_i \frac{P_i}{|T_i(x)|} \leq \gamma < 1
\]

hold. Then there exist constants \(C, \beta < \infty\) such that for each \(n \in \mathbb{Z}_+\) and for any
function \(h \in \mathbf{BV}\) the following Lasota—Yorke inequality holds:

\[
\text{var}(P^n h) \leq C^n \text{var}(h) + \beta ||h||,
\]
from which (as usual) it follows that for each nonnegative function \( h \in L^1 \) with \( \| h \| = 1 \) the limit
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} P_T^k h =: h_T,
\]
exists and
\[
P_T h_T = h_T, \quad \var(h_T) \leq \text{Const},
\]
while with respect to the absolutely continuous invariant measure \( \mu_T \) with the density \( h_T \), the correlations decay exponentially.

Now we are able to finish the analysis of Example 1.2. Observe that for \( p > \frac{1}{2} \) we have
\[
\prod_k \lambda_{T_k}^p = 2^p \cdot \left( \frac{1}{2} \right)^{1-p} = 2^{2p-1} > 1,
\]
which yields condition (3.5) and hence our regularity assumption. Therefore for \( p > 1/2 \) there exists an absolutely continuous \( T \)-invariant measure. On the other hand, for \( \frac{2}{3} < p < 1 \) condition (3.4):
\[
\sum_k \frac{p_k}{|T_k(x)|} = \frac{p}{2} + 2(1 - p) = 2 - \frac{3}{2} p < 1
\]
holds for a. a. \( x \in [0, 1] \). Thus we can apply Theorem 3.2, whence the boundedness of the density of the invariant measures follows.

4. Perron—Frobenius spectrum (PF-spectrum)

In the previous sections, we have restricted the analysis of statistical features of random maps to the properties of their invariant measures and, more specifically, Sinai—Bowen—Ruelle (SBR) measures. From a more general point of view the SBR measure is the eigenfunction of the Perron—Frobenius (transfer) operator of our random map in a suitable Banach space corresponding to the leading eigenvalue (1). Therefore it is very natural to extend the analysis of the dynamics to the complete spectrum of this operator.

It is worth noting that the interest to the PF-spectrum is based to a large extent on the fact that the subleading elements of the spectrum define the rate of mixing (convergence to the SBR measure, correlation decay, etc.).

4.1. Definition of the PF-spectrum. Let us start with a short description of objects related to the notion of spectrum which we shall need further. Let \( P: \mathcal{B} \to \mathcal{B} \) be a bounded linear operator in a complex Banach space \( (\mathcal{B}, \| \cdot \|_B) \). As usual, we denote by \( \text{sp}_B(P) \) its spectrum, which is defined as the complement to the set of regular elements, i.e. to the points \( z \in \mathbb{C} \) such that the resolvent \( (zI - P)^{-1} \) of the operator is defined in the entire space and hence is bounded. The maximal (by the absolute value) element of the spectrum is called the spectral radius:
\[
\rho_B(P) := \sup \{ |z| : z \in \text{sp}_B(P) \}.
\]
As it is well known [8], the spectral radius can be calculated by the following formula:

$$\rho_B(P) = \lim_{n \to \infty} \|P^n\|_B^{1/n}.$$  

Browder [5] introduced the notion of essential spectrum \(\text{ess}\,\text{sp}_B(P)\) of a bounded linear operator \(P\) as the union of elements of the spectrum \(z \in \text{sp}_B(P)\) such that at least one of the following properties holds:

1. The region of values of the operator \(zI - P\) is not bounded in \(B\).
2. \(\bigcup_{n \geq 0} \ker((zI - P)^n)\) is infinite dimensional.
3. \(z\) is a limit point of the spectrum \(\text{sp}_B(P)\).

Outside the essential spectrum only a countable number of isolated eigenvalues of the operator \(P\) may occur. Naturally the essential spectral radius \(\rho_{B,\text{ess}}(P)\) of the operator \(P\) is defined as the minimal nonnegative number such that all elements of the spectrum \(\text{sp}_B(P)\) outside the disc \(\{z \in \mathbb{C}: |z| \leq \rho_{B,\text{ess}}(P)\}\) are isolated eigenvalues of finite multiplicity. It turns out [20] that the essential spectral radius can be calculated by a formula similar to the one for the usual spectral radius:

$$\rho_{B,\text{ess}}(P) = \lim_{n \to \infty} \|P^n\|_{B,\text{ess}}^{1/n},$$  \hspace{1cm} (4.1)

but for the special seminorm:

$$\|P\|_{B,\text{ess}} := \inf \{\|P - K\|_B: \text{the map } K: B \to B \text{ is a compact operator}\}.$$  

Clearly, for each \(\varepsilon > 0\) the set

\[\text{sp}(P) \cap \{z \in \mathbb{C}: |z| \geq \rho_{B,\text{ess}}(P) + \varepsilon\}\]

consists of a finite number of eigenvalues \(\lambda_1, \lambda_2, \ldots, \lambda_{M(\varepsilon)}\) of the operator \(P\). Schematically, on the complex plane the spectrum can be represented as a disk of the radius \(\rho_{B,\text{ess}}(P)\) centered at the origin (describing the essential spectrum) and a (no more than countable) collection of points between this disk and the circle of the radius \(\rho_B(P)\) also centered at the origin.

### 4.2. Spectrum for the case of contracting on average random maps. In Section 2 it was shown that a contracting on average random map \(\overline{T}\) possesses only one invariant measure \(\mu_{\overline{T}}\) to which the sequence of iterations \(\{\overline{T}^n\mu\}\) converges (exponentially fast in the Hutchinson metrics) for each probabilistic initial measure \(\mu\).

From the point of view of dynamical system theory, the next question after this is the analysis of the spectrum of this convergence. The problem here is that typically the limit measure \(\mu_{\overline{T}}\) is not absolutely continuous, which rules out the description of the transfer operator \(P_{\overline{T}}\) in a space of reasonably ‘good’ functions, for example, in the space of functions of bounded variation. To overcome this difficulty, we shall study the action of the operator \(P_{\overline{T}}\) in a much larger space of generalized functions equipped with a norm induced by the Hutchinson metrics.

Let \((X, \rho)\) be a \(d\)-dimensional smooth manifold with a finite collection of continuous maps \(\{T_i\}\) from \(X\) into itself having bounded Lipschitz constants \(\Lambda_{T_i}\), and let \(\{p_i\}\) be a collection of probabilities, defining the random map \(\overline{T}\). Recall that
the contraction on average means that

$$\Lambda_T := \sum_i p_i \Lambda_{T_i} < 1.$$ 

Before defining our space of generalized functions, we need first to define the class of test-functions $\varphi: X \to \mathbb{R}^1$. For this purpose we introduce the following functionals:

$$H_\alpha(\varphi) := \sup_{x, y \in X, \rho(x, y) \leq \nu} \frac{|\varphi(x) - \varphi(y)|}{\rho^\alpha(x, y)},$$

$$V_\alpha(\varphi) := H_\alpha(\varphi) + |\varphi|_\infty,$$

where $|\varphi|_\infty := \text{ess sup} |\varphi|$, and the constant $\nu \in (0, 1]$. The first of these functionals is the Hölder constant with the exponent $\alpha < 1$, while the second one for each finite nonnegative value of the parameter $a$ is the norm of $\alpha$-Hölder functions on $X$ in the Banach space, which we shall denote by $C^\alpha$. Without loss of generality we shall assume that the diameter of the phase space $\sup_{x, y \in X} \rho(x, y)$ is not greater than 1. Note that the restriction $\rho(x, y) \leq \nu$ is introduced only to be able to work with the exponential map on general smooth manifolds: in the case of a flat torus this restriction can be omitted (or simply one can set $\nu = 1$).

Consider now the space of generalized functions $F$ on $X$ with the norm defined in terms of the test-functions from the space $C^\alpha$:

$$\|h\|_{(\alpha)} := \sup_{V_\alpha(\varphi) \leq 1} \int h \varphi.$$

The proof that the functional $\|\cdot\|_{(\alpha)}$ is indeed a norm in this space is standard and we leave it for the reader.

Denote by $F_\alpha$ the closure of the set of bounded in the norm $\|\cdot\|_{(\alpha)}$ generalized functions from $F$.

**Lemma 4.1.** We have $F_\beta \subseteq F_\alpha$ for any numbers $0 < \alpha \leq \beta \leq 1$, and for each function $\varphi \in C^\beta$ the inequality $V_\beta(\varphi) \leq V_\alpha(\varphi)$ holds.

