CLASSIFICATION OF GORENSTEIN TORIC DEL PEZZO VARIETIES IN ARBITRARY DIMENSION

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Dedicated to the memory of Professor V.A. Iskovskih

Abstract. An \( n \)-dimensional Gorenstein toric Fano variety \( X \) is called Del Pezzo variety if the anticanonical class \( -K_X \) is an \( (n-1) \)-multiple of a Cartier divisor. Our purpose is to give a complete biregular classification of Gorenstein toric Del Pezzo varieties in arbitrary dimension \( n \geq 2 \). We show that up to isomorphism there exist exactly 37 Gorenstein toric Del Pezzo varieties of dimension \( n \) which are not cones over \( (n-1) \)-dimensional Gorenstein toric Del Pezzo varieties. Our results are closely related to the classification of all Minkowski sum decompositions of reflexive polygons due to Emiris and Tsigaridas and to the classification up to deformation of \( n \)-dimensional almost Del Pezzo manifolds obtained by Jahnke and Peternell.

Key words and phrases. Toric varieties, Fano varieties, lattice polytopes.

Introduction

A projective algebraic variety \( X \) is called Gorenstein Fano variety if the anticanonical divisor \( -K_X \) is an ample Cartier divisor. An \( n \)-dimensional Gorenstein Fano variety is called Gorenstein Del Pezzo variety if

\[
-K_X = (n-1)L
\]

for some ample Cartier divisor \( L \) on \( X \). Smooth Del Pezzo \( n \)-folds have been classified up to deformation by Iskovskih and Fujita [Isk77], [Isk78], [Isk80], [Fuj80], [Fuj81], [Fuj84]. A classification of Gorenstein Del Pezzo surfaces is known due to Demazure [Dem80].

All Gorenstein Del Pezzo varieties of dimension \( n \geq 4 \) have been classified up to deformation by Fujita [Fuj90a], [Fuj90b]. A smooth \( n \)-fold \( X \) is called an almost Del Pezzo manifold if there exists a \( K_X \)-trivial birational morphism \( \varphi: X \to X' \), where \( X' \) is a Gorenstein Del Pezzo variety. It is well-known that if \( n = 2 \) then the minimal resolution \( \tilde{X} \) of a Gorenstein Del Pezzo surface \( X \) is always an almost Del Pezzo surface [DuV34]. Such a surface \( \tilde{X} \) can be obtained from \( \mathbb{P}^2 \) or \( \mathbb{P}^1 \times \mathbb{P}^1 \) by
blowing up points which are not necessary in general position. Recently all almost Del Pezzo manifolds of arbitrary dimension \( n \) were classified up to deformation by Jahnke and Peternell [JP08].

In this paper we consider \( n \)-dimensional Gorenstein Fano varieties \( X \) which are toric varieties, i.e., \( X \) contains an \( n \)-dimensional algebraic torus \( T \) as an open dense subset such that the group multiplication \( T \times T \to T \) extends to a regular action \( T \times X \to X \) [Oda88], [Fu93]. Let \( M \cong \mathbb{Z}^n \) be the lattice of characters of \( T \) and \( M_R := M \otimes \mathbb{R} \) the corresponding real vector space. A natural \( T \)-action on the space \( H^0(X, \mathcal{O}(-K_X)) \) splits into a finite direct sum of \( 1 \)-dimensional weight subspaces parametrized by a finite subset of characters \( \{ \chi_i \} \subset M \). In this situation, the convex hull \( \Delta \) of the set \( \{ \chi_i \} \) is a special \( n \)-dimensional polytope in \( M_R \) which is called reflexive polytope [Bat94]. The Gorenstein toric Fano variety \( X \) is uniquely determined up to an isomorphism of the \( n \)-dimensional lattice \( M \) by the corresponding reflexive polytope \( \Delta \). It is known that for any given dimension \( n \) there exist only finitely many \( n \)-dimensional reflexive polytopes up to a lattice isomorphism. This implies that there exist only finitely many Gorenstein toric Fano varieties of dimension \( n \) up to biregular isomorphism (see [Bat82]). If \( N := \text{Hom}(M, \mathbb{Z}) \) is the dual lattice and \( N_R = \text{Hom}(M_R, \mathbb{R}) \) the dual real vector space, then for every reflexive polytope \( \Delta \subset M_R \) there exists a dual reflexive polytope \( \Delta^* \subset N_R \). Using an isomorphism \( N \cong M \), we obtain that up to isomorphism all reflexive polytopes satisfy a nice combinatorial duality, which plays important role in Mirror Symmetry [Bat94] (see also [CK99]).

