DG-CATEGORIES AND SIMPLICIAL BAR COMPLEXES

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Dedicated to Pierre Deligne

Abstract. We prove that the DG category $K_{\mathcal{C}_A}$ of DG complexes in $\mathcal{C}_A$ associated to a DGA $A$, is homotopy equivalent to that of comodules over the bar complex of $A$. We introduce simplicial bar complexes to give the homotopy equivalence. As an application, we show that the category of comodules over the 0-th cohomology of the bar complex of the Deligne algebra is equivalent to that of variations of mixed Tate Hodge structures on an algebraic variety.


Key words and phrases. Bar complex, DG-category, Deligne cohomology.

1. Introduction and Conventions

1.1. Introduction. In the paper [C], Chen defined a bar complex of an associative differential graded algebra which computes the real homotopy type of a $C^\infty$-manifold. There he proved that the Hopf algebra of the dual of the nilpotent completion of the group ring $\mathbb{R}[\pi_1(X, p)]^\wedge$ of the fundamental group $\pi_1(X, p)$ is canonically isomorphic to the 0-th cohomology of the bar complex of the differential graded algebra $A^\bullet(X)$ of smooth differential forms on $X$.

There are several methods to construct mixed Tate motives over a field $K$. First construction is due to Bloch and Kriz [BK1], based on the bar complex of the differential graded algebra of the Bloch cycle complex. They defined the category of mixed Tate motives as the category of comodules over the cohomology of the bar complex. Another construction is due to Hanamura [H]. In the book of Kriz–May [KM], they used a homotopical approach to define the category of mixed Tate motives. Hanamura used some generalization of a complex to construct the derived category of mixed Tate motives. One can formulate this construction of a complex in the setting of DG category which is called DG complex in this paper. DG complexes are called twisted complexes in a paper of Bondal–Kapranov [BK2]. In the paper [Ke2], similar notion of perfect complexes are introduced. The notion of DG category is also useful to study cyclic homology. (See [Ke2], [Ke1].)

In this paper, we study two categories, the category of comodules over the bar complex of a differential graded algebra $A$ and the category of DG complexes of

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a DG category arising from the differential graded algebra. Roughly speaking, we show that these two categories are homotopy equivalent (Theorem 6.4). We use this equivalence to construct a certain coalgebra which classifies nilpotent variations of mixed Tate Hodge structures on an algebraic varieties $X$ (Theorem 10.7). This coalgebra is isomorphic to the coordinate ring of the Tannakian category of mixed Tate Hodge structures when $X = \text{Spec}(\mathbb{C})$.

The bar construction is also used to construct the motives associated to rational fundamental groups of algebraic varieties in [DG]. They used another type of bar construction due to Beilinson. In this paper, we adopt this bar construction, called simplicial bar construction, for differential graded algebras. Simplicial bar complexes depend on the choices of two augmentations of the differential graded algebras. If these two augmentations happen to be equal, then the simplicial bar complex is quasi-isomorphic to the classical reduced bar complex defined by Chen.

Let me explain this in the case of the DGA of smooth differential forms $A^*$ of a smooth manifold $X$. To a point $p$ of $X$, we can associate an augmentation $\epsilon_p : A^* \rightarrow \mathbb{R}$. In Chen’s reduced bar complexes, the choice of the augmentation, reflects the choice of the base point $p$ of the rational fundamental group. On the other hand, simplicial bar complex depends on two augmentations $\epsilon_1$ and $\epsilon_2$. The cohomology of the simplicial bar complex of $A^*$ with respect to the two augmentations arising from two points $p_1$ and $p_2$ of $X$ is identified with the nilpotent dual of the linear hull of paths connecting the points $p_1$ and $p_2$.

By applying simplicial bar construction with two augmentations of cycle DGA arising from two realizations, we obtain the dual of the space generated by fundamental groups of algebraic varieties in $\mathbb{C}$. Using this formalism, we treat the trivial isomorphism between two realization functors. A comparison theorem gives a formalism for dealing with the dual of the space generated by functors of the differential graded algebra of the Deligne complex of an algebraic variety $X$.

1.2. Conventions. Let $\mathbb{k}$ be a field. Let $\mathcal{C}$ be a $\mathbb{k}$-linear abelian category with a tensor structure. The category of complexes in $\mathcal{C}$ is denoted as $\mathcal{K}$. For objects $A = (A^i, \delta_A)$ and $B = (B^i, \delta_B)$ in $\mathcal{K}$, we define tensor product $A \otimes B$ as an object in $\mathcal{K}$ by the rule $(A \otimes B)^i = \bigoplus_{i+j=p} A^i \otimes B^j$. The differential $d_{A \otimes B}$ on $A^i \otimes B^j$ is defined by $d_{A \otimes B} = \delta_A \otimes 1_B + (-1)^i 1_A \otimes \delta_B$. An element

$$\text{Hom}_{\mathcal{K}}(A, B) = \prod_i \text{Hom}_{\mathcal{C}}(A^i, B^{i+p})$$

is called a homogeneous homomorphism of degree $p$ from $A$ to $B$ in $\mathcal{K}$. Let $A, A', B, B' \in \mathcal{K}$ and $\varphi = (\varphi_i) \in \text{Hom}_{\mathcal{K}}(A, B)$ and $\psi = (\psi_j) \in \text{Hom}_{\mathcal{K}}(A', B')$, we define $\varphi \otimes \psi \in \text{Hom}_{\mathcal{K}}(A \otimes A', B \otimes B')$ by setting $(\varphi \otimes \psi)_{i+j} = (-1)^q \varphi_i \otimes \psi_j$ on $A^i \otimes A'^j \rightarrow B^{i+p} \otimes B'^{j+q}$. (To remember this formula, the rule $(\varphi \otimes \psi)(a \otimes a') = (-1)^{\deg(a) \deg(\psi)} \varphi(a) \otimes \psi(a')$ is useful.) An object $M$ in $\mathcal{C}$ is regarded as an object in $\mathcal{K}$ by setting $M$ at degree zero part. For a complex $A \in \mathcal{K}$, we define the complex $A[i]$ by $A[i]^j = A^{i+j}$, where the differential is defined through this isomorphism.
The shift \( k[i] \) of the unit object \( k \) is defined in this manner. The homogeneous morphism \( k[i] \to k[j] \) of degree \( i-j \), whose degree \(-i\) part \( k[i]^{-i} = k \to k[j]^{-j} = k \) is defined by the identity map, is denoted as \( t_{ji} \). The degree \(-i\) element \( "1" \) of \( k[i] \) is denoted as \( e^i \). For an object \( B \in KC \), the tensor complex \( B \otimes k[i] \) is denoted as \( Be^i \). For objects \( A, B \in KC \) and \( \varphi \in \text{Hom}^0_{KC}(A, B) \), a homomorphism \( \varphi \otimes t_{ji} \in \text{Hom}_{KC}(Ae^i, Be^j) \) is a degree \((p+i-j)\) homomorphism. As a special case, for \( \varphi \in \text{Hom}^0_{KC}(A, B) \), the map \( \varphi \otimes t_{-1,i} \in \text{Hom}^1_{KC}(Ae^i, Be^{i-1}) \) is a degree one element. It is denoted as \( \varphi \otimes t \) for simplicity. The differential of \( M \) can be regarded as a degree one map from \( M \) to itself. Therefore \( d \) can be regarded as an element in \( \text{Hom}^1_{KC}(M, M) \).

An object in \( KK \) can be considered as a double complex in \( C \). Let \( (\cdots \to A^{•,i} \xrightarrow{d} A^{•,i+1} \to \cdots) \) be an object in \( KK \). Since \( d \) is a homomorphism of complex, we have

\[
d \otimes t \in \text{Hom}^1_{KC}(A^{•,j}e^{-j}, A^{•,j+1}e^{-j-1})
\]

is a degree one element in \( KC \). Let \( \delta \otimes 1 \) be the differential of \( A^{•}e^{-i} \in KC \). The summation of \( \delta \otimes 1 \) for \( i \) is also denoted as \( \delta \otimes 1 \). Then the degree one map \( \delta \otimes 1 + d \otimes t \) becomes a differential on the total graded object. Here \( \delta \otimes 1 \) is called the inner differential and \( d \otimes t \) is called the outer differential. Note that for an “element” \( a \in A^{ij} \), we have \((d \otimes t)(a \otimes e^{-j}) = (-1)^i d(a) \otimes e^{-j-1} \), which coincides with the standard sign convention of the associated simple complex of a double complex.

The resulting complex is called the associated simple complex of \( A^{••} \in KK \) and denoted by \( s(A) = s(A^{••}) \). For objects \( A = A^{••}, B^{••} \in KK \), the tensor product \( A \otimes B \) is defined as an object in \( KK \). Then we have a natural isomorphism in \( KK \)

\[
\nu: s(A) \otimes s(B) \simeq s(A \otimes B) \tag{1.1}
\]

defined by \( \nu(ae^{-j} \otimes be^{-j'}) = (-1)^{ij'} (a \otimes b)e^{-j-j'} \) for \( ae^{-j} \in A^{ij}e^{-j}, be^{-j'} \in B^{ij'}e^{-j'} \). This isomorphism is compatible with the natural associativity isomorphism.

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2. DG Categories

**2.1. Definition of DG categories and examples.** Let \( k \) be a field. A DG category \( C \) over \( k \) consists of the following data.

1. A class of objects \( \text{ob}(C) \).
2. A complex \( \text{Hom}^•(A, B) = (\text{Hom}^•(A, B), \partial) \) of \( k \) vector spaces for every objects \( A \) and \( B \) in \( \text{ob}(C) \).

We sometimes impose the following shift structure on \( C \).

3. Bijective correspondence \( T : C \to C \) for objects in \( C \). An object \( T^k(A) \) in \( C \) is denoted as \( Ae^k \) for \( k \in \mathbb{Z} \).

with the following axioms.
(1) For three objects $A$, $B$ and $C$ in $\mathcal{C}$, the composite
\[
\text{Hom}^\bullet(B, C) \otimes \text{Hom}^\bullet(A, B) \to \text{Hom}^\bullet(A, C)
\]
is defined as a homomorphism of complexes over $\mathbb{k}$.

(2) The above composite homomorphism is associative. That is, the following diagram of complexes commutes.
\[
\begin{array}{c}
\text{Hom}^\bullet(C, D) \otimes \text{Hom}^\bullet(B, C) \otimes \text{Hom}^\bullet(A, B) \\
\downarrow \\
\text{Hom}^\bullet(C, D) \otimes \text{Hom}^\bullet(A, B) \\
\downarrow \\
\text{Hom}^\bullet(A, D)
\end{array}
\]

(3) There is a degree zero closed element $\text{id}_{A}$ in $\text{Hom}^0(A, A)$ for each $A$ which is a left and right identity under the above composite homomorphism.

If we assume the shift structure $\mathcal{T}$, the following sign convention should be satisfied.

(4) There is a natural isomorphism of complexes
\[
\text{Hom}^\bullet(A, B)[-i + j] \simeq \text{Hom}^\bullet(Ae^i, Be^j): \varphi \mapsto \varphi \otimes t_{ji}
\]
satisfying the rule $(\varphi \otimes t_{ji}) \circ (\psi \otimes t_{kj}) = (-1)^{(i-j)\deg(\psi)}(\varphi \circ \psi) \otimes t_{jk}$. (It is compatible with the formal commutation rule for $\psi$ and $t_{ji}$.)

**Definition 2.1.** (1) Let $\mathcal{C}$ be a DG category and $a, b$ objects in $\mathcal{C}$. A closed morphism $\varphi: a \to b$ of degree 0 (i.e., $\partial \varphi = 0$) is called an isomorphism if there is a closed morphism $\psi$ of degree zero such that $\psi \circ \varphi = 1_a$, $\varphi \circ \psi = 1_b$.

(2) Let $\mathcal{C}_1, \mathcal{C}_2$ be DG categories. A DG functor $F$ is a collection $\{F(a)\}$ of objects in $\mathcal{C}_2$ indexed by objects in $\mathcal{C}_1$ and a collection $\{F_{a,b}\}$ of homomorphisms of complexes $F_{a,b}: \text{Hom}^\bullet(a, b) \to \text{Hom}^\bullet(F(a), F(b))$ indexed by $a, b \in \mathcal{C}_1$ which preserves the composites, the identities and the degree shift operator $A \mapsto A[1]$.

(3) A DG functor $F: \mathcal{C}_1 \to \mathcal{C}_2$ is essentially surjective, if for any $x \in \mathcal{C}_2$, there exists an object $y \in \mathcal{C}_1$ such that $x$ is isomorphic to $f(y)$ in $\mathcal{C}_2$.

(4) A DG functor $F: \mathcal{C}_1 \to \mathcal{C}_2$ is said to be homotopy equivalent if and only if it is essentially surjective, and the induced map
\[
H^i(F_{a,b}): H^i(\text{Hom}^\bullet(a, b)) \to H^i(\text{Hom}^\bullet(F(a), F(b))
\]
is an isomorphism for all $i \in \mathbb{Z}$ and $a, b \in \mathcal{C}_1$.

**Remark 2.2.** The definition of an isomorphism between two objects is stronger than that in the homotopy category $[\mathcal{C}]$ of $\mathcal{C}$ defined in [To, p. 17]. Therefore the definition of essential surjectivity is stronger than that of quasi-essential surjectivity defined in [To, p. 19]. As a consequence, the condition for homotopy equivalence is stronger than that for quasi-equivalence defined in [To, p. 18, Definition 2.3].

**Example 2.3.** Let $\text{Vec}_k$ be a category of $k$-vector spaces. The category of complexes of $k$-vector spaces is denoted as $K\text{Vec}_k$. Then $K\text{Vec}_k$ becomes a DG category by setting
\[
\text{Hom}^\bullet_{K\text{Vec}_k}(A, B) = \prod_i \text{Hom}(A^i, B^{i+p})
\]
for complexes $A = A^\bullet$ and $B = B^\bullet$. The differential $\partial \varphi$ of an element $\varphi = (\varphi_i)_i \in \text{Hom}_k^{p}(A, B)$ is defined by the formula

$$\partial(\varphi)_i = d_B \circ \varphi_i - (-1)^p \varphi_{i+1} \circ d_A. \quad (2.1)$$

Therefore $\varphi \in \text{Hom}_k^{p}(A, B)$ is a homomorphism of complexes if and only if $\partial(\varphi) = 0$. Two homomorphisms of complexes $\varphi$ and $\psi$ are homotopic to each other by the homotopy $\theta$ if and only if $\varphi - \psi = \partial(\theta)$ with $\theta \in \text{Hom}_k^{p+1}(A, B)$.

**Definition 2.4** (DG category associated with a DGA). Let $A = A^\bullet$ be a unitary associative differential graded algebra (denoted as DGA for short) over a field $k$. Then the structure of complex on $A$ induces a natural DG functor $\text{End}_k^A$ as follows.

(1) An object of $\text{End}_k^A$ is a copy of $A^\bullet$, and the multiplication rule is given by the composite of the following homomorphisms of complexes:

$$\text{Hom}_k^A(V, W) \xrightarrow{\otimes} \text{Hom}_k^A(V^\bullet, A^\bullet \otimes W^\bullet).$$

Here the structure of complex on $A^\bullet \otimes W^\bullet$ is defined in Section 1.2 and that on $\text{Hom}_k^A(V^\bullet, A^\bullet \otimes W^\bullet)$ is given by the rule (2.1).

(2) For two objects $V = V^\bullet$ and $W = W^\bullet$, the set of morphisms $\text{Hom}_k^A(V, W)$ is defined by the complex

$$\text{Hom}_k^A(V, W) = \text{Hom}_k^{A^\bullet}(V^\bullet, A^\bullet \otimes W^\bullet).$$

Remark 2.5. (1) Let $A \to A'$ be DGA’s. A homomorphism $\varphi: A \to A'$ of complexes is called a DGA homomorphism if preserves multiplication and unit. A DGA homomorphism $\varphi: A \to A'$ induces a natural DG functor $\varphi: \text{End}_k^A \to \text{End}_k^{A'}$.

(2) Let $C$ be a DG category and $M$ be an object of $C$. Then $\text{End}_k^C(M, M)$ becomes an (associative) DGA. We have $\text{End}_k^C(M, M) = (A^\bullet)^{\otimes}$ as DGA’s. Note that $(A^\bullet)^{\otimes}$ is a copy of $A^\bullet$ as a complex and the multiplication rule is given by $a \circ b = (-1)^{\deg(a) \deg(b)} (b \cdot a)^\circ$.

**2.2. DG complexes.** We introduce the notion of complexes in the setting of a DG category, which is called DG complex.

**Definition 2.6.** (1) Let $C$ be a DC category. A pair $(\{M_i\}_{i \in \mathbb{Z}}, \{d_{ij}\}_{i \geq j})$ consisting of (a) a series of objects $\{M_i\}_{i \in \mathbb{Z}}$ in $C$ indexed by $\mathbb{Z}$, and (b) a series of morphisms $d_{ij} \in \text{Hom}_C^{j-i+1}(M_i, M_j)$
indexed by $i > j$ in $\mathbb{Z}$ is called a DG complex in $\mathcal{C}$ if it satisfies the following equality.

$$\partial(d_{ij}) + \sum_{i > p > j} (-1)^{(i-p)(p-j+1)} d_{ip} \circ d_{pj} = 0.$$  \hfill (2.2)

The uncomfortable sign in this condition will be simplified by using $d^\#_{ij} = d_{ij} \otimes t_{-i,-j} \in \text{Hom}^1_\mathcal{C}(M^q e^{-q}, M^p e^{-p})$. Then the condition will be

$$\partial(d^\#_{ij}) + \sum_{i > p > j} d^\#_{ip} \circ d^\#_{pj} = 0.$$  \hfill (2.3)

(2) Let $M = (M^\bullet, d_M)$ and $N = (N^\bullet, d_N)$ be DG complexes in $\mathcal{C}$. We set $\text{Hom}^i_\mathcal{K}(M, N) = \text{lim}_{\alpha} \bigoplus_{q - \alpha \leq r} \text{Hom}_\mathcal{C}^{i+q-r}(M^q, N^r)$.

