MATHEMATICS OF CHAOS

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ABSTRACT. In September 2008, V.I. Arnold was awarded the prestigious Shaw Prize. This prize was established in 2002 by the Shaw Prize Foundation named after philanthropist Sir Run Run Shaw. Here we publish the lecture given by V.I. Arnold at the awards ceremony in Hong Kong and a short Autobiography (CV) prepared for this occasion.

Key words and phrases. Chaos, prime vectors, continued fractions, random numbers.

INTRODUCTION

Alan Turing claimed that it’s an error to think that mathematics is a chain of rigorous deductions leading from one statement to another by logical reasoning. The present talk confirms his opinion, describing the experiments and the hints as more important ingredients of mathematics (as of other natural sciences) than logical deductions.

The numbers of population in different countries of the world start from the digit 1 almost five times more frequently than from the digit 9. The areas of the countries provide the same peculiarity (absent, however, for the lengths of the rivers or for the altitudes of the mountains). This strange effect is explained in ergodic theory of dynamical systems and by Malthusian model.

A random fraction is irreducible with probability $6/\pi^2$, which is approximately 0.61. This discovery has lead Euler to the zeta-function: $\zeta(2) = \pi^2/6$.

A statistical study of the topological structure of the mountain region maps provides the Taylor expansion of the $\tan^{-1}$ function (and its generalization, obtained by the recent Givental methods of quantum fields’ mirror theory).

“Random” permutations provided by Galois field tables have statistical properties similar to those permutations which are generated by the list of the phone numbers of the academy’s members.

Some other “turbulent” objects of number theory (including the Young diagrams of the cycles of dynamical systems and the periods of continued fractions, whose Gauss–Kuz’min statistics influences the stability property of the Solar system) behave differently, and the laws of this “arithmetical turbulence” have been discovered only recently.

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All these phenomena of chaotic behavior of objects in natural sciences are explained in the following in an elementary way that should be accessible to a non-expert without any preliminary background.

1. Vector Divisibility

Remember, we were talking about irreducible fractions. Let’s consider a slightly more general problem, but first we have to give a definition.

**Definition.** Let’s call a vector $v \in \mathbb{Z}^m$ divisible if and only if all its components have a common divisor greater than 1.

**Example.** The following vectors are divisible:

$$\begin{pmatrix} 4 \\ 6 \end{pmatrix} = 2 \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 3 \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$ 

The following theorem shows the probability of a vector to be non-divisible.

**Theorem** (Arnold, 2008). The prime (=non-divisible) vectors of the crystalline lattice $\mathbb{Z}^m$ of dimension $m > 1$ are distributed in this lattice uniformly with constant density

$$\rho = \frac{1}{\zeta(m)},$$

where

$$\zeta(m) = \frac{1}{1^m} + \frac{1}{2^m} + \frac{1}{3^m} + \cdots$$

(this “zeta-function” being defined and studied by L. Euler).

**Example.** In the case of the plane, $m = 2$, the density of the set of non-divisible vectors of the integer lattice $\mathbb{Z}^2$ equals $\rho = 6/\pi^2 \simeq 0.61$.

The following picture shows all the divisible vectors in a square $14 \times 14$. There are 117 prime vectors among these 196 — the corresponding cells are gray, and $\frac{117}{196} \simeq 0.59$.

Note that for $m = 1$ (the case of prime numbers) the density $\rho$ is not constant, declining logarithmically (as it has been discovered by Legendre and Tchebyshev).
2. Continued Fractions

The next objects of our discussion are continued fractions. Let’s define them.

**Definition.** The continued fraction \([a_0, a_1, \ldots, a_n]\) is the rational number

\[
x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ddots + \cfrac{1}{a_n}}} = \frac{p_n}{q_n},
\]

where \(a_0\) is an integer and all \(a_k\) for \(k > 0\) are positive integers. A similar expression with an infinite sequence of integers \(a_k\) corresponds to any real number \(x\). An infinite continued fraction is just the limit for \(n \to \infty\) of finite ones \(p_n/q_n\).

