Abstract. We consider a discrete-time random walk $X_t$ on $\mathbb{Z}$ with transition probabilities
$$P(X_{t+1} = x+u \mid X_t = x, \xi) = P_0(u) + c(u; \xi(t, x)),$$
depending on a random field $\xi = \{\xi(t, x): (t, x) \in \mathbb{Z} \times \mathbb{Z}\}$. The variables $\xi(t, x)$ take finitely many values, are i.i.d. and $c(u; \cdot)$ has zero average. Previous results show that for small stochastic term the CLT holds almost surely, with dispersion independent of the field. Here we prove that the first correction in the CLT asymptotics is a term of order $T^{-1/4}$ depending on the field, with asymptotically gaussian distribution as $T \rightarrow \infty$.

2000 Math. Subj. Class. 60J15, 60F05, 60G60, 82B41.

Key words and phrases. Random walk, random environment, Central Limit Theorem.

1. Introduction

The model that we consider here is a general model of random walk in a random environment that changes with time, previously studied in the papers [4, 2, 6]. Let $X_t \in \mathbb{Z}^\nu$, $t \in \mathbb{Z}$, be a discrete-time random walk for which the probability of jumping at time $t$ depends on a random field $\xi = \{\xi(t, x): (t, x) \in \mathbb{Z}^{\nu+1}\}$ taking values in a finite set $S$, i.e., $\xi(t, x) \in S$:

$$P(X_{t+1} = y \mid X_t = x, \xi) = P_0(y-x) + c(y-x; \xi(t, x)). \quad (1.1)$$

We assume that the variables $\xi(t, x)$, $(t, x) \in \mathbb{Z}^{\nu+1}$, are i.i.d. with some non-degenerate distribution $\pi = \{\pi(s): s \in S\}$. It is not restrictive to assume that the random term has zero average $\langle c(u; \cdot) \rangle_\pi = 0$, so that $P_0$ gives the average transition probabilities, and $\sum_u c(u; s) = 0$ for all $s \in S$. We are interested in the asymptotic behavior of the displacement $X_t$ as $t \rightarrow \infty$ for a fixed configuration of the environment $\xi$.

The results proved in the papers [4, 2, 6], for small randomness, and under some standard additional assumptions, include the Central Limit Theorem (CLT) for the displacement $X_t - X_0$ almost everywhere in $\xi$, and a kind of local theorem, with the...
leading term of the asymptotics depending on the environment [6]. The model has strong analogies with a directed polymer model that has been studied by several authors (see [5] and references therein). The analogy holds for polymer models in dimension \( \nu \geq 3 \) in the small randomness region.

The random walk in random environment considered in the paper [8] is also of the form (1.1), in which \( P_0 \) is the standard random walk and the stochastic term is given by a random drift. The results of that paper indicate that the small randomness condition is not needed for the Central Limit Theorem.

In the present paper, we consider the behavior of the corrections to the CLT. Some results on the behavior of higher order terms in the CLT expansion for large times were obtained in the paper [4]. They depend on the environment and the traditional expansion in inverse powers of \( T^{1/2} \) is reduced to only a finite number of terms, more precisely it holds up to the term of order \( T^{-k/2} \), where \( k = \lfloor (\nu - 1)/2 \rfloor \) is the largest integer smaller that \( \nu/2 \). In particular for \( \nu = 1, 2 \) there is no such expansion. In the paper [1], it was shown that the random corrections to the first two cumulants for \( \nu = 1, 2 \) are asymptotically Gaussian for large times when properly normalized.

We consider here the case \( \nu = 1 \), and we show that the correction to the CLT is a term of order \( T^{-1/4} \), depending on the environment, which, if normalized, tends as \( T \to \infty \) to a Gaussian random variable. It is worth to remark that the random fluctuation due to the environment is of the order of the square root of the inverse of the macroscopic space scale (which is \( T^{1/2} \)), as usual in statistical mechanics.

More precisely, let \( f \) be a smooth function and consider, for \( \xi \) fixed, the usual normalized average. We separate the average from the random part as follows

\[
\sum_x P(X_T = x \mid X_0 = 0, \xi) f\left( \frac{x - b T}{\sqrt{T}} \right) = \sum_x P^T_0(x) f\left( \frac{x - b T}{\sqrt{T}} \right) + T^{-1/4} \hat{Q}_T(f|\xi).
\]

(1.2)

The main result of the present paper consists in proving that the “correction” \( \hat{Q}_T(f|\xi) \) tends, as \( T \to \infty \), to a limiting Gaussian variable which can be represented as an integral:

\[
\int_0^1 ds \int du K_\sigma(s, u) \zeta(s, u) \int dv K_\sigma(1 - s, v) f'(u + v),
\]

(1.3)

where \( \sigma^2 = \sum_u P_0(u)(u - b)^2 \) is the dispersion of the averaged random walk,

\[
K_\sigma(s, u) = \frac{e^{-s^2/(2\sigma^2)}}{\sqrt{2\pi\sigma^2 s}} \]

is the heat kernel, and \( \zeta(s, u) \) is the space-time white noise:

\[
\langle \zeta(s, u) \zeta(s', u') \rangle = \delta_{s,s'}\delta_{u,u'}.
\]

The representation (1.3) shows that the space-time environment acts essentially “point-wise” in space and time.
2. Assumptions and Formulation of the Results

The measure space \((\Omega, \varphi)\) of the environment is the product of copies of \((S, \pi)\) over \(\mathbb{Z}^2\): \(\Omega = S^{\mathbb{Z}^2}\) and \(\varphi = \pi^{\mathbb{Z}^2}\). Averages with respect to \(\varphi\) or \(\pi\) will be denoted by \(\langle \cdot \rangle\).

For \(\xi \in \Omega\) fixed, the random walk transition probabilities are given by (1.1). We assume the following properties.

I. \(\{P_0(u) + c(u; s) : u \in \mathbb{Z}^n\}\) is a probability distribution on \(\mathbb{Z}\) for all \(s \in S\).

II. \(\langle c(u; \cdot) \rangle = 0\) which implies that \(\{P_0(u) : u \in \mathbb{Z}^n\}\) is also a probability distribution on \(\mathbb{Z}\), and \(\sum_u c(u; s) = 0\) for all \(s \in S\).

III. \(P_0\) and \(c\) are finite range, i.e., there is a constant \(D > 0\) such that \(P_0(u) = 0\) and \(c(u; s) = 0\) for all \(s \in S\) if \(|u| > D\).

IV. \(\{P_0(u) : u \in \mathbb{Z}^n\}\) are the transition probabilities of a completely irreducible random walk ("average random walk"), i.e., the characteristic function

\[\tilde{p}_0(\lambda) = \sum_u P_0(u) e^{i \lambda u}, \quad \lambda \in S^1,\]

is such that \(|\tilde{p}_0(\lambda)| < 1\) for \(\lambda \neq 0\). This implies that the term \(\sigma^2\) which appears in the expansion \(\log \tilde{p}_0(\lambda) = ib\lambda - \frac{1}{2} \sigma^2 \lambda^2 + \cdots\) at \(\lambda = 0\) is positive.