(3) Let $i \in \mathbb{Z}$. For an element $\varphi \in \text{Hom}^i_\mathcal{K}(M, N)$, we define a map $D(\varphi) \in \text{Hom}^{i+1}_\mathcal{K}(M, N)$ as follows. For an element

$$\varphi = (\varphi_{r,q}) \in \bigoplus_{q - \alpha \leq r} \text{Hom}_\mathcal{C}^{i+q-r}(M^q, N^r),$$

we set $\varphi^\#_{r,q} = \varphi_{r,q} \otimes t_{-r,-q} \in \text{Hom}_\mathcal{C}^{i-q-\alpha}(M^q e^{-q}, N^r e^{-r})$ and define

$$D(\varphi)^\#_{r,q} = \partial(\varphi^\#_{r,q}) + \sum_{-\alpha \leq r' < r} d^\#_{r,r'} \circ \varphi^\#_{r',q} - (-1)^i \sum_{q < q' \leq r + \alpha} \varphi^\#_{r,q'} \circ d^\#_{q',q} \in \text{Hom}^{i+1}_\mathcal{K}(M^q e^{-q}, N^r e^{-r}).$$

This map defines a homomorphism $D: \text{Hom}^i_\mathcal{K}(M, N) \to \text{Hom}^{i+1}_\mathcal{K}(M, N)$, and the space $\text{Hom}^i_\mathcal{K}(M, N)$ becomes a complex of $k$-vector spaces.

(4) Let $(L, d_L)$, $(M, d_M)$ and $(N, d_N)$ be DG complexes. We define the composite

$$\text{Hom}^i_\mathcal{K}(M, N) \otimes \text{Hom}^j_\mathcal{K}(L, M) \to \text{Hom}^{i+j}_\mathcal{K}(L, N),$$

by the relation

$$(\psi \otimes \varphi)^\#_{p,q} = \sum_r \psi^\#_{r,p} \circ \varphi^\#_{r,q} \in \text{Hom}^{i+j}_\mathcal{K}(L^q e^{-q}, N^p e^{-p}).$$

We can check that the the above composite is a well defined homomorphism of complexes and associative.

(5) Let $M = (\{M^i\}, \{d_{ij}\})$ be a DG complex. If $M^i = 0$ except for a finite number of $i$’s, it is called a bounded DG complex.

It is easy to check the following proposition.

**Proposition 2.7.** Via the above differentials and the composites, the category of DG complexes in the DG category $\mathcal{C}$ becomes a DG category.

**Definition 2.8 (DG category of DG complexes).** The DG category of DG complexes defined as above is denoted as $\mathcal{K}$. The full sub DG category of bounded DG complexes in $\mathcal{K}$ is denoted as $\mathcal{K}^b$. 

3. **Topological Motivation for Simplicial Bar Complex**

### 3.1. The space of marked paths

Before introducing simplicial bar complexes, we give a topological motivation for them. Let \( X \) be a connected \( C^\infty \) manifold with finite dimensional homology. For a sequence of integers \( \alpha = (\alpha_0 < \cdots < \alpha_n) \in \mathbb{Z}^{n+1} \), we define an \( \alpha \)-marking of \( \mathbb{R} \) by

\[
\gamma_{\alpha_0 \leq t_{\alpha_0, \alpha_1} \leq t_{\alpha_1, \alpha_2} \leq \cdots \leq t_{\alpha_{n-1}, \alpha_n} \leq t_{\alpha_n, +} \in \mathbb{R}.
\]

For two indices \( \alpha = (\alpha_0 < \cdots < \alpha_n) \) and \( \beta = (\beta_0 < \cdots < \beta_m) \), we write \( \alpha < \beta \) if \( \alpha \subset \beta \). Let \( \alpha, \beta \) be two indices such that \( \alpha < \beta \). For an \( \alpha \)-marking \( t \), we define a \( \beta \)-marking \( u = i_{\alpha, \beta}(t) \) by setting \( u_\beta \beta_j = t_{\alpha_i, \alpha_{i+1}} \), where \( \alpha_i \leq \beta_j \) and \( \beta_j + 1 \leq \alpha_{i+1} \).

For example, if \( \alpha = (0, 2) \), \( \beta = (0, 1, 2, 3) \), then we have

\[
\alpha_0 = 0 < \alpha_1 = 3 \quad \beta_0 = 0 < \beta_1 = 1 < \beta_2 = 2 < \beta_3 = 3
\]

and we have \( u_{-0} = t_{-0} \), \( u_{0, 1} = u_{1, 2} = t_{0, 2} \), \( u_{2, 3} = u_{3, +} = t_{2, +} \). The intervals \( (\alpha, \beta) \) are denoted as \( I_{\alpha, \beta} \) and \( t_{\alpha, \beta} \) are called the beginning and the ending point, respectively.

![Diagram showing intervals and marked paths](image)

**Figure 1.**

We define the set of \( \alpha \)-marked paths \( P_\alpha(X) \) by

\[ P_\alpha(X) = \{(\gamma, t) : t \text{ is an } \alpha \text{-marking of } \mathbb{R}, \gamma : \mathbb{R} \to X \text{ is a path such that } \gamma(I_{\alpha, -}(t)), \gamma(I_{\alpha, +}(t)) \text{ are constants} \}. \]

For a marked path \( (\gamma, t) \in P_{\alpha_0, \ldots, \alpha_n} \), \( \gamma(I_{\alpha, -}(t)) \) and \( \gamma(I_{\alpha, +}(t)) \) are called the beginning and the ending point, respectively. By evaluating the map \( t_{\alpha_0, \ldots, \alpha_{n+1}} \), we have an evaluation map \( e_{\alpha_0, \ldots, \alpha_n} \)

\[ P_{\alpha_0, \ldots, \alpha_n}(X) \to X^{n+2} : (\gamma, t) \mapsto (\gamma(t_{\alpha_0, \alpha_1}), \gamma(t_{\alpha_0, \alpha_2}), \ldots, \gamma(t_{\alpha_0, \alpha_{n+1}}), \gamma(t_{\alpha_n, +})) \]

The product \( X^{n+2} \) of \( (n + 2) \)-copy of \( X \) appeared in \( e_{\alpha_0, \ldots, \alpha_n} \) is denoted as

\[ X_{\alpha_0, \ldots, \alpha_n} = X^{\alpha_0} \times X^{\alpha_1} \times \cdots \times X^{\alpha_n}.
\]

The coordinates of \( X_{\alpha_0, \ldots, \alpha_n} \) are denoted as

\[ (\tau_{\alpha_0, \alpha_1}, \tau_{\alpha_0, \alpha_2}, \ldots, \tau_{\alpha_0, \alpha_{n+1}}, \tau_{\alpha_n, +}). \]

We define \( p_\alpha : X_\alpha \to X \times X \) by \( t \mapsto (\tau_{\alpha, \alpha_0}, \tau_{\alpha, +}) \). We introduce a boundary map \( P_\alpha(X) \to P_\beta(X) \) for indices \( \alpha < \beta \) using the map \( i_{\alpha, \beta} \) defined as above. The map \( f : X_\alpha \to X_\beta \) is defined by \( (f(\tau))_{\beta_j \beta_{j+1}} = \tau_{\alpha, \alpha_{j+1}} \) by choosing \( \alpha_i \leq \beta_j \) and...
\[ \beta_{i+1} \leq \alpha_{i+1}. \] Note that \( f \) is a composite of diagonal maps and it is compatible with the maps \( p_\alpha, p_\beta \). We have the following commutative diagram for evaluation maps:

\[
e_{\alpha} : P_\alpha(X) \longrightarrow X_\alpha \\
e_{\beta} : P_\beta(X) \longrightarrow X_\beta
\]

3.2. Chain complexes associated to the system of marked paths. The system \( \{P_\alpha\}_\alpha \) and \( \{X_\alpha\}_\alpha \) induces the following homomorphisms of double complexes of singular chains by taking suitable summation with suitable sign:

\[
C_\bullet(P_\bullet) : \prod_{\alpha_0} C_\bullet(P_{\alpha_0}(X)) \longrightarrow \prod_{\alpha_0<\alpha_1} C_\bullet(P_{\alpha_0,\alpha_1}(X)) \longrightarrow \prod_{\alpha_0<\alpha_1<\alpha_2} C_\bullet(P_{\alpha_0,\alpha_1,\alpha_2}(X)) \longrightarrow \ldots
\]

\[
C_\bullet(X_\bullet) : \prod_{\alpha_0} C_\bullet(X_{\alpha_0}) \longrightarrow \prod_{\alpha_0<\alpha_1} C_\bullet(X_{\alpha_0,\alpha_1}) \longrightarrow \prod_{\alpha_0<\alpha_1<\alpha_2} C_\bullet(X_{\alpha_0,\alpha_1,\alpha_2}) \longrightarrow \ldots
\]

Moreover, \( \mathbb{R} \)-valued \( C^\infty \) forms on \( X_\bullet \) give rise to the following double complex:

\[
A^\bullet(X_\bullet) : \bigoplus_{\alpha_0} A^\bullet(X_{\alpha_0}) \leftarrow \bigoplus_{\alpha_0<\alpha_1} A^\bullet(X_{\alpha_0,\alpha_1}) \leftarrow \bigoplus_{\alpha_0<\alpha_1<\alpha_2} A^\bullet(X_{\alpha_0,\alpha_1,\alpha_2}) \leftarrow \ldots
\]

The following proposition is a consequence of Proposition 4.1.

**Proposition 3.1.** (1) The associated simple complexes \( C_\bullet(P_\bullet) \) and \( C_\bullet(X_\bullet) \) of \( C_\bullet(P_\bullet) \) and \( C_\bullet(X_\bullet) \) are acyclic.

(2) The associated simple complex \( A^\bullet(X_\bullet) \) of \( A^\bullet(X_\bullet) \) is acyclic.

**Remark 3.2.** Let \( p, q \) be points of \( X_\bullet \) and \( \epsilon_\alpha, \epsilon_\beta : A^\bullet(X) \rightarrow \mathbb{R} \) be augmentations obtained by evaluation maps at \( p \) and \( q \), respectively. Then the cohomology of

\[
\mathbb{R} \otimes_{A^\bullet, \epsilon_\alpha} A^\bullet(X_\bullet) \otimes_{A^\bullet, \epsilon_\beta} \mathbb{R}
\]

is canonically isomorphic to the dual of the nilpotent completion of \( \mathbb{R}[\pi_1(X, p, q)] \), where \( \pi_1(X, p, q) \) is the set of homotopy classes of paths connecting \( p \) and \( q \).

4. Simplicial Bar Complexes

**4.1. Definition of the simplicial bar complex.** Under the notation of the last section, \( A^\bullet(X_{\alpha_0,\ldots,\alpha_n}) \) is quasi-isomorphic to \( A^\bullet(\alpha^{n+2})(X) = A^\bullet(X) \otimes_{\mathbb{R}} \cdots \otimes_{\mathbb{R}} A^\bullet(X) \) and the restriction map \( A^\bullet(X_\beta) \rightarrow A^\bullet(X_\alpha) \) is compatible with a composite of multiplication maps of \( A \). In this section, we define the simplicial bar complex of a differential graded algebra \( A \) by imitating the definition of \( A^\bullet(X_\bullet) \).

Let \( k \) be a field and \( A = A^\bullet \) be a DGA (associative but might not be graded commutative) over \( k \). We define the free simplicial bar complex \( B(A) \) of \( A \) as follows. For a sequence of integers \( \alpha = (\alpha_0 < \cdots < \alpha_n) \), we define \( B_\alpha(A) \) as a copy of \( A^{\otimes (n+2)} \) which is written as

\[
B_\alpha(A) = A^{\otimes_{\alpha_0}} A^{\otimes_{\alpha_1}} \cdots A^{\otimes_{\alpha_{n-1}}} A^{\otimes_{\alpha_n}} A.
\]
whose element will be written as \( x_0^\alpha_0 \otimes x_1^\alpha_1 \otimes \cdots \otimes x_n^\alpha_n \otimes x_{n+1}^\alpha \). The length of the index \( \alpha \) is defined by \( n \) and denoted as \( |\alpha| \). For an index \( \beta = (\beta_0, \ldots, \beta_{n-1}) \), we denoted \( \beta < \alpha \) if \( \beta \subset \alpha \). We define a homomorphism \( d_{\beta,\alpha} : B_\alpha(A) \to B_\beta(A) \) for \( \beta = (\alpha_0, \ldots, \alpha_i, \ldots, \alpha_n) \) by

\[
d_{\beta,\alpha}(x_0^\alpha_0 \otimes \cdots \otimes x_i^\alpha_i \otimes x_{i+1}^\alpha_{i+1} \otimes \cdots \otimes x_n^\alpha_{n+1}) = (-1)^{n-i} x_0^\alpha_0 \otimes \cdots \otimes x_i^\alpha_i \otimes x_{i+1}^\alpha_{i+1} \otimes \cdots \otimes x_n^\alpha_{n+1},
\]

where \( \cdot \cdot \cdot \) denotes the multiplication for the DGA \( A^\bullet \). We define a homomorphism of complexes \( d_B : \bigoplus_{|\alpha|=n} B_\alpha(A) \to \bigoplus_{|\beta|=n-1} B_\beta(A) \) by

\[
d_B = \sum_{|\alpha|=n, |\beta|=n-1, \beta < \alpha} d_{\beta,\alpha}.
\]

Then \( d_B^2 = 0 \) and we have the following double complex \( B(A) \):

\[
\cdots \to \bigoplus_{|\alpha|=2} B_\alpha(A) \to \bigoplus_{|\alpha|=1} B_\alpha(A) \to \bigoplus_{|\alpha|=0} B_\alpha(A) \to 0.
\]

The free simplicial bar complex is defined as the associated simple complex of the above double complex and is denoted as \( B(A) \). The inner differential of \( B_\alpha(A) \) is denoted as \( \delta \).

**Proposition 4.1.** The complex \( B(A) \) is an acyclic complex.

**Proof.** Let \( B_{>N}(A) \) be the sub double complex of \( B(A) \) defined by

\[
\cdots \to \bigoplus_{N<\alpha_0<\alpha_1} B_{\alpha_0,\alpha_1}(A) \to \bigoplus_{N<\alpha_0} B_{\alpha_0,\alpha_1}(A) \to \bigoplus_{N<\alpha_0} B_{\alpha_0}(A) \to 0.
\]

Then we have the natural inclusion \( i : B_{>N}(A) \to B_{>N-1}(A) \). We define a map \( \theta : B_{>N}(A) \to B_{>N-1}(A)[-1] \)

\[
\theta : \bigoplus_{N<\alpha_0<\cdots<\alpha_i} B_{\alpha_0,\ldots,\alpha_i}(A) \to \bigoplus_{N-1<\beta<\cdots<\alpha_i} B_{\beta,\alpha_0,\ldots,\alpha_i}(A)
\]

by

\[
\theta(1 \otimes x_1^\alpha_1 \otimes \cdots \otimes x_n^\alpha_n \otimes 1) = 1 \otimes 1 \otimes x_1^\alpha_1 \otimes \cdots \otimes x_n^\alpha_n \otimes 1.
\]

Then we have \( \theta \circ d + d \circ \theta = i(x) \). Therefore the double complex (for the differential \( d \)) \( B(A) = \lim_N B_{>N}(A) \) is an acyclic complex. Therefore its associated simple complex is also acyclic.

**Remark 4.2.** Let \( X \) be a \( C^\infty \) manifold. Let \( A = A^\bullet(X) \) be the DGA of \( C^\infty \) forms. Since \( X_\alpha \) is isomorphic to the product of copies of \( X \), the natural homomorphism \( B_\alpha(A) \to A^\bullet(X_\alpha) \) of complexes are quasi-isomorphism by Künneth formula. The homomorphisms \( B_\alpha(A) \to B_\beta(A) \) and \( A^\bullet(X_\alpha) \to A^\bullet(X_\beta) \) are compatible with the above quasi-isomorphisms. As a consequence, the natural homomorphism \( B(A) \to A^\bullet(X_\bullet) \) is a quasi-isomorphism. Proposition 3.1 follows from Proposition 4.1.
We define left $A$ right $A$ action ($A$-$A$ action for short) on the complex $B_\alpha(A)$ by

$$A \otimes B_\alpha(A) \otimes A \rightarrow B_\alpha$$

$$y \otimes (x_0 \otimes x_1 \ldots x_n \otimes x_{n+1}) \otimes z \rightarrow (yx_0) \otimes x_1 \ldots x_n \otimes (x_{n+1}z).$$

Since the differentials of $B(A)$ are $A$-$A$ homomorphisms, $B(A)$ is an $A$-$A$ module.

