ON SOLUTIONS FOR THE KADOMTSEV–PETVIASHVILI I EQUATION

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To the memory of I. G. Petrovskii on the occasion of his 100th anniversary

ABSTRACT. Oscillatory integral techniques are used to study the well-posedness of the KP-I equation for initial data that are small with respect to the norm of a weighted Sobolev space involving derivatives of total order no larger than 2.


Key words and phrases. Kadomtsev–Petviashvili equation, initial value problem, well-posedness, oscillatory integrals.

1. Introduction

We consider the initial value problem (IVP) for the Kadomtsev–Petviashvili equation

\[
\begin{align*}
\frac{\partial}{\partial t} u + \partial_x^3 u + \beta \partial_x u^2 + \gamma \partial_y u = 0, \\
u(x, 0) = u_0(x), \\
(x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R},
\end{align*}
\]

where \( u = u(t, x, y) \) is a scalar unknown function, \( \beta \) is a real constant and \( \gamma = \pm 1 \).

The KP equation models \[9\] the propagation along the \( x \)-axis of nonlinear dispersive long waves on the surface of a fluid with a slow variation along the \( y \)-axis. KP arises as a universal model in wave propagation and may be viewed as a 2d generalization of the KdV equation. If we denote by \( \partial_x^{-1} \) the antiderivative with respect to the variable \( x \), then we can rewrite the evolution equation in (1) as

\[
\frac{\partial}{\partial t} u + \partial_x^3 u + \gamma \partial_x^{-1} \partial_y^2 u + \beta \partial_x u^2 = 0. \tag{2}
\]

We will be using (2) in the rest of the paper. This equation is of dispersive type and the strength of the dispersive effect depends on the sign of \( \gamma \). To see this, we recall that the solution of the linear problem associated with (2) can be written as

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we go back to

\[\delta\text{antiderivative. For well-posedness results, we recall the work of Saut [16], proved existence of global weak solutions for initial data in } H^s\text{. It is well known, in particular from the work of Stein [18], that the curvature of the surface plays an important role in obtaining good estimates for the restricted Fourier transform. The dispersive function also defines the intensity of the smoothing effect. Kenig, Ponce and Vega [10] proved that if }\]

\[u(x_1, \ldots, x_n, t) = \int_{\mathbb{R}^n} e^{i(t\phi(x_1, \ldots, x_n) + \sum_{i=1}^n x_i \xi_i)} \hat{u}_0(\xi_1, \ldots, \xi_n) d\xi_1 \ldots d\xi_n\]

for a generic dispersive function \(\phi(x_1, \ldots, x_n)\) satisfying \(|\nabla \phi(x)| \geq C \sum_{i=1}^n |\xi_i|^{|\gamma|}\), for \(\delta_i > 0\), it follows that if \(u_0 \in L^2\) then \(\partial_t^{1/2} u\) is a function in some \(L^p\) space. If we go back to (2), we see that

\[|\nabla_{(\xi, \nu)} \phi(\xi, \lambda)| \geq C \begin{cases} |\xi| & \text{if } \gamma = -1, \\ |\xi|^2 & \text{if } \gamma = 1. \end{cases}\]  

We call the equation (2) KP-I if \(\gamma = -1\) and KP-II if \(\gamma = 1\). It is clear then from (4) that the linear solution for the KP-I equation gains in general no more that \(\partial_t^{1/2}\) smoothness, while the one for the KP-II gains the full derivative \(\partial_t\).

The first result regarding the well-posedness for a KP type equation with low regularity is due to Ukai [22]. He used a standard energy method that does not recognize the type I or II of the equation. His result provides local well-posedness for initial data and their antiderivatives in \(H^s, s \geq 3\). Faminski [4] observed a better smoothing effect in the KP-II evolution and used this to prove well-posedness results. Bourgain performed a Fourier analysis [2] of the term \(\partial_t u^2\) in the KP-II equation in which the derivative is recovered in a nonlinear way. The result obtained gave local well-posedness of KP-II for initial data in \(L^2\). Since the \(L^2\) norm is conserved during the KP-II evolution, the \(L^2\) local result may be iterated to prove global well-posedness. Takaoka [19] and Takaoka and Tzvetkov [20] improved Bourgain’s result by proving local well-posedness in an anisotropic Sobolev space \(H^{1/2 + \epsilon, 0}_{x, y}\). For the KP-I equation, the situation is more delicate. There are several results on local and global existence of solutions, but not a satisfactory well-posedness theory for data with no more than two derivatives. Fokas and Sung [5], and Zhou [23], obtained global existence for small data via inverse scattering techniques. Schwarz [17] proved existence of weak global periodic solutions with small \(L^2\) data. The smallness condition was subsequently removed [3]. Tom [21] proved existence of global weak solutions for initial data in \(H^1\) together with their antiderivative. For well-posedness results, we recall the work of Saut [16], Isaza, Mejía, and Stallbohm [6] and finally the work of Iório and Nunes [8]. The last two authors use the quasi-linear theory of Kato, together with parabolic regularization, to prove local well-posedness with data and their antiderivatives in \(H^s, s > 2\).
limitation $s > 2$ is needed in order to ensure that $\partial_x u \in L^{\infty}$, an essential assumption for the proof. Molinet, Saut and Tzvetkov \cite{14} also proved that if one is willing to assume more regularity for the initial data (at least three derivatives in the $x$ variable and two in the $y$ variable need to be in $L^2$), then global well-posedness holds.

In this paper we use a method involving oscillatory integrals to prove that for small initial data $u_0$ in a certain weighted Sobolev space, defined with at most two derivatives, the IVP (1) is globally well-posed. The fact that we had to use “weights” in the definition of our space agrees with some recent counterexamples of Molinet, Saut and Tzvetkov \cite{15}. These counterexamples suggest “that any iterative method applied to the integral formulation of the KP-I equation always fails” when the initial data are only in anisotropic Sobolev spaces.

The method of proof that we adopt here follows the approach used by Kenig, Ponce and Vega to treat the Schrödinger IVP with derivative in the non-linearity \cite{11}. Here the situation is more complex due to the anisotropic nature of the problem. A weaker version of the so-called “smoothing effect estimates” and “maximal function estimates”, appeared in the work of Isaza, Mejía, and Stallbohm \cite{7}, but these estimates were not strong enough to complete a fixed point argument.

In the rest of this section, we introduce some notation and definitions. Then, in Section 2, we state the main theorem. In Section 3, we present the estimates related to the maximal function associated to (3). Section 4 is devoted to the smoothing effect estimates and Section 5 to group estimates. Finally, in Section 6, we present the main steps of the proof of the well-posedness theorem via the fixed point argument. The paper also has a short appendix on some consequences of the fractional Leibnitz rule.

**Notation.** We denote the Fourier transform of a function $f(x, y)$ by
\[
F(f)(\xi, \lambda) = \hat{f}(\xi, \lambda) = \int_{\mathbb{R}^2} e^{i(\xi x + \lambda y)} f(x, y) \, dx \, dy
\]
and the inverse Fourier transform by
\[
F^{-1}(g)(\xi, \lambda) = \hat{g}(x, y) = \int_{\mathbb{R}^2} e^{-i(\xi x + \lambda y)} g(\xi, \lambda) \, d\xi \, d\lambda.
\]
(We systematically ignore various “$2\pi$-constants”.) The symbol $\langle y \rangle = (1 + y^2)^{1/2}$ is often used as well as the operators $D_x^\sigma$ and $D_y^\gamma$, which are defined through the Fourier transform as the multiplier operators
\[
F(D_x^\sigma f)(\xi, \lambda) = |\xi|^\sigma \hat{f}(\xi, \lambda), \quad F(D_y^\gamma f)(\xi, \lambda) = |\lambda|^\gamma \hat{f}(\xi, \lambda).
\]
We denote by $H^{\sigma, \gamma}_{\langle y \rangle}$ the closure of the space of Schwartz functions with respect to the norm
\[
\|f\|_{H^{\sigma, \gamma}_{\langle y \rangle}} = \|(\langle y \rangle)^\alpha D_x^\sigma D_y^\gamma f\|_{L^2_{xy}}. \tag{5}
\]
\footnote{The precise “smallness” condition can be found in Theorems 2.1 and 2.2.}
\footnote{The precise definition of the norm in this space can be found in (9).}
We remove the dot on the indices $\sigma$ and $\gamma$ if we replace $D^2_x$ by $(1 + D^2_x)$ and $D^2_y$ by $(1 + D^2_y)$, respectively. We will also use a variety of mixed $L^p$ norms. For example, the space $L^p_t L^q_x L^r_y$ is the space of functions equipped with the norm
\[ \|f\|_{L^p_t L^q_x L^r_y} = \left( \int \left( \int \left( \int |f(x, y, t)|^q \, dy \right)^{q/p} \, dx \right)^{r/q} \, dt \right)^{1/r}. \]

We will also write $L^p_t L^q_x L^r_y$ and $L^2_{g(x,y)} dx dy$ to indicate the space of $L^2$ functions with respect to the measure $g(x, y) \, dx \, dy$.

Next, we define some projection operators that will appear throughout the paper. The operator $P_\pm : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ is defined as
\[ P_\pm u(\xi, \lambda) = \chi_{\{||\xi|| \geq |\lambda|\}} \hat{u}(\xi, \lambda), \]
and $P_- = I - P_+$. It will become clear later that the estimates on the solution of the linear problem associated to (8) would be easier to obtain if one could assume that all frequencies $\xi$ are far from zero. To place ourselves in this setting, we will use the projection operator $Q : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ defined by the formula
\[ Q_u(\xi, \lambda) = \chi_{\{||\xi|| \geq 1\}} \hat{u}(\xi, \lambda). \]

We use the notation $A \lesssim B$ to indicate that there exists a constant $m \neq 0$ such that $A \leq mB$. We will use the abbreviations L.H.S. and R.H.S. to refer to the left-hand side and right-hand side of inequalities or equations.

2. The main theorem

We now consider the KP-I initial value problem
\[ \begin{cases} \partial_t u + \partial_x^2 u - \partial_x^{-1} \partial_y^3 u + \beta \partial_x u^2 = 0, \\
u(x, 0) = u_0(x), \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}, \end{cases} \] (8)
where $u = u(t, x, y)$ is a scalar unknown function and $\beta$ is a real constant.

Using the operator $Q$ defined at the end of Section 1, which basically selects the large frequencies, we define the space $^3 Z_0$. We say that $u_0 \in Z_0$ if
\[ \|u_0\|_{Z_0} = \|u_0\|_{H^2} + \|(1 - Q) u_0\|_{H^{-\gamma_0, \gamma_0}} + \|(1 - Q) u_0\|_{H^{-\sigma_1, 2}} + \|(1 - Q) u_0\|_{H^{-\sigma_2, 2}} \]
\[ + \|Q u_0\|_{H^{-\gamma_3, \gamma_3}} + \|Q u_0\|_{H^{\sigma_3, 4}} + \|Q u_0\|_{H^{\gamma_4, 4}} < \infty, \] (9)
here $\alpha = \gamma_0 = \sigma_0 = \gamma_3 = 1/2 + \epsilon, \sigma_1 = 1/4 + \epsilon, \sigma_2 = \sigma_3 = 3/4 + \epsilon, \sigma_4 = \gamma_1 = \gamma_4 = 1 + \epsilon, \gamma_2 = 3/2 + \epsilon$, where $\epsilon > 0$ is small.

The first result we present is a global well-posedness statement for small initial data:

**Theorem 2.1.** For any $T > 0$, there exists $\delta > 0$ such that for any $u_0 \in Z_0$ with
\[ \max(\|\langle y \rangle^\alpha Q u_0\|_{L^2}, \|\langle y \rangle^\alpha D_x^{-\sigma_0} (1 - Q) u_0\|_{L^2}, \|D_x^{-\sigma_3} (1 - Q) Q u_0\|_{L^2}) \leq \delta, \] (10)
}\[\footnote{See also Definition 6.2.} \]
there exists a unique solution $u(x, y, t)$ for the IVP (8) in the interval $[0, T]$, satisfying

$$u \in C([0, T], Z_0) \cap Z_T,$$

where the space $Z_T$ is defined in Definition 6.1. Moreover, for any $T' \in (0, T)$, there exists $\rho > 0$ such that the map $\bar{u}_0 \to \bar{u}$ from $\{u_0 \in Z_0 : \|u_0 - u_0\|_{Z_0} \leq \rho\}$ into $C([0, T'], Z_0) \cap Z_{T'}$ is Lipshitz.

In the second theorem, we relax (10) a little, but we only obtain a local well-posedness result:

**Theorem 2.2.** There exists $\delta > 0$ such that for any $u_0 \in Z_0$ with

$$\|(y)^{\alpha}(1 + D_y)^{\gamma}u_0\|_{L^2} \leq \delta,$$

there exists $T = T(\|u_0\|_{Z_0})$ and a unique solution $u(x, y, t)$ for the IVP (8) in the interval $[0, T]$, satisfying

$$u \in C([0, T], Z_0) \cap Z_T.$$

Moreover, for any $T' \in (0, T)$, there exists $\rho > 0$ such that the map $\bar{u}_0 \to \bar{u}$ from $\{u_0 \in Z_0 : \|u_0 - u_0\|_{Z_0} \leq \rho\}$ into $C([0, T'], Z_0) \cap Z_{T'}$ is Lipshitz.