As the random term will have to satisfy a smallness condition, we introduce a factor \(\epsilon \in (0, 1)\) and write \(\epsilon c\) instead of \(c\). Observe that if \(c\) satisfies the assumptions above then \(\{P_0(u) + \epsilon c(u; s) : u \in \mathbb{Z}^n\}\) is a probability distribution for all \(\epsilon \in [0, 1]\).

There is no loss of generality in assuming that the random walk starts at the origin at time \(0\). Proceeding as in [4, 6] we see that the probability for the random walk to start at the origin and to be at \(x\) at time \(T\) is written as \(P(X_T = x | X_0 = 0, \xi) = P_T^x(x) + Q_T(x | \xi)\), where \(P_T^x\) denotes the convolution \(P_T^x = P_0 * P_0 * \cdots * P_0\), and the random term is

\[Q_T(x | \xi) := P(X_T = x | X_0 = 0, \xi) - P_T^x(x) = \sum_{0 \leq t_1 \leq t_2 \leq T-1} \sum_y P_{t_1}^y(y_1) \times M_s(t_2 - t_1, y_2 - y_1; \xi(t_1, y_1)) h^{T-t_2}(x - y_2; \xi(t_2, y_2)).\]

Here \(h(t; y) = (c(\cdot; s) * P_0^{t-1}(\cdot))(y)\), \(\xi(t, y)(\tau, z) = \xi(\tau - t, z - y)\) is the shifted environment, and

\[M_s(t, y; \xi) = \sum_{B: (0,0) \rightarrow (t, y)} e^{[B]} M_B^s(\xi), \quad M_B^s(\xi) = \prod_{i=1}^{n-1} h^{z_i}(z_i; \xi(t_i, y_i)).\]

where the first sum is over the possible subsets of points \(B = \{(t_1, y_1), \ldots, (t_n, y_n)\}\) of a trajectory starting at \((0, 0)\) and ending at \((t, y)\), the quantities \(\tau_i = t_{i+1} - t_i > 0, z_i = y_{i+1} - y_i\), denote the differences between subsequent points, and \(n = |B|\). Moreover, \((t_i(B), y_i(B)) := (t_1, y_1)\) and \((t_f(B), y_f(B)) := (t_n, y_n)\) are the initial and final point of \(B\), respectively, and the notation \(B: (t, x) \rightarrow (t', x')\) means that the sum is restricted to those \(B\) for which \(t_1(B) = t, y_1(B) = x, t_f(B) = t', y_f(B) = x'\). For \(t = 0\) we use the conventions \(P_0^0(y) = \delta_{y,0}\), and \(M_s(0, y; \xi) = \epsilon \delta_{y,0}\).
By Lemma (A.1) of [4], which is valid for \( \nu = 1 \) as well, setting \( k^t(x) = \max_u |h^t(x; s)| \), we have the following inequalities, for some \( C > 0 \) and all \( t > 0 \):

\[
\sum_{y \in \mathbb{Z}} (P^0_t(y))^2 \leq \frac{C}{t^{1/2}}, \quad \sum_{y \in \mathbb{Z}} (k^t(y))^2 \leq \frac{C}{t^{1/2}}. \quad (2.3)
\]

Let \( f \) be a smooth function, \( f \in C^{1+\alpha} \), with \( \alpha \in (0, 1) \), the norm in \( C^{1+\alpha} \) being \( \|f\|_{1+\alpha} = \|f\|_\infty + \|f'\|_\infty + \sup_{x \neq z} |f(x) - f(z)| / |x - z|^{1+\alpha} \). The conditional average of \( f \) is written as

\[
\sum_x P(X_T = x \mid X_0 = 0, \xi) f\left( \frac{x - bT}{\sqrt{T}} \right) = \sum_x P^0_T(x) f\left( \frac{x - bT}{\sqrt{T}} \right) + Q_T(f|\xi),
\]

and we are interested in the random term

\[
Q_T(f|\xi) = \sum_x Q_T(x|\xi) f\left( \frac{x - bT}{\sqrt{T}} \right). \quad (2.4)
\]

**Theorem.** Under the assumptions above, if \( \epsilon \) is small enough, the normalized functionals \( \tilde{Q}_T(f|\xi) := T^{1/4} Q_T(f|\xi) \) tend in distribution as \( T \to \infty \) to a centered Gaussian variable with dispersion

\[
M \int_0^1 ds \int du K^2_\alpha(s, u) \left( \int dv K_\alpha(1 - s, v) f'(u + v) \right)^2,
\]

where the constant \( M \) will be determined later (see the remark following the proof of Proposition 3 below).

The proof is based on several intermediate results stated as lemmas and propositions. In the course of the proof the notation const will denote several constants, which may depend on \( \epsilon \).

Taking into account that \( \sum_u c(u; s) = 0 \), setting \( b(s) = \sum_u u c(u; s) \), one finds for any real \( y \) and some \( u_*, |u_*| \leq D \),

\[
\sum_u c(u; s) f\left( \frac{y + u}{\sqrt{T}} \right) = \frac{y}{\sqrt{T}} \sum_u \frac{c(u; s)}{\sqrt{T}} f'\left( \frac{y + u}{\sqrt{T}} \right) = \frac{1}{\sqrt{T}} \left( b(s) f'\left( \frac{y}{\sqrt{T}} \right) + r_T(y; s) \right),
\]

where \( \langle r_T(y; \cdot) \rangle = 0 \), and for all \( y, s \) we have \( |r_T| \leq \text{const} T^{-\alpha/2} \|f\|_{1+\alpha} \). Hence the quantities

\[
\delta_T(t, y; s) := \sum_x h^t(x; s) f\left( \frac{y + x}{\sqrt{T}} \right) - \frac{b(s)}{\sqrt{T}} \sum_z P^{t-1}_0(z) f'\left( \frac{y + z}{\sqrt{T}} \right)
\]

\[
= \sum_z P^{t-1}_0(z) \left[ \sum_u c(u; s) f\left( \frac{y + u + z}{\sqrt{T}} \right) - \frac{b(s)}{\sqrt{T}} f'\left( \frac{y + z}{\sqrt{T}} \right) \right] \quad (2.6a)
\]

have also zero average and satisfy for all \( y, s \) the inequality

\[
|\delta_T(t, y; s)| \leq \text{const} \frac{\|f\|_{1+\alpha}}{T^{(1+\alpha)/2}}. \quad (2.6b)
\]
Taking the leading term, setting $M_B^2(\xi) = M_B^2(\xi)b(\xi(t_f(B)), y_f(B))$ and

$$\mathcal{M}(t, y|\xi) = \mathcal{M}_a(t, y|\xi)b(\xi(t), y),$$

we get the functionals

$$Q_T^{(1)}(f|\xi) = \frac{1}{\sqrt{T}} \sum_{T_1+T_2+T_3=T-1}^{\infty} P_0^{T_1}(y_1)$$

$$\times \mathcal{M}_a(t_2, y_2 - y_1|\xi(t_1, y_1)) P_0^{T_3}(x - y_2)f'\bigg(\frac{x - bT}{\sqrt{T}}\bigg).$$

Let $\tilde{Q}_T^{(1)}(f|\xi) = T^{1/4}Q_T^{(1)}(f|\xi)$ be the normalized functionals. Lemma 2 below shows that the difference $\tilde{Q}_T(f|\xi) - \tilde{Q}_T^{(1)}(f|\xi)$ vanishes in $L_2$-norm. The lemma is based on the following estimate.