We introduce a bar filtration $F^\bullet_b$ on $B(A)$ as follows:

$$F^i_bB(A) : \cdots ightarrow 0 \rightarrow \bigoplus_{|\alpha|=i} B_\alpha(A) \rightarrow \cdots \rightarrow \bigoplus_{|\alpha|=0} B_\alpha(A) \rightarrow 0.$$

Then we have the following spectral sequence

$$E_1^{-i,p} = \bigoplus_{|\alpha|=i} B_\alpha((H^\bullet(A))^p) \Rightarrow E_\infty^{-i+p} = H^{-i+p}(B(A)).$$

This spectral sequence is called the bar spectral sequence.

### 4.2. Augmentations and coproduct

Let $\epsilon_1$, $\epsilon_2$ be two augmentations of $A = A^\bullet$, i.e., DGA homomorphisms $\epsilon_i : A \rightarrow k$, where $k$ is the trivial DGA. We define $B(\epsilon_1 \mid A \mid \epsilon_2)$ by the associated simple complex of the double complex

$$B(\epsilon_1 \mid A \mid \epsilon_2) = k \otimes_{A, \epsilon_1} B(A) \otimes_{A, \epsilon_2} k.$$

Therefore the degree $-n$ part $B(\epsilon_1 \mid A \mid \epsilon_2)_n$ of $B(\epsilon_1 \mid A \mid \epsilon_2)$ is isomorphic to the sum $\bigoplus_{|\alpha|=n} B_\alpha(\epsilon_1 \mid A \mid \epsilon_2)$, where

$$B_\alpha(\epsilon_1 \mid A \mid \epsilon_2) = k \otimes A^{\alpha_0} \otimes \cdots \alpha_{n-1} \otimes A \otimes k$$

and the differential (outer differential) is given by

$$d(i \otimes x_1 \otimes \cdots \otimes x_n \otimes 1) = \epsilon_1(x_1) \otimes x_2 \otimes \cdots \otimes x_n \otimes 1$$

$$+ \sum_{i=1}^{n-1} (-1)^i 1 \otimes x_1 \otimes \cdots \otimes x_i x_{i+1} \otimes \cdots \otimes x_n \otimes 1$$

$$+ (-1)^n 1 \otimes x_1 \otimes \cdots \otimes x_{n-1} \otimes \epsilon_2(x_n).$$

As in the last subsection, we have the following bar spectral sequence

$$E_1^{-i,p} = \bigoplus_{|\alpha|=i} B_\alpha(\epsilon_1 \mid H^\bullet(A) \mid \epsilon_2)^p \Rightarrow E_\infty^{-i+p} = H^{-i+p}(B(\epsilon_1 \mid A \mid \epsilon_2)).$$

We introduce a coproduct structure on $B(A)$. Let $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ be augmentations on $A$. Let $\alpha = (\alpha_0 < \cdots < \alpha_n)$ be a sequence of integers. We define $\Delta_{\epsilon_2}$

$$\Delta_{\epsilon_2} : B(A) \rightarrow (B(A) \otimes \epsilon_2) \otimes (k \otimes \epsilon_2 B(A))$$

by

$$\Delta_{\epsilon_2}(x_0 \otimes x_1 \otimes \cdots \otimes x_n \otimes x_{n+1}) = \sum_{i=0}^n (x_0 \otimes \cdots \otimes x_i \otimes 1) \otimes (1 \otimes x_{i+1} \otimes \cdots \otimes x_{n+1}).$$
for an element in $B_n(A)$. This is a morphism in the category $KKVec_k$. We introduce an $A$-$A$ module structure on the right hand side of (4.2) by
\[ y_1 \cdot (x_0 \otimes \cdots \otimes x_i \otimes 1) \otimes (1 \otimes x_{i+1} \otimes \cdots \otimes x_{n+1}) \cdot y_2 = ((y_1 x_0 \otimes \cdots \otimes x_i \otimes 1) \otimes (1 \otimes x_{i+1} \otimes \cdots \otimes (x_{n+1} y_2)) \]
for $y_1, y_2 \in A$. We define the double complex $\Delta$ where the outer differential augmentation ideal. We define the double complex $\Delta$ where the coproduct of $A$.

**Proposition 4.3.** (1) The homomorphism $\Delta_{x_2}$ is a homomorphism of complex of $KKVec_k$ and compatible with the $A$-$A$ action. Therefore we have the following homomorphism in $KKVec_k$:
\[ \Delta_{x_1,x_2,x_3} : B(\epsilon_1 | A | \epsilon_1) \rightarrow B(\epsilon_1 | A | \epsilon_2) \otimes B(\epsilon_2 | A | \epsilon_3). \]
Using the isomorphism (1.1), we have the following morphism in $KKVec_k$:
\[ \Delta_{x_1,x_2,x_3} : B(\epsilon_1 | A | \epsilon_1) \otimes B(\epsilon_2 | A | \epsilon_2) \rightarrow B(\epsilon_1 | A | \epsilon_1) \otimes B(\epsilon_2 | A | \epsilon_2). \]

(2) Let $\epsilon$ be an augmentation of $A$. For $\alpha = (\alpha_0)$, let $\epsilon_{\alpha_0} : B_{\alpha_0} = k \otimes k \rightarrow k$ be the natural map. Then $u = \sum_n \epsilon_{\alpha_0}$ defines a homomorphism of differential graded coalgebras $u : B(\epsilon | A | \epsilon) \rightarrow k$.

(3) The following composite maps are identity:
\[ B(\epsilon_1 | A | \epsilon_2) \xrightarrow{\Delta_{x_1,x_2,x_3}} B(\epsilon_1 | A | \epsilon_1) \otimes B(\epsilon_1 | A | \epsilon_2) \xrightarrow{u \otimes 1} B(\epsilon_1 | A | \epsilon_2), \]
\[ B(\epsilon_1 | A | \epsilon_2) \xrightarrow{\Delta_{x_1,x_2,x_3}} B(\epsilon_1 | A | \epsilon_1) \otimes B(\epsilon_2 | A | \epsilon_2) \xrightarrow{1 \otimes u} B(\epsilon_1 | A | \epsilon_2), \]
where $\Delta_{x_1,x_2,x_3}$ is defined in (1).

**Definition 4.4.** (1) The map $\Delta_{x_1,x_2,x_3}$ defined in (1) of Proposition 4.3 is called the coproduct of $B(A)$. It is easy to see that the coproduct is coassociative. Thus $B(A)$ forms a DG coalgebroid over the set Spaug($A$) of augmentations of $A$.

(2) The map $u : B(\epsilon | A | \epsilon) \rightarrow k$ defined in (2) of Proposition 4.3 is called the counit. In general, for an associative differential graded coalgebra $B$, the map $u$ satisfying the properties as (4.3) is called a counit.

5. **Comparison to Chen’s Theory**

In this section, we compare the simplicial bar complex defined as above and the reduced bar complex defined by Chen. Let $A = A^\ast$ be a differential graded algebra over a field $k$ and $\epsilon : A \rightarrow k$ be an augmentation. Let $I = \text{Ker}(\epsilon : A \rightarrow k)$ be the augmentation ideal. We define the double complex $B_{\text{red}}(A, \epsilon)$ as
\[ B_{\text{red}}(A, \epsilon) : \cdots \rightarrow I \otimes I \xrightarrow{d_B, \text{red}} I \otimes I \rightarrow I \rightarrow k \rightarrow 0, \]
where the outer differential $d_B, \text{red} : I \otimes I \rightarrow I \otimes (I^{(i-1)})$ is defined by
\[ d_B, \text{red} : [x_1 | \cdots | x_i] \mapsto \sum_{p=1}^{i-1} (-1)^p [x_1 | \cdots | x_p x_{p+1} | \cdots | x_i]. \]
Here an element $x_1 \otimes \cdots \otimes x_i$ in $I \otimes I$ is denoted as $[x_1 | \cdots | x_i]$. The total complex $B_{\text{red}}(A, \epsilon)$ is called the Chen’s reduced bar complex.
We define a degree preserving linear map $B(\epsilon | A \epsilon) \rightarrow B_{\text{red}}(A, \epsilon)$ by

$$1 \otimes x_1 \otimes \cdots \otimes x_{n-1} \otimes 1 \mapsto [\pi(x_1)] \cdots [\pi(x_{n-1})],$$

(5.2)

where $\pi(x) = x - \epsilon(x) \in I$ for an element $x \in A$.

**Lemma 5.1.** The above map is a homomorphism of complexes of $KVec_k$.

We have the following theorem.

**Theorem 5.2.** The homomorphism of associated simple complexes $B(\epsilon | A \epsilon) \rightarrow B_{\text{red}}(A, \epsilon)$ obtained form the map (5.2) is a quasi-isomorphism.

**Proof.** First, we define the bar filtration $F^{-i}b_{B_{\text{red}}}(A, \epsilon)$ on the double complex $B_{\text{red}}(A, \epsilon)$ by

$$F^{-i}b_{B_{\text{red}}}(A, \epsilon): \cdots \rightarrow 0 \rightarrow I \otimes I \rightarrow \cdots \rightarrow I \rightarrow k \rightarrow 0.$$

Then the bar filtration on $B(\epsilon | A \epsilon)$ and that on $B_{\text{red}}(A, \epsilon)$ are compatible and we have the following homomorphisms of spectral sequences:

$$E_1^{-i,p} = \bigoplus_{|\alpha|=i} B_{\alpha}(\epsilon | H^\bullet(A) \epsilon)^p \Rightarrow E_\infty^{-i,p} = H^{-i+p}(B(\epsilon | A \epsilon))$$

$$'E_1^{-i,p} = (H^\bullet(I)^{\otimes p})^p \Rightarrow 'E_\infty^{-i,p} = H^{-i+p}(B_{\text{red}}(A, \epsilon))$$

We prove that the vertical arrow coincides from $E_2$-terms. $E_1$-terms are the following complexes of graded vector spaces:

$$\cdots \rightarrow \bigoplus_{\alpha_0 < \alpha_1 < \alpha_2} B_{\alpha_0,\alpha_1,\alpha_2} \rightarrow \bigoplus_{\alpha_0 < \alpha_1} B_{\alpha_0,\alpha_1} \rightarrow \bigoplus B_{\alpha_0} \rightarrow 0$$

$$\cdots \rightarrow H^\bullet(I)^{\otimes 2} \rightarrow H^\bullet(I)^{\otimes 1} \rightarrow k \rightarrow 0$$

Here we used the abbreviation

$$B_{\alpha_0,\ldots,\alpha_n} = k^{\alpha_0} \otimes H^\bullet(A) \otimes \cdots \otimes H^\bullet(A) \otimes k.$$  

We show that the vertical homomorphism of complexes are quasi-isomorphism. We have

$$B_{\alpha_0,\ldots,\alpha_n} = k^{\alpha_0} \otimes (H^\bullet(I) \oplus k) \otimes \cdots \otimes (H^\bullet(I) \oplus k) \otimes k$$

$$= \bigoplus_{k=0}^n \bigoplus_{c \in C(n,k)} H^\bullet(I)^{\otimes k},$$

where $C(n, k)$ is the subset of $\{1, \ldots, n\}$ of cardinality $k$. Since the images of $H^\bullet(I) \otimes H^\bullet(A)$ and $H^\bullet(A) \otimes H^\bullet(I)$ under the multiplication is contained in $H^\bullet(I)$,
the following filtration $G$ on $(E^{-i,1}, d_1)$ and $(E^{-i,1}, d_1)$, defines a subcomplex:

$$G^{-i}E^{-i,1} = \bigoplus_{k=0}^{l} \bigoplus_{c \in C(n,k)} H^*(I)^{\otimes k},$$

Thus the associated graded complexes of $K$ from the definitions. The statement (3) follows from the statement (1) and (2). We prove the proposition by the induction on $l$.

**Proof.**

**Proposition 5.3.** (1) The map $\epsilon_1: K_{1,l} \rightarrow k$ is an isomorphism.

(2) The natural inclusion $K_{m,l} \rightarrow K_{m+1,l}$ is a quasi-isomorphism for $m \geq 1$.

(3) The complex $K_{m,l} \rightarrow k$ is an acyclic complex for $l \leq m$.

**Proof.** We prove the proposition by the induction on $l$. The statement (1) is direct from the definitions. The statement (3) follows from the statement (1) and (2). We prove the statement (2) for $l$ assuming the statement (3) for $l-1$. The cokernel of the complex $\text{Coker}(K_{m,l} \rightarrow K_{m+1,l})$ is isomorphic to

$$\cdots \rightarrow \bigoplus_{|\alpha| = k+1, \alpha \in [1,m+1], \alpha_k = m+1} \bigoplus_{c \in C(k-1,l-1)} kt'_{\alpha,c} \left( \bigoplus_{\alpha \in [1,m+1], \alpha_k = m+1} \bigoplus_{c \in C(k-1,l-1)} kt''_{\alpha,c} \right) \rightarrow \cdots,$$

where

$$t'_{\alpha,c} = 1 \otimes u_1 \otimes \cdots \otimes u_k \otimes 1 \otimes 1,$$

$$t''_{\alpha,c} = 1 \otimes u_1 \otimes \cdots \otimes u_k \otimes x \otimes 1,$$

with $u_i = \begin{cases} x & \text{if } i \in c, \\ 1 & \text{if } i \notin c. \end{cases}$
Thus it is isomorphic to the cone of a complex homomorphism $K_{m,l} \to K_{m,l-1}$ by considering the both cases for $u_k = 1, x$. Since $K_{m,l} \to k$ and $K_{m,l-1} \to k$ is quasi-isomorphism by the induction hypothesis for $(m, l)$ and $(m, l-1)$, we have the statement (2).

End of proof of Theorem 5.2. Since the diagram (5.3) is obtained from (5.4) by tensoring $H^\bullet(I^{\otimes l})$ and taking the inductive limit on $m$, the homomorphism of associated graded complex of $E_1^{-i, \bullet} \to E_1^{-i, \bullet}$ is a quasi-isomorphism by Proposition 5.3. This proves the theorem.

□

Remark 5.4. Since the spectral sequence associated to the filtration $G$ is isomorphic from $E_2$ terms, the induced filtrations on $H_i(B(\epsilon \mid A \mid \epsilon))$ and $H_i(B_{red}(A, \epsilon))$ are equal.

6. Simplicial Bar Complex and the DG Category $KC_A$

6.1. Definition of comodules over DG coalgebras. In this section, we prove that the DG category of comodules over the simplicial bar complex $B(\epsilon \mid A \mid \epsilon)$ is homotopy equivalent to the DG category $KC_A$.

Let $B$ be a coassociative differential graded coalgebra over $k$ with a counit $u: B \to k$. The comultiplication $B \to B \otimes B$ is written as $\Delta_B$. We define $B$-comodules below. We use the terminology $B$-comodules in stead of DG $B$-comodules for simplicity, when $B$ is a differential graded coalgebra.

Definition 6.1 (DG category of $B$ comodules). (1) A $k$-complex $M$ with a homomorphism $\Delta_M$ of complexes

$$\Delta_M : M \to B \otimes M$$

is called a (left) $B$-comodule if the following properties hold.

(a) (Coassociativity.) The following diagram commutes:

$$
\begin{array}{ccc}
M & \xrightarrow{\Delta_M} & B \otimes M \\
\Delta_M \downarrow & & \downarrow 1 \otimes \Delta_M \\
B \otimes M & \xrightarrow{1 \otimes \Delta_B} & B \otimes B \otimes M.
\end{array}
$$

(b) (Counitarity.) The composite homomorphism $M \xrightarrow{\Delta_M} B \otimes M \xrightarrow{u \otimes 1} M$ is the identity of $M$.

(2) Let $M, N$ be $B$-comodules. We define the complex of homomorphisms $\text{Hom}_B^\bullet(M, N)$ by the associated simple complex of $\text{Hom}_{B\text{-com}}^\bullet(M, N)$ defined by

$$
\text{Hom}_{B\text{-com}}^\bullet(M, N) : \text{Hom}_{KVec_k}(M, N) \xrightarrow{d_B} \text{Hom}_{KVec_k}(M, B \otimes N) \xrightarrow{d_B} \text{Hom}_{KVec_k}(M, B \otimes B \otimes N) \xrightarrow{d_B} \cdots;
$$

where

$$\text{Hom}_{KVec_k}(M, B^{\otimes n} \otimes N) \xrightarrow{d_B} \text{Hom}_{KVec_k}(M, B^{\otimes (n+1)} \otimes N)$$
is defined by
\[ d_H \varphi = (-1)^{n+1}(1_B \otimes \varphi) \circ \Delta_M \]
\[ \quad + \sum_{i=1}^{n}(-1)^{n-i+1}(1_B^{\otimes i-1} \otimes \Delta_B \otimes 1_B^{\otimes (n-i)} \otimes 1_N) \circ \varphi + (1_B^{\otimes n} \otimes \Delta_N) \circ \varphi \]  
(6.2)

(3) Let \( L, M, N \) be \( B \)-comodules. We define the composite morphism
\[ \mu: \text{Hom}_{B\text{-com}}(M, N) \otimes \text{Hom}_{B\text{-com}}(L, M) \rightarrow \text{Hom}_{B\text{-com}}(L, N) \]
by \( \sum_{i,j\geq 0} \mu_{ij} \), where the morphism
\[ \mu_{ij}: \text{Hom}_{KVec_k}(M, B^{\otimes i} \otimes N)e^{-i} \otimes \text{Hom}_{KVec_k}(L, B^{\otimes j} \otimes M)e^{-j} \rightarrow \text{Hom}_{KVec_k}(L, B^{\otimes (i+j)} \otimes M)e^{-i-j} \]
is defined by \( \mu_{ij}(f \otimes g) = (1_B^{\otimes j} \otimes f) \circ g \).