**Proof.** Indeed,

$$\frac{|\varphi(x) - \varphi(y)|}{\rho^\beta(x, y)} = \frac{|\varphi(x) - \varphi(y)|}{\rho^\alpha(x, y)} \rho^{\beta - \alpha}(x, y) \leq H_\alpha(\varphi).$$

Extending the standard definition of the Perron—Frobenius operator to the action in the space of generalized functions, we get the representation

$$\int P_T h \cdot \varphi = \int h \cdot \sum_i p_i \varphi \circ T_i =: \int h \cdot (\varphi \circ T)$$

for each test-function from $\varphi \in C^\alpha$.

Let us fix some constants $0 < \alpha < \beta \leq 1, 0 < a < \infty$, whose exact values we shall define later. Our first aim is to derive a version of the Lasota—Yorke inequality for the action in the space $F_\alpha$. For $q \in [0, 1]$ define a function

$$\Lambda_T(q) := \sum_i p_i \Lambda_{T_i}^q,$$

which we shall need in this derivation.
Hence, 
\[ \sum \]

On the other hand, since 
which implies inequality (4.3) and for any \( \beta \in (0, 1] \) and a function \( h \in F_{\beta} \).

To prove the second inequality (4.4) we need more delicate estimates.

Introduce the following notation: 
\[ B_{\delta}(x) := \{ y \in X : \rho(x, y) \leq \delta \}, \quad B_{\delta} := \{ \xi \in T \cdot X : |\xi| \leq \delta \}, \]

Lemma 4.2. The superposition of any pair of random maps \( \mathcal{G} = \{ G_i, p_i \} \) and \( \mathcal{G}' = \{ G'_i, p'_i \} \) acting on the same manifold \( (X, \rho) \) and satisfying the Lipschitz condition is again the random map \( \mathcal{G} \circ \mathcal{G}' := \{ G'_i \circ G_j, p'_i p_j \} \), and \( \Lambda_{\mathcal{G} \circ \mathcal{G}'}(q) \leq \Lambda_{\mathcal{G}}(q) \cdot \Lambda_{\mathcal{G}'}(q) \) for each \( q \in [0, 1] \).

Proof. Indeed,
\[ \sum \]

\[ \Lambda_{\mathcal{G} \circ \mathcal{G}'}(q) = \sum_i \sum_j p'_i p_j \Lambda^{q}_{G'_i \circ G_j} \leq \sum_i \sum_j p'_i p_j \Lambda^{q}_{G'_i} \cdot \Lambda^{q}_{G_j} \]
\[ = \left( \sum_i p'_i \Lambda^{q}_{G'_i} \right) \cdot \left( \sum_j p_j \Lambda^{q}_{G_j} \right) = \Lambda_{\mathcal{G}'}(q) \cdot \Lambda_{\mathcal{G}}(q). \]

\[ \square \]

Theorem 4.1. For each number \( \kappa > 2 \) and for any \( h \in F_{\alpha} \) and \( n \in \mathbb{Z}_+ \), the Lasota—Yorke inequality holds:
\[ \| P_{\mathcal{T}} h \|_{(\alpha)} \leq \kappa \Lambda_{\mathcal{T}}^{\alpha}(\alpha) \| h \|_{(\alpha)} + \text{Const} \cdot (\kappa - 2)^{-1/\alpha} \| h \|_{(\beta)}. \] (4.2)

Proof. We start from the proof of the following two inequalities:
\[ \| P_{\mathcal{T}} h \|_{(\beta)} \leq \| h \|_{(\beta)}, \] (4.3)
\[ \| P_{\mathcal{T}} h \|_{(\alpha)} \leq \kappa \Lambda_{\mathcal{T}}^{\alpha}(\alpha) \| h \|_{(\alpha)} + \text{Const} \cdot (\kappa - 2)^{-1/\alpha} \| h \|_{(\beta)}. \] (4.4)

By the definition of the Hölder constant we have:
\[ \frac{\rho^\alpha(T_i x, y) - \rho^\alpha(T_i y)}{\rho^\alpha(x, y)} \frac{\rho^\alpha(T_i x, T_i y) - \rho^\alpha(T_i y)}{\rho^\alpha(x, T_i y)} \leq \Lambda_{\mathcal{T}}^{\alpha} H_{\alpha}(\varphi). \]

Hence,
\[ \sum_i p_i |\rho^\alpha(T_i x, T_i y) - \rho^\alpha(T_i y)| \leq \sum_i p_i \Lambda_{\mathcal{T}}^{\gamma} H_{\alpha}(\varphi). \]

Thus,
\[ H_{\alpha} \left( \sum_i p_i (\varphi \circ T_i) \right) \leq \Lambda_{\mathcal{T}}(\alpha) H_{\alpha}(\varphi) < H_{\alpha}(\varphi). \]

On the other hand, since 
\[ |\varphi \circ T_i|_{\infty} \leq |\varphi|_{\infty}, \]
we have
\[ \sum_i p_i |\varphi \circ T_i|_{\infty} \leq |\varphi|_{\infty}. \]

Therefore for each \( \beta \in (0, 1] \) we have
\[ V_{\beta} \left( \sum_i p_i |\varphi \circ T_i| \right) \leq V_{\beta}(\varphi), \]
which implies inequality (4.3) for each \( \beta \in (0, 1] \) and a function \( h \in F_{\beta} \).
Thus, $B_δ(x)$ is the ball of radius $δ$ centered at the point $x$ in the space $X$, while $B_δ$ is the ball of radius $δ$ centered at the origin in the tangent space $T_xX$. For each point $x ∈ X$ consider the exponential map

$$Ψ_x := \exp_x : B_δ ⊆ T_xX → X.$$  

Choosing $δ > 0$ small enough we can always assume that $Ψ_x B_δ ⊆ B_ν(x)$.

Denoting by $m$ the Lebesgue measure on $X$, we introduce the following smoothing operator

$$Q_δ ϕ(x) := \frac{∫_{Ψ_x B_δ} ϕ(y) m(dy)}{m(Ψ_x B_δ)} = \frac{∫_{B_δ} ϕ(Ψ_x z) · JΨ_x(z) dz}{∫_{B_δ} JΨ_x(z) dz},$$

where $JΨ_x$ is the Jacobian of the map $Ψ_x$.

Let us estimate the H"older constant of the function $Q_δ ϕ$:

$$|Q_δ ϕ(x) - Q_δ ϕ(y)|$$

$$≤ \left| ∫_{B_δ} JΨ_x(z) dz - ∫_{B_δ} JΨ_y(z) dz \right| · \frac{∫_{B_δ} ϕ(Ψ_x z) · JΨ_y(z) dz}{∫_{B_δ} JΨ_x(z) dz}$$

$$+ \frac{1}{∫_{B_δ} JΨ_x(z) dz} \left| ∫_{B_δ} ϕ(Ψ_x z) · JΨ_x(z) dz - ∫_{B_δ} ϕ(Ψ_y z) · JΨ_y(z) dz \right|$$

$$≤ 2(∫_{B_δ} JΨ_x(z) dz) \cdot |ϕ|_∞ \cdot \frac{|B_δ \setminus Ψ_x^{-1}Ψ_y B_δ| + |Ψ_x^{-1}Ψ_y B_δ \setminus B_δ|}{|B_δ|}$$

$$≤ \frac{1}{δ} C_1 |ϕ|_∞ ρ(x, y),$$

where the constant $C_1$ depends only on the properties of the manifold $X$.

Now we estimate how much the operator $Q_δ$ differs from the identical operator:

$$|Q_δ ϕ(x) - ϕ(x)| ≤ \frac{∫_{B_δ} |ϕ(Ψ_x(z)) - ϕ(x) · JΨ_x(z) dz}{∫_{B_δ} JΨ_x(z) dz} ≤ C_2 δ^α H_α(ϕ),$$

where the constant $C_2$ also depends only on the properties of the manifold $X$.