Using the operator $Q$ again, we transform (8) into a system with unknowns $Qu = u_1$ and $(Id - Q)u = u_2$:

$$\begin{align*}
\begin{cases}
\partial_t u_1 + \partial_x^2 u_1 - \partial_x^{-1} \partial_y^2 u_1 + \beta Q \partial_x (u_1^2 + u_2^2 + 2u_1 u_2) = 0, \\
\partial_t u_2 + \partial_x^2 u_2 - \partial_x^{-1} \partial_y^2 u_2 + \beta (Id - Q) \partial_x (u_1^2 + u_2^2 + 2u_2 u_1) = 0, \\
u_1(x, 0) = Q u_0(x) = u_0, \\
u_2(x, 0) = (Id - Q) u_0(x) = v_0,
\end{cases}
\end{align*}$$

(11)

Now observe that for any function $f$, $(Id - Q)\partial_x f$ and $f$ have similarly sized Fourier transforms supported on low frequencies. Hence, the effective nonlinear term for the problem is concentrated in the first equation, that is the term $Q \partial_x (u_1^2 + u_2^2 + 2u_2 u_1)$. On the other hand, the dispersive function for the first equation is $\phi(\xi, \lambda) = \chi(|\xi| > 1) (\xi^3 + \lambda^2 / \xi)$, which is not singular.

### 3. The maximal function estimate

In this section we prove a maximal function type estimate for the solution of the linear initial value problem associated to (8). Consider the problem

$$\begin{align*}
\begin{cases}
\partial_t u + \partial_x^2 u - \partial_x^{-1} \partial_y^2 u = 0, \\
u(x, 0) = u_0(x),
\end{cases}
\end{align*}$$

(12)

and denote by $U(t)u_0$ its solution. It is easy to see that

$$U(t)u_0(x) = \int_{\mathbb{R}^2} e^{i(t(\xi^3 + \lambda^2 / \xi) + (\xi x + \lambda y))} u_0(\xi, \lambda) d\xi d\lambda.$$  

(13)

We have the following theorems:
Theorem 3.1. For any $\sigma > 3/4$, $\gamma > 1/2$ and $\theta > 1$,
\[
\left( \sum_{s=-\infty}^{\infty} \sup_{s \leq r \leq s + 1} \sup_{|t| \leq 1} |QU(t)u_0(x, y)|^2 \right)^{1/2} \lesssim \|(1 + D_x)^\sigma (1 + D_y)^\gamma Q u_0\|_{L^2(\mathbb{R}^2)},
\]
(14)
\[
\left( \sum_{r=-\infty}^{\infty} \sup_{r \leq x < r + 1} \sup_{|t| \leq 1} |QU(t)u_0(x, y)|^2 \right)^{1/2} \lesssim \|(1 + D_x)^\sigma (1 + D_y)^\gamma Q u_0\|_{L^2(\mathbb{R}^2)} + \|(1 + D_y)^\theta Q u_0\|_{L^2(\mathbb{R}^2)}. \tag{15}
\]

For the low frequency solution we have:

Theorem 3.2. For $\sigma > 1/4$, $\gamma > 1/2$ and $\theta > 1$,
\[
\left( \sum_{s=-\infty}^{\infty} \sup_{s \leq r \leq s + 1} \sup_{|t| \leq 1} |(Id - Q)U(t)u_0(x, y)|^2 \right)^{1/2} \lesssim \|D_x^{-\sigma} (1 + D_y)^\gamma (Id - Q) u_0\|_{L^2(\mathbb{R}^2)}, \tag{16}
\]
\[
\left( \sum_{r=-\infty}^{\infty} \sup_{r \leq x < r + 1} \sup_{|t| \leq 1} |(Id - Q)U(t)u_0(x, y)|^2 \right)^{1/2} \lesssim \|D_x^{-\sigma} (1 + D_y)^\theta (Id - Q) u_0\|_{L^2(\mathbb{R}^2)}. \tag{17}
\]

The proof of these theorems follows arguments presented in [11]. We use the following lemma:

Lemma 3.3. Let $\phi(x, \lambda) = \xi^3 + \lambda^2/\xi$ and let $\theta_{k,j}$ be a $C^\infty$ function supported on the set $\Delta_{k,j} = \{\xi \sim 2^k$ and $|\lambda/\xi| \sim 2^j$, for $k \in \mathbb{Z}$, $j \in \mathbb{Z}\}$. Then the function
\[
I_{k,j}^t(x, y) = \int_{\mathbb{R}^2} e^{i(t(\phi(x, \lambda) + (x, y)(\xi, \lambda))} \theta_{k,j}(\xi, \lambda) \, d\xi \, d\lambda \tag{18}
\]
satisfies
\[
|I_{k,j}^t(x, y)| \leq H_{k,j}(|x|, |y|), \quad \text{for } |t| \leq 1
\]
and
\[
\sum_{-\infty < s < \infty} \sup_{r} H_{k,j}(|r|, |s|) \lesssim 2^{\alpha(k,j)}, \tag{19}
\]
\[
\sum_{-\infty < r < \infty} \sup_{s} H_{k,j}(|r|, |s|) \lesssim 2^{\delta(k,j)}, \tag{20}
\]
where $\alpha(k, j) = \frac{5}{2}k + j$ if $k \geq 0$ and $\alpha(k, j) = j + 1$ if $k \leq 0$, and
\[
\delta(k, j) = \begin{cases} \frac{5}{2}k + j & \text{if } k \geq \max(0, j), \\ \frac{3}{2}k + j & \text{if either } j \geq k \geq 0, \text{ or } k < 0 \text{ and } k \geq 2j, \\ 2k + j & \text{if } k < \min(0, 2j). \end{cases}
\]
Proof. We subdivide the interval of time $[0, 1]$ into dyadic intervals of size $[2^{\sigma-1}, 2^{\sigma}]$, for $\sigma \leq 0$. Then we rescale the integral $I_{k,j}$ by setting $t = \rho 2^{\sigma}$, $\rho \in [1/2, 1]$, and

$$2^{\sigma/3} \xi = \zeta \quad \text{and} \quad 2^{2\sigma/3} \lambda = \mu.$$  \hfill (21)

Then

$$I_{k,j}(x, y) = \int_{\mathbb{R}^2} 2^{-\sigma} e^{i(\rho \phi(\zeta, \mu) + (2^{-\sigma/3} x, 2^{-2\sigma/3} y)(\zeta, \mu))} \theta_{\tilde{k}, \tilde{j}}(\zeta, \mu) \, d\zeta \, d\mu = 2^{-\sigma} I_{\tilde{k}, \tilde{j}}(\tilde{x}, \tilde{y}),$$  \hfill (22)

where

$$\tilde{k} = k + \sigma/3, \quad \tilde{j} = j + \sigma/3,$$

$$\tilde{x} = 2^{-\sigma/3} x, \quad \tilde{y} = 2^{-2\sigma/3} y.$$  \hfill (23)

For simplicity we drop the “tilde” from $k, j, x$ and $y$ and we keep in mind that again $k, j \in \mathbb{Z}$. We start by proving (20). We make the decomposition

$$I_{k,j}(x, y) = I_{k,j}^0(x, y) + I_{k,j}^1(x, y) + I_{k,j}^2(x, y),$$

for an appropriate $C \gg 1$ to be determined later. A different decomposition will be used later when we address (20).

**Estimate of $K_{k,j}^0$.** Here we simply have

$$|K_{k,j}^0(x, y)| \leq \chi_{\{|x| \leq 1\}} \int_{\mathbb{R}^2} \theta_{k,j}(\zeta, \mu) \, d\zeta \, d\mu = 2^{2k+j} \chi_{\{|x| \leq 1\}} = H^0_{k,j}(|x|, |y|).$$  \hfill (25)

Define the function

$$\Psi_{x,y}(\zeta, \mu) = \rho \phi(\zeta, \mu) + (x, y) \cdot (\zeta, \mu)$$

for $\rho \in [1/2, 1]$. In the rest of the proof we will drop $\rho$. Compute

$$\nabla \Psi_{x,y}(\zeta, \mu) = [3\zeta^2 - \mu^2/\xi^2 + x, 2\mu/\zeta + y]$$  \hfill (26)

and

$$\frac{\partial^2}{\partial \mu} \Psi = 2/\zeta, \quad \frac{\partial}{\partial \mu} \theta_{k,j} \sim 2^{-k-j} \theta_{k,j},$$

$$\frac{\partial^2}{\partial \zeta} \Psi = 6\zeta + \mu^2/\zeta^3, \quad \frac{\partial}{\partial \zeta} \theta_{k,j} \sim 2^{-k-j} \theta_{k,j}.$$  \hfill (27)

We now choose the constant $C$ in the defining condition of $K_{k,j}^1$, $j = 1, 2$, so that, in the next region under consideration $K_{k,j}^1$, we have

$$|\partial \zeta \Psi_{x,y}(\zeta, \mu)| \gtrsim |x|.$$
Estimate of $K^1_{k,j}$. Observe that in this case we can assume that $\max(k, j) \geq 0$. (Otherwise we are in the region where $|x| \leq 1$, which was considered above.) We integrate by parts with respect to $\xi$ twice and we use (28) to obtain

$$|K^1_{k,j}(x, y)| \lesssim \int \left( \frac{\partial_\xi \Psi}{|\xi|^2} + \frac{\partial_\xi \Psi\xi}{|\xi|^3} + \frac{\partial_\xi \Psi_{\xi\xi}}{|\xi|^3} + \frac{\partial_\xi \Psi_{\xi\xi\xi}}{|\xi|^4} \right) d\xi d\mu$$

$$\lesssim 2^{2k+j} \left( \frac{2^{2k}}{\max(|x|^2, 1)} + \frac{2^k \max(2^k, 2^{2j-k})}{\max(|x|^2, 1)} + \frac{\max(1, 2^{2j-k})}{\max(|x|^2, 1)} \right)$$

$$\lesssim 2^{2k+j} \chi_{\{|x| > C \max(1, 2^{2j}, 2^{2k})\}} = H^1_{k,j}(|x|, |y|), \quad (29)$$

Estimate of $K^2_{k,j}$. In this case $\partial_\xi \Psi$ could be zero and no integration by parts could be performed. We need the following lemma.

Lemma 3.4. Assume that $\mu$ is fixed and that there exists $\zeta_0 = \zeta_0(\mu) \in \Delta_{k,j}$ such that $\partial_\xi \Psi(\zeta_0, \mu) = 0$. Then

$$|I_{k,j}(x, y)| = \left| \int e^{i(\zeta^3 + \mu/3 - (x, y)(\zeta, \mu))} \theta_{k,j}(\xi, \mu) d\mu d\zeta \right| \lesssim 2^{\beta(k, j)}, \quad (30)$$

where $\beta(k, j) = \frac{k}{2} + j$ if $k \geq j$ and $\beta(k, j) = \frac{3k}{2}$ if $k < j$.

Proof. We use the Van der Corput lemma (see for example Corollary of Proposition 2 Chap. VIII, in [18]). We first make a change of variables so that $\Delta_{k,j}$ is transformed into $\Delta_{1,1}$, that is, we set $(\xi, \tau) = (2^{-k} \xi, 2^{-j} \tau)$. With the new variables we have

$$\Psi(\xi, \tau) = 2^{3k} \xi^3 + 2^{2j-k} \tau^2 / \xi + (2^k x, 2^j y) \cdot (\xi, \tau) = m_{k,j} \tilde{\Psi}(\xi, \tau),$$

where $m_{k,j} = \max(2^{3k}, 2^{2j-k})$. Then

$$|K^2_{k,j}(x, y)| \sim 2^{2k+j} \left| \int e^{im_{k,j} \tilde{\Psi}(\xi, \tau)} \theta_{1,1}(\xi, \tau) d\tau d\xi \right|. \quad (31)$$

It is easy to check that $\|\tilde{\Psi}\|_{C^1} \leq C$ and that

$$|\partial_\xi \tilde{\Psi}| = m_{k,j}^{-1} |\xi| \left( 2^{3k} 6 + 2^{2j-k+1} \frac{\tau^2}{\xi^4} \right) \sim 1, \quad (32)$$

hence by integrating first with respect to $\xi$ using Van der Corput lemma, and then with respect to $\tau$, we obtain

$$|K^2_{k,j}(x, y)| \sim 2^{2k+j} (m_{k,j})^{-1/2} = 2^{\beta(k, j)} \chi_{\{|x| \leq \max(1, 2^{j}, 2^{2j})\}} = H^2(|x|, |y|), \quad (33)$$

where $\beta(k, j) = k/2 + j$ if $k \geq j$ and $\beta(k, j) = 3k/2$ if $k < j$ and $\max(k, j) > 0$. □