**Proposition 1.** For $\epsilon$ small enough, there is a constant $C(\epsilon)$ such that

$$\sum_{y} \langle \mathcal{M}_a^2(t, y|\cdot) \rangle \leq \frac{\epsilon^2 C(\epsilon)}{(t + 1)^{3/2}}.
$$

**Proof.** Let $b = \max_{s} |b(s)|$. It is easily seen, due to the orthogonality of the terms $M_B^2(\xi)$ for different $B$, that for $\epsilon$ small enough there is a constant $C(\epsilon) > 0$ such that

$$\sum_{y} \langle \mathcal{M}_a^2(t, y|\cdot) \rangle \leq (eb)^2 \sum_{n=1}^{\infty} \epsilon^{2n} \sum_{t_1 + \cdots + t_n = t, t_j > 0} \prod_{k=1}^{n} (k^\delta(x_i))^2 \leq \frac{\epsilon^2 C(\epsilon)}{(t + 1)^{3/2}}.$$ (2.8)

This follows, as similar inequalities in [4], from the second inequality (2.3), iterating the estimate $\sum_{t_1+\cdots+t_n=t}^{\infty} |T - t_1|^{-\alpha} \leq K(a)T^{-\alpha}$, valid for $a > 1$ and some constant $K(a) > 0$. □

**Lemma 2.** For $\epsilon$ small enough, there is a constant $C$ such that

$$\langle (\tilde{Q}_T(f|\xi) - \tilde{Q}_T^{(1)}(f|\xi))^2 \rangle \leq C \frac{||f||^2_{1+\alpha}}{T^{\alpha}}.$$ (2.9)

**Proof.** By (2.6a) the difference $\tilde{Q}_T(f|\xi) - \tilde{Q}_T^{(1)}(f|\xi)$ is written as

$$T^{1/4} \sum_{t=0}^{T-1} \sum_{t_1+t_2=t} \sum_{y_1, y_2} P_0^{T_1}(y_1) \mathcal{M}_a(t_2, y - y_1|\xi(t_1, y_1)) \delta(t - t, y - bT; \xi(t, y)).$$

Using the orthogonality of the terms $\mathcal{M}_a(t_2, y - y_1|\xi(t_1, y_1)) \delta(t - t, y - bT; \xi(t, y))$ for different choices of the pairs $(t_1, y_1)$ and $(t, y)$, and the obvious relation

$$\langle \mathcal{M}_a^2(t, y|\cdot) \rangle = \langle b^2(\cdot) \rangle \langle \mathcal{M}_a^2(t, y|\cdot) \rangle,$$

one finds that the left-hand side of (2.9) is bounded by

$$C_1^2 \frac{||f||^2_{1+\alpha}}{T^{1/2+\alpha}} \sum_{t=0}^{T-1} \sum_{t_1+t_2=t} \sum_{y_1, y_2} (P_0^{T_1}(y_1))^2 \langle \mathcal{M}_a^2(t_2, y_2|\cdot) \rangle \leq C \frac{||f||^2_{1+\alpha}}{T^{\alpha}}.$$
This follows from the first inequality (2.3) and the estimate
\[
\sum_{t=1}^{T} \sum_{t_1+t_2=t} \frac{1}{t_1^{1/2} t_2^{1/2}} \leq \sum_{t=1}^{T} \frac{\text{const}}{t^{1/2}} \leq \text{const} T^{1/2}.
\]

\[\square\]

Proposition 1 also implies the following result.

**Proposition 3.** If \( \epsilon \) is small enough, as \( T \to \infty \), the functionals
\[
\mathcal{E}_T(\xi) = \sum_{t=0}^{T-1} \sum_y \mathcal{M}_2(t, y|\xi)
\]
converge to a limiting functional \( \mathcal{E}(\xi) \) both in \( L_2 \) as well as \( \varphi \)-almost everywhere.

**Proof.** Using the orthogonality of \( \mathcal{M}_2(t, y|\xi) \) for different \((t, y)\) and Proposition 1 we find, for \( T' > T \),
\[
\langle (\mathcal{E}_{T'}(\xi) - \mathcal{E}_T(\xi))^2 \rangle \leq \sum_{t=T+1}^{T'-1} \sum_y \langle \mathcal{M}_2^2(t, y) \rangle \leq \sum_{t=T+1}^{T'-1} \text{const} \frac{T'}{(t+1)^{3/2}}
\]
\[
\leq \text{const} \left( \frac{1}{T^{1/2}} - \frac{1}{T'^{1/2}} \right).
\]

By the result in Appendix A of [6] (see also the Appendix of [3]), Proposition 3 is proved. \[\square\]

From now on we set for brevity
\[
H_T(t, y) = \sum_z P_0^{T-t-1}(z) f' \left( \frac{y + z - bT}{\sqrt{T}} \right).
\]
(2.10)

Let \( T_1 = \lfloor T^{\beta} \rfloor \), for some \( \beta \in (0, 1) \), and \( T_* = \lfloor \log_+ T \rfloor \), where \( \log_+ T = \max\{1, \log T\} \), and consider the functional:
\[
\hat{Q}_T^{(2)}(f|\xi) = \frac{1}{T_1^{1/4}} \sum_{t_1=0}^{T-T_1} \sum_{t_2=t_1}^{T_1} \sum_{y_1, y_2} P_0^{t_1}(y_1) \mathcal{M}_2(t_2-t_1, y_2-y_1|\xi(t_1, y_1)) H_T(t_2, y_2)
\]
(2.11)

which differs from \( \hat{Q}_T^{(1)}(f|\xi) \) in that the terms with large \( t_1 \) and large \( t_2 - t_1 \) have been removed.

**Remark.** As we prove in the Appendix (formula (A.1)), the constant \( M \) which appears in the statement of the theorem is the mean of the square of the functional \( \mathcal{E} \).

**Lemma 4.** Under the same assumptions as for Lemma 2 and Proposition 3, we have
\[
\lim_{T \to \infty} \langle (\hat{Q}_T^{(1)}(f|\xi) - \hat{Q}_T^{(2)}(f|\xi))^2 \rangle = 0.
\]

**Proof.** We consider first the large \( t_1 \) values. Let
\[
\hat{Q}_T(f|\xi) = \frac{1}{T_1^{1/4}} \sum_{t_1=T-T_1+1}^{T-1} \sum_{t_2=t_1}^{T-1} \sum_{y_1, y_2} P_0^{t_1}(y_1) \mathcal{M}_2(t_2-t_1, y_2-y_1|\xi(t_1, y_1)) H_T(t_2, y_2).
\]
The terms $\mathcal{M}_2(t_2 - t_1, y_2 - y_1|\xi(t_1, y_1))$ are orthogonal for different choices of the pairs $(t_1, y_1), (t_2, y_2)$. Taking into account that $|H_T(t, y)| \leq \|f'\|_{\infty}$ and proceeding as in the proof of Lemma 2, we find

$$\langle (\hat{Q}_T(f|\xi))^2 \rangle \leq \frac{\text{const}}{\sqrt{T}} \|f'\|_{\infty}^2 \sum_{t=T/2}^{T} \frac{1}{t^{1/2}} \leq \text{const} \|f'\|_{\infty}^2 \frac{T_1}{T} \to 0,$$

where we make use of the following inequalities, valid for all $T, r > 0$ such that $r/T < \delta < 1$ and for some constant $c(\delta)$,

$$\sum_{t=T}^{T+r} t^{-1/2} \leq \text{const}((T + r)^{1/2} - T^{1/2}) \leq c(\delta) \frac{r}{T^{1/2}}. \quad (2.12)$$