**Proposition 6.2.** The category of \( B \)-comodules \( (B\text{-com}) \) forms a DG-category by setting

1. the complex of homomorphism as \( \text{Hom}_{B\text{-com}}(\bullet, \bullet) \), and
2. the composite homomorphism as \( \mu \). The composite is denoted as “\( \circ \).”

**Proof.** To show that the multiplication homomorphism \( \mu \) is a homomorphism of complexes, it is enough to consider the outer differentials. Let
\[ f \otimes g \in \text{Hom}_{KVec_k}(M, B^{\otimes i} \otimes N) \otimes \text{Hom}_{KVec_k}(L, B^{\otimes j} \otimes M) \]
and \( d_H \) be the outer differential. Then we have
\[ d_H(f \circ g) = (-1)^{n+1}(1_B^{\otimes (j+1)} \otimes f) \circ (1_B \otimes g) \circ \Delta_L \]
\[ + \sum_{p=1}^{n}(-1)^{n-p+1}(1_B^{\otimes (p-1)} \otimes \Delta_B \otimes 1_B^{\otimes (n-p)} \otimes 1_N) \circ (1_B^{\otimes j} \otimes f) \circ g \]
\[ + (1_B^{\otimes n} \otimes \Delta_N) \circ (1_B^{\otimes j} \otimes f) \circ g \]
\[ = (-1)^{i+j+1}(1_B^{\otimes (i+j+1)} \otimes f) \circ (1_B \otimes g) \circ \Delta_L \]
\[ + (-1)^{i} \sum_{p=1}^{j}(-1)^{j-p+1}(1_B^{\otimes (j+1)} \otimes f) \circ (1_B^{\otimes (p-1)} \otimes \Delta_B \otimes 1_B^{\otimes (j-p)} \otimes 1_N) \circ g \]
\[ + (-1)^{i}(1_B^{\otimes (j+1)} \otimes f) \circ (1_B^{\otimes j} \otimes \Delta_M) \circ g \]
\[ + (-1)^{i+1}(1_B^{\otimes (j+1)} \otimes f) \circ (1_B^{\otimes j} \otimes \Delta_M) \circ g \]
\[ + \sum_{q=1}^{i}(-1)^{i-q+1}(1_B^{\otimes (i+q-j-1)} \otimes \Delta_B \otimes 1_B^{\otimes (i-q)} \otimes 1_N) \circ (1_B^{\otimes j} \otimes f) \circ g \]
\[ + (1_B^{\otimes (i+j)} \otimes \Delta_N) \circ (1_B^{\otimes j} \otimes f) \circ g \]
\[ = (-1)^{i}f \circ d_H(g) + d_H(f) \circ g. \]

The associativity for the composite can be proved similarly. \( \square \)
Let \( A \) be a DGA and \( \epsilon: A \rightarrow k \) be an augmentation. Until the end of this subsection, let \( B = B(\epsilon | A | \epsilon) \) be the simplicial bar complex and \( B_{\text{red}} = B_{\text{red}}(A, \epsilon) \) be the reduced bar complex for the augmentation \( \epsilon \). Let \( \pi_\alpha: B \rightarrow k \otimes k \) be the projection.

**Definition 6.3** ((B-com)_\text{red}^\text{b}). Let \( S \subset \mathbb{Z} \) a subset of \( \mathbb{Z} \). A B-comodule \( M \) is said to be supported on \( S \) if and only if the composite

\[
M \xrightarrow{\Delta_M} B \otimes M \xrightarrow{\pi_\alpha \otimes 1_M} k \otimes k \otimes M
\]

is zero if \( \alpha \notin S \). A module \( M \) supported on a finite set \( S \) is called a bounded \( B \) comodule. For a \( B \) comodule \( M \), the composite map

\[
M \rightarrow B \otimes M \rightarrow B_{\text{red}} \otimes M
\]

defines a \( B_{\text{red}} \) comodules structure on \( M \), which is denoted as \( M_{\text{red}} \). The class of objects in the DG category \((B-\text{com})_{\text{red}^b}\) is defined by bounded \( B \) comodules and the complex of morphisms from \( M \) to \( N \) is defined by \( \text{Hom}_{B_{\text{red}}-\text{com}}(M_{\text{red}}, N_{\text{red}}) \).

The rest of this section is devoted to proving the following theorem.

**Theorem 6.4** (Main Theorem). DG categories \( K^b \mathcal{C}_A \) and \((B-\text{com})_{\text{red}^b}\) are homotopy equivalent.

### 6.2. Correspondences on objects

We construct a one to one correspondence \( \varphi \) from the class of objects of \((K^b \mathcal{C}_A)\) to that of objects of \((B-\text{com})_{\text{red}^b}\). In this section, we simply denote \( B_\alpha \) and \( B \) for \( B(\epsilon | A | \epsilon) \) and \( B(\epsilon | A | \epsilon) \), respectively.

#### 6.2.1. Definition of \( \varphi \): ob\((K^b \mathcal{C}_A) \rightarrow \text{ob}(B-\text{com})_{\text{red}^b}\)

Let \( M = (M^i, d_{ij}) \) be an object of \( K^b \mathcal{C}_A \), where \( M^i \in \mathcal{C}_A \). We denote \( s(M) \) by the graded vector space \( \bigoplus_j M^i e^{-j} \). In this rest of Section 6.2, we define a \( B \)-comodule structure \( \Delta = \Delta_M \) on \( s(M) \) of (6.1) in Definition 6.1.

The morphism \( d_{ij} \in \text{Hom}_{K^b \mathcal{C}_A}^{j-i+1}(M^j, A^i \otimes M^j) \) defines an element

\[
D_{ji} = d_{ij} \otimes t_{-i,-j} \in \text{Hom}_{K^b \mathcal{C}_A}^1(M^i e^{-j}, A^i \otimes M^j e^{-j})
\]

\[
= \text{Hom}_{K^b \mathcal{C}_A}^1(M^i e^{-j}, (k \otimes A^i \otimes k) \otimes M^j e^{-j})
\]

\[
= \text{Hom}_{K^b \mathcal{C}_A}^1(M^i e^{-j}, B_{ji} \otimes M^j e^{-j}).
\]

We consider the following linear map

\[
M^k e^{-k} \xrightarrow{D_{kj}} B_{kj} \otimes M^j e^{-j} \xrightarrow{1 \otimes D_{ji}} B_{kj} \otimes M^i e^{-i} \xrightarrow{\mu \otimes 1} B_{kj} \otimes M^i e^{-i},
\]

where \( \mu \) is the multiplication map. The condition (2.2) is equivalent to the relation

\[
\sum_{j : k < j < i} (\mu \otimes 1)(1 \otimes D_{kj}) D_{kj} + (d_{A \otimes 1} + 1 \otimes d_{M^j}) D_{kj} + D_{kj} d_{M^k} = 0
\]

in \( \text{Hom}_{K^b \mathcal{C}_A}^1(M^k e^{-k}, B_{kj} \otimes M^i e^{-i}) \).

We set \( s(M) = \bigoplus_i M^i e^{-i} \). Then the sum \( D = \sum_{j < i} D_{kj} \) defines an element in \( \text{Hom}_{K^b \mathcal{C}_A}^1(s(M), A^* \otimes s(M)) \). By composing \( \epsilon \otimes 1 \in \text{Hom}_{K^b \mathcal{C}_A}^0(A^* \otimes s(M), s(M)) \), we have

\[
D_\epsilon = (\epsilon \otimes 1) D \in \text{Hom}_{K^b \mathcal{C}_A}^1(s(M), s(M)).
\]
Lemma 6.5. We set \( d_M = \sum_i d_{M_i} \). Then \( \delta_M = d_M + D \epsilon \) defines a differential on \( s(M) = \bigoplus_i M_i e^{-i} \).

Proof. By (6.4), we have
\[
(\mu \otimes 1)(1 \otimes D) + (d_A \otimes 1 + 1 \otimes d_M) D = 0
\]
By the commutative diagram
\[
\begin{array}{c}
s(M) \xrightarrow{D} A^* \otimes s(M) \xrightarrow{\epsilon \otimes 1} B \otimes s(M) \\
\downarrow \epsilon \otimes 1 \quad \downarrow \epsilon \otimes 1 \\
A^* \otimes s(M) \xrightarrow{\epsilon \otimes 1} s(M)
\end{array}
\]
we have \( (\epsilon \otimes 1)(\mu \otimes 1)(1 \otimes D) = D^2 \epsilon \). Therefore we have \( (d_M + D \epsilon)^2 = 0 \).

Definition 6.6. Let \( \alpha = (\alpha_0 < \cdots < \alpha_n) \) be a sequence of integers. We define \( D_\alpha = D_{\alpha_0, \ldots, \alpha_n} \in \text{Hom}_A^n(M^{\alpha_0} e^{-\alpha_0}, B_\alpha \otimes M^{\alpha_n} e^{-\alpha_n}) \) inductively by the following composite homomorphism:
\[
M^{\alpha_0} e^{-\alpha_0} \xrightarrow{D_{\alpha_0, \ldots, \alpha_{n-1}} \epsilon^{-1}} B_{\alpha_0, \ldots, \alpha_{n-1}} \otimes M^{\alpha_{n-1}} e^{-\alpha_{n-1}} \xrightarrow{1 \otimes D_{\alpha_{n-1}, \alpha_{n}} \epsilon^{-1}} (B_{\alpha_0, \ldots, \alpha_{n-1}} \otimes A^*) \otimes M^{\alpha_n} e^{-\alpha_n} = B_{\alpha_0, \ldots, \alpha_n} \otimes M^{\alpha_n} e^{-\alpha_n}.
\]
By the isomorphism
\[
\text{Hom}_A^n(M^{\alpha_0} e^{-\alpha_0}, B_\alpha \otimes M^{\alpha_n} e^{-\alpha_n}) \xrightarrow{\varphi \mapsto (1 \otimes l^{-n} \otimes 1) \varphi} \text{Hom}_A^n(M^{\alpha_0} e^{-\alpha_0}, B_\alpha e^n \otimes M^{\alpha_n} e^{-\alpha_n}),
\]
the element corresponding to \( D_\alpha \) is denoted by \( D_\alpha \). We set
\[
\Delta = \sum_\alpha D_\alpha : s(M) \to B \otimes s(M).
\]

Proposition 6.7. (1) The map \( \Delta \) is a homomorphism of complexes.
(2) The above homomorphism \( \Delta \) defines a \( B \)-comodule structure on \( s(M) \).

Proof. For the proof of (2), the coassociativity and counitarity is easy to check, so we omit the proof. We prove the statement (1). We compute the right hand side
of the following equality:
\[ dD_{\alpha_0, \ldots, \alpha_n}(x) = (d_B \otimes t \otimes 1_M)D_{\alpha_0, \ldots, \alpha_n}(x) \]  
(6.6)
\[ + \sum_{i=0}^{n-1} (1_A^t \otimes d_A \otimes 1_A^\otimes(n-i-1) \otimes 1_M)D_{\alpha_0, \ldots, \alpha_n}(x) \]  
(6.7)
\[ + (1_A^\otimes n \otimes d_M)D_{\alpha_0, \ldots, \alpha_n}(x). \]  
(6.8)

Here \( d_B \) is the outer differential defined in (4.1). Then the term (6.6) is equal to
\[ (d_B \otimes t \otimes 1_M)D_{\alpha_0, \ldots, \alpha_n}(x) \]
(6.9)
\[ = (1_A^t \otimes 1_M)(\epsilon \otimes 1_A^\otimes(n-1) \otimes 1_M)D_{\alpha_0, \ldots, \alpha_n}(x) \]
(6.10)
\[ + (-1)^n(1_A^\otimes(n-1) \otimes t \otimes 1_M)(1_A^\otimes(n-1) \otimes \epsilon \otimes 1_M)D_{\alpha_0, \ldots, \alpha_n}(x). \]
(6.11)

We define a map
\[ E(\alpha_1, \ldots, \alpha_{i-1}; \alpha_i; \alpha_{i+1}, \ldots, \alpha_n) \in \text{Hom}_k^{n+1}(M^{\alpha_0}e^{-\alpha_0}, B_{\alpha_{i-1}, \alpha_i} \otimes M^{\alpha_i}e^{-\alpha_i}) \]
by the composite
\[ M^{\alpha_0}e^{-\alpha_0} \xrightarrow{D_{\alpha_{i-1}, \alpha_i}} B_{\alpha_{i-1}, \alpha_i} \otimes M^{\alpha_i}e^{-\alpha_i} \xrightarrow{1_A^\otimes d_M} B_{\alpha_{i-1}, \alpha_i} \otimes M^{\alpha_i}e^{-\alpha_i}. \]

Then by the relation (6.4), we have
\[ \sum_{\alpha_i < \beta < \alpha_{i+1}} (1_A^\otimes \mu \otimes 1_A^\otimes(n-i-1) \otimes 1_M)D_{\alpha_0, \ldots, \alpha_i, \beta, \alpha_{i+1}, \ldots, \alpha_n} \]
\[ = -E(\alpha_0, \ldots, \alpha_i; \alpha_{i+1}, \ldots, \alpha_n) - E(\alpha_0, \ldots, \alpha_{i+1}; \alpha_i, \alpha_{i+2}, \ldots, \alpha_n) \]
\[ + (-1)^n(1_A^\otimes(n-1) \otimes d_A \otimes 1_A^\otimes(n-i-1) \otimes 1_M)D_{\alpha_0, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_n} \]
and therefore the term (6.10) is equal to
\[ \sum_{i=1}^{n-1} \sum_{\alpha_0 < \cdots < \alpha_i < \alpha_{i+1} < \alpha_{i+2} < \cdots < \alpha_n} (-1)^i(1_A^\otimes(n-1) \otimes t \otimes 1_M) \]
\[ \times (1_A^\otimes(i-1) \otimes \mu \otimes 1_A^\otimes(n-i-1) \otimes 1_M)D_{\alpha_0, \ldots, \alpha_i, \alpha_{i+1}, \alpha_{i+2}, \ldots, \alpha_n} \]
(6.12)
\[ = \sum_{\alpha_0 < \cdots < \alpha_{n-1}} D_{\alpha_0, \ldots, \alpha_{n-1}} d_M \]  
(6.13)
\[ - \sum_{\alpha_0 < \cdots < \alpha_{n-1}} (1_A^\otimes(n-1) \otimes d_M)D_{\alpha_0, \ldots, \alpha_{n-1}} \]  
(6.14)
Since

\[(6.8) + (6.11) + (6.13) = 0, \quad (6.9) + (6.12) = \Delta \delta M,\]
\[(6.7) + (6.14) = 0,\]

we have \(\Delta \delta M = d\Delta.\) \(\square\)

6.2.2. Definition of \(\psi\): \(ob(B\text{-com})^{\text{red}} \to ob(K^b C_A).\) Let \(N\) be a \(B\)-comodule. By the homomorphism \(\Delta: N \to B \otimes N,\) we have degree preserving linear maps

\[
\Delta^{(0)}: N \to \bigoplus_{\alpha_0} k^{\alpha_0} \otimes N,
\]

\[
\Delta^{(1)}: N \to \bigoplus_{\alpha_0 < \alpha_1} k^{\alpha_0} A^{\alpha_1} \otimes N e,
\]

\[
\Delta^{(2)}: N \to \bigoplus_{\alpha_0 < \alpha_1 < \alpha_2} k^{\alpha_0} A^{\alpha_1} A^{\alpha_2} \otimes N e^2.
\]

The \(\alpha_0\)-component, the \((\alpha_0, \alpha_1)\)-component and the \((\alpha_0, \alpha_1, \alpha_2)\)-component of \(\Delta^{(0)}, \Delta^{(1)}\) and \(\Delta^{(2)}\) are denoted as \(p_{\alpha_0}, D_{\alpha_0, \alpha_1}\) and \(D_{\alpha_0, \alpha_1, \alpha_2}\), respectively.

Lemma 6.8. (1) The linear maps \(p_{\alpha_0}\) are complete projection orthogonal to each other. Moreover these orthogonal projections defines a direct sum decomposition of \(N\).

(2) We have the following equalities

\[
(1 \otimes p_{\alpha_1}) D_{\alpha_0, \alpha_1} = D_{\alpha_0, \alpha_1} = D_{\alpha_0, \alpha_1} p_{\alpha_0},
\]

\[
D_{\alpha_0, \alpha_1, \alpha_2} = (1 \otimes D_{\alpha_1, \alpha_2}) D_{\alpha_0, \alpha_1}.
\] (6.15)

Proof. Since the composite map \(N \xrightarrow{\Delta} B \otimes N \xrightarrow{\epsilon \otimes 1} N\) is the identity, we have \(\sum_{\alpha_0} p_{\alpha_0} = id_N\). For \(x \in N\), we have

\[
(\Delta_B \otimes 1)(\Delta^{(0)}_N(x)) = \sum_{\alpha_0} (\Delta_B \otimes 1)((1 \otimes 1) \otimes p_0(x))
\]

\[
= \sum_{\alpha_0} (1 \otimes 1) \otimes (1 \otimes 1) \otimes p_{\alpha_0}(x)
\]

and

\[
(1 \otimes \Delta^{(0)}_N)(\Delta^{(0)}_N(x)) = \sum_{\alpha_0} (1 \otimes \Delta^{(0)}_N)((1 \otimes 1) \otimes p_{\alpha_0}(x))
\]

\[
= \sum_{\alpha_0 < \alpha_1} (1 \otimes 1) \otimes (1 \otimes 1) \otimes p_{\alpha_1} p_{\alpha_0}(x).
\]

Thus by the coassociativity, \(p_{\alpha_0}\) is a complete system of orthogonal projection.