We now go back to the “tilde” notation and we define

$$\tilde{H}^1_{k,j}(x, y) = \sum_{i=0,1,2} \tilde{H}^1_{k,j}(x, y).$$
Using (24), (25), (29), and (33) we finally have that
\[
\begin{align*}
\sum_{r \in \mathbb{Z}} \sup_{s} H_{k,j}(|r|, |s|) &= 2^{-\sigma} \sum_{r \in \mathbb{Z}} \sup_{s} \tilde{H}_{k,j}(2^{-\sigma/3}|r|, 2^{-2\sigma/3}|s|) \\
&= 2^{-2\sigma/3} \sum_{r \in \mathbb{Z}} \sup_{s} H_{k,j}(|r|, |s|) \\
&\leq 2^{-2\sigma/3} 2^{\tilde{\delta}(k,j)},
\end{align*}
\]
where
\[
\tilde{\delta}(k, j) = \begin{cases} 
\frac{5k}{2} + j & \text{if } \bar{k} \geq \max(0, \bar{j}), \\
\frac{3k}{2} + 2j & \text{if } j \geq \max(0, \bar{k}) \text{ or } k < 0 \text{ and } k \geq 2j, \\
2k + j & \text{if } k, j \leq 0,
\end{cases}
\tag{34}
\]
which by (23) gives
\[
\sum_{r \in \mathbb{Z}} \sup_{s} H_{k,j}(|r|, |s|) \lesssim 2^{\sigma/3 + \delta(k,j)}, \quad \sigma \leq 0,
\]
where
\[
\delta(k, j) = \begin{cases} 
\frac{5k}{2} + j & \text{if } k \geq \max(0, j), \\
\frac{3k}{2} + 2j & \text{if } j \geq k \geq 0 \text{ or } k < 0 \text{ and } k \geq 2j, \\
2k + j & \text{if } k \leq \min(0, 2j).
\end{cases}
\tag{35}
\]
This proves (20). We now use similar ideas to prove (19). After we rescaled the time as we did above we write
\[
I_{k,j}(x, y) = I_{k,j}(x, y) \chi_{\{|y| \leq 1\}} + I_{k,j}(x, y) \chi_{\{|y| \geq \max(1, C2r)\}} + I_{k,j}(x, y) \chi_{\{|y| \leq 2r\}}
= K_{k,j}^3(x, y) + K_{k,j}^4(x, y) + K_{k,j}^5(x, y).
\]

**Estimate of** $K_{k,j}^3$. Clearly we can estimate
\[
|K_{k,j}^3(x, y)| \lesssim 2^{2k+j} \chi_{\{|y| \leq 1\}} = H_{k,j}^3.
\tag{36}
\]

**Estimate of** $K_{k,j}^4$. Observe first that in this case we can assume that $j \geq 0$. We then use the fact that for $C \gg 1$
\[
|\partial_{\mu} \Psi| \gtrsim |y|,
\]
and we can integrate by parts with respect to $\mu$ twice and use (27) to obtain
\[
|K_{k,j}^4(x, y)| \lesssim \int \left( \frac{|\theta_{\mu\mu}|}{|\Psi|^{3/2}} + \frac{|\theta_{\mu\mu\mu}|}{|\Psi|^{5/3}} + \frac{|\theta_{\mu\mu\mu}|}{|\Psi|^{7/3}} + \frac{|\theta_{\mu\mu}}{|\Psi|^{5/2}} \right) d\zeta d\mu \\
\lesssim 2^{-2j} \chi_{\{|y| > C \max(1, 2r)\}} = H_{k,j}^4(|x|, |y|).
\tag{37}
\]
Estimate of $K_{k,j}^5$. In this case $\partial_t \Psi$ could be zero and no integration by parts could be performed. Instead we use an argument similar to the one presented during the proof of Lemma 3.4. Rescale the function $\Psi$ using variables $\zeta$ and $\tau$ and define $m_{k,j} = 2^{2j-k}$. Then

$$|K_{k,j}^5(x, y)| \sim 2^{2k+j}(m_{k,j})^{-1/2} = 2^{3k/2} \chi_{(1<|x| \leq 2^r)} = H^5(|x|, |y|).$$

(38)

Using (24), (36), (37), and (38) we finally have that

$$\sum_{s \in \mathbb{Z}} \max_{r \in \mathbb{Z}} H_{k,j}(|r|, |s|) = 2^{-\sigma} \sum_{r \in \mathbb{Z}} \max_{s \in \mathbb{Z}} \hat{H}_{k,j}(|r|, 2^{-s/3}|s|)$$

$$= 2^{-\sigma/3} \sum_{r \in \mathbb{Z}} \max_{s \in \mathbb{Z}} H_{k,j}(|r|, |s|)$$

$$\leq 2^{-\sigma/3} 2^\delta(k,j),$$

where

$$\delta(k, j) = \max(2\tilde{k} + \tilde{j}, -\chi_{\{j \geq 0\}^2} \tilde{j}, \chi_{\{j \geq 0\}^2}(5/2\tilde{k} + \tilde{j})).$$

(39)

After we replace $\tilde{k}$ with $k$ and $\tilde{j}$ with $j$, we have

$$\sum_{s \in \mathbb{Z}} \max_{r \in \mathbb{Z}} H_{k,j}(|r|, |s|) \leq 2^{-\sigma/3} 2^{\alpha(k,j)},$$

where $\alpha(k, j) = \frac{5}{2} k + j$ if $k \geq 0$ and $\alpha(k, j) = j + 1$ if $k \leq 0$ and this proves (19). □

The proofs of Theorems 3.1 and 3.2 follow the basic steps of the proof of Theorem 3.2 in [11].

Proof of Theorems 3.1 and 3.2. Here we prove only (14); (15) will follow from similar arguments and Lemma 3.3. Let $\theta_{k,j}$ be as in the proof of Lemma 3.3. We define

$$F(U_{k,j}(t) u_0)(\xi, \lambda) = e^{i\phi(L, \lambda)} \theta_{k,j}(\xi, \lambda) \hat{u}_0(\xi, \lambda)$$

(40)

with $k \in \mathbb{N}$ and $j \in \mathbb{Z}$. It suffices to show that

$$\left( \sum_{s = -\infty}^{s = 0} \max_{s \leq x \leq s+1} \left| \int \sum_{x = -\infty}^{x = 1} \int \left| U_{k,j}(t) u_0(x, y) \right|^2 \right| \right)^{1/2} \lesssim 2^{\alpha(k,j)} \|u_0\|_{L^2(\mathbb{R}^2)},$$

(41)

where $\alpha(k, j)$ is defined in Lemma 3.3. We recall that the dual operator to $U_{k,j}(t)$, with $t \in [-1, 1]$, is the operator $T^*$ such that

$$T^* g(t, x, y) = \int_{-1}^{1} U_{k,j}(t) g(t, x, y) dt.$$

By duality it suffices to show that

$$\left\| \int_{-1}^{1} U_{k,j}(t) g(t, \cdot) dt \right\|_{L^2} \lesssim 2^{\alpha(k,j)} \left( \sum_{s = -\infty}^{s = 0} \left( \int_{-1}^{1} \int_{\mathbb{R}} \left| g(x, y, t) \right| dx \ dt \ dt \ dy \right)^{1/2} \right)^{1/2}.$$

(42)
Using P. Tomas’ argument on L.H.S. of (42), it follows that

\[
\left\| \int_{-1}^{1} U_{k,j}(t) g(t, \cdot) \, dt \right\|_{L^2}^2 = \int_{\mathbb{R}^2} \int_{-1}^{1} \left( \int_{-1}^{1} U_{k,j}(t - \tau) g(\tau, x, y) \, d\tau \right) g(t, x, y) \, dt \, dx \, dy
\]

\[
\lesssim \sum_s \sup_{s \leq y < s + 1} \int_s^{s+1} \int_{\mathbb{R}} \left( \int_{-1}^{1} U_{k,j}(t - \tau) g(\tau, x, y) \, d\tau \right) g(t, x, y) \, dt \, dx \, dy
\]

\[
\lesssim \sum_s \left( \sup_{s \leq y < s + 1} \sup_{|t| \leq 1} \left\| \int_{-1}^{1} U_{k,j}(t - \tau) g(\tau, x, y) \, d\tau \right\|^2 \right)^{1/2}
\]

\[
\times \left( \sum_s \left( \int_{s}^{s+1} \int_{-1}^{1} |g(t, x, y)| \, dt \, dx \, dy \right)^2 \right)^{1/2}.
\]

So in order to prove (41) it suffices to show that

\[
\left( \sum_{s = -\infty}^{\infty} \sup_{s \leq y < s + 1} \sup_{|t| \leq 1} \left\| \int_{-1}^{1} U_{k,j}(t - \tau) g(\tau, x, y) \, d\tau \right\|^2 \right)^{1/2}
\]

\[
\lesssim 2^{2\alpha(k,j)} \left( \sum_{s = -\infty}^{\infty} \left( \int_{s}^{s+1} \int_{-1}^{1} |g(x, y, t)| \, dx \, dy \, dt \right)^2 \right)^{1/2}.
\] (43)

We observe that

\[
\left| \int_{-1}^{1} U_{k,j}(t - \tau) g(\tau, x, y) \, d\tau \right| \lesssim \int H_{k,j}(z, w) \int_{-1}^{1} |g(x - z, y - w, \tau)| \, dz \, dw \, d\tau
\]

and by Young’s inequality,

\[
\sup_x \left| \int_{-1}^{1} U_{k,j}(t - \tau) g(\tau, x, y) \, d\tau \right| \lesssim \int \sup_z H_{k,j}(z, w) \left( \int_{-1}^{1} |g(z, y - w, \tau)| \, d\tau \right) \, dz \, dw.
\]

If we partition \( w \) we can continue with

\[
\lesssim \sum_{s = -\infty}^{\infty} \sup_{s \leq w < s + 1} H_{k,j}(|z|, |w|) \int_{s}^{s+1} \int_{-1}^{1} |g(z, y - w, \tau)| \, d\tau \, dw \, dz.
\]
Then we partition $y$ with $s$ and take the $L^2$ norm, to find
\[
\lesssim \left( \sum_{s=\infty}^{\infty} \left( \sup_{s \leq y < s+1} \sum_{z \leq w < s+1} H_{k,j}(|z|, |w|) \right) \right)^{1/2}
\]
\[
\lesssim \left( \sum_{s=\infty}^{\infty} \left( \sum_{z \leq w < s+1} H_{k,j}(|z|, |w|) \right) \right)^{1/2}
\]

using the Minkowski inequality with respect to the sum on $\bar{s}$, we continue with
\[
\lesssim \sum_{s=\infty}^{\infty} \sup_{s \leq y < s+1} H_{k,j}(|z|, |w|) \left( \sum_{\sigma=\infty}^{\infty} \left( \int_{\mathbb{R}} \int_{\sigma}^{\sigma+1} \int_{-1}^{1} |g(z, y, \tau)| d\tau dy dz \right)^2 \right)^{1/2}
\]
\[
\lesssim \sum_{s=\infty}^{\infty} \sup_{s \leq y < s+1} H_{k,j}(|z|, |w|) \left( \sum_{\sigma=\infty}^{\infty} \left( \int_{\mathbb{R}} \int_{\sigma}^{\sigma+1} \int_{-1}^{1} |g(z, y, \tau)| d\tau dy dz \right)^2 \right)^{1/2}
\]
\[
\lesssim 2^{2\alpha(k,j)} \left( \sum_{\sigma=\infty}^{\infty} \left( \int_{\mathbb{R}} \int_{\sigma}^{\sigma+1} \int_{-1}^{1} |g(z, y, \tau)| d\tau dy dz \right)^2 \right)^{1/2},
\]

where in the last step we used (19). The proof is then complete. $\square$

Remark 3.5. If one wants to prove the classical $L^2_{x,y}$ estimate of the maximal function, the price to pay is formally an extra derivative with respect to the $y$ variable.

During the proof of the well-posedness result presented in Theorem 2.1 we will need a weighted maximal function estimate. The precise statement is presented in the following theorems

**Theorem 3.6.** For any $\sigma > 3/4$, $\gamma > 1/2$ and $\theta > 1$, there exists $\alpha > 1/2$ such that
\[
\left( \sum_{s=\infty}^{\infty} \sup_{s \leq y < s+1} \sup_{|t| \leq 1} \langle y \rangle^\alpha |QU(t)u_0(x, y)|^2 \right)^{1/2} \lesssim \|\langle y \rangle^\alpha (1 + D_x)^\sigma (1 + D_y)^\gamma u_0\|_{L^2(\mathbb{R}^2)}
\]
\[
+ \| (1 + D_x)^{\sigma-1/2} (1 + D_y)^{\gamma+1/2} u_0 \|_{L^2(\mathbb{R}^2)}, \quad (44)
\]
\[
\left( \sum_{r=\infty}^{\infty} \sup_{r \leq y < r+1} \sup_{|t| \leq 1} \langle y \rangle^\alpha |QU(t)u_0(x, y)|^2 \right)^{1/2} \lesssim \|\langle y \rangle^\alpha (1 + D_x)^\sigma (1 + D_y)^\gamma u_0\|_{L^2(\mathbb{R}^2)}
\]
\[
+ \| (1 + D_x)^{\sigma-1/2} D_y^{\gamma+1/2} u_0 \|_{L^2(\mathbb{R}^2)} + \| (1 + D_y)^{\gamma+1/2} u_0 \|_{L^2(\mathbb{R}^2)} \quad (45)
\]
For low frequencies we have:

**Theorem 3.7.** For any \( \sigma > 1/4, \gamma > 1/2 \) and \( \theta > 1 \), there exists \( \alpha > 1/2 \) such that

\[
\left( \sum_{k=-\infty}^{\infty} \sup_{s \leq y < s + 1} \sup_{t \leq 1} \left\| (\text{Id} - Q) U(t) u_0(x, y) \right\|^2 \right)^{1/2} \lesssim \|y\|^{\alpha} D_x^{-\gamma} (1 + D_y)^{\gamma} u_0 \|L^2(\mathbb{R}^2) + \|D_x^{-\theta} (1 + D_y)^{\sigma} u_0 \|L^2(\mathbb{R}^2),
\]

(46)

\[
\left( \sum_{r=-\infty}^{\infty} \sup_{r \leq x < r + 1} \sup_{t \leq 1} \left\| (\text{Id} - \tilde{Q}) U(t) u_0(x, y) \right\|^2 \right)^{1/2} \lesssim \|y\|^{\alpha} D_x^{-\gamma} (1 + D_y)^{\gamma} u_0 \|L^2(\mathbb{R}^2) + \|D_x^{-\theta} (1 + D_y)^{\sigma} u_0 \|L^2(\mathbb{R}^2).
\]

(47)

Here we prove only the inequalities (44) of Theorem 3.6 and (46) of Theorem 3.7, the rest follows with similar arguments.