On the other hand,

$$H_α \left( ∑_i p_i(ϕ ◦ T_i) - Q_δ \left( ∑_i p_i(ϕ ◦ T_i) \right) \right)$$

$$≤ H_α \left( ∑_i p_i(ϕ ◦ T_i) \right) + H_α \left( Q_δ \left( ∑_i p_i(ϕ ◦ T_i) \right) \right)$$

$$≤ 2Λ_τ(α) H_α(ϕ).$$

Thus,

$$V_α \left( ∑_i p_i(ϕ ◦ T_i) - Q_δ \left( ∑_i p_i(ϕ ◦ T_i) \right) \right) ≤ 2Λ_τ(α) H_α(ϕ) + C_2 δ^α H_α(ϕ).$$
Gathering the obtained estimates and using Lemma 4.1, we come to
\[
\|P^T h\|_{(\alpha)} \leq \sup_{V_{\alpha}(\varphi) \leq 1} \int h \cdot Q_\delta \left( \sum_i p_i (\varphi \circ T_i) \right) \\
+ \sup_{V_{\beta}(\varphi) \leq 1} \int h \cdot \sum_i p_i (\varphi \circ T_i - Q_\delta (\varphi \circ T_i)) \\
\leq \sup_{V_{\alpha}(\varphi) \leq 1} V_{\alpha} \left( Q_\delta \left( \sum_i p_i (\varphi \circ T_i) \right) \right) \|h\|_{(\beta)} \\
+ \sup_{V_{\alpha}(\varphi) \leq 1} V_{\alpha} \left( \sum_i p_i \left( \varphi \circ T_i - Q_\delta (\varphi \circ T_i) \right) \right) \|h\|_{(\alpha)} \\
\leq C_1 \|h\|_{(\beta)} + (2 \Lambda_T (\alpha) + C_2 \delta^\alpha) \|h\|_{(\alpha)}.
\]
Therefore, choosing the value of the parameter \(\delta\) such small that \(C_2 \delta^\alpha = \kappa - 2\), we get inequality (4.4).

Now according to Lemma 4.2 and the above inequalities we get
\[
\|P^T h\|_{(\alpha)} = \|P^T \alpha h\|_{(\alpha)} \leq \kappa \Lambda_T (\alpha) \|h\|_{(\alpha)} + \text{Const} \delta^{-1} \|h\|_{(\beta)} \\
\leq \kappa \Lambda_T (\alpha) \|h\|_{(\alpha)} + \text{Const} \cdot (\kappa - 2)^{-1/\alpha} \|h\|_{(\beta)},
\]
which finishes the proof of inequality (4.2). □

**Lemma 4.3.** The function \(\Lambda_T (\cdot)\) is convex, takes values strictly less than 1 in the interval \((0, 1]\), and under the condition
\[
\prod_i \Lambda_{T_i}^{p_i} < 1
\]
its unique point of minimum either lies inside this interval, or is larger than 1.

**Proof.** By the definition of the contraction on average, we have \(\Lambda_T (1) = \Lambda_T < 1\). On the other hand, \(\Lambda_T (0) = 1\), and
\[
\frac{d^2}{dq^2} \Lambda_T (q) = \sum_i p_i \Lambda_{T_i}^{p_i} \cdot (\ln \Lambda_{T_i})^2 > 0.
\]
Thus the function \(\Lambda_T (q)\) is strictly convex and for any \(q \in (0, 1]\) it is less than 1. Let us show that this function strictly decreases at 0. Indeed,
\[
\frac{d}{dq} \Lambda_T (0) = \sum_i p_i \ln \Lambda_{T_i}.
\]
On the other hand, from the contraction on average
\[
\sum_i p_i \Lambda_{T_i} < 1
\]
using the convexity of the logarithmic function, we get:
\[
\sum_i p_i \ln \Lambda_{T_i} < \ln 1 = 0,
\]
which proves that \( \frac{d}{dq} \Lambda_T(0) < 0 \). It remains to check the last statement.

\[
\frac{d}{dq} \Lambda_T(1) = \sum_i p_i \Lambda_{T_i} \cdot \ln \Lambda_{T_i} = \sum_i \ln \Lambda_{T_i} = \ln \left( \prod_i \Lambda_{T_i} \right).
\]

Thus, due to inequality (4.5), the unique (due to the strict convexity of the function \( \Lambda_T(\cdot) \)) solution of the equation \( \frac{d}{dq} \Lambda_T(q) = 0 \) (the point of minimum of the function \( \Lambda_T(\cdot) \)) belongs to the interval \((0, 1)\). On the other hand, if inequality (4.5) does not hold, this solution is greater than 1. \( \square \)

A typical behavior of the function \( \Lambda_T(q) \) is shown on Fig. 2. Denote by \( \bar{q} \) the value of the parameter \( q \) corresponding to the unique (due to the strict convexity) minimum of the function \( \Lambda_T(q) \). The value \( \bar{q} \) is positive, since \( \frac{d}{dq} \Lambda_T(0) < 0 \). The position of \( \bar{q} \) with respect to 1 is defined by the sign of the derivative of the function \( \Lambda_T(q) \) at 1 which might be both positive and negative. For example, if all the maps are contractive, then this sign is negative and the function \( \Lambda_T(q) \) strictly decreases on the interval \([0, 1]\). On the other hand, the map in Example 1.2 satisfies condition (4.5) if \( 0 < p < 1/5 \), and in this case \( \bar{q} = \frac{1}{2} \log_2(1/p - 1) \). We describe the properties of the function \( \Lambda_T(q) \) in such detail because the fact that it can grow in the vicinity of the point 1 plays an important role in further calculations.

**Lemma 4.4.** The unit disk in the strong norm \( \| \cdot \|_{(\alpha)} \) is a compact set in the weak norm \( \| \cdot \|_{(\beta)} \).

The proof of this statement follows immediately from standard results on the enclosure of the spaces of Hölder functions.

By the Ionescu-Tulcea and Marinescu Theorem [14], the above statements imply the quasicompactness of the operator \( P_T \) and the validity of the following estimate of its essential spectral radius, based on the Nussbaum Theorem [20].

**Lemma 4.5** [12]. The essential spectral radius of the operator \( P_T: \mathcal{F}_\alpha \to \mathcal{F}_\alpha \) belongs to the disk of radius \( \Lambda_T(\alpha) \) centered at zero.
Note that the estimates leading to the Lasota—Yorke type inequalities depend sensitively on the choice of the value of the parameter \( \alpha \). Thus, it is reasonable to choose the value of \( \alpha \) which yields the smallest (and hence the best) available estimate of the essential spectral radius. Normally (compare to \([4]\)) this value is equal to 1 which is unavailable since we consider only Hölder continuous test functions. However, in our case Lemma 4.3 shows that under condition (4.5) the optimal value of \( \alpha \) can be strictly less than 1.

An immediate corollary to Lemma 4.5 is the existence of a constant \( \gamma \in [\Lambda_{\mathcal{T}}(\alpha), 1) \) such that the set \( \text{sp}( \mathcal{P}_{\mathcal{T}} ) \setminus \{|z| \leq \gamma\} \) consists of a finite number of peripheral eigenvalues \( r_1, \ldots, r_N \) of finite multiplicity. Denote by \( P_1, \ldots, P_N \) the corresponding spectral projectors, and set \( P := 1_{\mathcal{F}_n} - \sum_{j=1}^{N} P_j \). Then \( \text{rank}(P_j) < \infty \), \( \mathcal{P}_{\mathcal{T}} P_j = r_j P_j \) (\( j = 1, \ldots, N \)), and the spectral radius of the operator \( \mathcal{P}_{\mathcal{T}} P \) does not exceed \( \gamma \).

Besides,

- If \( |r| = 1 \), the operator

\[
P_r := \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} r^{-k} \mathcal{P}_r = \sum_{j=1}^{N} \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \left( \frac{r_j}{r} \right)^k P_j = \begin{cases} P_j, & \text{if } r = r_j, \\ 0, & \text{otherwise} \end{cases}
\]

is well defined in the \( \|\cdot\|^{(\alpha)} \)-norm. In particular, \( P_j = P_{r_j} \), and since \( \int |P_{r_j} f| \leq \int |f| \) for all \( f \in \mathcal{C}^1(\mathcal{X}, \mathbb{R}^1) \), the operators \( P_j \) can be extended continuously to the entire space \( \mathbb{L}^1 \).

- For any function \( f \in P_{r_j} \mathcal{F}_\alpha \) there is a finite Borel signed measure \( \mu_f \) on \( \mathcal{X} \) such that \( \langle f, \varphi \rangle = \int \varphi \, d\mu_f \) for all \( \varphi \in \mathcal{C}^1(\mathcal{X}, \mathbb{R}^1) \). \( r_1 := 1 \in \text{sp}(\mathcal{P}_{\mathcal{T}}) \), \( \mu := \mu_{P_{r_1}} \) is a positive measure, \( \mu(\mathcal{X}) = m(\mathcal{X}) \), and all signed measures \( \mu_f \) are absolutely continuous with respect to \( \mu \).