The contribution for large $t_2 - t_1$,

$$\hat{Q}_T'(f|\xi) = \frac{1}{T^{1/4}} \sum_{t_1=0}^{T-T_1} \sum_{t_2=t_1+T}^{T-1} \sum_{y_1, y_2} P_{01}^T(y_1) \mathcal{M}_2(t_2 - t_1, y_2 - y_1|\xi(t_1, y_1)) H_T(t_2, y_2)$$

is treated in the same way. We find

$$\langle (\hat{Q}_T'(f|\xi))^2 \rangle \leq \text{const} \|f'\|_{\infty}^2 \sum_{t_1=0}^{T-T_1} \frac{1}{(t_1 + 1)^{1/2}} \sum_{t' = T_1}^{T-t_1} \frac{1}{(t' + 1)^{3/2}}$$

$$\leq \text{const} \frac{\|f'\|_{\infty}^2}{(TT_1)} \sum_{t_1=1}^{T} \frac{1}{t_1^{1/2}},$$

which implies that the $L_2$-norm of $\hat{Q}_T'$ falls off as $T^{-1/4}$.

\[ \square \]

**Lemma 5.** As $T \to \infty$, $\langle (\hat{Q}_T(f|\xi))^2 \rangle$ tends to the expression (2.5) with $M = (\xi^2)$.

For the proof, see Appendix.

It remains to be proved that the CLT holds for $\hat{Q}_T^{(2)}(f|\xi)$. We have

$$\hat{Q}_T^{(2)}(f|\xi) = \frac{1}{T^{1/4}} \sum_{t=0}^{T-T_1} \mathcal{E}_T(t|\xi) \quad (2.13a)$$

where the terms appearing in the sum are orthogonal and are written as

$$\mathcal{E}_T(t_1|\xi) = \sum_{t_1 = t_1}^{t_1 + T_1} \sum_{y_1, y_2} P_{01}^T(y_1) \mathcal{M}_2(t_2 - t_1, y_2 - y_1|\xi(t_1, y_1)) H_T(t_2, y_2). \quad (2.13b)$$

If $t_1 < t_1'$ and $t_1' - t_1 > T_1$, the quantities $\mathcal{E}_T(t_1|\xi)$ and $\mathcal{E}_T(t_1'|\xi)$ are independent, so that we can use the Bernstein method, which consists in approximating $\hat{Q}_T^{(2)}(f|\xi)$ by a sum of independent terms, by throwing away some “corridors”. Let $\gamma, \delta$ be positive numbers, $0 < \gamma < \delta < 1$, and $r = [T^\gamma], s = [T^\delta], \kappa = [T/(r + s)]$. We define the intervals $I_k$ and the “corridors” $J_k, k = 1, \ldots, \kappa$, as follows. $I_k$ is the integer interval with first point $(k-1)(r+s)$ and last point $kr+(k-1)s-1$, and $J_k$ has first point $kr+(k-1)s$ and last point $k(r+s)-1, k = 1, \ldots, \kappa$. Let furthermore $R$
denote the interval with first point $\kappa(r+s)$ and last point $T-1$ (it may be empty), and consider the quantity

$$\tilde{Q}_T^I(f|\xi) = \frac{1}{T^{1/4}} \sum_{t \in J_k, J_k \cup R} E_T(t|\xi).$$

**Lemma 6.** Under the assumptions above, as $T \to \infty$, $\tilde{Q}_T^I(f|\xi) \to 0$ in $L_2$-norm.

**Proof.** The estimates above imply $\langle E_T^2(t|\cdot) \rangle \leq \text{const} \|f'\|_\infty^2 t^{-1/2}$, so that

$$\left( \sum_{t \in J_k} E_T(t|\xi) \right)^2 \leq \text{const} \|f'\|_\infty^2 \sum_{t=kr+(k-1)s}^{k(r+s)-1} \frac{1}{t^{1/2}} \leq \text{const} \|f'\|_\infty^2 \left[(k(r+s)-1)^{1/2} - (kr+(k-1)s)^{1/2}\right] \leq \text{const} \|f'\|_\infty^2 \frac{s}{(kr)^{1/2}}.$$ 

Here we applied inequality (2.12), as $s/r \to 0$. Summing over $k$ from 1 to $\kappa$ and taking into account the factor $\frac{1}{\sqrt{T}}$ in front, we get a quantity of the order $\frac{s^{1/2}}{(\kappa T)^{1/2}} = O(T^{-\gamma+\delta})$, which tends to 0 as $T \to \infty$. The same happens for the contribution of the last interval $R$. □

By Lemma 6, the limiting distribution of $\tilde{Q}_T$ is the same as that of $\tilde{Q}_T^I = \tilde{Q}_T^{(2)} - \tilde{Q}_T^{I'}$, which can be written as a sum of independent variables

$$\tilde{Q}_T(f|\xi) = \frac{1}{T^{1/4}} \sum_{j=1}^t A_T^{(j)}(\xi), \quad A_T^{(j)}(\xi) = \sum_{t \in I_j} E_T(t|\xi).$$

We will prove the CLT for $\tilde{Q}_T(f|\xi)$ by establishing a Lyapunov condition. For this, we need an $L_4$ estimate for quantities of the type

$$A_{t_1,t_2}^{(T)}(\xi) = \sum_{t=t_1}^{t_2} E_T(t|\xi),$$

where it is understood that $t_2 + T_* < T$.

**Proposition 7.** If $\epsilon$ is small enough, there is a constant $K(\epsilon) > 0$ such that

$$\langle (A_{t_1,t_2}^{(T)})^4 \rangle \leq \epsilon^4 K(\epsilon) \|f'\|_\infty^4 (\sqrt{T_2} - \sqrt{T_1})^2.$$  (2.14)

**Proof.** We have

$$M_T(t_2 - t_1; y_2 - y_1; \xi(t_1,y_1)) = \sum_{B: \xi(t_1,y_1) \rightarrow (t_2,y_2)} \epsilon^{|B|} M_B^{f_\epsilon}(\xi).$$