If \( X \) is an \( n \)-dimensional toric variety such that \( -K_X = rL \) for an ample divisor \( L \) then, up to a translation by an element of \( M \), the reflexive polytope \( \Delta \) is isomorphic to an \( r \)-multiple of another lattice polytope \( P \), i.e., \( \Delta \cong rP \), where all vertices of \( P \) are contained in \( M \). Such polytopes \( P \) are called Gorenstein polytopes of index \( r \) [BR07], [BN08]. For example, the standard \( n \)-dimensional basic lattice simplex is a Gorenstein polytope of index \( n + 1 \) and the standard \( n \)-dimensional lattice unit cube is a Gorenstein polytope of index \( 2 \). The number \( d := n + 1 - r \) is called the degree of an \( n \)-dimensional Gorenstein polytope \( P \) (see also [Bat06]). The name “degree” of an \( n \)-dimensional lattice polytope \( P \) is motivated by the following fact: if

\[
E_P(t) := \sum_{k=0}^{\infty} |kP \cap M| t^k
\]

is the Ehrhart series of \( P \), then \( h_P(t) := (1-t)^{n+1} E_P(t) \) is a polynomial of degree \( d \).

The purpose of our paper is to give a complete classification up to isomorphism of all Gorenstein toric Del Pezzo varieties in arbitrary dimension \( n \geq 2 \). This is equivalent to a classification of all \( n \)-dimensional Gorenstein polytopes of index \( r = n - 1 \), or, equivalently, of degree \( 2 \). In the case \( n = 2 \), Gorenstein polygons of index \( 1 \) are exactly reflexive polygons and their classification is well-known (it was obtained independently in [Bat85], [Rab91], [Koe91]). There exist exactly 16 reflexive polygons (see Figure 1).

It is important to note that if \( X \) is an \( n \)-dimensional Gorenstein Del Pezzo variety then the cone \( X' \) over \( X \) is an \((n+1)\)-dimensional Gorenstein Del Pezzo variety. If in addition \( X \) is toric, then \( X' \) is also toric and the \((n+1)\)-dimensional Gorenstein
polytope $P'$ corresponding to $X'$ is a lattice pyramid $\Pi(P)$ over the $n$-dimensional Gorenstein polytope $P$ corresponding to $X$. This shows that it is enough to classify only those Gorenstein polytopes of index $r = n - 1$ which are not pyramids over low dimensional ones.

Using a computer program, 3- and 4-dimensional reflexive polytopes have been classified by Kreuzer and Skarke [KS98], [KS00]. Their computer calculation resulted in 4319 polytopes in dimension 3 and 473,800,776 polytopes in dimension 4. Using the lists of 3- and 4-dimensional polytopes, Kreuzer has found 31 3-dimensional Gorenstein polytopes of index 2 and 36 4-dimensional Gorenstein polytopes of index 3. In this paper, we give an independent proof of this fact without using computer classifications.

Our complete classification of $n$-dimensional Gorenstein polytopes of degree 2 (i.e., of index $r = n - 1$) gives the following more precise result:

<table>
<thead>
<tr>
<th>Dimension $n$</th>
<th>Number of polytopes</th>
<th>Number of non-pyramids</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>3</td>
<td>31</td>
<td>15</td>
</tr>
<tr>
<td>4</td>
<td>36</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>37</td>
<td>1</td>
</tr>
<tr>
<td>$\geq 6$</td>
<td>37</td>
<td>—</td>
</tr>
</tbody>
</table>
We remark that the classification of \( n \)-dimensional Gorenstein polytopes of index \( r = n - 2 \) is a much harder problem. This problem in case \( n = 3 \) is equivalent to the classification of all 3-dimensional reflexive polytopes [KS98]. Using the computer classification of 4-dimensional reflexive polytopes [KS00], Kreuzer has found exactly 5363 reflexive polytopes \( \Delta \) of dimension 4 which are isomorphic to \( 2P \) for some lattice polytope \( P \). This shows that there exist exactly 5363 Gorenstein polytopes \( P \) of dimension 4 and index 2. In particular, there exist exactly \( 1044 = 5363 - 4319 \) Gorenstein 4-dimensional polytopes of index 2 which are not pyramids over 3-dimensional reflexive polytopes. However, all these facts seem to be very difficult to check without computer.