One can interpret these statements as follows: for \( f, \varphi \in \mathcal{C}^1(\mathcal{X}, \mathbb{R}^1) \) and \( |r| = 1 \), we have

\[
\langle P_r f, \varphi \rangle = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} r^{-k} \langle \mathcal{P}_r f, \varphi \rangle = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} r^{-k} \int \varphi \circ \mathbb{T}^k \cdot f.
\]  

Hence, \( \langle P_r f, \varphi \rangle \leq |\varphi|_\infty \cdot \int |f| \), so \( P_r f \) can be extended by continuity to a continuous linear functional on \( \mathcal{C}^0(\mathcal{X}, \mathbb{R}^1) \), and by the Riss Theorem there is a measurable \( \mu_{P_r} \) such that \( \langle P_r f, \varphi \rangle = \int \varphi \, d\mu_{P_r} \). If \( r = 1 \) and \( f, \varphi \geq 0 \), then \( \int \varphi \, d\mu_{P_1} = \langle P_1 f, \varphi \rangle \geq 0 \) and \( (P_1 f, 1) = \int f \) according to (4.7). \( \mu_{P_{r_1}} \) is a positive measure and \( r = 1 \) is an eigenvalue of the operator \( \mathcal{P}_{\mathcal{T}} \). Finally, it follows from (4.7) that for any \( \varphi \geq 0 \) we have

\[
|\langle P_{r_j} f, \varphi \rangle| \leq \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int \varphi \circ \mathbb{T}^k \cdot |f| \leq |f|_\infty \langle P_{r_1} 1, \varphi \rangle = |f|_\infty \int \varphi.
\]

Besides, \( \mu_{P_{r_j}} \) is absolutely continuous with respect to \( \mu \).

It remains to show that \( P_j \mathcal{F}_\alpha = V_j := P_j(\mathcal{C}^1(\mathcal{X}, \mathbb{R}^1)) \). Since \( V_j \subseteq P_j \mathcal{F}_\alpha \) and they are finite dimensional linear subspaces in \( \mathcal{F}_\alpha \), this statement immediately follows from the denseness of the space \( \mathcal{C}^1(\mathcal{X}, \mathbb{R}^1) \) in \( \mathcal{F}_\alpha \).

- \( \mathbb{T}^* \mu = \mu \), since \( \int \varphi \, d(\mathbb{T}^* \mu) = \int \varphi \circ \mathbb{T} \, d\mu = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int \varphi \circ \mathbb{T}^{k+1} = \int \varphi \, d\mu \) for all \( \varphi \in \mathcal{C}^1(\mathcal{X}, \mathbb{R}^1) \).
If $r = 1$ is a simple eigenvalue and there are no other eigenvalues equal to 1 by the absolute value, then $P_1 f = \langle f, 1 \rangle \cdot P_1 1$ for all $f \in F_\alpha$. This follows immediately from the fact that $\langle P_1 f, 1 \rangle = \langle f, 1 \rangle$ (see (4.7)).

$\mu$ is a SBR measure, since for each $\varphi \in C^1(X, \mathbb{R})$

$$
\lim_{n \to \infty} \int \varphi \, d \left( \frac{1}{n} \sum_{k=0}^{n-1} T^k m \right) = \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \int \varphi \circ T^k = \langle P_1 1, \varphi \rangle = \int \varphi \, d\mu.
$$

So far the only isolated eigenvalue that we were able to identify explicitly was the unit eigenvalue (the leading one). Let us discuss the following argument. According to Lemma 4.5, the essential spectral radius of the operator $P_T$ is not larger than $\Lambda_T(\alpha)$. On the other hand, Lemma 2.2 guarantees the convergence to the limit measure with the rate at least $1 > \Lambda_T > \Lambda_T(\alpha)$ (by Lemma 4.3). Thus, if there is a function $f \in F_\alpha$ such that $P_T f = \Lambda_T f$, then $\Lambda_T$ is an isolated eigenvalue.

**Example 4.1.** $T_1(x) := 1/2 + \text{sign}(x - 1/2) \cdot [1/2 - |2 \text{sign}(x - 1/2) \cdot (x - 1/2) - 1/2|]$, where $\text{sign}(x)$ is the sign of the number $x$, and $T_2(x) := x/2$.

The maps $T_1$ are shown on Fig. 3. Note that this random map is contractive on average when $0 < p < 1/3$. By construction, the map $T_1$ has three unstable fixed points $c = 1/6$, $1/2$, and $1-c = 5/6$ and the point $1/2$ is the only fixed point of the contractive map $T_2$. Besides, the trajectories of the random map starting at points $c$ and $1-c$ are symmetric and consist of a countable number of points $\{2^{-k} - 2^{-n} \cdot c \}_n$ and $\{2^{-k} + 2^{-n} \cdot (1 - c) \}_n$ respectively. Consider a family of generalized functions

$$
f_{\bar{a}}(x) := \sum_{k=0}^{n} a_k \cdot 1_{2^{-1} - 2^{-k} \cdot c}(x) - \sum_{k=0}^{n} a_k \cdot 1_{2^{-1} + 2^{-k} \cdot (1 - c)}(x),
$$

parameterized by the sequence of coefficients $\bar{a} = \{a_k\}_k$. Due to the previous remark, this family is invariant with respect to the action of $P_T$, and our aim is to find a sequence $\bar{a}$ such that $P_T f_{\bar{a}} = \Lambda_T f_{\bar{a}}$. 

**Figure 3.** Example of a random map with a nontrivial isolated eigenvalue
Lemma 4.6. In Example 4.1 for each $0 < p < 1/5$ there is a sequence $\bar{a} = \bar{a}(p)$ such that $\mathbf{P}_T f_{\bar{a}} = \Lambda_T f_{\bar{a}}$. However, $\|f_{\bar{a}}\|_{(\alpha)} = \infty$ for any $\alpha \in \left(\frac{1}{2} \log_2(1/p - 1), 1\right)$.

Proof. Observe that $\Lambda_T = (1 + 3p)/2 < 1$. Therefore, rewriting the eigenvalue relation in terms of the weights $a_k$, we obtain the following recurrent relations:

$$\frac{1 + 3p}{2} a_0 = pa_0 + pa_1,$$

$$\frac{1 + 3p}{2} a_k = (1 - p)a_{k-1} + pa_{k+1}$$

for each $k \geq 1$. Solving these equations with respect to the variables with higher indices we get:

$$a_1 = \frac{1 + p}{2p} a_0, \quad a_{k+1} = \frac{1 + 3p}{2p} a_k - \frac{1 - p}{p} a_{k-1} \quad \text{for} \quad k \geq 1.$$

The eigenvalues of the matrix $A = \begin{pmatrix} \frac{1 + 3p}{2p} & \frac{1 - p}{p} \\ 1 & 0 \end{pmatrix}$, controlling the growth of the coefficients, are equal to 2 and $(1 - p)/(2p)$ respectively. Now, since for $0 < p < 1/5$ the second eigenvalue is greater and since $a_1/a_0 = (1 + p)/(2p) > 2$, we deduce that the constants $a_k$ grow as $((1 - p)/(2p))^k$ as $k \to \infty$.

It remains to estimate the $\| \cdot \|_{(\alpha)}$-norm of the generalized function $f_{\bar{a}}$:

$$\|f_{\bar{a}}\|_{(\alpha)} = 2c^\alpha \sum_{k=0}^{\infty} a_k 2^{-k\alpha} < \infty$$

if and only if $((1 - p)/(2p))2^{-k\alpha} < 1$. On the other hand, $(1 - p)/(2p) > 2$ for $0 < p < 1/5$, which contradicts the convergence. □

4.3. Spectrum for the case of expanding on average random maps. Let $T_i$ for each $i$ be a map from the unit interval into itself and let the following condition hold:

$$\gamma := \sup_x \sum_i \frac{p_i}{\left| T_i(x) \right|} < 1,$$

where the supremum is taken over all points $x \in X$ where the derivatives of the maps $T_i$ are well defined. Then, according to Theorem 3.2, the Lasota—Yorke inequality is valid:

$$\var(\mathbf{P}^n_{T} h) \leq C\gamma^n \var(h) + \beta\|h\|.$$

Therefore all known results on the spectral properties of the Perron—Frobenius operator obtained for piecewise expanding maps based on similar inequalities remain valid as well (see for example the detailed discussion in [1]).