**Proof.** Let \( \tilde{Q} \) be the operator such that \( F(\tilde{Q}f)(\xi, \lambda) = \chi_{\{ |\lambda/\xi| \geq 1 \}} \hat{f}(\xi, \lambda) \). Assume \( w_0(\xi, \lambda) = w_0(|\lambda|\xi) \), where \( w_0(r) \) is a smooth characteristic function of the interval \([-2, 2]\). We write \( QU(t) u_0 = (\text{Id} - \tilde{Q}) QU(t) u_0 + \tilde{Q}QU(t) u_0 \). Next we define the multiplier operator \( P_k \) such that \( F(P_k f)(\xi) = \psi(2^{k} \xi) \hat{f}(\xi) \), where \( \psi \) is a smooth characteristic function of \([1/2, 2]\), and \( \sum_{k \geq 0} P_k = Q \). We write \( (\text{Id} - \tilde{Q}) QU(t) u_0 = \sum_{k \geq 0} (\text{Id} - \tilde{Q}) QU(t) P_k u_0 \). We then define the operator

\[
W^z_k = |y|^{\alpha} (\text{Id} - \tilde{Q}) QU(t) P_k
\]

for \( z \in \mathbb{C} \) and we use complex interpolation. Assume \( \text{Re} \ z = 0 \), we use the argument presented to prove (14) and we write

\[
\left( \sum_{k=-\infty}^{\infty} \sup_{s \leq y < s + 1} \sup_{t \leq 1} \left\| (\text{Id} - \tilde{Q}) QU(t) P_k u_0(x, y) \right\|^2 \right)^{1/2} \lesssim \|u_0\|L^2(2^{\alpha(k,j)} dx dy),
\]

(48)

where \( \alpha(k, j) = 5k/2 + j+ \). Assume now that \( \text{Re} \ z = 1 \). We have

\[
F(y(\text{Id} - \tilde{Q}) QU(t) P_k u_0)(\xi, \lambda) \sim \partial_\lambda \left( \frac{e^{it(\xi^3 + \lambda^2/\xi)}}{2 \xi} w_0(\lambda/\xi) \psi(\xi) \hat{u}_0 \right)(\xi, \lambda)
\]

\[
\sim \frac{e^{it(\xi^3 + \lambda^2/\xi)}}{2 \xi} \left( \frac{\lambda}{\xi} \hat{u}_0 + 2^{-k} \bar{u}_0(\lambda/\xi) \psi(\xi) \hat{u}_0 + w_0(\lambda/\xi) \psi(\xi) \partial_\lambda \hat{u}_0 \right)
\]

and again, using the argument presented to prove (14), we have

\[
\left( \sum_{k=-\infty}^{\infty} \sup_{s \leq y < s + 1} \sup_{t \leq 1} \left\| y(\text{Id} - \tilde{Q}) QU(t) P_k u_0(x, y) \right\|^2 \right)^{1/2} \lesssim \|u_0\|L^2(2^{\alpha(k,j)} dx dy).
\]

(49)

We then interpolate between (48) and (49) and sum over \( k \) to obtain (44) in this case.
Next define $T_{k,j}$, $k \in \mathbb{N}$, $j \in \mathbb{N}$, such that $F(T_{k,j}(f))(\xi, \lambda) = \theta_{k,j}\hat{f}(\xi, \lambda)$, where $\theta_{k,j}$ was defined in Lemma 3.3, and $\sum_{k,j} T_{k,j}^2 = 1$. We first prove, again by interpolation, that for any $\alpha \in [0, 1]$

$$
\left( \sum_{s=-\infty}^{\infty} \sup_{s \leq y < s+1} \sup_{|t| \leq 1} \sup_{x} |y|^\alpha |\hat{Q}(t)T_{k,j}u_0(x, y)|^2 \right)^{1/2} \lesssim \|u_0\|_{L^2(2^{n(k,j)} \langle 2^{2j} + y^2 \rangle dx dy)},
$$

(50)

where $\alpha(k, j) = 5k/2 + j + 1$. Define the operator

$$W_{k,j} = |y|^2 \hat{Q}(t)T_{k,j}.$$

Then for $\text{Re} \ z = 0$, the argument presented to prove (14) gives

$$
\left( \sum_{s=-\infty}^{\infty} \sup_{s \leq y < s+1} \sup_{|t| \leq 1} \sup_{x} |y|^\alpha |\hat{Q}(t)T_{k,j}u_0(x, y)|^2 \right)^{1/2} \lesssim \|u_0\|_{L^2(2^{n(k,j)} \langle 2^{2j} + y^2 \rangle dx dy)},
$$

(51)

For $\text{Re} \ z = 1$ we have

$$F(y\hat{Q}(t)T_{k,j}u_0)(\xi, \lambda) \sim \partial_\xi (e^{it(\xi^2 + \lambda^2/\xi)} \hat{u}_0)(\xi, \lambda)
\sim e^{it(\xi^2 + \lambda^2/\xi)} \left( \frac{\lambda^2}{\xi} + \partial_\xi \right) \hat{u}_0(\xi, \lambda)
\sim 2^j F(\hat{Q}(t)T_{k,j}u_0)(\xi, \lambda) + F(\hat{Q}(t)T_{k,j}y \hat{u}_0)(\xi, \lambda).
$$

(52)

Again, the argument presented to prove (14) yields

$$
\left( \sum_{s=-\infty}^{\infty} \sup_{s \leq y < s+1} \sup_{|t| \leq 1} \sup_{x} |y|^\alpha |\hat{Q}(t)T_{k,j}u_0(x, y)|^2 \right)^{1/2} \lesssim \|u_0\|_{L^2(2^{n(k,j)} \langle 2^{2j} + y^2 \rangle dx dy)},
$$

(53)

Combining (51) and (53) and using complex interpolation we obtain (50). Now using (50) we have

$$
\left( \sum_{s=-\infty}^{\infty} \sup_{s \leq y < s+1} \sup_{|t| \leq 1} \sup_{x} |y|^\alpha |\hat{Q}(t) T_{k,j}^2 u_0(x, y)|^2 \right)^{1/2} \lesssim \sum_{k,j} \left( \sum_{s=-\infty}^{\infty} \sup_{s \leq y < s+1} \sup_{|t| \leq 1} \sup_{x} |y|^\alpha |\hat{Q}(t) T_{k,j}^2 u_0(x, y)|^2 \right)^{1/2}
\lesssim \sum_{k,j} \|T_{k,j} u_0\|_{L^2(2^{n(k,j)} \langle 2^{2j} + y^2 \rangle dx dy)}
\lesssim \sum_{k,j} \|T_{k,j} u_0\|_{L^2(2^{n(k,j)} \langle 2^{2j} \rangle dx dy)} + \|T_{k,j} u_0\|_{L^2(2^{n(k,j)} \langle 2^{2j} \rangle dx dy)}.
$$

For $\alpha = 1/2+$ we have $\frac{\alpha(k,j)}{2} + \alpha j = \frac{(1-k)}{4} k + (1+j) j$ and

$$
\sum_{k,j} \|T_{k,j} u_0\|_{L^2(2^{n(k,j)} \langle 2^{2j} \rangle dx dy)} \lesssim \|D_x^{1/4} D_y^{1/4} u_0\|_L^2.
$$

(54)
To treat the term involving $|y|^\alpha$ we need a commutator estimate. Recall that here $j \geq 0$. Then for $\alpha > 1/2$, $\sigma = 3/4$, $\gamma = 1/2+$ we have

$$\|T_{k,j}u_0\|_{L^2(2^{(k-j)}|y|^{2\sigma} \, dx \, dy)} \sim 2^{(k+j)} \|\alpha_{\alpha} T_{k,j}(D_x^{3/4} + D_y^{1/2+})u_0\|_{L^2}.$$  

Write $g(x, y) = D_x^{3/4} + D_y^{1/2+}u_0$, then by (7.1) it follows

$$\|\alpha_{\alpha} T_{k,j}g\|_{L^2} \sim \|D_x^{\gamma}(\theta_{k,j}\hat{g})\|_{L^2} \leq \|D_x^{\gamma}(\theta_{k,j}\hat{g}) - D_x^{\gamma}(\theta_{k,j}\hat{g})\|_{L^2} + \|D_x^{\gamma}(\theta_{k,j}\hat{g})\|_{L^2} \leq \|\theta_{k,j}\|_{L^{\infty}} \|D_x^{\gamma}(\hat{g})\|_{L^2},$$

where, in the last step, we used Sobolev’s theorem. Now, it is not hard to show that $\|\theta_{k,j}\|_{L^{\infty}} \leq 2^{(1/2-\alpha)(j+k)}$. In this case $j, k \geq 0$, hence $\|\theta_{k,j}\|_{L^{\infty}} \leq C$, but in the proof of (46) this “small” factor must be considered. From here we obtain

$$\sum_{k,j} \|T_{k,j}u_0\|_{L^2(2^{(k-j)}|y|^{1+} \, dx \, dy)} \leq \|D_x^{3/4} + D_y^{1/2+}u_0\|_{L^2} + \|\alpha_{\alpha} T_{k,j}u_0\|_{L^2} \sim 2^{(1/2+\alpha)(j+k)} + \|\alpha_{\alpha} T_{k,j}u_0\|_{L^2}.$$  

Then combining (54) and (55) we obtain (44).

We now pass to the proof of (46). A weaker version of this can be simply obtained following the argument above by substituting (53) with

$$\left(\sum_{s=-\infty}^{\infty} \sup_{s \leq y < s+1} \sup_{|t| \leq 1} \sup_{x} |\hat{Q}(Id - Q)U(t)T_{k,j}u_0(x, y)|^2 \right)^{1/2} \lesssim \|u_0\|_{L^2(2^{(k-j)}(2^{s}+y^2) \, dx \, dy)},$$

where $\alpha = j+$. But one can do better by repeating the proof of (16) directly with $2^s \hat{Q}U(t)T_{k,j}u_0$, that is the first contribution in (52). In this case, (39) is changed by

$$\delta(k, j) = \max\{2k + 2j, -\chi_{j \geq 0}, \chi_{j \geq 0}(5/2k + 2j),\}$$

which translate into $\tilde{\alpha}(k, j) = j+$, if $2k + 2j < j$ and $\tilde{\alpha}(k, j) = 2k + 2j$, if $2k + 2j \geq j$. The result then follows by interpolation. 

4. THE SMOOTHING EFFECT ESTIMATES

In this section we prove some basic estimates that describe the smoothing effect associated to the IVP (12). As mentioned in the introduction, the smoothing effect is linked to the curvature of the surface $S = \{(\xi, \lambda, \xi^3 + \lambda^2/\xi)\}$. We show that in the region where the surface is of quadratic type (i.e., $|\lambda|/|\xi| \gg |\xi|$), the dispersion behaves like in the Schrödinger equation and we do not gain more that $\partial_x^{1/2}$ smoothness. On the other hand, in the region where the surface is of cubic type (i.e., $|\lambda|/|\xi| \ll |\xi|$), the dispersion behaves like in the KdV equation and we gain the full derivative $\partial_x^j$. 