As the variables $\xi(t,x)$ are i.i.d., and $\langle c(u, \cdot) \rangle = 0$, averages of the type $\langle \prod_{k=1}^4 M_{B_k}^{f_\epsilon} \rangle$ all vanish unless the sets $B_j$ have the “covering property”: $B_j \subseteq \bigcup_{i \neq j} B_i$, $j = 1, \ldots, 4$. Let $\mathcal{C}_4$ be the class of 4-tuples of sets $B = \{B_1, \ldots, B_4\}$ with the covering property. An element $B \in \mathcal{C}_4$ is identified as a finite subset $B = \bigcup_{j=1}^4 B_j$ of $\mathbb{Z}^{\nu+1}$, each point $V \in B$ being provided with a “specification” $\ell_V = \{j: V \in B_j\}$, a subset of the labels $\{1, 2, 3, 4\}$ of cardinality $|\ell_V| \geq 2$. $S = \{\ell_V: V \in B\}$ is the
“specification of $B$”. A one-to-one correspondence between the elements of $\mathcal{C}_4$ and
the pairs $(B, S)$ is obtained by imposing the following conditions on $(B, S)$:

i) If $V, V' \in B$ are two distinct points with the same time coordinate, then

$$\ell_V \cap \ell_{V'} = \emptyset;$$

ii) all labels appear at least once, i.e., $\bigcup_{V \in B} \ell_V = \{1, \ldots, 4\}.$

Condition i) implies that there are no more than two vertices with a given time coordinate. $\mathcal{C}_4$ is identified with the collection of the pairs $(B, S)$ satisfying the conditions above.

To any element $B = (B, S) \in \mathcal{C}_4$ one can associate a graph $\mathfrak{G} = (B_0, \mathcal{L})$ where the set of vertices is obtained by adding the origin to $B$, $B_0 = B \cup \{0\}$, and $\mathcal{L}$ is the set of bonds, which is determined as follows. For each vertex $V = (t, x) \in B$ and each label $j \in \ell_V$ we consider the class of vertices $V_j := \{V' = (t', x'), j \in \ell_{V'}, t' > t\}$ with the same label and larger time. If $V_j$ is not empty, we draw a bond connecting $V$ to the vertex $V_x \in V_j$ with the minimal time coordinate (which is unique, by condition i). In other words, we connect vertices with the same label in the order of increasing time. The graph is completed by adding a bond for each label $j = 1, \ldots, 4$, connecting the origin to the initial point of the sets $B_j$. This set of 4 bonds is denoted $\mathcal{L}^*$, so that $\mathcal{L} = \mathcal{L}^* \cup \mathcal{L}'$.

Denoting by $\mathcal{C}_4^{t_1, t_2}$ the collection of $B \in \mathcal{C}_4$ for which $t_i(B_k) \in [t_1, t_2]$, $k = 1, \ldots, 4$, and by $N(B) = \sum_{V \in B} |\ell_V| = \sum_{j=1}^4 |B_j|$ the “cardinality” of $B$, we find

$$\left\langle \left( A_{t_1, t_2}^{(T)} \right)^4 \right\rangle \leq b^4 \|f\|_4^4 \sum_{B \in \mathcal{C}_4^{t_1, t_2}} e^{N(B)} S(\mathfrak{G}_B). \quad (2.15)$$

Here $\mathfrak{G}_B$ is the graph associated to $B$ and

$$S(\mathfrak{G}) = \prod_{b \in \mathcal{L}^*} \pi_e(b) \prod_{b \in \mathcal{L}'} \pi(b),$$

where $\mathcal{L} = \mathcal{L}^* \cup \mathcal{L}'$ is the set of bonds of the graph $\mathfrak{G}$ and the weights are as follows: if $b \in \mathcal{L}$ and $b = (V, V')$ with $V = (t, x)$ and $V' = (t', x')$, $t' > t$, then $\pi(b) = k^{t'-t}(x' - x)$, whereas if $b \in \mathcal{L}^*$, then $b = (0, V)$ for some $V = (t, x)$, and $\pi_e(b) = F_0^e(x)$.

We say that a graph is “disconnected” if after removing the vertex at the origin it splits into two connected components. This is only possible if the 4-tuple $B$ is made of two equal pairs of sets, say $B_i = B_{i+1} = B_1$ and $B_{i+1} = B_i = B_2$, with no common points $B_1 \cap B_2 = \emptyset$. For such graphs any bond $b = (V, V')$, $V, V' \in B_j$, $j = 1, 2$ appears twice (“double bond” or “simple loop”). As the contributions of $B_1, B_2$ factorize, the total contribution of such graphs can be estimated as in Proposition 1.

Let $\mathfrak{G} = (B_0, \mathcal{L})$ be a graph. The bonds $(V_j, V_{j+1}) \in \mathcal{L}$ make up an “increasing path” of $\mathfrak{G}$ with the initial vertex $V_1$ and the final vertex $V_{n+1}$ if the vertices are time-ordered: $V_j = (t_j, x_j), j = 1, \ldots, n, t_j < t_{j+1}$. An increasing path is a connected subgraph, with $V_1$ as the “initial” and $V_{n+1}$ as the “final” vertex. An increasing path is maximal if there is no increasing path obtained by adding to it any other bond of $\mathcal{L}$.
Lemma 8. Any connected graph \( \mathfrak{G} = (B_0, \mathcal{L}) \) coming from elements of \( B \in \mathfrak{C}_4 \) with the property that each vertex \( V \in B \) has multiplicity \( |\ell_V| = 2 \) can be represented as a union of four increasing paths \( L_j, j = 1, \ldots, 4 \), starting at the origin and such that:

i) they are bond-disjoint, i.e., each bond \( b \in \mathcal{L} \) belongs to one path only;

ii) if \( \hat{B}_j \) is the set of vertices of \( L_j \) which are in \( B \), then one has \( \hat{B}_1 \cup \hat{B}_2 = \hat{B}_3 \cup \hat{B}_4 = B \) and \( \hat{B}_1 \cap \hat{B}_2 = \hat{B}_3 \cap \hat{B}_4 = \emptyset \).

Proof. Property ii) means of course that the two paths of each pair \((L_1, L_2)\) and \((L_3, L_4)\) intersect only at the origin.

At each vertex \( V \in B \) there are always two “incoming” bonds (connected to vertices with a lower time coordinate) and there may be “outgoing” bonds (connected to vertices with a larger time coordinate). If there are one or none outgoing bonds, we say that \( V \) is a final vertex of \( \mathfrak{G} \), more precisely a 1f-vertex if there is one outgoing bond and a 2f-vertex if there is no such a bond. The final vertices clearly correspond to the end-points of the sets \( B_j \). \( \mathfrak{G} \) can either have two 2f-vertices and no 1f-vertex, or one 2f-vertex and two 1f-vertices. Vertices that are neither initial nor final are called “m-vertices” and have two incoming and two outgoing bonds.

We label the vertices of \( B \) in the lexicographic order, first according to time, and then according to the space coordinates. The “initial” (“final”) time \( T_i \) \((T_f)\) of \( \mathfrak{G} \) is the time coordinate of \( V_1 \) \((V_{|B|})\). For a given time \( T_i \leq t_* < T_f \) the number of paths \( n(t_*) \) that continue beyond \( t_* \) is the number of bonds \( b = (V, V') \) with \( V = (t, x), V' = (t', x') \) and \( t \leq t_* < t' \). Under the assumptions above, \( 2 \leq n(t_*) \leq 4 \).