We prove the second statement. For \(x \in N\), we have

\[
(\Delta_B \otimes 1)(D_{\alpha_0, \alpha_1}(x)) = (1 \otimes 1) \otimes D_{\alpha_0, \alpha_1}(x) + D_{\alpha_0, \alpha_1}(x) \otimes (1 \otimes 1)
\]

\[
\in (k^{\alpha_0} \otimes k) \otimes (k^{\alpha_0} A^{\alpha_1} k) \otimes N \oplus (k^{\alpha_0} A^{\alpha_1} k) \otimes (k^{\alpha_0} k) \otimes N.
\]
and

\[(1 \otimes \Delta^{(0)})(D_{\alpha_0, \alpha_1}(x)) = \sum_{\gamma} (1 \otimes p_\gamma)(D_{\alpha_0, \alpha_1}(x)) \otimes (1 \otimes 1) \in (k \otimes A \otimes_1 N) \otimes (k \otimes k) = (k \otimes A \otimes_1 k) \otimes (k \otimes k) \otimes N \]

\[(1 \otimes \Delta^{(1)})((1 \otimes 1) \otimes p_{\alpha_0}(x)) = \sum_{\beta, \gamma} (1 \otimes 1) \otimes D_{\beta, \gamma}(p_{\alpha_0}(x)) \in (k \otimes k) \otimes (k \otimes A \otimes k) \otimes N.\]

By comparing the direct sum component, we have the lemma.

As for the equality (6.15), we consider the part of \((\Delta_B \otimes 1)\Delta = (1 \otimes \Delta)\Delta\). The component of \((\Delta_B \otimes 1)\Delta(x)\) is equal to \(D_{\alpha_0, \alpha_1, \alpha_2}(x)\). On the other hand, this component of \((1 \otimes \Delta)\Delta\) is equal to \((1 \otimes D_{\alpha_0, \alpha_1, \alpha_2})D_{\alpha_0, \alpha_1, \alpha_2}\).

Now we construct the correspondence \(\psi\). Let \(M^i\) be the direct sum component of the complex \(Ne^i\) associated to the projector \(p_i\). We assume that \(N\) is a bounded \(B\) comodule. We make an object \(\{\{M^i\}, \{d_{ji}\}\}\) of \(K^bC_A\) from the sequence of complex \(M^i\). By (2) of the above lemma, \(D_{\alpha_0, \alpha_1}\) is regarded as a degree zero linear map \(M^{\alpha_0}e^{-\alpha_0} \rightarrow (A \otimes M^{\alpha_1})e^{-\alpha_1+1}\). Therefore \(D_{\alpha_0, \alpha_1}\) defines an element of \(\text{Hom}_{B}^1(M^{\alpha_0}e^{-\alpha_0}, M^{\alpha_1+1}e^{-\alpha_1})\).

**Proposition 6.9.** The pair \((M^{\alpha_0}, d_{\alpha_0, \alpha_1})\) defined as above is a bounded DG complex in \(C_A\).

**Proof.** It is enough to show the relation (6.4). It is obtained by considering the \((k \otimes A \otimes k) \otimes M^{\alpha_2}\)-component of the equality \(\Delta(dx) = d(\Delta(x))\) restricted to \(M^{\alpha_0}\). \(\square\)

### 6.3. The functor \(\psi\): \((B\text{-com})^\text{red, b} \rightarrow K^bC_A\) on Morphisms

#### 6.3.1. \(B\)-comodules and morphisms in \(K^bC_A\)

In this section, we construct the functor \(\psi\): \((B\text{-com})^\text{red, b} \rightarrow K^bC_A\) for morphisms in Theorem 6.4. In this subsection, we set \(B_{\text{red}} = B_{\text{red}}(A, \epsilon)\).

**Definition 6.10.** (1) Let \(p\) be the projector \(B_{\text{red}} \rightarrow Ie\) and \(p_{n, \beta}\) be the homomorphism

\[p_{n, \beta}: B^\otimes_\text{red} \otimes Ne^{-n} \rightarrow I \otimes \cdots \otimes I \otimes N^\beta e^{-\beta}\]

defined by \(p_{n, \beta} = p \otimes \cdots \otimes p \otimes p_{\beta}\).

(2) Let \(M, N\) be objects in \((B\text{-com})^\text{red, b}, \psi(M) = \{M^i\}, \psi(N) = \{N^i\}\) the corresponding objects in \(K^bC_A\) and \(f = \sum_n f(n)\) an element in

\[\text{Hom}_{B_{\text{red}}-\text{com}}^i(M_{\text{red}}, N_{\text{red}}) = \oplus_n \text{Hom}_{K^bC_A}^i(M, B^\otimes_\text{red} \otimes Ne^{-n})\]
For integers $\alpha < \beta, n \geq 0$ we define a homomorphism
\[ \Psi(f)^{(n)}_{\alpha, \beta} \in \text{Hom}_{KVec_k}(M^\alpha e^{-\alpha}, A \otimes N^\beta e^{-\beta}) \]
by the composite
\[
M^\alpha e^{-\alpha} \rightarrow M \xrightarrow{f^{(n)}} B^\otimes_n \otimes N e^{-n} \xrightarrow{p_n, A} I \otimes \cdots \otimes I \otimes N^\beta e^{-\beta} \xrightarrow{(\text{product}) \otimes 1} A \otimes N^\beta e^{-\beta}.
\]

(3) Let $f$ an element in $\text{Hom}_{B_{\text{red}}^{\bullet}}(M_{\text{red}}, N_{\text{red}})$. We define an element $\psi(f)$ of $\text{Hom}_{KCA}(\psi(M), \psi(N))$ by
\[
\psi(f)^{(n)}_{\alpha, \beta} = \sum_{n \geq 0} \Psi(f)^{(n)}_{\alpha, \beta}.
\]

**Proposition 6.11.** (1) Let $M, N$ be objects $(B\text{-com})^{\text{red,b}}$. Then the map
\[
\psi: \text{Hom}_{B_{\text{red}}^{\bullet}}(M_{\text{red}}, N_{\text{red}}) \rightarrow \text{Hom}_{KCA}(\psi(M), \psi(N))
\]
is a homomorphism of complexes and quasi-isomorphism.

(2) The above map is compatible with composite, that is, the following diagram commutes:
\[
\begin{array}{ccc}
\text{Hom}_{B_{\text{red}}^{\bullet}}(M_{\text{red}}, N_{\text{red}}) & \xrightarrow{\psi} & \text{Hom}_{KCA}(\psi(M), \psi(N)) \\
\psi \downarrow & & \downarrow \\
\text{Hom}_{KCA}(\psi(L), \psi(M)) & \xrightarrow{\psi \otimes \psi} & \text{Hom}_{KCA}(\psi(L), \psi(N)) \\
\end{array}
\]

Proof. Let $M, N$ be objects in $(B\text{-com})^{\text{red,b}}$ and set $\psi(M) = \{M_i, \partial M_{i,j}\}$ and $\psi(N) = \{N_i, \partial M_{i,j}\}$. Then we have $\partial M = \partial M_{i,j} = d_{M_{i,j}} + \delta M_{i,j}$, where
\[
d_{M_{i,j}} = p_j \circ \partial \circ t_i: M^i \rightarrow M \xrightarrow{d} M \rightarrow M^j
\]
\[
\delta M = \delta M_{i,j} = (p \otimes p_j) \circ \Delta_{M_{\text{red}}} \circ t_i: M^i \rightarrow M \xrightarrow{\Delta_{M_{\text{red}}}^\otimes} B_{\text{red}} \otimes M \rightarrow I \otimes M^j.
\]
Let $d_H$ be the outer differential (6.2) for $\text{Hom}_{B_{\text{red}}^{\bullet}}(M, N)$ and
\[
d_{B_{\text{red}}^{\otimes}}(f) = \sum_{i=1}^{n} (1^\otimes(i-1) \otimes d_{B_{\text{red}}} \otimes 1^\otimes(n-i) \otimes N) \circ f \in \text{Hom}(M, B_{\text{red}}^{\otimes} \otimes N),
\]
where $d_{B_{\text{red}}}$ is defined in (5.1). To prove the compatibility of differential for the map $\psi$, it is enough to show the following equality:
\[
\psi(d_H(f) + d_{B_{\text{red}}^{\otimes}}(f))_{pq} = \sum_{p < k < q} (\mu \otimes 1_{N^q}) \{(1_A \otimes \delta_{N, kq}) \psi(f)_{pq} + (-1)^i(1_A \otimes \psi(f)_{kq}) \delta_{M, pk}\}.
\]
We check this equality. We set
\[ H^i_{t_1, \ldots, t_k} = \bigoplus_{i_0} \text{Hom}^i(M, I^\otimes i_0 \otimes \cdots \otimes I^\otimes i_k \otimes N), \]
\[ H^i_{\ast, t} \rightarrow \bigoplus_{i_0} H^i_{i_0 \ast, t} \rightarrow \bigoplus_{i_0, i_1} H^i_{i_0 i_1 \ast, t} \rightarrow \]
We remark that the image of \( H^i_{\ast, t} \) under the total differential is contained in
\[ H^i_{\ast, t}, \quad H^i_{p \ast, t}, \quad H^i_{s \ast, t}, \quad H^i_{t \ast, t}, \quad H^i_{s - t, t}, \quad H^i_{s - i, t}, \quad H^i_{s - i, t - i, t}. \]
The \( H^i_{s - t, t} \) part of the image image of \( H^i_{t_1, \ldots, t_k} \) under the total differential is zero if \((i_1, \ldots, i_k) \neq (1, \ldots, 1) \) or \((1, \ldots, 2, \ldots, 1)\).
(b) Let
\[ f = \left( m \rightarrow \sum_{t} [a_1^{(t)}] \otimes \cdots \otimes [a_p^{(t)}] \otimes b_{pq}^{(t)} \otimes \cdots \otimes [a_k^{(t)}] \otimes n^{(t)} \right) \in H^i_{1 \ast, \ldots, 1}. \]
Since \( \psi \) is non-zero only on the \( H^i_{1 \ast, \ldots, 1} \) component, \( \psi(f) = 0 \) and
\[ \psi(df)(m) = (-1)^p (\psi(f_1)(m) - \psi(f_2)(m)), \quad (6.16) \]
where
\[ f_1 = \left( m \rightarrow \sum_{t} [a_1^{(t)}] \otimes \cdots \otimes [a_p^{(t)}] \otimes \cdots \otimes [a_k^{(t)}] \otimes n^{(t)} \right), \]
\[ f_2 = \left( m \rightarrow \sum_{t} [a_1^{(t)}] \otimes \cdots \otimes [a_p^{(t)}] \otimes b_{pq}^{(t)} \otimes \cdots \otimes [a_k^{(t)}] \otimes n^{(t)} \right). \]
Therefore the right hand side of (6.16) is zero.
On the other hand,
\[ df(m) = (1^\otimes k \otimes \delta_N)(f(m)) - (-1)^p (1 \otimes f)(\delta_M(m)) \]
\[ + \sum_{t} \{ [da_1^{(t)}] \otimes [a_2^{(t)}] \otimes [a_k^{(t)}] \otimes n^{(t)} - \cdots \pm [a_1^{(t)}] \otimes [a_2^{(t)}] \otimes [da_k^{(t)}] \otimes n^{(t)} \}
\[ + d_N(f(m)) - (-1)^p (d_M(m)) \]
and we have \( df = \psi d \) on \( H^i_{1 \ast, \ldots, 1} \).
As for the compatibility of composite, it is enough to check the equality:
\[ \psi(f \circ g)_{pq} = \sum_{p < k < q} (\mu \otimes I_{N^0})(1_A \otimes \psi(f)_{kj})\psi(g)_{jk}. \]
This equality also follows from the definition of \( \psi \).
To show that the homomorphism \( \psi \) is a quasi-isomorphism, we introduce a finite filtration \( \text{Fil}^0 \) on \( N \) by \( \text{Fil}^0 N = \bigoplus_{i \geq 0} N^i e^{-1} \). Since the quotient \( \text{Fil}^0 N/\text{Fil}^{0 + 1} N \)
has a trivial $B$-coaction, we may assume that the action on $N$ is trivial by considering the long exact sequence for the short exact sequence

$$0 \to \text{Fil}^0 N \to \text{Fil}^0 N \to \text{Fil}^0 N/\text{Fil}^{i+1} N \to 0.$$  

By similar argument, we may assume that $M$ has also trivial $B$-coaction. Thus the proposition is reduce to the case where $M = M^0 e^{-\alpha} N = N^0 e^{-\beta} e = Q$. In this case, we can check by direct calculation. □

7. $H^0(B(\epsilon|A|\epsilon))$-Comodules for Connected DGA

A DGA $A^\bullet$ (resp. DG coalgebra $B^\bullet$) is said to be connected if $H^i(A^\bullet) = 0$ for $i < 0$ and $H^0(A^\bullet) = k$ (resp. $H^i(B^\bullet) = 0$ for $i < 0$ and $H^0(B^\bullet) = k$). Let $A^\bullet$ be a connected DGA and $\epsilon: A^\bullet \to k$ be an augmentation of $A^\bullet$. Then by the bar spectral sequence, the DG coalgebra $B(\epsilon|A|\epsilon)$ is connected.

Lemma 7.1. Let $A$ be a connected DGA. Then there exists a subalgebra $\overline{A}$ of $A$ such that the natural inclusion $\overline{A} \to A$ is a quasi-isomorphism and

\begin{align*}
(p) \quad A^{-i} &= 0 \text{ for } i < 0, \quad \text{(positivity condition)} \\
(r) \quad \overline{A}^0 &= k \quad \text{(rigidity condition)}.
\end{align*}

Proof. Let $B^1$ and $Z^1$ be the image and the kernel of $d$ in $A^1$. Then we have $B^1 \subset Z^1 \subset A^1$. We choose a splitting $Z^1 = B^1 \oplus L$. We set $\overline{A}^0 = k$, $\overline{A}^1 = L$ and $\overline{A}^i = A^i$ for $i > 1$. Then $\overline{A}$ is a sub DGA of $A$ which is quasi-isomorphic to $A$ and satisfies the conditions of the lemma. □

Definition 7.2. (1) We define $K^0 C_A$ (resp. $K^{\leq 0}C_A, K^{\geq 0}C_A$) as the full subcategory of $K^0 C_A$ of objects $M = (M^i, d_{i,j})$ where $M^i$ is of the form $M^{-i+1} e^i$ (resp. $\bigoplus_{i+j \leq 0} M^{j+1} e^{-j}$, $\bigoplus_{i+j \geq 0} M^{j+1} e^{-j}$) for $k$-vector spaces $M^{-i+1}$.

(2) We define an additive category $H^0 K^0 C_A$ as follows. The class of objects of $H^0 K^0 C_A$ is the same as in $K^0 C_A$. For $a, b \in K^0 C_A$, we define the morphism

$$\text{Hom}_{H^0 K^0 C_A}(a, b) = H^0(\text{Hom}_{K^0 C_A}(a, b)).$$

Definition 7.3. Let $A^\bullet$ be a differential graded algebra.

(1) A pair $(M, \nabla)$ of a free $A^\bullet$-module $M$ and a $k$-linear map $\nabla: M \to M \otimes_{A^\bullet} A^1$ is called an $A^\bullet$-connection if $\nabla(am) = a\nabla(m) + da \cdot m$. An $A^\bullet$-connection is said to be integrable if $\nabla \circ \nabla = 0$.

(2) An $A^\bullet$-connection $(M, \nabla)$ is said to be trivial if it is generated by horizontal sections.

(3) An $A^\bullet$-connection $(M, \nabla)$ is called nilpotent if there exists a finite filtration by connections $F^p M$ such that $Gr^p_M$ is a trivial connection. The category of integrable nilpotent connections is denoted as $(INC_A)$. Morphism is a $A^0$ homomorphism compatible with connections.

(4) A homomorphism $F \in \text{Hom}_{INC_A}(M, N)$ is homotopically trivial if there is a map $h: M \to N \otimes_{A^\bullet} A^{-1}$ of $A^\bullet$-modules such that $F = \nabla \circ h + h \circ \nabla$. (See the
By localizing homotopically trivial homomorphisms, we have a category \((\text{HINC}_A)\) of homotopy integrable nilpotent \(A^*\)-connections. By the definition, if the condition (p) is satisfied, then \((\text{INC}_A)\) and \((\text{HINC}_A)\) are equivalent.

**Proposition 7.4.** Let \(A^*\) be a connected DGA with an augmentation \(\epsilon\). The category \((\text{HINC}_A)\) of homotopy integrable nilpotent \(A^*\)-connections is equivalent to the category \(\text{H}^0\text{K}^0\text{C}_A\).