It’s easy to see that if the element \(a_{n+1}\) of the continued fraction representing \(x\) is large, the difference \(|x - [a_0, a_1, \ldots, a_n]|\) is very small. In this case the continued fraction \([a_0, a_1, \ldots, a_n]\) provides an excellent rational approximation for \(x\).

**Example.** For \(\pi\) the corresponding fraction is \(\pi = [3, 7, 15, 1, 292, \ldots]\). The number 15 is big enough to provide a nice (Archimedes) approximation

\[
\frac{p_3}{q_3} = \frac{22}{7}.
\]

The number 292 is very big, providing an excellent approximation

\[
\left| \pi - \frac{p_3}{q_3} \right| = \left| \pi - \frac{355}{113} \right| \approx 3 \cdot 10^{-7}.
\]

There is an important question one should study about continued fractions. It can be motivated by the following discovery by H. Poincaré:

The multitude of the strong resonances being dangerous for the Solar System stability, one needs to study the statistics of the elements of the continued fractions for the ratios \(\omega_i/\omega_j\) of the periods of different periodical motions \(\omega_i\) in celestial mechanics. Large elements provide strong resonant perturbations, and therefore it’s important to understand how frequently the elements of these continued fractions are large.
Resonance:
\[
\frac{\omega_1}{\omega_2} \simeq \frac{120.5''}{299''} \simeq \frac{2}{5}
\]
[\(\omega\) being the number of angular seconds, run in 24 hours]. The resonance provides a large “secular” perturbation of the motion of one planet by the attraction of the other.

The first step towards this problem is to study continued fractions for random numbers. The complete answer is given by the following theorem.

**Theorem** (R.O. Kuz’min, 1928). The frequency \(f_k\) of the elements \(a_j = k\) in the continued fraction of a random real number \(x\) is given by the formula

\[
f_k = c \ln \left[ 1 + \frac{1}{k(k+2)} \right], \quad \text{where} \quad c = \frac{1}{\ln 2}.
\]

The percentages for small \(k\) are approximately

\[
\begin{array}{ccccccccccc}
    k & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
100f_k & 41 & 17 & 9 & 6 & 4 & 3 & 2 & 1.8 & 1.4
\end{array}
\]

The most frequent elements are those equal to 1. They form \(f_1 \simeq 41\%\) of all the elements of the continued fraction of a random number. The frequencies of large values \(k\) are small. When \(k \to \infty\) the frequencies \(f_k\) decrease as \(c/k^2\) for a random number.

For the golden ratio all the elements in the continued fraction except \(a_0\) are equal to 1:

\[
x = \frac{\sqrt{5} - 1}{2} = \frac{1}{1 + \frac{1}{1 + \cdots}} = [0, 1, 1, 1, \ldots].
\]

This shows that the golden ratio \((x \simeq 0.618)\) is not a “random number”.

The theorem above holds not for every number, but for *almost every*: this means that the exceptions (including the golden ratio) form a *set of Lebesgue measure zero* on the axis \(x\).
Now consider a nice class of continued fractions that are (eventually) periodic. Eventually means that if one cuts some of the leading terms of the fraction, it becomes periodic in the obvious sense. There is a nice theorem describing them.

**Theorem (Lagrange).** The continued fraction of a real number $x$ is periodic if and only if $x$ is a root of some quadratic equation

$$rx^2 + px + q = 0$$

with integer coefficients $(r, p, q) \in \mathbb{Z}^3$.

In 1993 V.I. Arnold conjectured that the frequencies of the elements $k$ of the periods of the (non-random) roots of the quadratic equations (1) have in average the same Kuz’min values $f_k$, as the frequencies for the random real numbers’ continued fractions. Additionally, this statement holds for the equations (1) with $r = 1$.

These “Arnold’s conjectures” have been proved recently in Khabarovsk by V.A. Bykovsky with his students. The averaging is provided here by the arithmetical means along the integer points of the growing balls in the space of equations. At the same time we have the following theorem.

**Theorem (Arnold, 2007).** The sequences forming the periods of the continued fractions of the roots of quadratic equations

$$x^2 + px + q = 0$$

with integer coefficients have (in the average) a statistics that differs from the Kuz’min statistics $\{f_k\}$ (in spite of the fact that their elements and their subsets verify the Kuz’min statistics, according to the Bykovsky’s theorem that proves Arnold’s conjectures).