It follows from a recent result of Haase, Nill and Payne [HNP08, Cor. 3.2] together with the main result in [Bat06] that every \( n \)-dimensional Gorenstein polytope \( P \) of degree \( d \) is a pyramid if

\[
 n \geq 4d \left( \frac{2d + V - 1}{2d} \right),
\]

where

\[
 V = (2d - 1)^{2d-1} \cdot \left((2d - 1)!!\right)^{2d} \cdot 14(2d - 1)^{2d}.
\]

In [Nil08, Prop. 1.5] Nill has shown that the same statement holds true already for \( n \geq (V - 1)(2d + 1) \).

However, these inequalities are still too far from being sharp. It would be nice to know exactly the maximal dimension \( n \) (depending on \( d \)) such that every \( n \)-dimensional Gorenstein polytope of degree \( d \) is a pyramid. In this connection we suggest the following:

**Conjecture 0.1.** Every \( 3d \)-dimensional Gorenstein polytope \( P \) of degree \( d \) is a pyramid. Moreover, there exists up to an isomorphism a unique \((3d-1)\)-dimensional Gorenstein polytope \( \Theta_d \) of degree \( d \) which is not a pyramid.

Conjecture 0.1 is easy to check for \( d = 1 \). Our classification verifies Conjecture 0.1 in the case \( d = 2 \), but it remains open for \( d \geq 3 \). A precise description of the special Gorenstein polytope \( \Theta_d \) for arbitrary degree \( d \) is given in Example 1.22.

Let us summarize the ideas used in our classification of \( n \)-dimensional Gorenstein polytopes of degree 2.

First of all we use a natural partial order on the set of \( n \)-dimensional Gorenstein polytopes of degree 2: for two lattice polytopes \( P_1 \) and \( P_2 \), we write \( P_1 \preceq P_2 \) if \( P_1 \) is isomorphic to a lattice subpolytope of \( P_2 \). We say that an \( n \)-dimensional Gorenstein polytope \( P \) of degree 2 is minimal if there does not exist an \( n \)-dimensional Gorenstein subpolytope \( P' \preceq P \) of degree 2 such that \( P' \neq P \). For \( n \)-dimensional Gorenstein polytopes of fixed degree \( d \) there exists a combinatorial duality which generalizes the polar duality for \( n \)-dimensional reflexive polytopes [BN08]. Using this duality, we obtain that if \( P \) is a minimal \( n \)-dimensional Gorenstein polytope of degree 2, then the dual Gorenstein polytope \( P^* \) is a maximal one, i.e., \( P^* \) is not contained in a strictly larger \( n \)-dimensional Gorenstein polytope of degree 2.

Since every \( n \)-dimensional Gorenstein polytope is contained in a maximal one, in the classification of all \( n \)-dimensional Gorenstein polytopes of degree 2 the first step
is to find all minimal ones among them. After that, using the duality $P \leftrightarrow P^*$, we immediately obtain all maximal $n$-dimensional Gorenstein polytopes of degree 2. Finally, all remaining Gorenstein polytopes can be found as lattice subpolytopes of the maximal ones.

Let $P$ be an $n$-dimensional lattice polytope. The integer

$$\text{Vol}_n(P) := n! \text{(volume of } P),$$

we call the lattice normalized volume of $P$. In the classification of minimal Gorenstein polytopes of degree 2 the following statement is very important: if $P$ is minimal, then $\text{Vol}_n(P) \leq 4$. The proof of this technical statement consists of several steps and uses the complete classification of $n$-dimensional lattice polytopes whose $h^*$-polynomial is of degree $\leq 1$ \cite{BN07}. As we already remarked, the classification of all $n$-dimensional Gorenstein polytopes $P$ of degree 2 is equivalent to the classification of all $n$-dimensional lattice polytopes $P$ having the Ehrhart series $E_P(t) = 1 + a_P t + t^2 (1 - t)^{n+1}$, $a_P \in \mathbb{Z}_{\geq 0}$, where $a_P = \text{Vol}_3(P) - 2$. If $X_P$ is the corresponding $n$-dimensional Gorenstein toric Del Pezzo variety and $-K_{X_P} = (n - 1)L$ for some Cartier divisor $L$, then the integer $a_P + 2 = \text{Vol}_2(P)$ equals the intersection number $L^n$ and the Ehrhart series $E_P(t)$ is exactly the Hilbert–Poincaré series of the homogeneous coordinate ring of $X_P$:

$$S_P := \bigoplus_{k=0}^{\infty} H^0(X_P, \mathcal{O}(kL)).$$

The graded Gorenstein ring $S_P$ can be seen as a semigroup algebra of the set of lattice points in the $(n+1)$-dimensional cone $C_P$ over the Gorenstein polytope $P$. If $P$ is a Gorenstein polytope such that $\text{Vol}_3(P) < 4$ (i.e., $a_P < 2$), then one can apply algebraic ideas from \cite{Bat06} and classify all possible Gorenstein rings $S_P$ (and the corresponding Gorenstein polytopes $P$) by an enumeration of all binomial relations among the minimal generators of $S_P$.

Let us say some words about the way of presentation of the results of our classification. It is not difficult to draw pictures of all 3-dimensional Gorenstein polytopes of degree 2. However, the same way of presentation can not be applied in dimension $n \geq 4$. It was shown in \cite[Th. 2.6]{BN08} that if a $d$-dimensional reflexive polytope $\Delta$ is a Minkowski sum of $r$ lattice polytopes $\Delta_1, \ldots, \Delta_r$:

$$\Delta = \Delta_1 + \cdots + \Delta_r,$$

then the $(d+r-1)$-dimensional Cayley polytope $P := \Delta_1 \ast \cdots \ast \Delta_r$ is a Gorenstein polytope of degree $d$ (or, of index $r$). In particular, a Minkowski sum decomposition $\Delta = \Delta_1 + \cdots + \Delta_r$ of a 2-dimensional reflexive polytope $\Delta$ defines an $(r+1)$-dimensional Gorenstein polytope $P = \Delta_1 \ast \cdots \ast \Delta_r$ of degree 2. The classification of all Minkowski sum decompositions of reflexive polygons was recently obtained by Emiris and Tsigaridas \cite{ET06}. Our classification shows that all Gorenstein polytopes of degree 2 and of dimension $n \geq 4$ are Cayley polytopes corresponding to different Minkowski sum decompositions of 2-dimensional reflexive polytopes.
The same statement holds true for almost all (except one) 3-dimensional Gorenstein polytopes of degree 2.

The paper is organized as follows:

In Section 1 we review basic definitions and properties of Gorenstein polytopes. The combinatorial duality of Gorenstein polytopes $P \leftrightarrow P^*$ and their constructions as Cayley polytopes corresponding to Minkowski summands of reflexive polygons will be of our primary interest.

In Section 2 we investigate the graded Gorenstein $K$-algebra $S_P$ associated with a Gorenstein polytope $P$ of degree $d$. Let $A_P := S_P / \langle y \rangle$ be the Artinian Gorenstein $K$-algebra obtained from $S_P$ as quotient modulo an ideal $\langle y \rangle$ generated by a regular sequence of homogeneous elements $y_1, \ldots, y_n$ of degree 1. If the degree of $P$ is 2 then $1 + ap t + t^2$ is the Hilbert–Poincaré polynomial of $A_P$ and the graded Gorenstein $K$-algebras $A_P$ can be classified up to isomorphism (see Proposition 2.9). This classification allows to determine all possible binomial relations in the $K$-algebra $S_P$. If $a_P \leq 2$, then we find a complete list of Gorenstein polytopes $P$ which are not pyramids (see Theorem 2.10).

In Section 3 we describe general combinatorial properties of Gorenstein polytopes of degree 2. In particular, we prove some useful equalities for 3-dimensional Gorenstein polytopes of degree 2. It is well known that if $\Delta$ and $\Delta^*$ are dual to each other then

$$\text{Vol}_3(\Delta) + \text{Vol}_3(\Delta^*) = 12.$$ 

This remarkable property of reflexive polygons was recently investigated by many mathematicians [PR00], [HS02], [RST05]. We show that there exists a similar relation between the number 12 and dual to each other 1-dimensional faces $E \subset P$, $E^* \subset P^*$ of the 3-dimensional Gorenstein polytopes $P$, $P^*$ of degree 2

$$\sum_{E \subset P, \dim E = 1} \text{Vol}_2(E) \cdot \text{Vol}_2(E^*) = 12.$$ 

(see Proposition 3.4).