**Proof.** We define a functor \(F: \text{K}^0\text{C}_A \to (\text{INC}_A)\) as

\[
F(M) = \left( \bigoplus_i M^{-i,i} \otimes A^0, \nabla \right)
\]

for an object \(M = (M^i, d_{i,j}) \in \text{K}^0\text{C}_A\) with \(M^i = \bigoplus_j M^{-i,j}\). Here \(\nabla\) is defined by \((\sum d_{i,j}) \otimes 1 + 1 \otimes d\). By restricting the above functor, we have a functor \(\text{H}^0\text{K}^0\text{C}_A \to (\text{INC}_A)\). By the definition of \((\text{INC}_A)\), the following natural homomorphisms are isomorphisms:

\[
\text{Z}^0\text{Hom}_{\text{K}^0\text{C}_A}(M, N) \to \text{Hom}_{\text{INC}_A}(F(M), F(N)),
\]

\[
\text{H}^0\text{Hom}_{\text{K}^0\text{C}_A}(M, N) \to \text{Hom}_{\text{INC}_A}(F(M), F(N)).
\]

Thus the functor \(F: \text{K}^0\text{C}_A \to (\text{INC}_A)\) and \(\overline{F}: \text{H}^0\text{K}^0\text{C}_A \to (\text{HINC}_A)\) are fully faithful functors. We show the essentially surjectivity. Let \(\nabla: M \to A^1 \otimes M\) be a nilpotent integrable \(A\)-connection. Let \(F^*M\) be a nilpotent filtration of \(M\) for the connection \(\nabla\). We choose a splitting \(M \simeq \bigoplus \text{Gr}^i_F(M, i = F^i(M)/F^{i+1}(M))\). Let \(M^{-i,i} = \text{Gr}^i_F(M)\) and \(\nabla_{ij}: M^{-i,i} \to A^1 \otimes M^{-j,j}\) be the corresponding component of \(\nabla\). We set

\[
M^i = M^{-i,i} \text{ and } d_{i,j} = \nabla_{ij} \in \text{Hom}_{\text{K}^0\text{C}_A}(M^i, M^j).
\]

Then \((\{M^i\}, d_{ij})\) is an object of \(\text{K}^0\text{C}_A\).

We define the following homomorphism (1) by taking the fiber of \(F(M)\) at the augmentation \(\epsilon\):

\[
\text{Z}^0\text{Hom}_{\text{K}^0\text{C}_A}(M, N) \xrightarrow{(1)} \text{Hom}_k(F(M) \otimes \epsilon k, F(N) \otimes \epsilon k)
\]

\[
= \text{Hom}_k(\bigoplus M^{-i,i}, \bigoplus N^{-j,j}).
\]

**Lemma 7.5.** The image of (1) is identified with \(\text{H}^0\text{Hom}_{\text{K}^0\text{C}_A}(M, N)\).

**Proof.** Since the image of \(\text{B}^0\text{Hom}_{\text{K}^0\text{C}}(M, N)\) under the functor (1) is zero, the functor (1) factors through

\[
\text{H}^0\text{Hom}_{\text{K}^0\text{C}_A}(M, N) \xrightarrow{(2)} \text{Hom}_k\left( \bigoplus M^{-i,i}, \bigoplus N^{-j,j} \right).
\]
We show the injectivity of (2) by the induction of the nilpotent length of the corresponding connections.

Let $M$ and $N$ be $k$-vector spaces. Since $A$ is connected, the natural homomorphism

$$H^0\Hom_{\mathcal{K}^0_{\mathcal{C}_A}}(M, N) \to \Hom_{\mathcal{K}^0_{\mathcal{C}_A}}\left(\bigoplus_i M^{-i,i}, \bigoplus_j N^{-j,j}\right)$$

is an isomorphism. Let $N^{> i}$ and $N^{< i}$ be the stupid filtration of $N$ as DG complex in $\mathcal{C}_A$ and the quotient of $N$ by $N^{> i}$. Since $A$ is connected, we have the following left exact sequences

$$0 \to H^0\Hom_{\mathcal{K}^0_{\mathcal{C}_A}}(M, N^{> i}) \to H^0\Hom_{\mathcal{K}^0_{\mathcal{C}_A}}(M, N) \to H^0\Hom_{\mathcal{K}^0_{\mathcal{C}_A}}(M, N^{< i}),$$

$$0 \to H^0\Hom_{\mathcal{K}^0_{\mathcal{C}_A}}(M^{< i}, N) \to H^0\Hom_{\mathcal{K}^0_{\mathcal{C}_A}}(M, N) \to H^0\Hom_{\mathcal{K}^0_{\mathcal{C}_A}}(M^{> i}, N).$$

By induction on the nilpotent length of $N$ and $M$, we have the lemma by 5-lemma.

**Theorem 7.6.** Let $A$ be a connected DGA. The category $H^0\mathcal{K}^0_{\mathcal{C}_A}$ is equivalent to the category of nilpotent $H^0(\mathcal{B}(\epsilon|A|\epsilon))$-comodules.

**Proof.** By Lemma 7.1, we choose a quasi-isomorphic sub DGA $A'$ such that $A'^{-i} = 0$ for $i < 0$ and $A'^0 \simeq k$. In this case, the condition (p) is satisfied and the categories $(\text{INC}_{A'})$ and $(\text{HINC}_{A'})$ are equivalent. Thus we have the following commutative diagram of categories:

$$
\begin{array}{ccc}
(\text{INC}_{A'}) & \xrightarrow{\alpha} & (\text{INC}_A) \\
\downarrow & & \downarrow \\
(\text{HINC}_{A'}) & \xrightarrow{\alpha'} & (\text{HINC}_A) \\
\downarrow F_{A'} & & \downarrow F_A \\
(H^0(\mathcal{B}_{A'})\text{-com}) & \xrightarrow{\alpha''} & (H^0(\mathcal{B}_A)\text{-com}).
\end{array}
$$

We know that $\alpha'$ is fully faithful and $\alpha''$ is an equivalence. Thus it is enough to show that $F_A$ is fully faithful and $F_{A'}$ is an equivalence.

We show the full faithfulness of $F_A$. Let $\mathcal{N}$, $\mathcal{M}$ be $\mathcal{B}$-comodules corresponding to the nilpotent integrable $A$ connections $M$ and $N$. We set $H^0 = H^0(\mathcal{B}(\epsilon|A|\epsilon))$. By the induction of the nilpotent length, we show that the natural map

$$H^i\Hom_{\mathcal{B}\text{-com}}(\mathcal{M}, \mathcal{N}) \to H^i\Hom_{H^0(\mathcal{B})\text{-com}}(H^0(\mathcal{M}), H^0(\mathcal{N}))$$

is an isomorphism for $i = 0$ and injective for $i = 1$ (7.1) using 5-lemma. To prove this, it is enough to prove in the case where $\mathcal{M} = \mathcal{N}$ are trivial connections $k$. We consider two complexes

$$\Hom^\bullet_{\mathcal{B}\text{-com}}(k, k): k \to B \to B \otimes B \to \ldots,$$

$$\Hom^\bullet_{H^0(\mathcal{B})\text{-com}}(k, k): k \to H^0(B) \to H^0(B) \otimes H^0(B) \to \ldots.$$
and the spectral sequence for stupid filtration for the first double complex:

\[ E_1^{p,q} = (H(B)^{\otimes p})^{p+q} \Rightarrow H^{p+q}(\text{Hom}_{\mathcal{B}^{\text{com}}}(k, k)) \]

Since \( H^i(B) = 0 \) for \( i < 0 \) by the connectedness condition, we have

\[ E_1^{0,0} = E_2^{0,0} = k, \quad E_2^{1,0} = H^1(\text{Hom}_{\mathcal{B}^{\text{com}}}(k, k)), \]

and

\[ E_2^{i,0} \simeq H^i(\text{Hom}_{\mathcal{B}^{\text{com}}}(k, k)). \]

Therefore, we have the statement (7.1) for \( \mathcal{M} = \mathcal{N} = k \). Therefore the functor \( F_A \) is a fully faithful functor.

We introduce a universal integrable nilpotent connection on \( H^0(B(\epsilon | A' | \epsilon)) \) to show the essential surjectivity of \( (\text{HINC}_A) \to (H^0(\mathcal{B}_A))^{\text{com}}) \). We use the isomorphism \( H^0(B_{\text{red}}(A', \epsilon)) \simeq H^0(B(\epsilon | A' | \epsilon)) \) between simplicial bar complex and Chen’s reduced bar complex. By the conditions (p) and (r), \( H^0(B_{\text{red}}(A', \epsilon)) \) is identified with a subspace of \( B_{\text{red}}(A', \epsilon)^0 \). The coproduct on \( B_{\text{red}}(A', \epsilon) \) induces a connection

\[ H^0(B_{\text{red}}(A', \epsilon)) \to (A')^1 \otimes H^0(B_{\text{red}}(A', \epsilon)) \]

on \( H^0(B_{\text{red}}(A', \epsilon)) \). For an \( H^0 \)-comodule \( (M, \Delta_M) \), We define \( \mathcal{M} \) as the kernel of the following map:

\[ H^0 \otimes M \xrightarrow{\Delta_H \otimes 1_M - 1_H \otimes \Delta_M} H^0 \otimes H^0 \otimes M. \]

Then \( \mathcal{M} \) has a structure of \( A' \)-connection and \( F_{A'}(\mathcal{M}) = M \). This proves the essentially surjectivity.

Since the category of nilpotent \( H^0 \)-comodules is stable under taking kernels and cokernels, we have the following corollary.

**Corollary 7.7.** (1) If \( A^* \) is a connected DGA, then \( \mathcal{A} = H^0K^0\mathcal{C}_A \) is an abelian category.

(2) Let \( A \) and \( A' \) be connected DGA’s and \( \varphi: A \to A' \) a quasi-isomorphism of DGA. Then the associated categories \( (\text{HINC}_A) \to (\text{HINC}_{A'}) \) are equivalent.

8. Patching of DG Categories

In this section, we consider patching of DG categories and their bar complexes. A typical example for patching appears as van Kampen’s theorem. Let \( X \) be a manifold which is covered by two open sets \( X_1 \) and \( X_2 \). Suppose that \( X_{12} = X_1 \cap X_2 \) is connected. Then the fundamental group \( \pi_1(X) \) is isomorphic to the amalgam product \( \pi_1(X_1) \ast \pi_1(X_2) \). We can interpret this isomorphism as an equivalence of two categories. The first category \( \text{Loc}(X) \) is a category of local systems on \( X \) and the second is a category \( \text{Loc}(X_1) \times_{\text{Loc}(X_{12})} \text{Loc}(X_2) \) of triples \((\mathcal{L}_1, \mathcal{L}_2, \varphi)\), where \( \mathcal{L}_1, \mathcal{L}_2 \) are local systems on \( X_1 \) and \( X_2 \) and \( \varphi \) is an isomorphism of local systems \( \varphi: \mathcal{L}_1 \mid_{X_{12}} \to \mathcal{L}_2 \mid_{X_{12}} \).

We must be careful in the nilpotent version. Let \( \text{Loc}^{\text{nil}}(X) \) be the category of nilpotent local systems. Then the natural functor

\[ \text{Loc}(X)^{\text{nil}} \to \text{Loc}(X_1)^{\text{nil}} \times_{\text{Loc}(X_{12})^{\text{nil}}} \text{Loc}(X_2)^{\text{nil}} \]
is not an equivalence in general. For example, if $X_{12}$ is contractible, then there might be no filtration on $L_1|X_{12} = L_2|X_{12}$ which induces a nilpotent filtration $F_1$ on the local system $L_1$ on $X_1$ and that on the local system $L_2$ on $X_2$.

**Definition 8.1.** (1) Let $\mathcal{A}$ be an abelian category. $\mathcal{A}$ is said to be nilpotent, if

(a) there is an object $1_\mathcal{A}$, and

(b) for any object $M$ in $\mathcal{A}$, there is a filtration $N^\bullet$ on $M$ such that the associated graded quotient $Gr^p_N(M)$ is a direct sum of copies of $1_\mathcal{A}$.

This filtration is called a nilpotent filtration of $M$.

(2) Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be nilpotent abelian categories and $\psi: \mathcal{A}_1 \to \mathcal{A}_2$ be an exact additive functor. The functor $\psi$ is said to be nilpotent if for any nilpotent filtration $N^\bullet$ on an object $M$, $\psi(N^\bullet)$ is a nilpotent filtration on $\psi(M)$.

(3) Let $\mathcal{A}_2, \mathcal{A}_2, \mathcal{A}_{12}$ be a nilpotent abelian category and $F_1: \mathcal{A}_1 \to \mathcal{A}_{12}$ and $F_2: \mathcal{A}_2 \to \mathcal{A}_{12}$ be nilpotent functors. We define a fiber product $(\mathcal{A}_1 \times_{\mathcal{A}_{12}} \mathcal{A}_2)^{nil}$ as the full sub category of $(\mathcal{A}_1 \times_{\mathcal{A}_{12}} \mathcal{A}_2)$ consisting of objects $(M_1, M_2, \varphi)$ such that there exist nilpotent filtrations $N_1$ and $N_2$ on $M_1$ and $M_2$ and an isomorphism $\varphi: F_2(M_2) \to F_1(M_1)$ such that $\varphi(F_2(N_2)) = F_1(N_1)$.

(4) Let $A_1, A_2, A_{12}$ be DGCA’s and $u_1: A_1 \to A_{12}$ and $u_2: A_2 \to A_{12}$ be homomorphisms of DGCA’s. We define a fiber product $\tilde{A} = A_1 \times_{A_{12}} A_2$ as $\tilde{A}^p = A_1^p \oplus A_2^p \oplus A_{12}^{p-1}$. We introduce a product of elements $a = (a_1, a_2, a_{12})$ and $b = (b_1, b_2, b_{12})$ of $\tilde{A}^p$ and $\tilde{q}$ as

$$a \cdot b = (a_1 \cdot b_1, a_2 \cdot b_2, u_1(a_1) \cdot b_{12} + (-1)^{deg b_2} a_{12} \cdot u_2(b_2)).$$

By setting $s(A_1 \times_{A_{12}} A_2) = A_1 \oplus A_2 \oplus A_{12}e^{-1}$, this rule can be written as the following simpler rule

$$(a_1 + a_2 + a_{12}e^{-1})(b_1 + b_2 + b_{12}e^{-1}) = a_1b_1 + a_2b_2 + u_1(a_1)b_{12}e^{-1} + a_{12}e^{-1}u_2(b_2).$$

Let $A_1, A_2$ and $A_{12}$ be connected DGCA’s, $u_1: A_1 \to A_{12}$ and $u_2: A_2 \to A_{12}$ homomorphisms of DGCA’s and $\tilde{A} = A_1 \times_{A_{12}} A_2$. Assume that there exists an augmentation $\epsilon: A_{12} \to \mathbb{k}$. Since the natural projection $p_i: \tilde{A} \to A_i$ ($i = 1, 2$) is a homomorphism of DGCA’s, we have functors

$$p_i: H^0K^0C_{\tilde{A}} \to H^0K^0C_{A_i},$$

$$p_{12}: H^0K^0C_{A_1} \to H^0K^0C_{A_{12}}$$

for $i = 1, 2$. Using the functorial property in Remark 2.5. Let $M = \bigoplus_i M^i e^i \in K^0C_{\tilde{A}}$ and

$$d_{ji} = d_{ji}^{(1)} \oplus d_{ji}^{(2)} \oplus d_{ji}^{(12)}: M^i \to \tilde{A}^1 \otimes M^j = (A_1^1 \oplus A_2^1 \oplus A_{12}^0) \otimes M^j$$

be the DG complex structure on $M$. Then by the condition for DG complex structure, we have

$$\partial d_{ik}^{(p)} + \sum_j d_{ij}^{(p)} \circ d_{jk}^{(p)} = 0 \quad (p = 1, 2),$$

$$(\partial d_{ik}^{(12)} + d_{ik}^{(1)} - d_{ik}^{(2)} + d_{ij}^{(1)} - d_{ij}^{(12)} - d_{ij}^{(2)})d_{jk}^{(2)} = 0.$$
Therefore the map
\[ \varphi: (M, u_1(d_{ij}^{(2)})) \to (M, u_2(d_{ij}^{(1)})) \]
defined by \( \varphi(\sum_{i} m_i) = \sum_{i} m_i + d_{ji}^{(12)}(m_i) \) is an isomorphism of DG complexes
\[ \varphi: p_{12}^2 p_{2}(M) \to p_{12}^1 p_{1}(M). \]
By associating \((p_1(M), p_2(M), \varphi)\) to \(M\), we have a functor
\[ H^0K^0\mathcal{C}_A \to (H^0K^0\mathcal{C}_A \times H^0K^0\mathcal{C}_{A_2})^{\text{nil}}. \] (8.1)

**Proposition 8.2.** Under the above situations, the natural functor (8.1) is an equivalence of categories.

**Proof.** The full faithfulness follows from the direct calculation. We show the essentially surjectivity. Since \(A_1\) is connected, \(H^0K^0\mathcal{C}_{A_1}\) is equivalent to the category \((HINC_{A_1})\) for \(* = 1, 2, 12\). Let \(M = (M_1, M_2, \varphi)\) be an object of \((HINC_{A_1} \times HINC_{A_{12}})