For instance, this difference follows from the fact of these periods being palindromic, which is described below.

**Definition.** A palindrome is a text which remains unchanged being read back: “ESOPE RESTE ICI ET SE REPOSE”.

**Theorem (Arnold, 2003; Aicardi and Pavlovskai, 2007).** The periods of the irrational roots of equations

$$x^2 + px + q = 0$$

with integer coefficients (and also of the irrational square roots of rational numbers) are palindromes.

**Example.** The root $x = (\sqrt{113} - 1)/2$ of the equation $x^2 + x = 28$ has the continued fraction of period 7. The periodic sequence

$$\ldots 1, 4, 2, 2, 4, 1, 9, 1, 4, 2, 2, 4, 1, 9, \ldots$$

(infinite in both sides) coincides with the infinite sequence “read back”:

$$\ldots 9, 1, 4, 2, 2, 4, 1, 9, 1, 4, 2, 2, 4, 1, \ldots$$

Therefore, the above period, consisting of 7 elements, is palindromic.
The theorem proves that the same palindromic property holds for the irrational roots of quadratic equations with integer coefficients of the two forms
\[ x^2 + px + q = 0, \quad vx^2 - u = 0. \]
The roots of some other quadratic equations also have this property.

**Theorem** (Arnold, 2007). *The frequencies \( F_k \) of the elements \( k \) of the continued fractions for the eigenvalues of the matrices \( A \in \text{SL}(2, \mathbb{Z}) \) (integer matrices \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) of determinant 1) differ from the Kuz’min frequencies \( f_k \).

**Example.** In the case \( k = 1 \) the frequencies are
\[ F_1 = 0.5, \quad f_1 = \frac{\ln 4/3}{\ln 2} \approx 0.41. \]

The averages \( \hat{T}(R) \) of the period lengths \( T(p, q) \) of the continued fractions of the real roots of equations \( x^2 + px + q = 0 \) (with integer coefficients \( p \) and \( q \)), averaged along the discs \( p^2 + q^2 \leq R^2 \) behave approximately like a linear function of the radius length \( R \):

Conjecturally, a better description of the growth of the period length \( T(p, q) \) (which is different along different directions in the plane \( (p, q) \)) is provided by the square root of the discriminant \( p^2 - 4q \) of the quadratic equation \( x^2 + px + q \) (the period length \( T \) depending only on the discriminant value).

### 3. Random Number Recognition

The problem is to understand whether the given \( n \) numbers are random. Look at the following two sequences:

\[
\begin{align*}
03, 09, 27, 81, 43, 29, 87, 61, 83, 49, 47, 41, 23, 69, 07; \\
37, 74, 11, 48, 85, 22, 59, 96, 33, 70, 07, 44, 81, 18, 55.
\end{align*}
\]

Both sequences (3) and (4) look rather random, but these are just a geometric and arithmetical progression formed by the residues modulo 100 (\( n = 15 \)).

It turns out that there exists an *objective measurement* of the randomness, provided by the Kolmogorov parameter \( \lambda \) (introduced by Kolmogorov in 1933 in his Italian article in the insurance journal Giorn. Ist. Ital. Attuari, vol. 4, no. 7).
measurement shows that the randomness probability is (approximately) 300 times higher for the geometrical progression \((3)\) than for the arithmetical one \((4)\).

The value \(\lambda\) of the Kolmogorov parameter is calculated using the distance \(\mathcal{F}\) between the observed distribution function of the sample of \(n\) observed values \(\{x_i\}\) and the theoretical distribution function (multiplied by \(n\)).