As we already mentioned, condition (3.5) does not imply the inequality of Lasota—Yorke type, and moreover there are examples when under condition 3.5 there is no exponential correlation decay. Therefore the question how to extend the description of the spectrum to this case remains open.
4.4. Stochastic stability. In this section we shall study random perturbations of random maps under consideration. Since under condition (3.4) perturbations of expanding on average random maps can be considered exactly as in the case of deterministic piecewise expanding maps (see [2, 3] and general discussion in [1]), we shall restrict ourselves only to the case of contracting on average systems.

As usual, under the randomly perturbed system we shall mean the superposition of the original system and a Markov process acting on the same phase space and defined by the family of transition operators $Q_\varepsilon$ (here $\varepsilon$ stands for the ‘size’ of perturbation).

To simplify the calculations we shall start from the case $X = \text{Tor}^d$ and then shall explain how the corresponding arguments should be changed in the case of a general smooth manifold.

Consider two families of operators: integral operators $Q_\varepsilon: F_\alpha \to C^1$ and the dual ones $Q_\varepsilon^*: C^1 \to C^1$:

$$Q_\varepsilon f(x) := \int q_\varepsilon(z, x) f(z) dz,$$
$$Q_\varepsilon^* \varphi(x) := \int q_\varepsilon(x, z) \varphi(z) dz$$

with the family of nonnegative kernels $q_\varepsilon(\cdot, \cdot)$, with respect to which we shall assume that for some $1 < M < \infty$ the following conditions hold:

$$\int q_\varepsilon(x, y) dy = 1, \quad q_\varepsilon(x, y) = 0 \quad \forall \rho(x, y) > \varepsilon,$$

(4.8)

$$\int |q_\varepsilon(x, z) - q_\varepsilon(y, z + y - x)| dz \leq M\rho(x, y),$$

(4.9)

We start the analysis of the operator $Q_\varepsilon^*$ with the following simple estimates.

**Lemma 4.7.** For each $\varphi \in C^\alpha$ we have

$$|Q_\varepsilon^* \varphi|_\infty \leq |\varphi|_\infty,$$

(4.10)

$$|Q_\varepsilon^* \varphi - \varphi|_\infty \leq \varepsilon^\alpha H_\alpha(\varphi).$$

(4.11)

**Proof.** The proof is straightforward:

$$|Q_\varepsilon^* \varphi(x)| = \left| \int q_\varepsilon(x, z) \varphi(z) dz \right| \leq \int q_\varepsilon(x, z) |\varphi(z)| dz \leq |\varphi|_\infty.$$  

$$|Q_\varepsilon^* \varphi(x) - \varphi(x)| = \left| \int q_\varepsilon(x, z) \varphi(z) dz - \varphi(x) \right| \leq \int q_\varepsilon(x, z) |\varphi(z) - \varphi(x)| dz$$

$$\leq \varepsilon^\alpha H_\alpha(\varphi),$$

since only the points $z \in B_\varepsilon(x)$ should be taken into account. \hfill \Box

Now let us estimate the norm of the operator $Q_\varepsilon$.

**Lemma 4.8.** Let $M_1(\varepsilon) := \max\{2\varepsilon^{\alpha/2}, M\varepsilon^{(1-\alpha)/2}\}$. Then $\|Q_\varepsilon\|_{(\alpha)} \leq 1 + M_1(\varepsilon)$.

**Proof.** Our aim is to show that $Q_\varepsilon^* \varphi$ is a valid test function and to estimate the values of $V_\alpha(Q_\varepsilon^* \varphi)$. We already estimated the supremum norm of the function $Q_\varepsilon^* \varphi$. To get the estimate of the Hölder constant, we consider two different situations:
when the points \( x, y \) are close, i.e. \( \rho(x, y) \leq \sqrt{\varepsilon} \), and the opposite case when they are far apart. In the first case, we proceed as follows:

\[
|Q_c^*\varphi(x) - Q_c^*\varphi(y)| = \left| \int q_c(x, z)\varphi(z)\,dz - \int q_c(y, z)\varphi(z)\,dz \right|
\]

\[
= \left| \int q_c(x, z)\varphi(z)\,dz - \int q_c(y, z + y - x)\varphi(z + y - x)\,dz \right|
\]

\[
\leq \left| \int q_c(x, z)\varphi(z)\,dz - \int q_c(x, z)\varphi(z + y - x)\,dz \right|
\]

\[
+ \left| \int q_c(x, z)\varphi(z + y - x)\,dz - \int q_c(y, z + y - x)\varphi(z + y - x)\,dz \right|
\]

\[
\leq \int q_c(x, z)|\varphi(z) - \varphi(z + y - x)|\,dz
\]

\[
+ \int |q_c(x, z) - q_c(y, z + y - x)| \cdot |\varphi(z + y - x)|\,dz
\]

\[
\leq \rho^\alpha(x, y)H_\alpha(\varphi) + M\rho(x, y)|\varphi|_\infty
\]

\[
\leq [H_\alpha(\varphi) + M\varepsilon^{(1-\alpha)/2}]\rho^\alpha(x, y).
\]

In the opposite case, when \( \rho(x, y) > \sqrt{\varepsilon} \), we shall proceed in a different way:

\[
|Q_c^*\varphi(x) - Q_c^*\varphi(y)| \leq |Q_c^*\varphi(x) - \varphi(x)| + |Q_c^*\varphi(y) - \varphi(y)| + |\varphi(x) - \varphi(y)|
\]

\[
\leq 2|Q_c^*\varphi - \varphi|_\infty + \rho^\alpha(x, y)H_\alpha(\varphi)
\]

\[
\leq 2\varepsilon^\alpha H_\alpha(\varphi) + \rho^\alpha(x, y)H_\alpha(\varphi)
\]

\[
\leq (1 + 2\varepsilon^{\alpha/2})\rho^\alpha(x, y)H_\alpha(\varphi).
\]

Hence we have

\[
V_\alpha(Q_c^*\varphi) = H_\alpha(Q_c^*\varphi) + |Q_c^*\varphi|_\infty
\]

\[
\leq (1 + 2\varepsilon^{\alpha/2})H_\alpha(\varphi) + M\varepsilon^{(1-\alpha)/2}|\varphi|_\infty
\]

\[
\leq (1 + \max\{2\varepsilon^{\alpha/2}, M\varepsilon^{(1-\alpha)/2}\})V_\alpha(\varphi).
\]

Thus, setting \( M_1(\varepsilon) := \max\{2\varepsilon^{\alpha/2}, M\varepsilon^{(1-\alpha)/2}\} \), we get

\[
\|Q_\varepsilon f\|_{(\alpha)} = \sup_{V_\alpha(\varphi)\leq 1} \int Q_\varepsilon f \cdot \varphi = \sup_{V_\alpha(\psi)\leq 1} \int f \cdot Q^*_\varepsilon \varphi
\]

\[
\leq \sup_{V_\alpha(\varphi)\leq 1} V_\alpha(Q^*_\varepsilon \varphi) \cdot \sup_{V_\alpha(\psi)\leq 1} \int f \cdot \psi
\]

\[
\leq (1 + M_1(\varepsilon)) \cdot \|f\|_{(\alpha)}.
\]

Observe that in several places we used the estimates of the supremum norms obtained in Lemma 4.7. □

**Lemma 4.9.** Let \( G : F_\alpha \to F_\alpha \) be a linear operator, and let \( G^* : C^1 \to C^1 \) be dual to it, i.e. \( \int Gf \cdot \varphi = \int f \cdot G^*\varphi \). Then for all \( f \in F_\beta \)

\[
\|Gf\|_{(\beta)} \leq \left( \sup_{V_\beta(\varphi)\leq 1} V_\alpha(G^*\varphi) \right) \cdot \|f\|_{(\alpha)}.
\]
Proof. Indeed,
\[ \|Gf\|_{(\beta)} = \sup_{V_\beta(\varphi) \leq 1} \int Gf \cdot \varphi = \sup_{V_\alpha(\varphi) \leq 1} \int f \cdot G^*\varphi \]
\[ \leq \left( \sup_{V_\beta(\varphi) \leq 1} V_\alpha(G^*\varphi) \right) \cdot \sup_{V_\alpha(\varphi) \leq 1} \int f \cdot \psi. \]
\[ \square \]

Lemma 4.10. We have
\[ \|Q_\varepsilon - I\| \equiv \|Q_\varepsilon - I\|_{(\beta - \alpha)} := \sup_{\|f\|_{(\alpha)} \leq 1} ||Q_\varepsilon f - f||_{(\beta)} \to 0 \text{ as } \varepsilon \to 0. \]