In Section 4 we prove that if a Gorenstein polytope $P$ of degree 2 is minimal then $\text{Vol}_3(P) \leq 4$. As it was explained above, together with the classification in Section 2 this result easily implies a complete classification of Gorenstein polytopes of degree $P$. Here we present the whole list of $d$-dimensional Gorenstein polytopes of degree 2 which are not pyramids (including some of their combinatorial properties).

In Section 5 we compare our combinatorial method with the birational method of Jahnke and Peternell used in their classification of almost Del Pezzo manifolds [JP08].

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1. General Properties of Gorenstein Polytopes

By a lattice $M$ of dimension $n$ we mean a free abelian group of rank $n$, $M \cong \mathbb{Z}^n$. We also consider the dual lattice $N = M^* = \text{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^n$ together with the
canonical pairing
\[
(\cdot, \cdot): M \times N \to \mathbb{Z}.
\]
Further, we define \( M_\mathbb{Q} := M \otimes \mathbb{Q}, N_\mathbb{Q} := N \otimes \mathbb{Q} \) together with the \( \mathbb{Q} \)-bilinear pairing \( (\cdot, \cdot): M_\mathbb{Q} \times N_\mathbb{Q} \to \mathbb{Q} \). The real vector spaces \( M_\mathbb{R} \) and \( N_\mathbb{R} \) are defined analogously. A polytope \( P \) in \( M_\mathbb{R} \) is the convex hull of a finite subset in \( M_\mathbb{R} \). We say that \( P \) is a lattice polytope if all its vertices lie in \( M \). Further, a face of \( P \) is a subset \( F \subseteq P \subseteq M_\mathbb{R} \) which minimizes on \( P \) some linear function \( f \in N_\mathbb{R} \), and a facet of \( P \) is a codimension-1 face of \( P \). By \( \partial P \) we denote the boundary of the polytope \( P \) and by \( \text{int}(P) \) the interior \( P \setminus \partial P \) of \( P \).

Let \( P \) be an \( n \)-dimensional lattice polytope in \( M_\mathbb{R} \). For any nonnegative integer \( k \) the number \( i(P, k) := |kP \cap M| \) is a polynomial of degree \( n \) in \( k \) [Ehr67]. This polynomial is called the Ehrhart polynomial of \( P \). Its values in negative integers are defined by the formula
\[
i(P, -k) = (-1)^n |\text{int}(kP) \cap M|.
\]
Moreover, one has
\[
E_P(t) := \sum_{k \geq 0} i(P, k)t^k = \frac{h_0^* + h_1^*t + \cdots + h_n^*t^n}{(1-t)^{n+1}}
\]
for some nonnegative integers \( h_i^* \) [Sta80]. The Ehrhart polynomial \( i(P, k) \) can be recovered from the integers \( h_i^* \) by the formula:
\[
i(P, k) = \sum_{j=0}^n h_j^* \binom{n+k-j}{n}.
\]

**Definition 1.1.** The polynomial
\[
h_i^*(t) := (1-t)^{n+1}E_P(t) \sum_{k \geq 0} |kP \cap M|t^n
\]
is called the \( h^*-polynomial \) of an \( n \)-dimensional lattice polytope \( P \). Its degree is called the degree of \( P \): \( \deg P := \max\{i: h_i^* \neq 0\}, 0 \leq \deg P \leq \dim P \). The vector \( h^*(P) = (h_0^*, h_1^*, \ldots, h_{\deg P}^*) \) is called the \( h^*-vector \) of \( P \).

**Remark 1.2.** It was proved by Stanley in [Sta80] that the numbers \( h_i^* = h_i^*(P) \) are monotone functions of \( P \): if \( P \) and \( P' \) are two lattice polytopes and \( P' \subseteq P \), then \( h_i^*(P') \leq h_i^*(P) \) for all \( i \).