Lemma 4.1. If $U(t)u_0$ is the solution of the IVP (12), then

$$\|\partial_x U(t)P_+Qu_0\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2_{x,y}}, \quad (58)$$

$$\|\partial_x^{1/2} U(t)P_-Qu_0\|_{L^\infty_t L^2_x} \lesssim \|u_0\|_{L^2_{x,y}}, \quad (59)$$

The inhomogeneous version of (59) becomes

$$\left\| \partial_x \int_0^t U(t - t')P_-Qf(x, y, t') \right\|_{L^\infty_y L^2_x} \leq c\|f\|_{L^1_y L^1_x}, \quad (60)$$

Proof. Our proof follows the proof of the one-dimensional KdV smoothing effect presented in [12]. To prove (58), we write

$$\partial_x U(t)P_+Qu_0(x, y) = \int_{|\xi| \geq 1/2} e^{it\xi} e^{i\xi x} e^{i\xi y} \chi_0(\xi, \lambda) d\xi d\lambda$$

Then, by the Plancherel formula,

$$\|\partial_x U(t)P_+Qu_0\|_{L^2_{x,y}} = \|e^{i\theta(\xi, \lambda)} \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) J^{-1}\|_{L^2_{x,y}}$$

To prove (59), we use a similar argument and write

$$\partial_x^{1/2} U(t)P_+Qu_0(x, y) = \int_{|\xi| \leq 1/2} e^{i\xi x} e^{i\xi y} \chi_0(\xi, \lambda) d\xi d\lambda$$

We make the change of variables $(\xi, \rho) = (\xi, \xi^2 + \lambda^2 / \xi)$, with Jacobian $J(\xi, \lambda) \gtrsim |\xi|$. We set $\lambda = \gamma(\xi, \rho)$ and we continue the chain of inequalities above with

$$\int e^{i\xi x + i\lambda^2 / \xi} \chi_0(\xi, \lambda) d\xi d\lambda$$

Then, by the Plancherel formula,

$$\|\partial_x^{1/2} U(t)P_-Qu_0\|_{L^2_{x,y}} = \|e^{i\gamma(\xi, \lambda)} \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) J^{-1}\|_{L^2_{x,y}}$$

To prove (59), we use a similar argument and write

$$\partial_x^{1/2} U(t)P_-Qu_0(x, y) = \int_{|\xi| \leq 1/2} e^{i\xi x} e^{i\xi y} \chi_0(\xi, \lambda) d\xi d\lambda$$

We make the change of variables $(\xi, \rho) = (\xi, \xi^2 + \lambda^2 / \xi)$, with Jacobian $J(\xi, \lambda) \gtrsim |\xi|$. We set $\lambda = \gamma(\xi, \rho)$ and we continue the chain of inequalities above with

$$\int e^{i\xi x + i\lambda^2 / \xi} \chi_0(\xi, \lambda) d\xi d\lambda$$

Then, by the Plancherel formula,

$$\|\partial_x^{1/2} U(t)P_-Qu_0\|_{L^2_{x,y}} = \|e^{i\gamma(\xi, \lambda)} \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) J^{-1}\|_{L^2_{x,y}}$$

To prove (59), we use a similar argument and write

$$\partial_x^{1/2} U(t)P_-Qu_0(x, y) = \int_{|\xi| \leq 1/2} e^{i\xi x} e^{i\xi y} \chi_0(\xi, \lambda) d\xi d\lambda$$

We make the change of variables $(\xi, \rho) = (\xi, \xi^2 + \lambda^2 / \xi)$, with Jacobian $J(\xi, \lambda) \gtrsim |\xi|$. We set $\lambda = \gamma(\xi, \rho)$ and we continue the chain of inequalities above with

$$\int e^{i\xi x + i\lambda^2 / \xi} \chi_0(\xi, \lambda) d\xi d\lambda$$

Then, by the Plancherel formula,

$$\|\partial_x^{1/2} U(t)P_-Qu_0\|_{L^2_{x,y}} = \|e^{i\gamma(\xi, \lambda)} \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) \chi_0(\xi, \lambda) J^{-1}\|_{L^2_{x,y}}$$

To prove (59), we use a similar argument and write

$$\partial_x^{1/2} U(t)P_-Qu_0(x, y) = \int_{|\xi| \leq 1/2} e^{i\xi x} e^{i\xi y} \chi_0(\xi, \lambda) d\xi d\lambda$$

We make the change of variables $(\xi, \rho) = (\xi, \xi^2 + \lambda^2 / \xi)$, with Jacobian $J(\xi, \lambda) \gtrsim |\xi|$. We set $\lambda = \gamma(\xi, \rho)$ and we continue the chain of inequalities above with

$$\int e^{i\xi x + i\lambda^2 / \xi} \chi_0(\xi, \lambda) d\xi d\lambda$$

Then, by the Plancherel formula,
In order to prove the inhomogeneous estimate (60), we first observe that the adjoint operator $(U(t)P_- Q)^* \colon L_y^1 L_x^2, t \to L_x^2, y$ is defined as

$$ (U(t)P_- Q)^* f(x, y, t) = \int_{-\infty}^\infty P_-(t)Qf(x, y, t)\, dt $$

and that the dual version of (60) becomes

$$ \left\| \partial_x^{1/2} \int_{-\infty}^\infty P_-(t)Qf(x, y, t)\, dt \right\|_{L_y^2, x} \leq \| f \|_{L_y^1 L_x^2, t}. \tag{61} $$

Now, following the argument on page 554 of [12], we see that it suffices to show that

$$ \left\| \partial_x \int_{-\infty}^\infty U(t - t') P_- f Q(x, y, t') \, dt' \right\|_{L_y^\infty L_x^2, t} \leq \| f \|_{L_y^1 L_x^2, t}. \tag{62} $$

To see this, we take a smooth function $g$ such that $\| g \|_{L_y^1 L_x^2, t} \leq 1$ and write

$$ \begin{align*}
\int_{-\infty}^\infty \int_{\mathbb{R}^2} \left( \partial_x \int_{-\infty}^\infty U(t - t') P_- f Q(x, y, t') \, dt' \right) g(x, y, t) \, dx \, dy \, dt & = \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{\mathbb{R}^2} \xi e^{i(t-t')\phi(\xi, \lambda)} \chi_{[1 \leq |\xi| < 2 \frac{|\lambda|}{|t|}]} \hat{f}(\xi, \lambda) \hat{g}(\xi, \lambda, t') \, d\xi \, d\lambda \, dt' \\
& = \int_{-\infty}^\infty \int_{-\infty}^\infty \int_{\mathbb{R}^2} \xi^{1/2} e^{-i\lambda \phi(\xi, \lambda)} \chi_{[1 \leq |\xi| < 2 \frac{|\lambda|}{|t|}]} \hat{f}(\xi, \lambda, t') \xi^{1/2} e^{i\phi(\xi, \lambda)} \hat{g}(\xi, \lambda, t) \, d\xi \, d\lambda \, dt' \\
& = \int_{\mathbb{R}^2} \int_{-\infty}^\infty U(\xi, t') P_-(t') f(y, x, t') \, dt' \partial_x^{1/2} \int_{-\infty}^\infty U(t) P_- Q g(x, y, t) \, dt \, dx \, dy \\
& \leq \int_{\mathbb{R}^2} \int_{-\infty}^\infty U(\xi, t') P_-(t') f(y, x, t') \, dt' \partial_x^{1/2} \int_{-\infty}^\infty U(t) P_- Q g(x, y, t) \, dt \left\| \partial_x^{1/2} \int_{-\infty}^\infty U(t) P_- Q g(x, y, t) \, dt \right\|_{L_y^2, x} \\
& \leq \| f \|_{L_y^1 L_x^2, y} \| g \|_{L_y^1 L_x^2, y}.
\end{align*} $$

5. The group estimates

In order to define a Banach space suitable for a contraction mapping theorem, we need to analyze how the group $U(t)$ acts in certain weighted anisotropic Sobolev spaces.

**Lemma 5.1.** Assume that $\langle y \rangle^\alpha g$, $(1 + D_x)^{-\alpha}(1 + D_y)^{\alpha} g \in L_x^2, y$ for $\alpha \in [0, 1]$. Then

$$ \| \langle y \rangle^\alpha U(t) Q g(x, y) \|_{L_y^2, x} \leq \| \langle y \rangle^\alpha g \|_{L_y^2, x} + \| (1 + D_y)^{\alpha}(1 + D_x)^{-\alpha} g \|_{L_y^2, x}. \tag{63} $$

For the small frequencies we also have

$$ \| \langle y \rangle^\alpha U(t)(\Id - \hat{Q}) g(x, y) \|_{L_y^2, x} \leq \| \langle y \rangle^\alpha g \|_{L_y^2, x} + \| (1 + D_y)^{\alpha} D_x^{-\alpha} g \|_{L_y^2, x}. \tag{64} $$

**Proof.** Here we only present the proof for (63), the one for (64) follows by similar arguments. We define $\hat{Q}$ to be the operator such that $\hat{Q} f(\lambda) = \chi_{|\lambda| \geq 1} \hat{f}(\lambda)$. (Note that this $\hat{Q}$ is different from the one used in the proof of Theorem 3.7.) Then assume $\psi_0(\lambda) = \psi_0(|\lambda|)$ is a smooth characteristic function of the interval $[-2, 2]$. 

\[ \text{□} \]
We consider first the operator $T^z = \langle y \rangle^2 U(t)(Id - \tilde{Q})Qg(x, y)$, for $z \in \mathbb{C}$. Clearly, for $\text{Re} \ z = 0$ we have
\[
\|U(t)(Id - \tilde{Q})Qg(x, y)\|_{L^2} \lesssim \|g\|_{L^2((dx, dy))}.
\]
(65)

On the other hand, when $\text{Re} \ z = 1$ we estimate
\[
F(yU(t)(Id - \tilde{Q})Qg)(\xi, \lambda) \sim \partial_\lambda \left(e^{it(\xi^2 + \lambda^2/\xi)} \psi_0(\lambda) \hat{g}\right)
\approx e^{it(\xi^2 + \lambda^2/\xi)} \left(\frac{\lambda}{\xi} \hat{g} + \omega'(\lambda) \hat{g} + \partial_\lambda \hat{g}\right) \chi_{\{\xi \geq 1\}}.
\]
It follows that for $\text{Re} \ z = 1$
\[
\sup_{|t| \leq 1} \|yU(t)(Id - \tilde{Q})Qg(x, y)\|_{L^2} \lesssim \|g\|_{L^2((1+y^2)dx, dy)}.
\]
(66)

Interpolating (65) and (66) we obtain
\[
\sup_{|t| \leq 1} \|(y)^\alpha U(t)(Id - \tilde{Q})Qg(x, y)\|_{L^2} \lesssim \|g\|_{L^2((1+y^2)^\alpha dx, dy)},
\]
for $\alpha \in [0, 1]$, and this proves (63) for $|\lambda| \leq 1$. To treat the region $|\lambda| > 1$ we introduce a Littlewood–Paley decomposition. Let $\psi_{k,j}$, $k, j \in \mathbb{N}$, be such that
\[
\psi_{k,j}(\xi, \lambda) = \psi(2^{-k} \xi, 2^{-j} \lambda)
\]
where $\psi(|r|, |s|)$ is a smooth characteristic function of the rectangle $[1/2, 2] \times [1/2, 2]$. Let $P_{k,j}$ be the multiplier operator such that
\[
F(P_{k,j}(f))(\xi, \lambda) = \psi_{k,j} \hat{f}(\xi, \lambda).
\]
(67)

We need some preliminary estimates:
\[
\left\|\langle y \rangle^\alpha \left(\sum_{k,j} |P_{k,j} f|^2\right)^{1/2}\right\|_{L^2} \lesssim \|\langle y \rangle^\alpha f\|_{L^2} + \|f\|_{L^2},
\]
(68)
\[
\left\|\langle y \rangle^\alpha \left(\sum_{k,j} |P_{k,j} f_{k,j}|^2\right)^{1/2}\right\|_{L^2} \lesssim \|\langle y \rangle^\alpha \left(\sum_{k,j} |f_{k,j}|^2\right)^{1/2}\right\|_{L^2},
\]
(69)
\[
\left\|\frac{1}{\langle y \rangle^\alpha} \left(\sum_{k,j} |P_{k,j} f|^2\right)^{1/2}\right\|_{L^2} \lesssim \left\|\frac{1}{\langle y \rangle^\alpha} f\right\|_{L^2},
\]
(70)
\[
\|\langle y \rangle^\alpha f\|_{L^2} \lesssim \|\langle y \rangle^\alpha \left(\sum_{k,j} |P_{k,j} f|^2\right)^{1/2}\right\|_{L^2}.
\]
(71)