Proof. Applying Lemma 4.9 to the operator \( G = Q_\varepsilon - I \), we see that the sufficient condition of the validity of the desired statement is the convergence of
\[ \sup_{V_\beta(\varphi) \leq 1} V_\alpha(Q_\varepsilon^*\varphi - \varphi) \to 0 \]
as \( \varepsilon \to 0 \). Let us prove this convergence. Observe that since \( \beta > \alpha \) and \( \varphi \in C^\beta \), we can get a stronger estimate compared to Lemma 4.7:
\[ |Q_\varepsilon^*\varphi(x) - \varphi(x)| = \left| \int q_\varepsilon(x, z)\varphi(z) \, dz - \varphi(x) \right| \]
\[ \leq \int q_\varepsilon(x, z)|\varphi(z) - \varphi(x)| \, dz \]
\[ \leq \varepsilon^\beta H_\beta(\varphi). \]

Applying now estimates similar to ones used in the proof of Lemma 4.8 and taking into account that we consider more smooth test-functions \( \varphi \in C^\beta \) in the case \( \rho(x, y) \leq \varepsilon \), we get
\[ |(Q_\varepsilon^*\varphi(x) - \varphi(x)) - (Q_\varepsilon^*\varphi(y) - \varphi(y))| \leq |Q_\varepsilon^*\varphi(x) - Q_\varepsilon^*\varphi(y)| + |\varphi(x) - \varphi(y)| \]
\[ \leq \rho^\beta(x, y)H_\beta(\varphi) + M\rho(x, y)|\varphi|_\infty + \rho^\beta(x, y)H_\beta(\varphi) \]
\[ \leq [2\varepsilon^{\beta - \alpha}H_\beta(\varphi) + M\varepsilon^{1 - \alpha}|\varphi|_\infty] \rho^\alpha(x, y). \]

While in the opposite case, when \( \rho(x, y) > \varepsilon \), using the same argument as in the proof of Lemma 4.8, we get
\[ |(Q_\varepsilon^*\varphi(x) - \varphi(x)) - (Q_\varepsilon^*\varphi(y) - \varphi(y))| \leq 2|Q_\varepsilon^*\varphi - \varphi|_\infty \leq 2\varepsilon^\beta H_\beta(\varphi) \]
\[ \leq 2\varepsilon^{\beta - \alpha}H_\beta(\varphi) \rho^\alpha(x, y). \]

Hence for \( \varphi \in C^\beta \)
\[ H_\alpha(Q_\varepsilon^*\varphi - \varphi) \leq 2\varepsilon^{\beta - \alpha}H_\beta(\varphi) + M\varepsilon^{1 - \alpha}|\varphi|_\infty, \]
which yields the following estimate
\[ V_\alpha(Q_\varepsilon^*\varphi - \varphi) \leq 2\varepsilon^{\beta - \alpha}H_\beta(\varphi) + M\varepsilon^{1 - \alpha}|\varphi|_\infty + \varepsilon^\beta H_\beta(\varphi) \]
\[ \leq 3\varepsilon^{\beta - \alpha}H_\beta(\varphi) + M\varepsilon^{1 - \alpha}|\varphi|_\infty \leq (3 + M\varepsilon^{1 - \beta})\varepsilon^{\beta - \alpha}V_\alpha(\varphi) \to 0 \]
as \( \varepsilon \to 0 \). \[ \square \]
The properties of the transition operator obtained above together with Theorem 4.1 under the additional assumption that $\Lambda_T(\alpha) < 1/2$ make it possible to use results about the spectral stability of transfer operators satisfying Lasota-Yorke type inequalities [15] and to obtain the following stability result.

**Theorem 4.2.** Let the conditions (4.8), (4.9) be satisfied and let $\Lambda_T(\alpha) < 1/2$ for some $\alpha \in (0, 1)$. Then all elements of the spectrum $\Sigma_{P_T}(P_T)$ outside the disk of radius $\Lambda_T(\alpha)$ are stochastically stable and the corresponding eigenproectors of the perturbed system converge to the genuine ones.

**Proof.** First let us show that the transfer operator for the stochastically perturbed system satisfies a Lasota—Yorke type inequality. A straightforward calculation shows that this operator is equal to $Q_\varepsilon P_T$. Combining the results of Lemma 4.8 and Theorem 4.1, we get for any $h \in F_\alpha$ that

$$
\|Q_\varepsilon P_T h\|_{(\alpha)} \leq (1 + M_1(\varepsilon))\|P_T h\|_{(\alpha)}
\leq (1 + M_1(\varepsilon))\kappa \Lambda_T(\alpha)\|h\|_{(\alpha)} + \text{Const} \cdot (\kappa - 2)^{-\gamma}\|h\|_{(\beta)}.
$$

Therefore, if $\Lambda_T(\alpha) < 2$ for some $0 < \alpha < 1$, then the number $\gamma := (1 + M_1(\varepsilon)) \cdot \kappa \Lambda_T(\alpha) < 1$ for the value of $\kappa > 2$ guaranteed by Theorem 4.1. Since all other assumptions of the abstract spectral stability result in [15] were already checked during the analysis of our Banach spaces of generalized functions, we come to the desired statement.

To proceed further, we need to generalize the notion of the periodic turning point, well known in the one-dimensional dynamics. Namely, a point $x \in X$ is called the periodic turning point for the map $T : X \to X$ if $T^n x = x$ for some $n \in \mathbb{Z}_+$ and the derivative of the map $T$ is not well defined at the point $x$.

**Definition 4.1.** A point $x \in X$ is called the periodic turning point for the random map $\overline{T} : X \to X$ if there is a finite collection of indices $i_1, i_2, \ldots, i_k$ such that the point $x$ is the periodic turning point for the deterministic map $T_{i_1} \circ T_{i_2} \circ \cdots \circ T_{i_k}$.

For example, the point $x = 1/2$ is the periodic turning point for the random map in Example 1.2 for any nontrivial distribution $(0 < p < 1)$.

**Theorem 4.3.** The assumption $\Lambda_T(\alpha) < 1/2$ can be replaced by the following one: either all the maps $T_i$ are bijective and $C^1$-differentiable, or they are piecewise $C^1$-differentiable and have no periodic turning points. Then all isolated eigenvalues are stochastically stable.

**Proof.** The key idea here is to consider another representation of the perturbed operator $(Q_\varepsilon P_T)^n = \tilde{Q}_\varepsilon(n) P_T$ and to show that the new operator $\tilde{Q}_\varepsilon(n)$ satisfies the same assumptions (4.8), (4.9) as the operator $Q_\varepsilon$ (except that the value of $\varepsilon(n)$ might be $1/\min\{\Lambda_T\}$ times larger). Therefore, choosing $n$ large enough, we can always get $\Lambda_{\tilde{T}}(\alpha) < 1/2$. This idea was first applied in the case of piecewise expanding maps in [2, 3] and then in the case of hyperbolic maps in [4]. If all maps $T_i$ are bijective, this can be done by a simple change of variables, while in the second case the construction is more involved but is completely similar to the one in [2, 3].

□
Now let us show what should be changed in the case of a general smooth manifold. Since locally in a neighborhood of a point \( x \in X \) one can introduce local coordinates by means of the exponential map \( \Psi_x \), the tangent linear space \( T_xX \) can be isometrically mapped into \( \mathbb{R}^d \). Let \( \nu > 0 \) be a number such that for each point \( x \in X \) the ball (in the metrics \( \rho \)) of radius \( \nu \) centered at this point belongs to the domain of values of the exponential map \( \Psi_x \). Note that we have already introduced the restriction on the distance between the points in the definition of the Hölder constant needed to be consent with the domain of definition of the exponential map.

In fact, the first difference appears only in the analysis of random perturbations, in particular, condition (4.9) should be rewritten as

\[
\left| q_{\epsilon}(x, y) - q_{\epsilon}(\Psi_x(\Psi_{-1}x(x) + t), \Psi_y(\Psi_{-1}y(y) + t)) \right| dy \leq \rho(x, \Psi_x(\Psi_{-1}x(x) + t)) M,
\]

where \( t \in \mathbb{R}^d \) and \( |t| \leq \nu \).

Assuming now that \( \epsilon < \nu \) and replacing the expressions of type \( z + y - x \) to

\[
\Psi_z(\Psi_{-1}z(z) + \Psi_{-1}y(y) - \Psi_{-1}x(x)),
\]

we obtain the same estimates as in the flat case (when \( X \) is the unit torus). Therefore all results of this section remain valid for the case of a general smooth manifold.