Suppose that we label the bonds of \( \mathcal{L} \) by labels \( \pm \) in such a way that pairs of incoming or outgoing bonds at any vertex \( V \in B \) have always different labels (we call this “no-cross labeling”). The bonds labeled in this way make up four maximal increasing paths of bonds with the same label, which we call “labeled paths”, and labeled paths with the same label never cross. There are exactly four of them, as each of them ends at a final vertex, the number of paths ending at \( V \) is two if \( V \) is 2f and one if it is 1f. No-cross labeling is however not enough: it might give one path with label +, say, which visits all vertices of \( B \), and three paths with label − with no common vertices. We will show that one can use a procedure for labeling which always gives two paths for each label. We call this “proper labeling”. Once proper labeling is done one takes as \( L_1, L_2 \) the two labeled paths with +, as \( L_3, L_4 \) the two labeled paths with − and the lemma is proved.

A double bond \( (V, V') \) (or “simple loop”) must be given two opposite labels. We assume that \( \mathfrak{G} \) has no simple loops, except for those made of bonds of \( \mathcal{L}' \), for, if there is one, we replace it by a single vertex with the appropriate bonds, and if proper labeling holds for the new graph it holds for the original one as well. We say that two vertices \( V, V' \in B \) are a “boundary pair” (b.p.) if there is no increasing path leading from one of them to the other one. By the no-loop condition, if one of them is 2f the other one is also 2f. If a graph has a boundary pair, no-cross labeling implies proper labeling, as there are clearly two paths for each label. A b.p. defines an “upper graph” made of all increasing paths starting at \( V \) and \( V' \), which is empty if they are both 2f, and a ”lower graph”, made of the paths ending at \( V, V' \). They
make up the whole of $\mathcal{S}$ and only have the vertex pair $V, V'$ in common. If both of them are separately labeled by the no-cross rule, the same holds for the whole graph. Any graph $\mathcal{S}$ is thus split into “components”, separated by b.p.’s. A graph that cannot be split into nonempty components is called “indecomposable”.

We apply an iterative procedure. Assume that no-cross labeling holds up to $V_k$, $k > 1$, i.e., all pairs of outgoing bonds and all pairs of incoming bonds at $V_j$, $j < k$, and possible pair of incoming bonds at $V_k$, have been labeled according to the no-cross rule. We want to extend the procedure up to the next vertex $V_{k+1}$. The bonds that enter $V_{k+1}$ either come out of $V_k$, or are “left-over bonds” from previous vertices, the number of which can be at most two. We say that we have a “standard case” (s.c.) at $V_k$ if no-cross labeling holds up to $V_k$ and moreover the left-over bonds, if they are two, have opposite labels. We will now show that if we have a s.c. at $V_k$, then either we have again a s.c. at $V_{k+1}$ or $V_k, V_{k+1}$ are a b.p.

Let $V_k = (t_k, x_k), V_{k+1} = (t_{k+1}, x_{k+1})$, and suppose we have a s.c. at $V_k$. If $V_k$ is 2f, then by the no-loop condition there are two left-over bonds with opposite sign that meet at $V_{k+1}$, which is also 2f, and there is nothing more to do. So we assume that there are outgoing bonds at $V_k$. We consider three cases.

A) $n(t_k) = 2$ (“2-path standard case”, or 2-s.c.). We have one bond, $b'$, out of $V_k$, and a left-over bond $b$ with label (say $+$), which have to meet at $V_{k+1}$. By the no-cross rule $b'$ takes the label $-$, and, by the no-loop condition, $V_{k+1}$ is 2f and we are finished.

B) $n(t_k) = 3$ (“3-path standard case”, or 3-s.c.). Observe that $V_{k+1}$ cannot be of type 2f.

B1) One bond, $b_*$, out of $V_k$ (i.e., $V_k$ is 1f) and two left-over bonds, let them be $b, b'$. We have two subcases. B1a) $b, b'$ meet at $V_{k+1}$. If $V_{k+1}$ is 1f, then $V_k, V_{k+1}$ are a b.p. If there are two bonds out of $V_{k+1}$, by giving $b_*$ an arbitrary label we are again in a 3-s.c. B1b) At $V_{k+1}$ $b_*$ meets one of the other two, say $b$. By the no-cross rule $b_*$ takes the same label as $b'$, and if $V_{k+1}$ is not final we are again in a 3-s.c. If $V_{k+1}$ is of type 1f we are in a 2-s.c.

B2) One left-over bond $b_*$, and two bonds $b, b'$ out of $V_k$. Then $b_*$ meets one of the other two, say $b$, at $V_{k+1}$, and by the no-cross rule $b'$ takes the same label as $b_*$, and $b$ the opposite one, and we are in the situation B1b) above.

C) $n(t_k) = 4$ (“4-path standard case”, or 4-s.c.). Let $b, b'$ be the left-over bonds, $b_*, b'_*$ be the bonds out of $V_k$. At $V_{k+1}$ one of the left-over bonds, say $b_*$ meets one of the bonds out of $V_k$, say $b'_*$, otherwise there is a loop or $V_k, V_{k+1}$ are a b.p. Then $b_*$ is given the same label as that of $b$, and $b'_*$ the opposite one. If $V_{k+1}$ is 2f, then $V_{k+2}$ is also 2f and the procedure is concluded. If $V_{k+1}$ is 1f, we are in a 3-s.c. If $V_{k+1}$ is m, we are again in a 4-s.c.

We now show how one labels an indecomposable component which starts at a boundary pair $V, V'$.

(i) (2-path graph.) $V, V'$ are both 1f. The graph is made of two bonds which meet at a final vertex of $B$, and by the no-cross rule they take opposite labels.

(ii) (3-path graph.) One of the vertices, say $V$, is of type m, and $V'$ is 1f. Let $b_*$ the bond out of $V'$ and $b, b'$ the bonds out of $V$. By giving the bond $b_*$ an
arbitrary label we are in the same 3-s.c. at point B2) above, where V plays the role of Vₖ.

(iii) (4-path graph.) V, V' are both m, and they are clearly the first two vertices (in the lexicographic order) of the upper component, so we relabel them V₁, V₂. Let b₁, b₁' be the bonds out of V₁, and b₂, b₂' those out of V₂. At V₃ one bond out of V₁, say b₁, has to meet a bond from V₂, say b₂'. We give b₁ the label +, b₂ the same, and the bonds with prime take the label −. If V₃ is final of type 2f, then V₄ also is, and we are finished. If V₃ is 1f, we are in a 3-s.c., and if V₃ is of type m, we are in a 4-s.c.

It is now easy to show how to assign proper labeling to a graph Φ. The incoming bonds of V₁, let them be b₁', b₂' ∈ L', are given opposite labels. There are two cases.

A) The two incoming bonds at V₂ are the remaining bonds of L'. They also take opposite labels, V₁, V₂ are a b.p., and the upper graph is a 4-path graph (case (iii) above) if both V₁ and V₂ are m, or a 3-path graph (case (ii) above) if one of them is 1f, or a 2-path graph is they are both 1f, and there is nothing to do if they are both 2f. By what is said above we can assign no-cross labeling up to the end, and, as we have clearly four labeled paths, we get proper labeling.