Kolmogorov divides the \(C_0\)-distance \(\mathcal{F}\) by the normalizing quantity \(\sqrt{n}\) (to obtain a quantity of order 1)

\[
\lambda_n = \frac{\mathcal{F}}{\sqrt{n}}.
\]

Kolmogorov proved that the probability of the observation of a value \(\lambda \leq \Lambda\) (for a sample of \(n\) independent values of \(x\)) reaches the limit for \(n \to \infty\) (and doesn’t depend on special properties of the random variable \(x\), provided that its distribution function is continuous). The universal limit is given by the formula

\[
\Phi(\Lambda) = \sum_{k=-\infty}^{\infty} (-1)^k e^{-2k^2\Lambda^2} \quad (\Lambda > 0).
\]

This Kolmogorov’s distribution function grows monotonically from \(\Phi(0) = 0\) (where all the derivatives of \(\Phi\) also vanish) to \(\Phi(\infty) = 1\):
The mean value of the parameter $\Lambda$ in the Kolmogorov’s distribution $\Phi$ is equal to

$$\Lambda_* = \sqrt{\frac{\pi}{2}} \ln 2 \simeq 0.867.$$ 

**Conclusion.** If the observed value $\lambda_n$ is much smaller (or much larger) than the mean value $\Lambda_*$, the probability of the randomness of the sample $\{x_i\}$ is small.

**Example.** In the Kolmogorov’s article “On a new confirmation of the Mendel’s laws” (Doklady Acad. of Sci. USSR 27 (1940), no. 1, 37–31) he explained that if the experiments of the students of Lysenko, “disproving” the Mendel’s law 1:3 (for recessive allele appearances in the second generation) were providing the proportion of observed cases closer to 1:3 than the proportion reported in their article, then this closeness would indicate the falsified nature of the reported numbers.

The next two theorems study the randomness of the sequences (3) and (4) in the terms of Kolmogorov parameter.

**Theorem** (Arnold, 2007). The arithmetical progression $\{at\}$ ($t = 1, 2, \ldots, n$) of the fractional parts $\{x\} = x - [x]$ provides the Kolmogorov parameter values $\lambda_n$ tending to 0 as $n \to \infty$, if the progression difference $a$ is a rational number $a = p/q$ ($p$ and $q$ being integers).

For some (exceptional) irrational values of $a$ the Kolmogorov randomness parameter $\lambda_n$ attains arbitrary large values: $\lambda_{n_j} > K > 0$ for some $n_j \to \infty$. The exceptional values form a set of Lebesgue measure zero.

**Theorem** (Arnold, 2007). For the geometrical progression $\{Aa^t\}$ ($t = 1, 2, \ldots, n$) of fractional parts the Kolmogorov randomness parameter $\lambda_n$ tends to 0 as $n \to \infty$, provided that the base number $a$ of the progression is rational.

This is a corollary of the Fermat’s and Euler’s theorem on the periodicity of the geometrical progressions of residues.

For most irrational values of the base number $a$ the Kolmogorov randomness parameter values $\lambda_n(A, a)$ do not tend neither to 0 nor to $\infty$ as $n \to \infty$: conjecturally, the values $\lambda_n(A, a)$ are (for large $n$) distributed asymptotically according to the Kolmogorov’s distribution $\Phi$ (provided, for instance, that the parameters $A$ and $a$ of the geometrical progression are distributed uniformly).

The “theoretical distribution” was supposed in the last theorem to be uniform (the uniformity of the distribution of the fractional parts of a generic geometrical progression was a classical conjecture in number theory), and I am unaware of its rigorous proof. (Some ideas, explaining the relations of this uniformity to the physics of the adiabatic invariants, are described in my book on the Galois fields in a more general form.)

V.I. Arnold was born on 12 VI 1937 at Odessa.

Studied at the Moscow University, 1954–1959.


The physical-mathematical sciences doctor, 1963, for the Thesis on the stability of the Hamiltonian systems, same Institute. The graduated studies were supervised by A.N. Kolmogorov.

Since 1965 V.I. Arnold worked as a professor at the chair of differential equations of the mathematical-mechanical faculty of the Moscow State University. Since 1986 he works also at the Steklov Mathematical Institute, Moscow. V.I. Arnold was elected member of the Russian Academy of Sciences in 1990.

V.I. Arnold served as the vice-president of the International Union of Mathematicians (1999–2003), being also the President of the Moscow Mathematical Society.


Being Moscow University’s professor 30 years, V.I. Arnold worked also as the professor at the University Paris-Dauphine from 1993 to 2005 (remaining now its honorary professor).