4.5. Finite rank approximations. Now let us discuss finite-dimensional approximations of transfer operators. Again, due to the same reason as in the previous section, we shall restrict the analysis to the case of contracting on average random maps.

Let \( \{\Delta_i\} \) be a finite partition of the phase space \( X \) into domains (cells) \( \Delta_i \) of diameter not greater than \( \delta > 0 \). For a point \( x \in X \), by \( \Delta_x \) we denote the element of the partition containing it. Under this notation, the so called Ulam approximation can be described as the operator

\[
\tilde{Q}_{\delta} f(x) := \frac{1}{|\Delta|} \int_{\Delta_x} f.
\]

Note that this operator is self-dual, i.e. \( \tilde{Q}_{\delta} = \tilde{Q}_{\delta}^* \). One can also easily check that the dimension of the space \( \tilde{Q}_{\delta} \mathcal{F}_\alpha \) coincides with the number of elements in the Ulam partition.

Lemma 4.11. We have \( \|Q_{\delta}\|_{(\alpha)} = \infty \).

Proof. Let a point \( y_0 \in X \) belong to the boundary between two elements of the Ulam partition, and let the points \( y(\epsilon) \) and \( y'(\epsilon) \) belong to neighboring elements of the partition both on the distance \( \epsilon \) from \( y_0 \). For the function

\[
f_{\epsilon}(x) := 1_{y(\epsilon)}(x) + 1_{y'(\epsilon)}(x),
\]

where \( 1_y \) means the \( \delta \)-function at the point \( y \), the following inequalities hold:

\[
\|f_{\epsilon}\|_{(\alpha)} \leq \text{Const } \epsilon^\alpha,
\]

\[
\|Q_{\delta} f_{\epsilon}\|_{(\alpha)} \geq \text{Const } > 0.
\]
Figure 4. A counterexample for the original Ulam construction

The first of these inequalities follows from the definition of the norm $\| \cdot \|_{(\alpha)}$, while the second one is a consequence of the fact that the function $Q_\delta f_\epsilon$ is the characteristic function of the union of two neighboring elements of the partition containing the points $y(\epsilon)$ and $y'(\epsilon)$. Thus, $\|Q_\delta f_\epsilon\|_{(\alpha)}/\|f_\epsilon\|_{(\alpha)} \to \infty$ as $\epsilon \to 0$.

This result shows that the original Ulam approximation scheme cannot be immediately applied to the spectral analysis of random maps.

What is still possible is that the leading eigenfunction—SBR measure can be stable for the class of maps we consider (the above example does not contradict this: the SBR measure is preserved). I believe there should be very deep reasons explaining the stability of the leading eigenfunction, while all others are not stable, however presently we do not have the adequate explanation. In the literature (see, for example, [1] and further references therein) the stability of the SBR measure is proven for the class of piecewise expanding maps. Moreover, numerous numerical studies confirm this stability for a much broader class of dynamical systems. To the best of our knowledge, the following simple example of a one-dimensional discontinuous map represent the first counterexample to the original Ulam hypothesis.

**Lemma 4.12.** The map

$$T x := \begin{cases} \frac{x}{4} + \frac{1}{2}, & \text{if } 0 \leq x < \frac{5}{12}, \\ -2x + 1, & \text{if } \frac{5}{12} \leq x < \frac{1}{2}, \\ \frac{x}{2} + \frac{1}{4}, & \text{otherwise}, \end{cases}$$

from the unit interval into itself is uniquely ergodic, but the leading eigenvector of the Ulam approximation $\Pi_{1/n} P_T$ does not converge weakly to the only $T$-invariant measure.

Observe that the situation when the SBR measure becomes unstable with respect to the Ulam scheme is indeed very exotic and the map in the corresponding example is not only discontinuous (see Fig. 4), but this discontinuity occurs in a periodic...
turning point (compare to instability results about general random perturbations in [3]).

Proof. Denote by \( v^{(n)} \) the normalized leading eigenvector of the matrix \( \Pi_{1/n} P_T \).
A straightforward calculation shows that for each \( n \in \mathbb{Z}_+ \) all entries of the vector \( v^{(2n+1)} \) are zeros except the first entry, which is equal to \( 1/3 \), and the \((n + 1)\)-th one, which is equal to \( 2/3 \). Compare this to the only invariant measure of the map \( T \), the unit mass at the point 1/2. \( \square \)

To overcome this difficulty, let us consider another ‘smoothed’ approximation scheme. In each element of the partition \( \{ \Delta_i \} \) (of diameter \( \leq \delta \)) we fix an arbitrary point (its ‘center’) \( x_i \in \Delta_i \). Now, for a given smooth enough kernel \( q_\varepsilon(\cdot, \cdot) \) satisfying the assumptions from the previous section, we define the following finite dimensional operator:

\[
Q_{\varepsilon, \delta} f(x) := \sum_i 1_{\Delta_i}(x) \int q_\varepsilon(z, x_i) f(z) \, dz.
\]

Observe that the dual operator is equal to:

\[
Q^*_{\varepsilon, \delta} \varphi(x) := \sum_i q_\varepsilon(x, x_i) \int_{\Delta_i} \varphi(z) \, dz.
\]

Indeed,

\[
\int Q_{\varepsilon, \delta} f(x) \cdot \varphi(x) \, dx = \int \sum_i 1_{\Delta_i}(x) \int q_\varepsilon(z, x_i) f(z) \, dz \cdot \varphi(x) \, dx = \int f(z) \sum_i q_\varepsilon(z, x_i) \left( \int 1_{\Delta_i}(x) \varphi(x) \, dx \right) \, dz = \int f(z) \sum_i q_\varepsilon(z, x_i) \int_{\Delta_i} \varphi(x) \, dx \, dz = \int f(z) \cdot Q^*_{\varepsilon, \delta} \varphi(z) \, dz.
\]

Lemma 4.13. Let, additionally to assumptions (4.8), (4.9), for any points \( x, y, z \in X \) the inequality

\[
|q_\varepsilon(x, y) - q_\varepsilon(x, z)| + |q_\varepsilon(y, x) - q_\varepsilon(z, x)| \leq M \varepsilon^{-d-1} \rho(y, z).
\]

hold. Then

\[
\|Q_{\varepsilon} f - Q_{\varepsilon, \delta} f\|_{(\alpha)} \leq 5M \varepsilon^{-d-1} \delta^{1-\alpha} \|f\|_{(\alpha)}.
\]

Proof.

\[
\int (Q_{\varepsilon} f - Q_{\varepsilon, \delta} f) \cdot \varphi = \int f \cdot (Q^*_{\varepsilon} \varphi - Q^*_{\varepsilon, \delta} \varphi).
\]

Denote

\[
\Phi(x) := Q^*_{\varepsilon} \varphi(x) - Q^*_{\varepsilon, \delta} \varphi(x) = \sum_i \int_{\Delta_i} (q_\varepsilon(x, z) - q_\varepsilon(x, x_i)) \varphi(z) \, dz.
\]
Then
\[ |\Phi|_{\infty} \leq |\varphi|_{\infty} \cdot \sup_x \sum_i \int_{\Delta_i} |q_\varepsilon(x, z) - q_\varepsilon(x_i, x)| \, dz \]
\[ \leq |\varphi|_{\infty} \cdot \sup_x \sum_i \int_{\Delta_i} (|q_\varepsilon(x, z) - q_\varepsilon(x, x_i)| + |q_\varepsilon(x, x_i) - q_\varepsilon(x_i, x)|) \, dz \]
\[ \leq 2M \varepsilon^{-d-1} \delta |\varphi|_{\infty}, \]

since \( x, z \in \Delta_i \), and hence, \( \max \{ \rho(z, x_i), \rho(x, x_i) \} \leq \delta \).