**Definition 1.3** [BN07, Def. 2.1]. We call an \( n \)-dimensional lattice polytope \( P \) (\( n \geq 1 \)) a Lawrence prism with heights \( \theta_1, \ldots, \theta_n \) if there exists a lattice basis \( e_1, \ldots, e_n \) of \( M \) and non-negative integers \( \theta_1, \ldots, \theta_n \) such that
\[
P = \text{conv}\{0, \theta_1e_n, e_1, e_1 + \theta_2e_n, \ldots, e_{n-1}, e_{n-1} + \theta_ne_n\}.
\]
The lattice vector \( e_n \) is called the direction of \( P \).

**Definition 1.4** [BN07, Def. 2.2]. We call an \( n \)-dimensional lattice polytope \( P \) (\( n \geq 2 \)) exceptional if there exists a lattice basis \( e_1, \ldots, e_n \) of \( M \) such that
\[
P = \text{conv}\{0, 2e_1, 2e_2, e_3, \ldots, e_n\}.
\]
The following statement was proved in [BN07]:

**Theorem 1.5.** Every $n$-dimensional lattice polytope $P$ of degree $\leq 1$ is either a basic simplex, or an exceptional simplex, or a Lawrence prism such that $\theta_1 + \cdots + \theta_n \geq 2$. The corresponding $h^*$-polynomial is equal respectively to $1$, $1 + 3t$, $1 + (\theta_1 + \cdots + \theta_n - 1)t$.

**Definition 1.6.** For any $r$ convex polytopes $\Delta_1, \ldots, \Delta_r$ in $M_\mathbb{R}$, their Minkowski sum is defined as

$$\Delta := \Delta_1 + \cdots + \Delta_r = \{x_1 + \cdots + x_r : x_i \in \Delta_i\}.$$ 

The polytopes $\Delta_i$ ($i = 1, \ldots, r$) are called Minkowski summands of $\Delta$. Here we do not require all polytopes $\Delta_1, \ldots, \Delta_r$ to have the maximal dimension $n = \text{dim } M_\mathbb{R}$.

Define the lattice $\tilde{M} := M \oplus \mathbb{Z}^r = M \oplus \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_r$, where $\{e_1, \ldots, e_r\}$ form a lattice basis of $\mathbb{Z}^r$. The Cayley polytope $\Delta_1 \ast \cdots \ast \Delta_r$ associated with $\Delta_1, \ldots, \Delta_r$ is the convex hull of $(\Delta_1, e_1), \ldots, (\Delta_r, e_r)$ in $\tilde{M}_\mathbb{R}$.

**Definition 1.7.** Let $\Delta \subset M_\mathbb{R}$ be a lattice polytope of dimension $n$. We set $\Delta_0 := \Delta$ and choose $r$ 0-dimensional lattice polytopes $\Delta_1, \ldots, \Delta_r$ (points). Then the polytope $\Delta_0 \ast \Delta_1 \ast \cdots \ast \Delta_r$ is called $r$-fold pyramid over $\Delta$ and we denote it by $\Pi^r(\Delta)$. If $r = 1$ we write simply $\Pi(\Delta)$ and call $\Pi(\Delta)$ pyramid over $\Delta$.

**Remark 1.8.** A Lawrence prism with heights $\theta_1, \ldots, \theta_n$ can be considered as Cayley polytope of $n$ 1-dimensional polytopes $[0, \theta_1], \ldots, [0, \theta_n] \subset \mathbb{R}$. An $n$-dimensional exceptional simplex is the $(n-2)$-fold pyramid over the lattice triangle $\text{conv}\{0, 2e_1, 2e_2\}$.

**Definition 1.9.** Consider a $d$-dimensional convex polytope $\Delta$ in $M_\mathbb{R}$ that contains the zero point $0 \in M_\mathbb{R}$ in its interior. The polytope

$$\Delta^* = \{y \in N_\mathbb{R} : \langle x, y \rangle \geq -1 \forall x \in \Delta\}$$

is called dual polytope of $\Delta$.

A $d$-dimensional lattice polytope $\Delta$ in $M_\mathbb{R}$ is called reflexive if $0 \in \Delta$ and the dual polytope $\Delta^*$ is again a lattice polytope. If $\Delta$ contains a single interior lattice point $m$ such that $\Delta - m$ is reflexive, we say that $\Delta$ is reflexive with respect to $m$.

**Remark 1.10.** If $\Delta$ is reflexive polytope, then the dual polytope $\Delta^*$ is again reflexive and $(\Delta^*)^* = \Delta$. The duality $\Delta \leftrightarrow \Delta^*$ is very important for Mirror Symmetry [Bat94], [CK99].