To prove (68), note that the square of the L.H.S. is $\sum_{k,j} \|\langle y \rangle^\alpha P_{k,j} f\|_{L^2}$. We estimate this by interpolation. Define the operator $T_{k,j}^z = \|y\|^2 \tilde{P}_{k,j}$. Then, when $\text{Re} \ z = 0$ we get the trivial bound $\|T_{k,j}^z f\|_{L^2}$. When $\text{Re} \ z = 1$ we get
\[
\|yP_{k,j} f\|_{L^2}^2 = \|\partial_\lambda (\psi_{k,j} f)\|^2_{L^2} \lesssim 2^{-2j} \|\psi_{k,j} f\|_{L^2}^2 + \|\psi_{k,j} \partial_\lambda f\|_{L^2}^2 \lesssim 2^{-2j} \|\tilde{P}_{k,j} f\|_{L^2}^2 + \|P_{k,j} (y f)\|_{L^2}^2.
\]
We now use the support properties of $\tilde{P}_{k,j}$ and $P_{k,j}$ on the Fourier transform to conclude that when $\text{Re } z = 0$
\[ \left\| |y|^2 \left( \sum_{k,j} |P_{k,j} f|^2 \right) \right\|_{L^2}^{1/2} \lesssim \| f \|_{L^2}, \tag{72} \]
and when $\text{Re } z = 1$
\[ \left\| |y|^2 \left( \sum_{k,j} |P_{k,j} f|^2 \right) \right\|_{L^2}^{1/2} \lesssim \| \langle y \rangle f \|_{L^2}. \tag{73} \]
Complex interpolation between (72) and (73) proves (68). We now prove (69). We use again interpolation. When $\text{Re } z = 0$:
\[ \left\| \sum_{k,j} P_{k,j} f_{k,j} \right\|_{L^2} = \sup_{\| \|_{L^2} \leq 1} \left\| \sum_{k,j} \int P_{k,j} f_{k,j} g \, dx \, dy \right\| = \sup_{\| \|_{L^2} \leq 1} \left\| \sum_{k,j} \int f_{k,j} P_{k,j} g \, dx \, dy \right\| \]
\[ \lesssim \left\| \left( \sum_{k,j} |f_{k,j}|^2 \right)^{1/2} \left( \sum_{k,j} |P_{k,j} g|^2 \right)^{1/2} \right\|_{L^2} \]
where, in the last step, we used Littlewood–Paley theory. When $\text{Re } z = 1$ we use similar arguments to show that
\[ \left\| \sum_{k,j} P_{k,j} f_{k,j} \right\|_{L^2} = \left\| 2^{-j} \tilde{P}_{k,j} f_{k,j} + \sum_{k,j} P_{k,j} \langle y f_{k,j} \rangle \right\|_{L^2} \]
\[ \lesssim \left\| \left( \sum_{k,j} |f_{k,j}|^2 \right) \right\|_{L^2} + \left\| \left( \sum_{k,j} |y f_{k,j}|^2 \right) \right\|_{L^2} \]
\[ \lesssim \left\| \langle y \rangle \left( \sum_{k,j} |f_{k,j}|^2 \right)^{1/2} \right\|_{L^2}. \]
We are now ready to prove (70). Let $g_{k,j}$ be functions such that
\[ \left\| \left( \sum_{k,j} |g_{k,j}|^2 \right)^{1/2} \langle y \rangle^\alpha \right\|_{L^2} \leq 1. \]
Then
\[ \left\| \frac{1}{\langle y \rangle^\alpha} \left( \sum_{k,j} |P_{k,j} f|^2 \right)^{1/2} \right\|_{L^2} = \sup_{g_{k,j}} \left\| \sum_{k,j} P_{k,j} f_{g_{k,j}} \rangle \, dx \, dy \right\| = \sup_{g_{k,j}} \left\| \sum_{k,j} P_{k,j} g_{k,j} \rangle f \, dx \, dy \right\| \]
\[ \lesssim \left\| \langle y \rangle^\alpha \sum_{k,j} P_{k,j} g_{k,j} \rangle \right\|_{L^2} \left\| \frac{1}{\langle y \rangle^\alpha} f \right\|_{L^2}. \]
We then use (69) to obtain the desired result. We now prove (71). Write \( f = \sum_{k,j} P_{k,j}^2 f \). Let \( g \) be such that \( \|(1 + |y|)^{-\alpha}g\|_{L^2} \leq 1 \). We estimate
\[
\int f(x, y)g(x, y) \, dx \, dy = \int \sum_{k,j} P_{k,j}^2 f g \, dx \, dy = \int \sum_{k,j} P_{k,j} f P_{k,j} g \, dx \, dy
\]
\[
\leq \int \left( \sum_{k,j} |P_{k,j} f|^2 \right)^{1/2} \left( \sum_{k,j} |P_{k,j} g|^2 \right)^{1/2} \, dx \, dy
\]
\[
\lesssim \left\| \langle y \rangle^\alpha \left( \sum_{k,j} |P_{k,j} f|^2 \right)^{1/2} \right\|_{L^2}^2 \left\| \langle y \rangle^{-\alpha} \left( \sum_{k,j} |P_{k,j} g|^2 \right)^{1/2} \right\|_{L^2}^2
\]
\[
\lesssim \left\| \langle y \rangle^\alpha \left( \sum_{k,j} |P_{k,j} f|^2 \right)^{1/2} \right\|_{L^2} \left\| \langle y \rangle^{-\alpha} g \right\|_{L^2}^2,
\]
where we used (70) in the last step. We are now ready to prove (63). Using the same ideas we presented above it is not hard to show that for any \( \alpha \in [0, 1] \)
\[
\sup_{|t| \leq 1} \left\| \langle y \rangle^\alpha P_{k,j} \tilde{Q} U(t) u_0 \right\|_{L^2} \lesssim \| u_0 \|_{L^2((2^{k-2\alpha} + |y|^{2\alpha}) \, dx \, dy)}.
\]
(74)
Then, by (71), we write
\[
\sup_{|t| \leq 1} \left\| \langle y \rangle^\alpha \tilde{Q} U(t) u_0 \right\|_{L^2} \lesssim \left\| \langle y \rangle^\alpha \left( \sum_{0 \leq k,j} |P_{k,j} U(t) u_0|^2 \right)^{1/2} \right\|_{L^2}^2
\]
\[
= \sum_{0 \leq k,j} \left\| \langle y \rangle^\alpha |P_{k,j} U(t) u_0| \right\|_{L^2}^2.
\]
(75)
Let \( \tilde{\psi}_{k,j} \) be such that \( \tilde{\psi}_{k,j} \psi_{k,j} = \psi_{k,j} \). We continue (75) with
\[
\sum_{0 \leq k,j} \left\| \langle y \rangle^\alpha |P_{k,j} \tilde{P}_{k,j} u_0| \right\|_{L^2}^2.
\]
Recall also that \( \| P_{k,j} u \|_{L^2}^2 \lesssim \| f \|_{L^2}^2 \). Thus, our basic estimate (74) gives
\[
\sup_{|t| \leq 1} \left\| \langle y \rangle^\alpha \tilde{Q} U(t) u_0 \right\|_{L^2}^2 \lesssim \sum_{0 \leq k,j} \left\| \langle y \rangle^\alpha |P_{k,j} \tilde{P}_{k,j} u_0| \right\|_{L^2}^2
\]
\[
\lesssim \sum_{0 \leq k,j} \| \tilde{P}_{k,j} u_0 \|_{L^2((2^{k-2\alpha} + |y|^{2\alpha}) \, dx \, dy)}^2
\]
\[
\lesssim \sum_{0 \leq k,j} \left( \| 2^{j-\alpha} \tilde{P}_{k,j} u_0 \|_{L^2}^2 + \| \langle y \rangle^\alpha \tilde{P}_{k,j} u_0 \|_{L^2}^2 \right)
\]
\[
\lesssim \| (1 + D_x)^{-\alpha} (1 + D_y)^\alpha u_0 \|_{L^2}^2 + \| \langle y \rangle^\alpha u_0 \|_{L^2}^2
\]
\[
\lesssim \| (1 + D_x)^{-\alpha} (1 + D_y)^\alpha u_0 \|_{L^2}^2 + \| \langle y \rangle^\alpha u_0 \|_{L^2}^2,
\]
where, in the last step, we used (68). This concludes the proof of (63) and hence the proof of the lemma. □
We recall the Strichartz estimates associated to the IVP (8) due to Ben-Artzi and Saut [1]:

**Proposition 5.2.** If $0 \leq \theta < 1$, and $(q, p) = \left(\frac{2}{1 + \theta}, \frac{2}{\theta}\right)$, then

$$
\|U(t)u_0\|_{L_t^q L_x^p} \lesssim \|u_0\|_{L^2}.
$$

(76)

In order to be able to estimate the intermediate orders of derivatives in the non-linear term of (8), we need (76) with $(q, p) = (2, \infty)$ and with the weight $\langle y \rangle^\alpha$, $\alpha = 1/2^+$. For this purpose we prove the following proposition:

**Proposition 5.3.** For any $\alpha \in [0, 1]$ and $t \in [0, 1]$, we have

$$
\|\langle y \rangle^\alpha QU(t)u_0\|_{L_t^q L_x^p} \lesssim T^{\delta(\epsilon)}(\|\langle y \rangle^\alpha ((1 + D_x)^e + (1 + D_y)^e)Qu_0\|_{L^2} + \|(1 + D_y)^{\alpha + e}Qu_0\|_{L^2}).
$$

(77)

For the low frequencies we have

$$
\|\langle y \rangle^\alpha (Id - Q)U(t)u_0\|_{L_t^q L_x^p} \lesssim T^{\delta(\epsilon)}(\|\langle y \rangle^\alpha (1 + D_x)^e u_0\|_{L^2} + \|D_x^{-\alpha - e}(1 + D_y)^{\alpha + 2e}(Id - Q)u_0\|_{L^2})
$$

(78)

for any $0 < \epsilon < 1$.

**Proof.** Here we prove only (77); (78) will follow from similar arguments. We first observe that

$$
\|\langle y \rangle^\alpha QU(t)u_0\|_{L_t^q L_x^p} \lesssim T^{\delta(\epsilon)}(\|\langle y \rangle^\alpha ((1 + D_x)^e + (1 + D_y)^e)\langle y \rangle^\alpha QU(t)u_0\|_{L_t^q L_x^p}),
$$

(79)

where $q(\epsilon) = \frac{2}{1 + \epsilon}, p(\epsilon) = \frac{2}{\epsilon}, \delta(\epsilon) = \frac{\epsilon}{2}$. Then by (94) we can continue with

$$
T^{\delta(\epsilon)}\|\langle y \rangle^\alpha ((1 + D_x)^e + (1 + D_y)^e)\langle y \rangle^\alpha QU(t)u_0\|_{L_t^q L_x^p},
$$

and from here on we can proceed by interpolation, like in the proof of Theorem 3.6, using (76).

\[\square\]

6. **Proof of Theorems 2.1 and 2.2: the fixed point argument**

In this section we give only an outline of the proof of Theorem 2.1. The basic argument is illustrated in many articles in the literature on dispersive equations. In particular, we refer the reader to [11]. The proof of Theorem 2.2 is similar and the main difference is explained in Remark 6.4 below.

We start by transforming (11) into the system of integral equations

$$
u_1 = \chi_{[0, T]}U(t)u_0 + \chi_{[0, T]} \int_0^t QU(t - t') \partial_x(u_1^2 + u_2^2 + 2u_1 u_2) dt',
$$

(80)

$$
u_2 = \chi_{[0, T]}U(t)u_0 + \chi_{[0, T]} \int_0^t (Id - Q)U(t - t') \partial_x(u_1^2 + u_2^2 + 2u_1 u_2) dt'.
$$

(81)

Then it is clear that a solution for (80) and (81) is a fixed point for the operator

$$
L(w, v) = (L_1(w, v), L_2(w, v)),
$$

(82)
where
\[
L_1(w, v) = \chi_{[0,T]} U(t) w_0 + \chi_{[0,T]} \int_0^t Q U(t-t') \partial_x (w^2 + v^2 + 2wv) \, dt',
\]
\[
L_2(w, v) = \chi_{[0,T]} U(t) v_0 + \chi_{[0,T]} \int_0^t (Id - Q) U(t-t') \partial_x (w^2 + v^2 + 2wv) \, dt'.
\]

We now define the Banach space \(X_T \times Y_T\) where we will find the fixed point for (82).

**Definition 6.1.** Let \(\sigma_1 > 3/4, \gamma_1 > 1/2, \sigma_2 > 1, \gamma_2 > 1\), and let \(\alpha = \alpha(\sigma_1, \sigma_2, \gamma_1, \gamma_2) = 1/2^+\) be the smallest \(\alpha\) that can be chosen in Theorems 3.6 and 3.7. Consider the norms
\[
\|v\|_i = \|v\|_{L^2}, \quad \|v\|_2 = \|\langle w \rangle^\alpha (1 + D_x)^{\beta_1} (1 + D_y)^{\beta_2} v \|_{L^\infty L^2_x},
\]
\[
\|v\|_3 = \|\langle w \rangle^\alpha (1 + D_y)^{\beta_2} v \|_{L^\infty L^2_y}, \quad \|v\|_4 = \|\langle w \rangle^\alpha (1 + D_x)^{\beta_1} \|_{L^\infty L^2_x},
\]
\[
\|v\|_5 = \|\partial_x P_+ (D_x^2 + D_y^2) v \|_{L^\infty L^2_y}, \quad \|v\|_6 = \|\partial_x P_+ (1 + D_x)^{\gamma_1} (1 + D_y)^{\gamma_2} v \|_{L^\infty L^2_x},
\]
\[
\|v\|_7 = \|\partial_x P_+ (D_x^2 + D_y^2) v \|_{L^\infty L^2_y}, \quad \|v\|_8 = \|\partial_x P_- (1 + D_x)^{\gamma_1} (1 + D_y)^{\gamma_2} v \|_{L^\infty L^2_x},
\]
\[
\|v\|_9 = \|\langle w \rangle^\alpha v \|_{L^2 L^\infty_t}, \quad \|v\|_10 = \|\langle w \rangle^\alpha v \|_{L^2 L^\infty_t},
\]
\[
\|v\|_{11} = \|\langle w \rangle^\alpha \partial_y v \|_{L^2 L^\infty_t}, \quad \|v\|_{12} = \|\langle w \rangle^\alpha \partial_x v \|_{L^2 L^\infty_t},
\]
and
\[
\|v\|_X = \max_{i=1,\ldots,12} (\|v\|_i).
\]

Note that norms 1, \ldots, 4 are related to the group estimates, norms 5, \ldots, 8 are related to the smoothing estimates and norms 9, 10, 11 are concerned the maximal function estimates. Below we denote by \(\| \cdot \|_{i,t}, i = 1, \ldots, 4\), the norm obtained from \(\| \cdot \|_i, i = 1, \ldots, 4\), when the time variable is left free.