\(HINC_{A_2})^{\text{nil}}\), where
\[ M_p = (M_p, \nabla_p): M_p \to A^1_p \otimes A^0_p M_p \in HINC_{A_p} \quad (p = 1, 2) \]
is a nilpotent integrable connection. By the definition of nilpotent fiber product, there exist filtrations \(F_1^*\) and \(F_2^*\) on \(M_1\) and \(M_2\) and an \(A_{12}^0\)-isomorphism \(\varphi: u_2(M_2) \to u_1(M_1)\) such that (1) \(\varphi\) is compatible with the connections on \(u_1(M_1)\) and \(u_2(M_2)\), and (2) the filtrations \(u_1(F_1^*)\) and \(u_2(F_2^*)\) are isomorphic via the isomorphism \(\varphi\). By the isomorphism, we have an identification \(k \otimes_{e, A_{12}^0} M_1 \simeq k \otimes_{e, A_{12}^0} M_2 = M^{(0)}\). The \(k\)-vector space \(M^{(0)}\) has a filtration \(F^*\) induced by \(u_1(F_1^*) = u_2(F_2^*)\). Using identification \(A_0^p \otimes_k M^{(0)} \simeq M_p\) we have a map \(\nabla_p: M^{(0)} \to A^1_p \otimes_k M^{(0)}\) for \(p = 1, 2\). On the other hand, the morphism \(\varphi\) defines a map \(\overline{\varphi}: M^{(0)} \subset u_2(M_2) \to u_1(M_1) \simeq M^{(0)} \otimes A_{12}^0\). Thus we have a map
\[ M^{(0)} \to (A^1_1 \oplus A^1_2 \oplus A^0_{12}) \otimes M^{(0)}. \]
By simple calculation, this map gives rise to an integrable nilpotent filtration. This nilpotent filtration gives rise to an object \((M_1, M_2, \varphi)\) of \((HINC_{A_1} \times HINC_{A_{12}})

\(HINC_{A_2})^{\text{nil}}\). \(\square\)

Let \(\epsilon: A_{12} \to k\) be an augmentation of \(A_{12}\). By composing the morphism \(\varphi_1: A_i \to A_{12}\), we have a augmentation \(\epsilon_i = \epsilon \circ \varphi_1: A_i \to k\) for \(i = 1, 2\). By composing the natural projection \(\tilde{A} \to A_i\), we have two augmentations \(\epsilon_i: \tilde{A} \to k\) for \(i = 1, 2\). We introduce a comparison copath morphism
\[ p(\epsilon): B(e_1 \mid \tilde{A} \mid e_2) \to k \]
connecting \(e_1\) and \(e_2\) associated to \(\epsilon\). Since
\[ B(e_1 \mid \tilde{A} \mid e_2)^0 = \bigoplus_{|\alpha|=k} B^{-k}_\alpha \]
and
\[ \bigoplus_{|\alpha| = 0} B^0_\alpha = \bigoplus_{\alpha_0} \mathbb{k} \otimes \mathbb{k}, \]
\[ \bigoplus_{|\alpha| = 1} B^1_\alpha = \bigoplus_{\alpha_0 < \alpha_1} \mathbb{k} \otimes \hat{A}^1 \otimes \mathbb{k} = \bigoplus_{\alpha_0 < \alpha_1} \mathbb{k} \otimes (A^1_0 \oplus A^1_1 \oplus A^1_2) \otimes \mathbb{k}. \]

We define a linear map \( p(\epsilon) : B(e_1 \mid A \mid e_2)^0 \to \mathbb{k} \) by the summation of (1) projection to the factor \( \mathbb{k}^{\alpha_0} \otimes \mathbb{k} \) over \( \alpha_0 \), and (2) the composite of \( \epsilon \) and the projection to \( \mathbb{k} \otimes \hat{A}^1 \otimes \mathbb{k} \) over \( \alpha_0 < \alpha_1 \). We can show that the composite
\[ B(e_1 \mid A \mid e_2)^{-1} \to B(e_1 \mid A \mid e_2)^0 \to \mathbb{k} \]
is zero. For example, \( 1 \otimes x \otimes 1 \in \mathbb{k} \otimes \hat{A}^1 \otimes \mathbb{k} \) goes to
\[ \epsilon_1(x) \otimes 1 + 1 \otimes dx \otimes 1 + 1 \otimes \varphi_1(x) \otimes 1 \in B(e_1 \mid A \mid e_2)^0, \]
which is an element of the kernel of \( p(\epsilon) \).

**Definition 8.3.** The map \( p(\epsilon) \) is called the comparison copath associated to \( \epsilon \).

### 9. The Graded Case

We consider a graded version of DGA. Let \( A = \bigoplus_{\alpha \geq 0} A_\alpha = \bigoplus_{\alpha \geq 0} A^\bullet_\alpha \) be a graded DGA over a field \( \mathbb{k} \). We assume that the image of \( A_0 \otimes A_0 \) under the multiplication map is contained in \( A_{p+q} \). We introduce a DG category \( \mathcal{C}^\text{gr}_A \) as follows. Objects are finite direct sum of the form \( V^\bullet(i), \) where \( V^\bullet \) is a complex of \( \mathbb{k} \)-vector space. For objects \( V^\bullet(i) \) and \( W^\bullet(j) \), we define
\[ \text{Hom}^\bullet_{\mathcal{C}^\text{gr}_A}(V^\bullet(i), W^\bullet(j)) = \begin{cases} \text{Hom}_{\mathcal{K}\text{Vec}_\mathbb{k}}(V^\bullet, A_{j-i} \otimes W^\bullet) & \text{if } j \geq i, \\ 0 & \text{if } j < i. \end{cases} \]

Then \( \mathcal{C}^\text{gr}_A \) becomes a DG category. The DG category of DG complexes in \( \mathcal{C}^\text{gr}_A \) is denoted as \( K\mathcal{C}^\text{gr}_A \). The category \( \mathcal{C}^\text{gr}_A \) has a tensor structure by \( V(p) \otimes W(q) = (V \otimes W)(p+q) \). The full subcategory of bounded DG complexes is denoted as \( K^0\mathcal{C}^\text{gr}_A \). By considering \( \mathbb{k} \) as a graded DGA in a trivial way, we have a DG-category \( \mathcal{C}^\text{gr}_\mathbb{k} \). For example, for \( \mathbb{k} \)-vector spaces \( V, W \), we have
\[ \text{Hom}_{\mathcal{C}^\text{gr}_\mathbb{k}}(V(p), W(q)) = \begin{cases} \text{Hom}(V, W) & \text{if } p = q, \\ 0 & \text{otherwise}. \end{cases} \]

The object \( V(p) \) is called the \( p \)-Tate twist of \( V \) and the Tate weight of \( V(p) \) is defined to be \( p \). The category \( \mathcal{C}^\text{gr}_\mathbb{k} \) is the category of formal Tate twist of \( \mathbb{k} \)-vector spaces.

**Definition 9.1.** Let \( M = \bigoplus M^\bullet_i \) be a graded complexes of \( \mathbb{k} \)-vector spaces. We introduce a homogenization \( M^h \) of \( M \) by \( \bigoplus (M_i \otimes \mathbb{k}(-i)) \) as an object in \( \mathcal{C}^\text{gr}_\mathbb{k} \). By this correspondence, the category of graded complex is equivalent to the category \( \mathcal{C}^\text{gr}_\mathbb{k} \). Under this identification, the object \( \mathbb{k}(p) \otimes \mathbb{k}(q) \) is identified with \( \mathbb{k}(p+q) \).
Using the above notations, for \( V^\bullet(p), W^\bullet(q) \in C^g_k \), we have
\[
\text{Hom}_c^g(V^\bullet(p), W^\bullet(q)) = \text{Hom}_c^g(V^\bullet(p), A^h \otimes W^\bullet(q)).
\]

Let \( \epsilon_0 \colon A_0 \to k \) be an augmentation of \( A_0 \) and \( \epsilon \colon A \to k \) a composite of \( \epsilon_0 \) and the natural projection. We define the homogeneous bar complex \( B^h(\epsilon \mid A \mid \epsilon) \) in \( KC_k \) as follows. Let \( \alpha = (\alpha_0 < \cdots < \alpha_n) \) be a sequence of integers. We define \( B^h_\alpha \in C^g_k \) by
\[
A^h \otimes A^h \otimes \cdots \otimes A^h \otimes k,
\]
and \( B^h = B^h(\epsilon \mid A \mid \epsilon) \in KC^g_k \) by the complex
\[
\cdots \to \bigoplus_{|\alpha|=1} B^h_\alpha \to \bigoplus_{|\alpha|=0} B^h_\alpha \to 0.
\]

The associated simple object in \( C^g_k \) is denoted as \( B^h = B^h(\epsilon \mid A \mid \epsilon) \). Then the Tate weight \( w \) part of \( B^h \) is equal to the sum of
\[
(k \otimes A_{p_1}^h \otimes \cdots \otimes A_{p_n}^h \otimes k) \otimes \mathbb{Q}(-p_1 - \cdots - p_n - 1),
\]
where \( w = -(p_1 + \cdots + p_n) \). The weight \( w \) part of \( B^h \) is denoted as \( B^h(w) \). We define the homogenized reduced bar complex \( B^h_{\text{red}} = B^h_{\text{red}}(A, \epsilon) \) in the same way.

Thus \( B^h, B^h_{\text{red}} \) are graded DG coalgebras in \( C_k \). We introduce DG category structure on bounded homogeneous \( B^h \) comodules.

**Definition 9.2.** (1) Let \( V^\bullet = \bigoplus V^\bullet(i) \) be an object in \( C^g_k \). A homomorphism \( \Delta : V^\bullet \to B^h \otimes V^\bullet \) in \( C^g_k \) (i.e., homomorphism of graded complex) is called a coproduct if it satisfies the coassociativity and counitality defined in Definition 6.1. An object \( V^\bullet \) in \( C^g_k \) equipped with a coproduct is called a homogeneous \( B^h \)-comodule. We define \( B^h_{\text{red}} \) comodules in the same way.

(2) Let \( V^\bullet \) and \( W^\bullet \) be homogeneous \( B^h \) comodules. We will say that a morphism \( f \) in \( \text{Hom}_c^g(V^\bullet, W^\bullet) \) is a \( B^h \) homomorphism if the diagram
\[
\begin{array}{ccc}
V^\bullet & \xrightarrow{f} & W^\bullet \\
\downarrow \Delta_V & & \downarrow \Delta_W \\
B^h \otimes V^\bullet & \xrightarrow{\Delta_{\text{red}} \otimes f} & B^h \otimes W^\bullet
\end{array}
\]
is commutative.

Let \( V = V^\bullet, W = W^\bullet \) be homogeneous \( B^h \) comodules. We define the complex of homomorphisms \( \text{Hom}_c^g(V, W) \) from \( V \) to \( W \) by the associated simple complex of
\[
0 \to \text{Hom}_c^g(V, W) \xrightarrow{d_1} \text{Hom}_c^g(V, B^h \otimes W) \xrightarrow{d_2} \text{Hom}_c^g(V, B^h \otimes^2 W) \xrightarrow{d_3} \cdots,
\]
where \( d_H \) is defined similarly to (6.2). We define the DG category \( (B^h\text{-com})^b \) as the category of bounded homogeneous \( B^h \)-comodules with \( \text{Hom}_{B^h\text{-com}}(\bullet, \bullet) \) as the complex of morphisms. The following proposition is a graded version of Theorem 6.4, and Theorem 7.6.
Proposition 9.3. (1) The DG category $K^b C^\text{gr}_A$ is homotopy equivalent to the category of bounded homogeneous $B^h(\epsilon | A| \epsilon)$ comodules.

(2) Assume that the graded DGA $A$ is connected. Then $H^0 K^0 C^\text{gr}_A$ is equivalent to the category of homogeneous $H^0(B^h(\epsilon | A| \epsilon))$ comodules. As a consequence, $H^0 K^0 C^\text{gr}_A$ is an abelian category.

Proof. (1) We set $B^h = B^h(\epsilon | A| \epsilon)$. We give a one to one correspondence

$$\text{ob}(B^h\text{-com})^b \rightarrow \text{ob}(K^b C^\text{gr}_k)$$

of objects. Let $V = \bigoplus_i V^i(i)$ be an object in $C^\text{gr}_k$ and $V \rightarrow B^h \otimes V$ be the comultiplication of $V$. We use the direct sum decomposition

$$B^h = \left( \bigoplus_{\alpha_0} k \otimes k \right) \oplus \left( \bigoplus_{\alpha_0 < \alpha_1} k \otimes h_{\alpha_1} \otimes h_{\alpha_0} \right) \oplus \bigoplus_{|\alpha| = 1 > 0} B^h_{\alpha_0} e_i.$$

Let $\pi_{\alpha_0}$ be the projection $B^h \rightarrow k \otimes k$. Let $p_{\alpha_0}$ be the composite of the map

$$V \xrightarrow{\Delta_v} B \otimes V \xrightarrow{\pi_{\alpha_0}} k \otimes V$$

and $V^{\alpha_0}$ be $\text{Im}(p_{\alpha_0}) e^{\alpha_0}$. Since the map $p_{\alpha_0}$ is homogeneous, $V^{\alpha_0}$ is an object in $C^\text{gr}_k$. By the assumption of boundedness, $V^{\alpha_0} = 0$ except for finite numbers of $\alpha_0$.

Let $\pi_{\alpha_0, \alpha_1}$ be the projection $B^h \rightarrow k \otimes h_{\alpha_1} \otimes h_{\alpha_0}$ and consider the composite $D_{\alpha_0, \alpha_1}$, by the composite

$$V \xrightarrow{\Delta_v} B \otimes V \xrightarrow{\pi_{\alpha_0, \alpha_1}} k \otimes h_{\alpha_1} \otimes h_{\alpha_0} \otimes V e.$$  

By the associativity condition, the morphism $D_{\alpha_0, \alpha_1}$ induces a morphism

$$\text{Hom}_C^1(V^{\alpha_0} e^{\alpha_0}, A^h \otimes V^{\alpha_1} e^{-\alpha_1}) \rightarrow \text{Hom}_A^1(V^{\alpha_0} e^{\alpha_0}, V^{\alpha_1} e^{-\alpha_1})$$

which is also denoted as $D_{\alpha_0, \alpha_1}$. This defines a one to one correspondence

$$\text{ob}(B^h\text{-com})^b \rightarrow \text{ob}(K^b C^\text{gr}_k).$$

For the construction and the proof of the inverse correspondence and the homotopy equivalence is similar to those in Section 6, so we omit the detailed proof.

(2) Using the following lemma, the proof of (2) is similar. \hfill \square

Lemma 9.4. Let $A$ be a connected graded DGA. Then there exists a graded subalgebra $\overline{A}$ of $A$ such that (1) the natural inclusion $\overline{A} \rightarrow A$ is a quasi-isomorphism, and (2) the conditions (p) and (r) in Lemma 7.1 are satisfied.

10. Deligne Complexes

10.1. Definition of Deligne algebras. Here we give an application of DG category to Hodge theory. Let $X$ be a smooth irreducible variety over $\mathbb{C}$ and $\overline{X}$ be a smooth compactification of $X$ such that $D = \overline{X} - X$ is a simple normal crossing divisor. Let $\mathcal{U} = \{ U_i \}_{i \in I}$ be an affine covering of $\overline{X}$ indexed by a totally ordered set $I$, and $\mathcal{U}_{an} = \{ U_{an,j} \}_{j \in J}$ be a topological simple covering of $X$ indexed by a totally ordered set $J$, which is a refinement of $\mathcal{U} \cap X$. Assume that the map $J \rightarrow I$ defining the refinement preserves the ordering of $I$ and $J$. 

Let $\Omega^\bullet_{\text{log}}(\log(D))$ be the sheaf of algebraic logarithmic de Rham complex of $X$ along the boundary divisor $D$. Let $F^\bullet$ be the Hodge filtration of $\Omega^\bullet_{\text{log}}(\log(D))$:

$$F^i : 0 \to \cdots \to 0 \to \Omega^i_{\text{log}}(\log(D)) \to \Omega^{i+1}_{\text{log}}(\log(D)) \to \cdots.$$ 

Then $F^\bullet$ is compatible with the product structure of $\Omega^\bullet_{\text{log}}(\log(D))$. The Čech complex $\check{C}(U, \Omega^\bullet_{\text{log}}(\log(D)))$ becomes an associative DGA by standard Alexander–Whitney associative product using the total order of $I$. We define

$$A_{F_{\text{DR}},i} = \check{C}(U, F^i \Omega^\bullet_{\text{log}}(\log(D))).$$

Then the product induces a morphism of complexes

$$A_{F_{\text{DR}},i} \otimes A_{F_{\text{DR}},j} \to A_{F_{\text{DR}},i+j}$$

and by this multiplication $A_{F_{\text{DR}}} = \bigoplus_{i \geq 0} A_{F_{\text{DR}},i}$ becomes a graded DGA which is called the algebraic de Rham DGA. We define the homogenized algebraic de Rham DGA $A_{F_{\text{DR}}}^h \in K^\text{gr} C_k$ as $\bigoplus_{i \geq 0} A_{F_{\text{DR}},i}(-i)$.

Let $\Omega^\bullet_{\text{an}}$ be the analytic de Rham complex and $\check{C}(U_{\text{an}}, \Omega^\bullet_{\text{an}})$ the topological Čech complex of $\Omega^\bullet_{\text{an}}$. Since the map $J \to I$ preserve the orderings, we have a natural quasi-isomorphism of DGA’s:

$$\check{C}(U, \Omega^\bullet_{\text{log}}(\log(D))) \to \check{C}(U_{\text{an}}, \Omega^\bullet_{\text{an}}).$$

We set $A_{\text{DR},an,i} = \check{C}(U_{\text{an}}, \Omega^\bullet_{\text{an}})$ and $A_{\text{DR},an} = \bigoplus_{i \geq 0} A_{\text{DR},an,i}$. Then $A_{\text{DR},an}$ becomes a graded DGA by Alexander–Whitney associative product, which is called the analytic de Rham DGA. We also define $A_{B,i} = \check{C}(U_{\text{an}}, (2\pi i)^i \mathbb{Q})$ and set $A_B = \bigoplus_{i \geq 0} A_{B,i}$, which is called the Betti DGA. It is a sub graded DGA of $A_{\text{DR},an}$. We define the homogenized analytic de Rham DGA $A_{\text{DR},an}^h$ and the Betti DGA $A_B^h$ as $A_{\text{DR},an}^h = \bigoplus_{i \geq 0} A_{\text{DR},an,i}(-i)$ and $A_B^h = \bigoplus_{i \geq 0} A_{B,i}(-i)$.