B) A bond out of V₁ is incoming at V₂. If V₁ is 1f, the bond out of it, b, has to meet a bond, say b₁' of L' at V₂ (which cannot be 2f). We give b a +, b₁' a −, and the remaining bond b₂' of L' takes a +. Then at V₂ we are in a 2-s.c. if V₂ is also 1f, and in a 3-s.c. if it is m. If V₁ is m with outgoing bonds b₁, b₁', and V₂ has b₂ as incoming, meeting there, as before with b₁, we give b₁ and b₁' a +, b₂' a −. Then, if V₂ is 2f, by the no-loop condition b₁ and b₁' meet at V₃, which is 2f, if V₂ is 1f we are in a 3-s.c. at V₂, and if V₂ is m we are in a 4-s.c.

As there are no more cases to consider, the lemma is proved.

Remark. The condition that all vertices have multiplicity two can be easily removed. Let V ∈ B be such that |ℓᵥ| > 2. If there is no bond out of V (i.e., V is final for three (|ℓᵥ| = 3) or four (|ℓᵥ| = 4) paths), then the procedure above can be carried out up to the end, with the only change that one (for |ℓᵥ| = 3) of the pairs (B₁, B₂), (B₃, B₁) or both (for |ℓᵥ| = 4) have V as common final vertex. If V is not final and |ℓᵥ| = 3, one can modify the graph by “taking one bond out of V”, i.e., one replaces two bonds (V₁, V), and (V, V₂) (where, as usual, vertices are in the time order) by a single bond (V₁, V₂). One then performs the labeling procedure as in the proof of the lemma, by which the bond (V₁, V₂) gets some label, say +, and will belong to one of the labeled paths Lₖ. As a final step we only have to modify Lₖ by replacing the bond (V₁, V₂) by the two bonds (V₁, V₁), (V₁, V₂). The new result is again that one of the pairs (B₁, B₂), (B₃, B₄) has V in common. When |ℓᵥ| = 4, one can “split the vertex into two ones” and label the graph as in the proof above. The two vertices may be both m or 1f, or one m and one 2f, or one m and one 1f. In all cases they split the graph, and it is easy to see that the result is that both pairs (B₁, B₂), (B₃, B₄) have V in common.

As a final result we have that any graph Φ is the union of four labeled paths Lₖ, j = 1, ..., 4 which have as vertex set the origin and a subset Bₖ ⊆ B, such that
\( \hat{B}_1 \cup \hat{B}_2 = \hat{B}_3 \cup \hat{B}_4 = B \). Moreover, the contribution of \( \Phi \) is written as

\[
S_\Phi = \prod_{j=1}^{4} S_{L_j}, \quad S_{L_j} = \pi_\ast(b_1^{(j)}) \prod_{i=2}^{n_j} \pi(b_i^{(j)}),
\]

where \( n_j = |\hat{B}_j| \) is the number of bonds of \( L_j \), labeled in the increasing time order as \( b_i^{(j)} \), \( i = 1, \ldots, n_j \). In what follows, as there is a one-to-one correspondence between \( L_j \) and \( \hat{B}_j \), we will write \( S_{\hat{B}_j} \) for \( S_{L_j} \).

**Conclusion of the proof of proposition.** Observe that all 4-tuples \( \mathcal{B} \) which give rise to a given graph \( \Phi = (B_0, \mathcal{L}) \) have the same cardinality \( N_\Phi = N(\mathcal{B}) \), since \( |\ell_V| \) is the number of incoming bonds at \( V \). Moreover, it is easy to see that for each graph \( \Phi \) there is a constant \( \kappa \) \((= (4!)^{1/4})\) such that there are at most \( \kappa N_\Phi \) 4-tuples from which it can come. (This comes from the fact that there are at most \( |\ell_V|! \) ways of exchanging the labels at each vertex.) Hence the sum over such \( \mathcal{B} \)'s is bounded by

\[
\sum_{B: \Phi = \Phi} e^{N(\mathcal{B})} S(\Phi) \leq (\epsilon \kappa)^{N_\Phi} S_\Phi = \prod_{j=1}^{4} (\epsilon \kappa)^{n_j} S_{\hat{B}_j},
\]

where we take into account that \( N_\Phi = \sum_{j=1}^{4} n_j \).

Let now \( B \) be fixed, and consider the contribution of all graphs with \( B \) as the vertex set. Setting \( S_{\hat{B}_j}^* = (\epsilon \kappa)^{n_j} S_{\hat{B}_j} \) we find

\[
\sum_{\mathcal{L}} (\epsilon \kappa)^{N_\Phi} S((B, \mathcal{L})) \leq \sum_{\{\hat{B}_1, \hat{B}_2\}} \sum_{\{\hat{B}_3, \hat{B}_4\}} \prod_{j=1}^{4} S_{\hat{B}_j}^* \leq 3^n \sum_{\{B_1, B_2\}} \sum_{B_1 \cup B_2 = B} (S_{B_1}^* S_{B_2}^*)^2, \tag{2.16}
\]

where we have used the elementary inequality

\[
\prod_{j=1}^{4} S_{\hat{B}_j}^* \leq \frac{1}{2} [(S_{\hat{B}_1}^* S_{\hat{B}_2}^*)^2 + (S_{\hat{B}_3}^* S_{\hat{B}_4}^*)^2],
\]

taking into account that the number of different pairs of subsets \( B_1, B_2 \) with \( B_1 \cup B_2 = B \) does not exceed \( \sum_{k} \binom{n}{k} 2^k = 3^n \).

As \( n \leq |B_1| + |B_2| \), setting \( S_B^2 = (\epsilon \kappa \sqrt{3}) \overline{|B|} S_B \), the sum of the left-hand side of (2.16) over all graphs with a given vertex set \( B \) is bounded by

\[
\sum_{\{B_1, B_2\}} \sum_{B_1 \cup B_2 = B} (S_{B_1}^* S_{B_2}^*)^2,
\]

and the sum over the \( B \)'s of the quantity at the left-hand side of inequality (2.15) is bounded by

\[
\left( \sum_{B: t_i(B) \in [t_1, t_2]} [S_B^2]^2 \right)^2.
\]
We can now fix the initial point of $B$, let it be $(t_*, x_*)$, and sum over all other points. Proceeding as in the proof of Proposition 1 we see that for $\epsilon$ small enough

$$
\sum_{B: t_1(B)=t_* \atop y(B)=x_*} S^2_B \leq (P^*_0(x_*)^2) \cdot \epsilon \sum_n e^{2n} \sum_{t_1, \ldots, t_n \geq 0} \prod_{i=1}^n (k^i(x_i))^2 \leq \epsilon^2 C_*(\epsilon) (P^*_0(x_*)^2)
$$

We have just to replace $\epsilon$ by $\epsilon_2 = \epsilon \kappa \sqrt{3}$:

$$
\langle (A^{(T)}_{i_1, i_2})^4 \rangle \leq \epsilon_2^4 b^4 C_*^2(\epsilon_2) \|f'\|_\infty \sum_{t, t' \in [i_1, i_2]} \sum (P^*_0(x_1) P^*_0(x_2))^2.
$$

It now suffices to apply the first estimate (2.3) to get the result. 

\[ \square \]