Recall that \(F\) denotes the Fourier transform. We define the spaces of functions
\[
X_T = \{ f(x, y, t) : t \in [0, T], F_x(f)(\xi) = 0 \text{ if } |\xi| \leq 1, \text{ and } \|f\|_X < \infty \},
\]
and
\[
Y_T = \{ f(x, y, t) : t \in [0, T], F_x(f)(\xi) = 0 \text{ if } |\xi| > 1, \text{ and } \|f\|_Y < \infty \},
\]
where
\[
\|f\|_Y = \max_{i=1,3,9,10,11,12} \|f\|_i.
\]
We combine \(X_T\) and \(Y_T\) to obtain the space
\[
Z_T = \{ f(x, y, t) : t \in [0, T], \|Qf\|_X + \|(Id - Q)f\|_Y < \infty \}.
\]

We also need a space for the initial data. For this we introduce the following definition:

**Definition 6.2.** For any \(g(x, y)\) we define
\[
\|g\|_{X,0} = \max_{i=1,\ldots,4} \|\xi_0 g\|_i.
\]
(where we ignore the $L^\infty_t$ part of the $i = 1, \ldots, 4$ norms, since we are considering initial data) and the corresponding space

$$X_0 = \{ g(x, y) : F_x(g)(\xi) = 0 \text{ if } |\xi| \leq 1, \text{ and } \|f\|_{X,0} < \infty \}. $$

We also define

$$\|g\|_{Y,0} = \max(\|g\|_{L^2}, \|\langle y \rangle^\alpha D_x^{-1/4} - D_y^1 g\|_{L^2}, \|\langle y \rangle^\alpha D_x^{-1/2} - D_y^{1/2} + g\|_{L^2}, \|D_x^{-3/4} - D_y^{3/2} + g\|_{L^2}) < \infty,$$

and the associated space

$$Y_0 = \{ g(x, y) : F_x(g)(\xi) = 0 \text{ if } |\xi| > 1, \text{ and } \|f\|_{Y,0} < \infty \}. $$

We combine $X_0$ and $Y_0$ to obtain the space

$$Z_0 = \{ g(x, y) : \|Qg\|_{X,0} + \|(Id - Q)g\|_{Y,0} < \infty \}. $$

We now show that if

$$\max(\|w_0\|_{X,0}, \|v_0\|_{X,0}) \ll \delta, \quad (86)$$

for an appropriate $\delta$, then the operator $L$ is a smooth contraction in a ball of $X_T \times Y_T$, centered at the origin of radius $R \sim \delta$. First, we observe that, by definition, $F_x(L_1(w, v))(\xi) = 0$ for $|\xi| \leq 1$ and $F_x(L_2(w, v))(\xi) = 0$ for $|\xi| > 1$. We start with the estimate of the norm of $L_1(w, v)$. We decided not to write explicitly the estimates for its linear term because they are basically contained in Section 5. We then start by estimating the nonlinear term of (83) containing the term $w^2$. From (63), (58), (60), (44), (45) and (77) we have

$$\left\|\int_0^t U(t-t')\partial_x w^2(t') dt'\right\|_{L_x^1(t)} \lesssim \int_0^T \|\partial_x w^2(t')\|_{1,t'} dt',$$

$$\left\|\int_0^t U(t-t')\partial_x w^2(t') dt'\right\|_{L_x^2(t)} \lesssim \int_0^T \left(\|\partial_x w^2(t')\|_{2,t'} + \|\partial_x w^2(t')\|_{1,t'}\right) dt',$$

$$\left\|\int_0^t U(t-t')\partial_x w^2(t') dt'\right\|_{L_x^3(t)} \lesssim \int_0^T \left(\|\partial_x w^2(t')\|_{3,t'} + \|\partial_x w^2(t')\|_{1,t'}\right) dt',$$

$$\left\|\int_0^t U(t-t')\partial_x w^2(t') dt'\right\|_{L_x^4(t)} \lesssim \int_0^T \left(\|\partial_x w^2(t')\|_{4,t'} + \|\partial_x w^2(t')\|_{1,t'}\right) dt',$$

$$\left\|\int_0^t U(t-t')\partial_x w^2(t') dt'\right\|_{L_x^5(t)} \lesssim \int_0^T \|\partial_x w^2(t')\|_{1,t'} dt',$$

$$\left\|\int_0^t U(t-t')\partial_x w^2(t') dt'\right\|_{L_x^6(t)} \lesssim \int_0^T \|\partial_x w^2(t')\|_{2,t'} dt',$$

$$\left\|\int_0^t U(t-t')\partial_x w^2(t') dt'\right\|_{L_x^7(t)} \lesssim \|\partial_x (D_x^2 + D_y^2)(w^2)\|_{L_x^{1,1}} \lesssim \left(\int \|\langle y \rangle^\alpha \partial_x w^2\|_{1,t}^2 dt\right)^{1/2}.$$
and here we use the previous two steps.

show that estimates similar to those presented above are available for the term with

Estimate of

\[ \| \int_0^t U(t-t') \partial_x w^2(t') \, dt' \|_{L^2_x} \]

\[ \lesssim \| \partial_x (D^\alpha_y D^\gamma_y) w^2(\alpha, \gamma) \|_{L^2_x} \lesssim \left( \int \| \partial_x w^2 \|_{L^2_t}^2 \right)^{1/2}, \]

\[ \int_0^t U(t-t') \partial_x w^2(t') \, dt' \]

\[ \lesssim \int_0^t \| \partial_x w^2(t') \|_{L^2} \, dt' + \int_0^t \| \partial_x w^2(t') \|_{L^1} \, dt', \]

\[ \int_0^t U(t-t') \partial_x w^2(t') \, dt' \]

\[ \lesssim \int_0^t \| \partial_x w^2(t') \|_{L^2} \, dt' + \int_0^t \| \partial_x w^2(t') \|_{L^1} \, dt', \]

It is now clear that the heart of the matter is reduced to obtaining good estimates for

\[ \| \langle y \rangle D^\alpha_x (\partial_x w^2) \|_{L^2_y}, \quad \| \langle y \rangle D^\alpha_y (\partial_x w^2) \|_{L^2_y}, \quad \| \langle y \rangle D^\alpha_y D^\gamma_y \partial_x w^2 \|_{L^2_y}. \]

**Estimate of** \[ \| \langle y \rangle D^\alpha_x (\partial_x w^2) \|_{L^2_y}. \]

We can write

\[ \| \langle y \rangle D^\alpha_x (\partial_x w^2) \|_{L^2_y} \sim \| \langle y \rangle D^\alpha_x (\partial_x w^2) \|_{L^2_y} + \| \langle y \rangle D^\alpha_y (\partial_x w^2) \|_{L^2_y}, \]

\[ \sim \| \langle y \rangle D^\alpha_x (P^+ w) w \|_{L^2_y} + \| \langle y \rangle D^\alpha_y (P^- w) w \|_{L^2_y}, \]

\[ + \| \langle y \rangle D^\gamma_y (\partial_x w) \partial_x w \|_{L^2_y} \lesssim \| w \|_5 \| w \|_{10} + \| w \|_7 \| w \|_9 + \| w \|_1 \| w \|_12. \]

**Estimate of** \[ \| \langle y \rangle D^\alpha_y (D^\gamma_y) \partial_x w^2 \|_{L^2_y}. \]

With similar arguments and (94) one obtains

\[ \| \langle y \rangle D^\alpha_y (D^\gamma_y) \partial_x w^2 \|_{L^2_y} \lesssim \| w \|_5 \| w \|_{10} + \| w \|_7 \| w \|_9 + \| w \|_1 (\| w \|_12 + \| v \|_11). \]

**Estimate of** \[ \| \langle y \rangle D^\alpha_y D^\gamma_y \partial_x w^2 \|_{L^2_y}. \]

Assume now that \( \sigma + \gamma \leq 2 \). Then from (63) and (94) it follows that

\[ \| \langle y \rangle D^\alpha_y D^\gamma_y \partial_x w^2 \|_{L^2_y} \lesssim \| D^\alpha_y D^\gamma_y (\langle y \rangle \partial_x w^2) \|_{L^2}, \]

\[ \lesssim \| D^\alpha_y ((\langle y \rangle \partial_x w^2)) \|_{L^2} + \| D^\gamma_y ((\langle y \rangle \partial_x w^2)) \|_{L^2}, \]

\[ \lesssim \| \langle y \rangle D^\alpha_y (\partial_x w^2) \|_{L^2} + \| \langle y \rangle D^\gamma_y (\partial_x w^2) \|_{L^2}, \]

and here we use the previous two steps.

We return to the estimates for the terms in (83) involving \( v \). It is not hard to show that estimates similar to those presented above are available for the term with
wev as long as $v \in Y_T$. Estimating the term involving $v^2$ is even easier, because 

$$QU(t-\tau')\partial_x(v^2) \sim U(t-\tau')(v^2),$$

that is the derivative is “inactive” in this part of the operator $L_1$. We then obtain that 

$$\|L_1(w, v)\|_X \lesssim \|w_0\|_{X, 0} + \|w\|_X^2 + \|v\|_Y^2.$$ \tag{87}

Notice that no factor of $T$ appears in front of $((\|w\|_X^2 + \|v\|_Y^2)$). This will force us to assume that $\|w_0\|_{X, 0}$ is small, in order to claim that the operator $L$ is indeed a contraction. We will return to this matter a bit later. We start the estimate for the operator $L_2(w, v)$. By unitarity of the linear flow in $L^2$ Sobolev spaces, we have 

$$\|\chi_{[0, T]}U(t)(Id - Q)v_0\|_1 \lesssim \|v_0\|_{H^2}.$$ 

Next, from (64) we have 

$$\|\chi_{[0, T]}U(t)(Id - Q)v_0\|_3 \lesssim \|(y)^\alpha D_y^2 v_0\|_{L^2} + \|D_x^{-\alpha}D_y^{\gamma+\alpha}v_0\|_{L^2}.$$ 

From (46) we have 

$$\|\chi_{[0, T]}U(t)(Id - Q)v_0\|_{10} \lesssim \|(y)^\alpha D_y^-\gamma(1 + D_y)^\gamma v_0\|_{L^2} + \|D_x^{-\gamma}(1 + D_y)^\gamma v_0\|_{L^2},$$

where $\gamma = 1/2^+, \gamma = 1/4^+, \theta = 1^+$. From (47), we have 

$$\|\chi_{[0, T]}U(t)(Id - Q)v_0\|_{10} \lesssim \|(y)^\alpha D_y^-\gamma(1 + D_y)^\gamma v_0\|_{L^2} + \|D_x^{-\gamma-1/2}(1 + D_y)^\gamma v_0\|_{L^2} + \|D_x^{-\gamma+1/2}(1 + D_y)^\gamma v_0\|_{L^2}.$$ 

From (78), we also have 

$$\|\chi_{[0, T]}U(t)(Id - Q)v_0\|_{11} \lesssim \|(y)^\alpha (1 + D_y)^{1+\epsilon} v_0\|_{L^2} + \|D_x^{-\alpha+\epsilon}(1 + D_y)^{1+\alpha+2\epsilon} v_0\|_{L^2},$$

and we require that $1 + \epsilon \leq \gamma_2$. Similarly, 

$$\|\chi_{[0, T]}U(t)(Id - Q)v_0\|_{12} \lesssim \|(y)^\alpha (1 + D_y)^{1+\epsilon} v_0\|_{L^2} + \|(1 + D_y)^{\alpha+2\epsilon} v_0\|_{L^2}.$$ 

Now we pass to the estimate of the nonlinear terms. For any $i = 1, 3, 9, 10, 11, 12$ and smooth $f$ and $h$ we have 

$$\left\|\chi_{[0, T]} \int_0^t U(t-\tau')\partial_x(fh)\, d\tau'\right\|_{L_{t,x'}} \lesssim \int_0^T \left(\|U(t-\tau')\partial_x(fh)(t')\|_{L_{t,x'}}\right) \, d\tau'$$

$$\lesssim T^{1/2}\|(D_x^2 + D_y^2)(fh)(t')\|_{L^2_{t,x'}} + T^{1/2}\|(y)^\alpha D_y^{2\gamma+\alpha}(fh)(t')\|_{L^2_{t,x'}}$$

$$\lesssim T^{1/2}\|(f)\|_{H_{11}}.$$ 

This and Definition 6.1 show that 

$$\|L_2(w, v)\|_X \lesssim \|v_0\|_{Y, 0} + T^{1/2}(\|w\|_X^2 + \|v\|_Y^2).$$ \tag{88}

Then, by (87) and (88), it is easy to conclude that $L$ is a contraction (and hence has a unique fixed point, if we assume (86), for some small $\delta$. This does not yet prove Theorems 2.1 and 2.2. But before we present the conclusion of the proofs we would like to make a few remarks.