**Definition 10.1** (Deligne algebra). We define the Deligne algebra $A_{\text{Del}} = A_{\text{Del}}(X)$ of $X$ as the fiber product $A_{F_{\text{DR}}} \times A_{\text{DR},an} A_B$. It is graded by $A_{\text{Del},i} = A_{F_{\text{DR}},i} \times A_{\text{DR},an,i} A_{B,i}$. (See Definition 8.1 for the definition of fiber product of DGA’s.)

**Example 10.2.** In the case of $X = \text{Spec} \mathbb{C}$, the complexes $A_{\text{Del}}$ and $A_{\text{Del}}^h$ are equal to

$$A_{\text{Del}}^0 = \mathbb{C} \oplus \bigoplus_{i \geq 0} (2\pi i)^i \mathbb{Q}, \quad A_{\text{Del}}^1 = \bigoplus_{i \geq 0} \mathbb{C},$$

$$A_{\text{Del}}^{0,0} = \mathbb{C} \oplus \bigoplus_{i \geq 0} (2\pi i)^i \mathbb{Q}(-i), \quad A_{\text{Del}}^{0,1} = \bigoplus_{i \geq 0} \mathbb{C}(-i).$$

The differentials are the natural maps. In this case, the bar spectral sequence degenerates at $E_1$ and $\text{Gr}(H^0(B(\text{Del} | A_{\text{Del}} | \text{Del}))$ is isomorphic to the tensor algebra generated by $\bigoplus_{i \geq 0} \mathbb{C}/(2\pi i)^i \mathbb{Q}$ over $\mathbb{Q}$.

**Remark 10.3.** (1) We can define $A_{\text{DR}}$ for a smooth variety over an arbitrary subfield $K$ of $\mathbb{C}$.

(2) Actually, $A_{\text{Del}}(X)$ depends on the choice of the compactification $\overline{X}$, and coverings $U$ and $U_{\text{an}}$. If we take another choice of a compactification $\overline{X}'$ and coverings...
\( \mathcal{U}' \) and \( \mathcal{U}_{an}' \), we can always chose a compactification \( \overline{X}' \) and coverings \( \overline{X}'' \) and \( \overline{X}_{an}'' \) and a morphism \( \pi : \overline{X}' \to X \) and \( \pi' : \overline{X}'' \to \overline{X} \) such that

(a) \( \pi' \) and \( \pi' \) are isomorphisms on \( X \), and
(b) \( \mathcal{U}'' \) and \( \mathcal{U}_{an}'' \) are refinement of \( \pi^{-1}(\mathcal{U}) \), \( \pi'(-1)(\mathcal{U}') \) and \( \pi^{-1}\mathcal{U}_{an}', \pi'(-1)\mathcal{U}_{an}' \).

Therefore the Deligne algebra of \( X \) for \( \overline{X} \), \( \mathcal{U} \), \( \mathcal{U}_{an} \) and that for \( \overline{X} \), \( \mathcal{U}' \), \( \mathcal{U}'_{an} \) are quasi-isomorphic to each other.

(3) We can replace the role \( \mathbb{Q} \) in \( \mathcal{A}_R \) by an arbitrary subfield \( F \) of \( \mathbb{R} \). The corresponding Deligne algebra is denoted as \( \mathcal{A}_{Del,F}(X) \).

Let \( p_{dR} \) (resp. \( p_B \)) be a \( \mathbb{C} \)-valued point of \( X \) (resp. a point in \( X(\mathbb{C})^{an} \)). Then we have an augmentation \( \epsilon_{dR}(p) \) (resp. \( \epsilon_B(p) \)) of \( \mathcal{A}_{dR} \) (resp. \( \mathcal{A}_B \)). The following proposition is a consequence of Proposition 9.3.

**Proposition 10.4.** The category \( H^0K^0\mathcal{C}^R_{\mathcal{A}_{Del}} \) is equivalent to the category of homogeneous \( H^0(B^h(\epsilon_B|\mathcal{A}_{Del}(X)|\epsilon)) \)-comodules.

### 10.2. Nilpotent variations of mixed Tate Hodge structures

In this subsection, we prove that the category of nilpotent variations of mixed Tate Hodge structures on a smooth irreducible algebraic variety is equivalent to that of comodules over \( H^0(B^h(\epsilon|\mathcal{A}_{Del}(X)|\epsilon)) \).

**Definition 10.5 (VMTHS).** Let \( X \) be a smooth irreducible algebraic variety over \( \mathbb{C} \). A triple \( (\mathcal{F}, \nabla, \mathcal{V}, \text{comp}) \) of the following data is called a variation of mixed Tate Hodge structure on \( X \).

1. A 4-tuple \( \mathcal{F} = (\mathcal{F}, \nabla, F^*, W_\bullet) \) is a locally free sheaf \( \mathcal{F} \) on \( X \) with a logarithmic connection
   \[
   \nabla: \mathcal{F} \to \Omega^1_X(\log(D)) \otimes \mathcal{F}
   \]
   and decreasing and increasing filtrations \( F^* \) and \( W_\bullet \) on \( \mathcal{F} \) with the following properties. (We assume that the filtration \( W \) is indexed by even integers.)
   
   (a) \( \nabla(F^i F) \subset \Omega^1_X(\log(D)) \otimes F^{i-1} \mathcal{F} \).
   
   (b) \( \nabla(W_{-2i} \mathcal{F}) \subset \Omega^1_X(\log(D)) \otimes W_{-2i} F \).

2. A pair \( \mathcal{V} = (\mathcal{V}, W_\bullet) \), where \( \mathcal{V} \) is a local system on \( X(\mathbb{C})^{an} \) and \( W_\bullet \) is a filtration on \( \mathcal{V} \).

3. An isomorphism of local systems
   \[
   \mathcal{V} \otimes \mathbb{C} \cong (\mathcal{F} \otimes \mathcal{O}_{X(\mathbb{C})^{an}})^{\nabla=0}
   \]
on \( X(\mathbb{C})^{an} \) compatible with the filtrations \( W_\bullet \) on \( \mathcal{F} \) and \( \mathcal{V} \).

4. The fiber of the associated graded object \( (\text{Gr}_{-2i} W_\bullet \mathcal{F}, \text{Gr}_{-2i} \mathcal{V}, \text{comp}) \) at \( x \) is isomorphic to a sum of Tate Hodge structures of weight \(-2i\) for all \( x \in X(\mathbb{C})^{an} \).

**Definition 10.6 (NVMTHS).**

1. A variation of mixed Tate Hodge structure is said to be constant if the local system \( \mathcal{V} \) is a trivial local system.

2. A constant variation of mixed Tate Hodge structure is said to be split if it is isomorphic to a sum of pure Tate Hodge structures.

3. A variation of mixed Tate Hodge structure is said to be nilpotent (denoted as NVMTHS for short) if there exists a filtration \( (N_x \mathcal{F}, N_x \mathcal{V}, \text{comp}) \) of \( (\mathcal{F}, \mathcal{V}, \text{comp}) \)
such that the associated graded object \((\Gr^N_p \mathcal{F}, \Gr^N_p \mathcal{V}, \text{comp})\) is a split constant variation of mixed Tate Hodge structures on \(X(\mathbb{C})^{an}\).

**Theorem 10.7.** The category \(H^0 K^0 C_{\text{Del}}^J\) is equivalent to NVMTHS(X). It is also equivalent to the category of homogeneous \(H^0(B^h(\epsilon_B | A_{\text{Del}}(\mathbb{C}) | \epsilon_B))\)-comodules.

By applying the above theorem for the case \(X = \text{Spec}(\mathbb{C})\), we have the following corollary.

**Corollary 10.8.** The category of mixed Hodge structures is equivalent to the category of homogeneous \(H^0(B^h(\epsilon_B | A_{\text{Del}}(\mathbb{C}) | \epsilon_B))\)-comodules.

### 10.3. Proof of Theorem 10.7.

We construct a nilpotent variation of mixed Tate Hodge structure for an object \(V = \{V^i\} \in K^0 C_{\text{Del}}^J\). We set \(V^i = V^{ip} e^j(p)\), \(V = \bigoplus_i V^i e^{-i}\) and \(V(p) = \bigoplus_i V^{ip} e^j(p)\). Then we have \(V = \bigoplus_i V^i e^{-i} = \bigoplus_p \mathcal{V}(p)\).

We define \(\nabla = D + 1 \otimes d\) defines an integrable \(A_{\text{Del}}^J\)-connection \(V \rightarrow A_{\text{Del}}^J \otimes V\). We write \(D = (\{\omega_s\}, \{f_{st}\}, \{\varphi_s\})\) according to the above direct sum decomposition, i.e., \(\rho_{st} \in \text{Hom}^0_{C_{\text{Del}}^J}(V, \bigoplus_i \Gamma(U_{an,s,t}, (2\pi i)^s \mathbb{Q}_B(-i)) \otimes V)\), etc. Then as an element of \(\text{Hom}^0_{C_{\text{Del}}^J}(V, A_{\text{Del}}^{h,2} \otimes V)\),
we have
\[ dD = \left\{ \{d\omega_s\}, \{-\omega_t + \omega_s + df_{st}\}, \{-f_{tu} + f_{su} - f_{st}\}, \{-\rho_{tu} + \rho_{su} - \rho_{st}\}, \{\omega_s + d\varphi_s\}, \{f_{st} - \rho_{st} + \varphi_t - \varphi_s\} \right\} \]

and
\[ D \circ D = \left\{ \{\omega_s\omega_s\}, \{-\omega_s f_{st} + f_{st}\omega_t\}, \{f_{st} f_{tu}\}, \{\rho_{st} \rho_{tu}\}, \{-\omega_s \varphi_s\}, \{-f_{st} \varphi_t + \varphi_s \rho_{st}\} \right\}. \]

Here the products are tensor products of composites of endomorphisms of \( V \) and exterior products of differential forms. Then by the integrability condition, we have \( dD = D \circ D \).

By this relation, we have cocycle relations
\[ (1 + f_{su}) = (1 + f_{st})(1 + f_{tu}), \quad (1 + \rho_{su}) = (1 + \rho_{st})(1 + \rho_{tu}), \quad \text{integrability of connections} \]
\[ d\omega_s = \omega_s \omega_s, \quad \text{and the following commutative diagrams:} \]
\[ \begin{array}{c}
\bigoplus_p V(p) \otimes \Gamma(U_{an,s,t}, (2\pi i)^{-p}Q) \\
\downarrow 1+\varphi_s \\
V \otimes \Gamma(U_{an,s,t}, \mathcal{O}_{an}) \\
\downarrow 1+f_{st} \\
V \otimes \Gamma(U_{an,s,t}, \mathcal{O}_{an})
\end{array} \quad \begin{array}{c}
\bigoplus_p V(p) \otimes \Gamma(U_{an,s,t}, (2\pi i)^{-p}Q) \\
\downarrow 1+\varphi_s \\
V \otimes \Gamma(U_{an,s,t}, \mathcal{O}_{an}) \\
\downarrow 1+f_{st} \\
V \otimes \Gamma(U_{an,s,t}, \mathcal{O}_{an})
\end{array} \quad \text{(10.2)}
\]
\[ \begin{array}{c}
V \otimes \Gamma(U_{s,t}, \mathcal{O}) \\
\downarrow \omega_t + 1 \otimes d \\
V \otimes \Gamma(U_{s,t}, \Omega^1) \\
\downarrow (1+f_{st}) \otimes 1 \\
V \otimes \Gamma(U_{s,t}, \Omega^1)
\end{array} \quad \begin{array}{c}
V \otimes \Gamma(U_{s,t}, \mathcal{O}) \\
\downarrow \omega_t + 1 \otimes d \\
V \otimes \Gamma(U_{s,t}, \Omega^1) \\
\downarrow (1+f_{st}) \otimes 1 \\
V \otimes \Gamma(U_{s,t}, \Omega^1)
\end{array} \quad \text{(10.3)}
\]
\[ \begin{array}{c}
V \otimes \Gamma(U_{an,s}, \mathcal{O}_{an}) \\
\downarrow 1+\varphi_s \\
V \otimes \Gamma(U_{an,s}, \Omega^1_{an}) \\
\downarrow 1+\varphi_s \\
V \otimes \Gamma(U_{an,s}, \Omega^1_{an})
\end{array} \quad \begin{array}{c}
V \otimes \Gamma(U_{an,s}, \mathcal{O}_{an}) \\
\downarrow 1+\varphi_s \\
V \otimes \Gamma(U_{an,s}, \Omega^1_{an}) \\
\downarrow 1+\varphi_s \\
V \otimes \Gamma(U_{an,s}, \Omega^1_{an})
\end{array} \quad \text{(10.4)}
\]

By the cocycle relations, we have a local system \( V \) and locally free sheaf \( \mathcal{F} \) by patching constant sheaves \( V \) and free sheaves \( V \otimes \mathcal{O} \) on \( U_{an,s} \) and \( U_s \) by the patching data \( 1 + \rho_{st} \) and \( 1 + f_{st} \). The connections \( \omega_s + 1 \otimes d \) define integrable connections on \( V \otimes \Gamma(U_{s,t}, \mathcal{O}) \) by the integrability condition. They are patched together to a global connection by the commutative diagram (10.3). The sheaf homomorphisms \( 1 + \varphi_s \) are patched together into a global homomorphism \( \text{comp} \) of sheaves \( V \rightarrow \mathcal{F} \otimes \mathcal{O}_{an} \) on \( X_{an} \) by the commutative diagram (10.2). Via the sheaf homomorphism \( \text{comp} \), the local system \( V \otimes \mathbb{C} \) is identified with the subshaf of horizontal sections of \( \mathcal{F} \otimes \mathcal{O}_{an} \) by the commutative diagram (10.4).
We introduce filtrations $F^*$ and $W^*$ on $U_s$ and $U_{an,s}$ by

$$
F^i \mathcal{F} = \bigoplus_{p \geq i} V(-p) \otimes \mathcal{O},
$$

$$
W_{2i} \mathcal{F} = \bigoplus_{p \leq i} V(-p) \otimes \mathcal{O},
$$

$$
W_{2i} \mathcal{V} = \bigoplus_{p \leq i} V(-p).
$$

The filtration $F$ gives rise to a well defined filtration on $\mathcal{F}$ since the patching data $f_{ij}$ for $\mathcal{F}$ is contained in $\Gamma(U_{s,t}, A_{F,\text{DR}}^{h,0}) = \Gamma(U_{s,t}, \mathcal{O}(0))$.

Since $\omega$ is contained in

$$
\bigoplus_p \text{Hom}_k(V(p), V(p) \otimes \Gamma(U_s, \Omega^1)) \\
\oplus \bigoplus_p \text{Hom}_k(V(p), V(p + 1) \otimes \Gamma(U_s, \Omega^1)(-1)),
$$

the Griffiths transversality condition in Definition 10.5 is satisfied. It defines a mixed Tate Hodge structure at any point $x$ in $X(\mathbb{C})^{\text{an}}$ by the definitions of two filtrations. Therefore $(\mathcal{F}, \mathcal{V}, \text{comp})$ defines a variation of mixed Tate Hodge structure on $X$.

We set $N^i V = \bigoplus_{i \leq p} V^p$. Then this filtration defines a filtration on the variation of mixed Tate Hodge structure $(\mathcal{F}, \mathcal{V}, \text{comp})$ and the associated graded variation is split constant mixed Tate Hodge structures on $X$. Thus we have the required nilpotent variation of mixed Tate Hodge structure $(\mathcal{F}, \mathcal{V}, \text{comp})$ on $X$.

We can construct an inverse correspondence by attaching an $A_{\text{Del}}$-connection to a nilpotent variation of mixed Tate Hodge structures as follows. We choose sufficiently fine Zariski covering $\mathcal{U} = \{U_i\}$ such that the restrictions of $F$ are free on each $U_i$. Using two filtrations $F$ and $W$, $\mathcal{F}$ splits into $\bigoplus_p F(p)$ as an $\mathcal{O}$ module. We choose a trivialization of $\mathcal{F}(p)|_{U_i}$. By taking a refinement $\mathcal{U}_{an}$ of $\mathcal{U} \cap X_{an}$, we may assume that $\mathcal{U}_{an}$ is a simple covering. By restricting $U_{an,s}$, we choose a trivialization $1 + \varphi_i$ of the local system $\mathcal{V}$. At last we choose a nilpotent filtration $N^*$ compatible with the connection and local system and a splitting of $N^*$. Using these data we construct an integrable nilpotent $A_{\text{Del}}$-connection $D = (\{\omega_s\}, \{f_{st}\}, \{\rho_{st}\}, \{\varphi_s\})$ by the commutative diagram (10.2), (10.3), (10.4).

By passing to homotopy equivalence classes of morphisms, we have the required equivalence of two categories.

References


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