**Proof of theorem.** The proof is an elementary application of the Lyapunov condition (see [7]), which amounts to checking that, as $T \to \infty$

$$
L_4(T) = \frac{1}{T} \sum_{j=1}^{K(T)} \langle (A^{(j)}(\xi))^4 \rangle \rightarrow 0. \tag{2.17}
$$

By the previous lemma, taking into account the inequalities (2.12), the left-hand side of (2.17) is bounded by

$$
\frac{\text{const}}{T} \|f'\|^4 \sum_{j=1}^{K(T)} \frac{r^2}{j(r + s)} \leq \text{const} \|f'\|^4 \frac{\log T}{T^{1-\gamma}} \rightarrow 0. \quad \Box
$$

**APPENDIX. PROOF OF LEMMA 5**

By Lemmas 2, 3, it suffices to prove that

$$
\langle (\hat{\mathcal{G}}_T^{(2)}(f|\xi))^2 \rangle \rightarrow (e^2) \int_0^1 ds \int du K_0^2(s, u) \left( \int dv K_0(1-s, v) f'(u+v) \right)^2. \tag{A.1}
$$

By the local CLT for $P_0$ (see, e.g., [9]) for all $t \geq 1$ and we have $f \in C^0$

$$
\left| \sum z \left( P^n_0(z) - \frac{e^{-\frac{(z-bt)^2}{2\sigma^2}}}{\sqrt{2\pi t\sigma}} \right) f \left( \frac{y+z-bT}{\sqrt{T}} \right) \right| \leq \text{const} \|f\|_\infty \frac{1}{t^{1/2}}.
$$

If $f \in C^\alpha$, with $\alpha \in (0, 1)$, approximating the Riemann sum by the corresponding integral, for all $1 \leq t < T$ we have

$$
\left| \sum z \frac{e^{-\frac{(z-bt)^2}{2\sigma^2}}}{\sqrt{2\pi t\sigma}} f \left( \frac{y+z-bT}{\sqrt{T}} \right) - \int e^{-\frac{x^2}{2\pi \sigma}} f \left( \frac{y-b(T-t)}{\sqrt{T}} + x \sqrt{\frac{t}{T}} \right) dx \right| 
\leq \text{const} \|f\|_\alpha \max \left\{ \frac{1}{t^{1/2}}, \frac{1}{T^{\alpha/2}} \right\},
$$

where $\|f\|_\alpha = \|f\|_\infty + \sup_{x \neq x'} \frac{|f(x)-f(x')|}{|x-x'|^\alpha}$.
Going back to the expression (2.10) for \( H_T(t, y) \), taking \( T = T - t_2 - 1 \) and considering that \( T - t_2 \geq T_1 - T_* \uparrow \infty \) as \( T \to \infty \), we see that in the expression of \( \hat{Q}_T^{(2)} \) we can replace \( H_T(t_2, y_2) \) by the quantity

\[
\hat{H}_T(t, y) = \int e^{-\frac{y^2}{2\pi \sigma^2}} f'(\frac{y - bt}{\sqrt{T}} + x \sqrt{1 - \frac{t}{T}}) \, dx
\]

\[
eq \int K_x \left( 1 - \frac{t}{T}, v \right) f' \left( \frac{y - bt}{\sqrt{T}} + v \right) \, dv,
\]

where we have changed variables in the integral: \( u = x\sqrt{1 - t/T} \).

The contribution for \( t_1 \) small, \( t_1 \leq T_1 = [T^\beta] \), \( \beta \in (0, 1) \),

\[
\hat{Q}_T^{(2)}(f|\xi) = \frac{1}{T^{1/2}} \sum_{t_1=0}^{T_1} \sum_{t_2=t_1+1}^{t_1+T_*} \sum_{y_1, y_2} P_t^*(y_1) \mathcal{M}_t(t_2 - t_1, y_2 - y_1 | \xi_{(t_1, y_1)}) \hat{H}_T(t_2, y_2),
\]

can be also neglected. In fact, proceeding as in the proof of Lemma 2, we have

\[
\langle (\hat{Q}_T^{(2)}(f|\xi))^2 \rangle \leq \text{const} \|f'\|_\infty^2 \sum_{t_1=1}^{T_1} \frac{1}{T^{1/2}} \leq \text{const} \|f'\|_\infty^2 \frac{T_1}{T}.
\]

One is left with the asymptotics of the quantity

\[
\frac{1}{\sqrt{T}} \sum_{t_1=T_1+1}^{T-T_1} \sum_{t_2=t_1+1}^{t_1+T_*} \sum_{y_1, y_2} (P_t^*(y_1))^2 \langle (\mathcal{M}_t(t_2 - t_1, y_2 - y_1 | \cdot))^2 \rangle (\hat{H}_T(t_2, y_2))^2.
\]

(A.2)

Since \( t_2 - t_1 \leq T_* \), and \( |y_2 - y_1| \leq DT_* \) (by the short range condition), one finds \( |\hat{H}_T(t_2, y_2) - \hat{H}_T(t_1, y_1)| \leq C \|f\|_{1+\alpha}(T_* / T)^\alpha \). Hence one can replace \( \hat{H}_T(t_2, y_2) \) by \( \hat{H}_T(t_1, y_1) \) and sum over \( t_2, y_2 \). By Proposition 3, the asymptotics of the quantity (A.2) is the same as that of

\[
\langle \hat{Q}^2 \rangle \frac{T - T_1}{\sqrt{T}} \sum_{t=T_1+1}^{T - T_1} (P_t^*(y))^2 (\hat{H}_T(t, y))^2.
\]

As \( T_1 \to \infty \), \( t \) is uniformly large, and one can again replace \( P_t^*(y) \) by the leading term in the asymptotic expansion of the local CLT, getting another Riemann sum. \( \hat{H}_T(t, y) \) is \( C^\alpha \) in \( y \) with the Hölder constant bounded by \( \|f\|_{1+\alpha} / T^{\alpha/2} \), and we find, for large \( T_1 \),

\[
\frac{1}{\sqrt{T}} \sum_y (P_t^*(y))^2 (\hat{H}_T(t, y))^2 \sim \frac{1}{\sqrt{Tt}} \sum_y e^{-\frac{(y - M)^2}{2\pi^2 \sigma^2}} (\hat{H}_T(t, y))^2
\]

\[
\sim \frac{1}{\sqrt{Tt}} \int dx \frac{e^{-\frac{x^2}{2\pi^2 \sigma^2}}}{\sqrt{t}} \left( \int K_x (1 - t/T, v) f'(x\sqrt{t/T} + v) \, dv \right)^2 = \frac{1}{T} G \left( \frac{t}{T} \right),
\]

with

\[
G(s) = \int du K^2_\sigma(s, u) \left( \int dv K_\sigma (1 - s, v) f'(u + v) \right)^2.
\]
Hence the required asymptotics is the same as that of the expression
\[
\langle \epsilon^2 \rangle = \sum_{t=T_1+1}^{T-T_1} \frac{1}{T} G\left( \frac{t}{T} \right).
\]
This is again a Riemann sum for the integral over the interval (0, 1) of a function which for small \( s \) behaves as \( 1/\sqrt{s} \).
Hence relation (A.1) holds and Lemma 5 is proved.

References