Remark 6.3. Based on the order of derivatives needed for the maximal function estimates in Theorems 3.1 and 3.6, one may guess that a local well-posedness theory could be obtained in a Sobolev space of order at most $3/2^+\epsilon$. In fact, one can prove that $\|D_x^{\alpha+2\epsilon}\partial_x(u)(\|y\|^\alpha u\|_{L^2_{x,y}}^2)$ can be bounded using the smoothing effect norms of type $\|\cdot\|_i$, for $i = 5, 6, 7, 8$, of order at most $3/2^+$, and maximal function norms.
of type $\| \cdot \|_j$, for $j = 9, 10$. But, unfortunately, the intermediate terms appearing in the Leibnitz rule cannot be handled in a simple way. Take for example the expression $\| D_y^\alpha (D_\gamma \partial_x (u \kappa (y)^\alpha)) \|_{L^2_{y,x}}$, $\gamma = 1/2^+$. To estimate this term one would need interpolation norms between $\| \cdot \|_i$, for $i = 5, 6, 7, 8$, of order at most $3/2 + \varepsilon$ and $\| \cdot \|_{j'}$, for $j = 9, 10$. Unfortunately, these are not available when the two functions involved are $D_y \partial_x P_{\pm} (u)$ and $P_{\pm} u$. One would need to prove directly the estimates for the intermediate norms of the linear solution $U (t) u_0$. We decided not to attempt this here, because we believe that the well-posedness results that one can obtain from the oscillatory integral theory would not be optimal (compare for example with [10] and [13]).

Remark 6.4. In the estimate of $L_2$, the low-frequency part of the operator $L$, a power $T^{1/2}$ appears. This is because we do not need the “smoothing effect” norm, the only norm that prevents a factor $T^\varepsilon$ from appearing. This also says that if one is interested only in a local well-posedness result, the smallness assumption on $\| u_0 \|_{Y,0}$ can be removed.

Now we go back to the conclusion of the proof of Theorems 2.1 and 2.2. We need to relax the smallness assumption (86). We will prove that for global well-posedness we only need the norms

$$\| \langle y \rangle^\alpha u_0 \|_{L^2}, \quad \| \langle y \rangle^\alpha D_x^{-1/2-\varepsilon} v_0 \|_{L^2}, \quad \| D_x^{-3/4-\varepsilon} v_0 \|_{L^2}$$

to be small. To do so, we rescale the solution $u$ by observing that if $u(x, y, t)$ solves (8) on $[0, T]$, then $u_{\rho} (x, y, t) = \rho^2 u (\rho x, \rho^2 y, \rho^3 t)$ also solves (8) with initial data $u_{\rho,0} (x, y) = \rho^2 u_0 (\rho x, \rho^2 y)$, on the interval $[0, \rho^{-3} T]$. We have the following lemma:

**Lemma 6.5.** Assume $\rho \in (0, 1)$ and $v_{\rho} (x, y) = \rho^2 u (\rho x, \rho^2 y)$. Then for any $\gamma \geq 0$ we have

$$\| D_x^\rho u_{\rho} \|_{L^2} \lesssim \rho^{1/2+\gamma} \| D_x^\rho u \|_{L^2}, \quad \| D_y^\rho u_{\rho} \|_{L^2} \lesssim \rho^{1/2+2\gamma} \| D_y^\rho u \|_{L^2}, \quad \| \langle y \rangle^\alpha D_x^\rho u_{\rho} \|_{L^2} \lesssim \rho^{1/2+\gamma-2\alpha} \| \langle y \rangle^\alpha D_x^\rho u \|_{L^2}. \quad (91)$$

If then $\gamma > 1$, we also have

$$\| \langle y \rangle^\alpha D_y^\gamma u_{\rho} \|_{L^2} \lesssim \rho^{1/2+2\gamma-2\alpha} \| \langle y \rangle^\alpha D_y^\gamma u \|_{L^2}. \quad (92)$$

**Proof.** First we observe that

$$D_y^\gamma u_{\rho} (\xi, \lambda) = \rho^{-1} |\lambda|^{\gamma} \hat{u} (\rho^{-1} \xi, \rho^{-2} \lambda).$$

We then have

$$\| D_y^\gamma u_{\rho} \|_{L^2}^2 = \| |\lambda|^{\gamma} \hat{u}_{\rho} \|_{L^2}^2$$

$$= \rho^{-2} \int |\lambda|^{2\gamma} |\hat{u}|^2 \rho^{-1} \xi, \rho^{-2} \lambda) d\xi d\lambda$$

$$\lesssim \rho^{1+2\gamma} \int |\lambda|^{2\gamma} |\hat{u}|^2 (\xi, \lambda) d\xi d\lambda$$

$$\lesssim \rho^{1+2\gamma} \| D_y^\gamma u \|_{L^2}^2,$$
and (90) follows. The same argument can be used to prove (89). We now prove (91) by complex interpolation. If \( z = i\beta \), then by (89)
\[
\| |y|^2 D^\alpha_x u_\rho \|_{L^2} \lesssim \rho^{1/2+2\alpha} \| D^\alpha_x u \|_{L^2}.
\]
If \( z = 1 + i\beta \), then by (89) we have
\[
\| |y|^2 D^\alpha_x u_\rho \|_{L^2} = \| |\xi|^\alpha \partial_\xi \hat{u}_\rho \|_{L^2} = \rho^{-2} \| D^\alpha_y (yu)_\rho \|_{L^2} \lesssim \rho^{1/2+2\alpha-2} \| y D^\alpha_x u \|_{L^2}.
\]
Interpolation gives immediately (91). We are now ready to prove (92). Again we use interpolation. As above, for \( z = i\beta \),
\[
\| |y|^2 D^\alpha_y u_\rho \|_{L^2} \lesssim \rho^{1/2+2\alpha} \| D^\alpha_y u \|_{L^2}.
\]
If \( z = 1 + i\beta \), then
\[
\| |y|^2 D^\alpha_y u_\rho \|_{L^2} = \| \partial_\lambda (|\lambda| \beta \hat{u}_\rho) \|_{L^2} = \| |\lambda|^{\gamma^{-1}} \beta \hat{u}_\rho \|_{L^2} + \| |\lambda| \gamma \partial_\lambda (\hat{u}_\rho) \|_{L^2} = \| D^\gamma^{-1} u_\rho \|_{L^2} + \rho^{-2} \| D^\gamma_y (yu)_\rho \|_{L^2} \lesssim \rho^{1/2+2\gamma-2} (\| D^\gamma_y^{-1} u \|_{L^2} + \| D^\gamma_y (yu) \|_{L^2}).
\]
To finish the chain of inequalities, we use (94) and we obtain
\[
\| |y|^2 D^\gamma_y u_\rho \|_{L^2} \lesssim (\rho^{1/2+2\gamma-2}) \| \langle y \rangle D^\gamma_y u \|_{L^2}.
\]
Interpolation gives immediately (92).

It is now easy to see from (89)–(92) that if we repeat the fixed point argument above for \( Qu_\rho = u_{1,\rho} \) and \((Id - Q)u_\rho = u_{2,\rho}\), and
\[
\max(\| \langle y \rangle^\alpha u_{0,\rho} \|_{L^2}, \| \langle y \rangle^\alpha D^\gamma_x^{-1/2-\epsilon} v_{0,\rho} \|_{L^2}, \| D^\gamma_x^{-\alpha/4-\epsilon} v_{0,\rho} \|_{L^2}) \ll \delta,
\]
then \( \max(\| u_{0,\rho} \|_{L^2}, \| v_{0,\rho} \|_{L^2}) \leq \delta \) and the fixed point argument can be applied for an appropriate small \( \delta \). This concludes the proof of Theorem 2.1. To conclude also the proof of Theorem 2.2, one needs to use Remark 6.4 and (89)–(92), as we did above.

Remark 6.6. One can also consider the modified KP-I initial value problem
\[
\begin{cases}
\partial_x (\partial_t u + \partial_x^3 u + \beta u^2 \partial_x u) + \partial_y^2 u = 0, \\
u(x, 0) = u_0(x), \quad (x, y) \in \mathbb{R}^2, \quad t \in \mathbb{R}.
\end{cases}
\]
Using the arguments presented above, one can prove a local well-posedness theorem in \( H^2 \) without introducing spaces involving the weight \( \langle y \rangle^\alpha \) and without assuming smallness of the initial data. In fact, the weight \( \langle y \rangle^\alpha \) was introduced above to transform a norm \( L^1_t L^1_x \) needed in the inhomogeneous smoothing effect (60), into an \( L^2 \) norm needed for the maximal function estimates (14) and (15). In modified KP-I this is not needed, because the square power in the nonlinearity \( u^2 \partial_x u \) takes us for free from the \( L^1 \) norm to the \( L^2 \). Then, based on (89) and (90), we can rescale the solution and remove the smallness assumption. The precise well-posedness
statement for (93) can be summarized in the theorem below. Define the space $W_0$ through the norm
\[ \|f\|_{W_0} = \|f\|_{H^2} + \|D_\gamma^{-}(1 + D_\gamma)^{\gamma} f\|_{L^2} + \|D_\gamma^{-\sigma}(1 + D_\gamma)^{\theta} f\|_{L^2}, \]
where $\gamma > 1/2$, $\sigma > 1/4$, $\theta > 1$.

**Theorem 6.7.** For any $u_0 \in W_0$ there exist $T = T(\|u_0\|_{W_0})$ and a unique solution $u$ for (93) such that $u(x, y, t) \in C([0, T], W_0)$. Moreover, the map $u_0 \mapsto u$ is continuous with respect to the initial data in the appropriate topology.

An adaptation of the arguments from [12] to the KP-I setting should, in principle, provide similar local well-posedness results for pure power generalizations (with nonlinearity $u^p u_x$, $p \in \mathbb{N}$) of KP-I. These extensions would lower the regularity required for existence in the energy method argument of Tom [21] and also provide uniqueness.

### 7. Appendix: the fractional Leibnitz rule

In this section we recall some known facts on fractional Leibnitz rule for one variable functions and some related results involving the weight $\langle y \rangle^\alpha$.

**Theorem 7.1.** Assume $0 < \sigma < 1$ and $1 < p < \infty$. Then
\[ \|D_\gamma (f g) - f D_\gamma (g) - g D_\gamma (f)\|_{L^p} \lesssim \|g\|_{L^\infty} \|D_\gamma (f)\|_{L^p}. \]

For the proof one can see [12]. We also need a lemma that relates fractional derivatives with the weight $\langle y \rangle^\alpha$.

**Lemma 7.2.** Assume $0 \leq \alpha \leq 1$, $0 < \gamma < 1$ and $1 < p < \infty$. Then
\[ \|\langle y \rangle^\alpha((1 + D_\gamma)\gamma f)\|_{L^p} = \|(1 + D_\gamma)\gamma (\langle y \rangle^\alpha f)\|_{L^p} + O(\|\langle y \rangle^\alpha f\|_{L^p}). \]

**Proof.** It suffices to prove the following estimate on the commutator:
\[ \|[\langle y \rangle^\alpha, (1 + D_\gamma)\gamma f]\|_{L^p} \lesssim \|\langle y \rangle^\alpha f\|_{L^p}. \]

We start by first proving that for any $\beta \in \mathbb{R}$
\[ \|D_\gamma (\langle y \rangle^\beta)\|_{L^\infty} \leq C. \]

By Proposition 1, page 241 in [18] we have that
\[ |K_\beta(\lambda)| = |F(\langle y \rangle^\beta)(\lambda)| \lesssim |\lambda|^{-1-N}, \]
for all $N \geq 0$. Then
\[ |D_\gamma (\langle y \rangle^\beta)| = \left| \int e^{-iy\lambda} |\lambda|^{\gamma} K_\beta(\lambda) \, d\lambda \right| \lesssim \int_{|\lambda| < 1} |\lambda|^{\gamma-1} \, d\lambda + \int_{|\lambda| \geq 1} |\lambda|^{\gamma-3} \, d\lambda \lesssim 1, \]
where in the last step we used (97) with $N = 0$ and $N = 2$. We are now ready to prove (95) by complex interpolation. For $z \in \mathbb{C}$, $0 \leq \text{Re} \, z \leq 1$, we define the
operator $T^z = [(y)^{2z}, (1 + D)^\gamma]$. Assume now $z = i\beta$. Then by Theorem 7.1
\[
\|T^z f\|_{L^p} \lesssim \|[(1 + D)^\gamma - D^{\gamma}f]\|_{L^p} + \|D^\gamma f\|_{L^p} 
\]
\[
\lesssim \|[(1 + D)^{2\gamma} - D^{2\gamma}f]\|_{L^p} + \|D^{2\gamma} f\|_{L^p} 
\]
where in the last step we used (7.1) and the fact that the multiplier $m(\lambda) = (|\lambda| + 1)^\gamma - |\lambda|^\gamma$ is a good Marcinkiewicz multiplier (see [18], page 245). Assume now that $z = 1 + i\beta$. It suffices to estimate
\[
\hat{T}^z f = (y)^{2\beta} y^2 (1 + D)^\gamma f - (1 + D)^\gamma (y^2 f)
\]

Observe that
\[
y^2 (1 + D^{\gamma}) f = M_2 f + M_1 y f + (1 + D)^\gamma (y^2 f)
\]
where $M_j$ are multiplier operators such that
\[
F(M_j f)(\lambda) = (1 + |\lambda|)^{\gamma-j} \hat{f}(\lambda).
\]
Using this representation, Theorem 7.1, (96) and again theory of multiplier operators, we have
\[
\|\hat{T}^{1+i\beta} f\|_{L^p} \sim \|f\|_{L^p} + \|y f\|_{L^p} + \|y^2 f\|_{L^p} \lesssim \|(1 + y^2) f\|_{L^p}.
\]
Then complex interpolation gives (95).

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