V.I. Arnold published several dozens of books. Examples are:

- Ergodic Problems of Classical Mechanics (with A. Avez);
- Ordinary Differential Equations;
- Mathematical Methods of Classical Mechanics;
- Geometrical Methods of theory of Ordinary Differential Equations;
- Catastrophes Theory;
- Singularities of Caustics and of Wave Fronts;
- Problems for Children from 5 to 15 years old;
- Huygens and Barrow, Newton and Hooke — first steps of calculus and of catastrophes theory;
- Yesterday and Long Ago;
- Contact Geometry and Wave Propagation;
- Lectures of Partial Derivatives equations;
- Pseudoperiodic Topology (with M. Kontsevitch and A. Zoritch);
- Mild and Soft Mathematical Models;
- Continued Fractions;
- Euler Groups and Geometric Progressions Arithmetics;
- Dynamics, Statistics and Projective Geometry of Galois Fields;
New Obscurantism and Russia’s Educational System;
Is Mathematics Needed at Highschools;
Geometry of Complex Numbers, Quaternions and Spins;
Experimental Mathematics;
What is Mathematics;
Experimental Discoveries of Mathematical Facts;
Science of Mathematics and Arts of Mathematicians;
Geometry.

The above list contains 10 universities textbooks.
Most known mathematical papers of V.I. Arnold deal with Hamiltonian systems (including the discovery of the “Arnold diffusion” and the creation of the symplectic topology).
Arnold’s articles on the “quantum catastrophes theory” include the studies of the bifurcations of the caustics, based on the Arnold’s discovery of an unexpected interrelations between the simple critical points of functions and simple Lie algebras (and also to Coxeter reflections’ groups).
The real algebraic geometry of plane curves was related by V. I. Arnold to the four-dimensional topology (and to quantum fields theory) — this discovery generated many studies of many mathematicians of the algebraic geometry part of the 16th problem of Hilbert.
Recent Arnold’s works on arithmetical turbulence provide unexpected statistical properties of the Young diagrams of the cycles of random permutations of \( N \to \infty \) points.
Many domains of modern mathematics, generated by the Arnold’s articles, include, for instance:

- Lagrange and Legendre cobordism theories (in symplectic and contact topologies);
- Statistics of the most frequent representations of finite groups;
- Ergodic theory of the segments’ permutations;
- Planetary dynamo theory (in magnetohydrodynamics);
- Statistics of the higher-dimensional continued fractions;
- Theory of singularities of the distribution of galaxies;
- Arnold’s discovery of the “strange duality” of Lobachevsky triangles (leading to the mirror symmetry theory of the quantum fields physics);
- Asymptotical statistics of the Fermat–Euler geometrical progressions of residues;
- Theory of the weak asymptotics (for the distributions of the solutions of Diophantine problems);
- Description of the boundary singularities of the optimal control problems (in terms of the geometry of icosahedron);
- Topological Galois theory (of radical insolvability for the algebraic equations of degrees \( \geq 5 \));
- Creation of the characteristic classes theories for the Braids and for the algebraic functions;
• Arnold’s discovery of the topological reasons of the divergences of the permutation theory’s series (including the classification of the neighbourhoods and in the orbits spaces of dynamical systems);
• Asymptotical study of irreducible representations frequencies (in the eigenspaces of the Laplacian on a symmetrical Riemannian manifold);
• Topological classification of the immersed smooth plane curves;
• Ergodic theory and projective geometry of Galois fields;
• Statistics of the convex polygons, whose vertices are integer points on the plane;
• Topological interpretation of the Maxwell’s multipole formula for the spherical harmonics;
• Palindromicity theory for the periodic continued fractions of the quadric irrationalities \((x^2 + px + q = 0)\);
• Arnold’s discovery of the validity of the Gauss–Kuz’min statistics for the random periodic continued fractions;
• Arnold’s discovery of the violation of the Gauss–Kuz’min statistics for the periodic continued fractions of eigenvalues of the random matrices (in \(\text{SL}(2, \mathbb{Z})\));
• Arnold’s invention of the characteristic class, involved in the quantization conditions;
• Arnold’s symplectic geometry theory of the Lagrange tore in completely integrable Hamilton systems;
• The ergodic and number-theoretical “Arnold’s cats” of physicists (F. Dyson, I. Persival, . . . ).