Let us estimate \( H_\alpha(\Phi) \). If \( \rho(x, y) \leq \delta \) then
\[ |\Phi(x) - \Phi(y)| \leq |Q_\varepsilon^* \varphi(x) - Q_\varepsilon^* \varphi(y)| + |Q_\varepsilon^* \delta \varphi(x) - Q_\varepsilon^* \delta \varphi(y)| \]
\[ \leq \int |q_\varepsilon(x, z) - q_\varepsilon(y, z)| \cdot |\varphi(z)| \, dz + \sum_i |q_\varepsilon(x, x_i) - q_\varepsilon(y, x_i)| \int_{\Delta_i} |\varphi| \]
\[ \leq M \varepsilon^{-d-1} \rho(x, y) |\varphi|_{\infty} + M \varepsilon^{-d-1} \rho(x, y) \cdot |\varphi|_{\infty} \]
\[ = 2M \varepsilon^{-d-1} \delta^{1-\alpha} \rho^\alpha(x, y) \cdot |\varphi|_{\infty}. \]

Otherwise, if \( \rho(x, y) > \delta \), we apply another estimate
\[ |\Phi(x) - \Phi(y)| \leq 2|\Phi|_{\infty} \leq 4M \varepsilon^{-d-1} \delta |\varphi|_{\infty} \leq 4M \varepsilon^{-d-1} \rho^\alpha(x, y) \cdot \delta^{1-\alpha} \cdot |\varphi|_{\infty}. \]

Thus,
\[ H_\alpha(\Phi) \leq 4M \varepsilon^{-d-1} \delta^{1-\alpha} \cdot |\varphi|_{\infty}, \]

and hence
\[ V_\alpha(\Phi) \leq 5M \varepsilon^{-d-1} \delta^{1-\alpha} \cdot V_\alpha(\varphi), \]

which yields the desired statement. \( \square \)

**Theorem 4.4.** Let the family of kernels \( \{ q_\varepsilon(\cdot, \cdot) \} \) satisfy conditions (4.8), (4.9), (4.12). Then
\[ \| Q_\varepsilon, \delta \|_{(\alpha)} \leq 1 + M_1(\varepsilon) + 5M \varepsilon^{-d-1} \delta^{1-\alpha}, \]
\[ \| Q_\varepsilon, \delta - 1 \| \leq (5M + 3 + M \varepsilon^{1-\beta})(\varepsilon^{-d-1} \delta^{1-\alpha} + \varepsilon^{3-\alpha}) \to 0 \]
as \( \varepsilon^{-d-1} \delta^{1-\alpha} + \varepsilon^{3-\alpha} \to 0. \)

Hence for the case \( \Lambda_\Upsilon(\alpha) < 1/2 \) the isolated eigenvalues and the corresponding eigenprojectors of the operator \( P_\Upsilon \) are stable with respect to the considered approximation.

**Proof.** According to Lemmas 4.8 and 4.13
\[ \| Q_\varepsilon, \delta \|_{(\alpha)} \leq \| Q_\varepsilon \|_{(\alpha)} + \| Q_\varepsilon, \delta - Q_\varepsilon \|_{(\alpha)} \leq 1 + M_1(\varepsilon) + 5M \varepsilon^{-d-1} \delta^{1-\alpha}, \]

which proves the first statement.

Similarly but using Lemma 4.10 instead of Lemma 4.8, we get
\[ V_\alpha(Q_\varepsilon, \delta \varphi - \varphi) \leq V_\alpha(Q_\varepsilon, \delta \varphi - Q_\varepsilon \varphi) + V_\alpha(Q_\varepsilon \varphi - \varphi) \]
\[ \leq (5M \varepsilon^{-d-1} \delta^{1-\alpha} + (3 + M \varepsilon^{1-\beta})\varepsilon^{3-\alpha}) \cdot V_\alpha(\varphi), \]

which finishes the proof. \( \square \)
In fact, one can consider the finite dimensional approximation defined by the two-parameter family of operators \( \{ Q_{\varepsilon, \delta} \} \varepsilon, \delta \) as a smoothed version of the original Ulam construction, which corresponds to the case \( \varepsilon = 0 \). Observe that in our approximations the relation between the parameters is completely different: it is necessary that \( \varepsilon \gg \delta \).

We consider also another (seeming more natural) finite rank approximation scheme. Denote by \( \Pi_3 \) the pure Ulam approximation operator corresponding to the partition into domains \( \{ \Delta_i \} \) whose diameters do not exceed \( \delta \):

\[
\Pi_3 f(x) := \frac{1}{|\Delta_i|} \int_{\Delta_i} f(s) \, ds,
\]

where \( \Delta_i \) stands for the element of the partition containing the point \( x \). Note that this operator is self adjoint. We shall approximate our transfer operator \( P_T \) by \( \Pi_3 Q T \). To study the properties of this approximation, we need as usual to analyze the properties of the adjoint operator, i.e. of the operator

\[
Q_{\varepsilon}^\ast \Pi_3 \varphi(x) = \int q_{\varepsilon}(x, z) \frac{1}{|\Delta_z|} \int_{\Delta_z} \varphi(s) \, ds \, dz.
\]

**Lemma 4.14.** \( \| Q_{\varepsilon} - \Pi_3 Q_{\varepsilon} \|_{(\alpha)} \leq (3+2M) \varepsilon^{-d-1} \delta^{\alpha(1-\alpha)} \to 0 \) as \( \varepsilon^{-d-1} \delta^{\alpha(1-\alpha)} \to 0 \).

**Proof.** Denote

\[
\Phi(x) := Q_{\varepsilon}^\ast \varphi(x) - Q_{\varepsilon}^\ast \Pi_3^\ast \varphi(x) = \int q_{\varepsilon}(x, z) \left( \varphi(z) - \frac{1}{|\Delta_z|} \int_{\Delta_z} \varphi(s) \, ds \right) \, dz
\]

\[
= \int q_{\varepsilon}(x, z) \frac{1}{|\Delta_z|} \int_{\Delta_z} (\varphi(z) - \varphi(s)) \, ds \, dz.
\]

Since \( \varphi \in C^\alpha \) and the diameter of the elements of the partition does not exceed \( \delta \), we have

\[
|\Phi|_\infty \leq \delta^\alpha H_\alpha(\varphi) \sup_x \int q_{\varepsilon}(x, z) \, dz = \delta^\alpha H_\alpha(\varphi).
\]

Now we are going to estimate the Hölder constant of the function \( \Phi \), which we shall do in two steps. First, we consider the case when \( \rho(x, y) \leq \delta^\alpha \):

\[
|\Phi(x) - \Phi(y)| \leq |Q_{\varepsilon}^\ast \varphi(x) - Q_{\varepsilon}^\ast \varphi(y)| + |Q_{\varepsilon}^\ast \Pi_3^\ast \varphi(x) - Q_{\varepsilon}^\ast \Pi_3^\ast \varphi(y)|
\]

\[
\leq \int |q_{\varepsilon}(x, z) - q_{\varepsilon}(y, z)| \cdot |\varphi|_\infty \, dz
\]

\[
+ \int |q_{\varepsilon}(x, z) - q_{\varepsilon}(y, z)| \cdot \frac{1}{|\Delta_z|} \int_{\Delta_z} |\varphi(s)| \, ds \, dz
\]

\[
\leq 2M \varepsilon^{-d-1} \rho(x, y)|\varphi|_\infty \leq 2M \varepsilon^{-d-1} \delta^{\alpha(1-\alpha)} |\varphi|_\infty \rho^\alpha(x, y).
\]

In the opposite case, when \( r(x, y) > \delta^\alpha \), we use a different estimate:

\[
|\Phi(x) - \Phi(y)| \leq 2|\Phi|_\infty \leq 2\delta^\alpha H_\alpha(\varphi) + 2\delta^\alpha(1-\alpha) H_\alpha(\varphi) \rho^\alpha(x, y).
\]
Thus
\[
V_\alpha(\Phi) \leq 2\delta^{\alpha(1-\alpha)}H_\alpha(\varphi) + 2M\varepsilon^{-d-1}\delta^{\alpha(1-\alpha)}|\varphi|_\infty + \delta^\alpha H_\alpha(\varphi)
\]
\[
\leq 3\delta^{\alpha(1-\alpha)}H_\alpha(\varphi) + 2M\varepsilon^{-d-1}\delta^{\alpha(1-\alpha)}|\varphi|_\infty
\]
\[
\leq (3 + 2M)\varepsilon^{-d-1}\delta^{\alpha(1-\alpha)}V_\alpha(\varphi) \to 0
\]
as \(\varepsilon^{-d-1}\delta^{\alpha(1-\alpha)} \to 0\).

Observe that the rate of convergence in this approximation is lower than the previous one, however the numerical application of the 2nd scheme is more straightforward.

**Corollary 4.15.** Again as in Theorem 4.3 and due to the same reason, the assumption \(\Lambda_T(\alpha) < 1/2\) can be replaced by either the bijectivity of the maps \(T_i\) or the absence of periodic turning points of the random map \(T\).

**References**


Institute for Information Transmission Problems, RAS, B. Karetny per. 19, Moscow 101447, Russia, and Observatoire de la Côte d’Azur, BP 4229, F-06304 Nice Cedex 4, France

E-mail address: blank@obs-nice.fr