**Proposition 1.11** [Hib92]. Let $\Delta \subset M_\mathbb{R}$ be a lattice polytope. Then $\Delta$ is reflexive if and only if for every nonnegative integer $k$ the number of lattice points in $k\Delta$ equals the number of lattice points in the interior $\text{int}((k + 1)\Delta)$ of $\Delta$.

**Corollary 1.12.** An $n$-dimensional lattice polytope $\Delta$ is reflexive if and only if

$$h_i^*(\Delta) = h_{n-i}^*(\Delta), \quad 0 \leq i \leq \frac{n}{2}.$$  

**Definition 1.13.** Let $r$ be a positive integer. An $n$-dimensional lattice polytope $P$ is called Gorenstein of index $r$ if $rP$ is a reflexive polytope with respect to some interior lattice point $m \in \text{int}(rP)$. 
Analogous to Proposition 1.11, one obtains:

**Proposition 1.14.** Let \( P \subset M_\mathbb{R} \) be an \( n \)-dimensional lattice polytope. Then \( P \) is a Gorenstein polytope of index \( r \) if and only if for every nonnegative integer \( k \) the number of lattice points in \( kP \) equals the number of lattice points in the interior \( \text{int}((k + r)P) \) of \( \Delta \). In this case, one has \( \deg P = n + 1 - r \).

**Corollary 1.15.** Let \( P \) be an \( n \)-dimensional lattice polytope with \( h^*-polynomial \) \( h^*_P(t) = \sum_i h^*_it^i \) and let \( d = \deg P \). Then \( P \) is a Gorenstein polytope of degree \( d = n + 1 - r \) if and only if \( h^*_i(P) = h^*_d-i(P), \quad 0 \leq i \leq d/2 \).

**Definition 1.16** [BB97]. Consider two \((n+1)\)-dimensional lattices \( M := M \oplus \mathbb{Z} \) and \( \overline{N} := N \oplus \mathbb{Z} \) which are dual to each other with respect to the natural extension of the pairing \( \langle *, * \rangle : M \times N \to \mathbb{Z} \) to \( M \times \overline{N} \to \mathbb{Z} \). For the latter we use the same notation \( \langle *, * \rangle \). Let \( M_\mathbb{R} \) and \( \overline{N}_\mathbb{R} \) be the corresponding real vector spaces.

A strongly convex \((n+1)\)-dimensional cone \( C \in M_\mathbb{R} \) with the vertex \( 0 = -C \cap C \) is called Gorenstein cone if there exists an element \( n_C \in N_\mathbb{R} \) such that \( \langle e_i, n_C \rangle = 1 \) for all primitive lattice generators \( e_i \) of \( C \). Note that \( n_C \) is uniquely determined.

If the cone \( C \) is Gorenstein, then the set \( P_C = \{ x \in C : \langle x, n_C \rangle = 1 \} \) is a convex \( n \)-dimensional lattice polyhedron, which is called support of \( C \).

On the other hand, if \( P \) is an \( n \)-dimensional lattice polytope in \( M_\mathbb{R} \), then \( C_P := \{(\lambda x, \lambda) : \lambda \in \mathbb{R}_{\geq 0}, x \in P \} \subset M_\mathbb{R} \) is a Gorenstein cone with the support polytope \( P \) (here we have \( n_C_P = (0, \ldots, 0, 1) \)).

**Definition 1.17** [BB97]. Define the dual cone as \( C^\vee := \{ y \in \overline{N}_\mathbb{R} : \langle x, y \rangle \geq 0 \ \forall x \in C \} \).

An \((n+1)\)-dimensional Gorenstein cone \( C \) is called reflexive if its dual cone \( C^\vee \) is Gorenstein as well. In this case we denote by \( m_{C^\vee} \subset M \) the lattice point such that the hyperplane \( \langle m_{C^\vee}, y \rangle \) contains all lattice points generating the dual cone \( C^\vee \). For a reflexive Gorenstein cone, the uniquely defined positive integer \( r_C = \langle m_{C^\vee}, n_C \rangle \) is called index of \( C \).

**Remark 1.18.** It is very important for our purpose that if \( C \) is a reflexive Gorenstein cone of index \( r \) then \( C^\vee \) is also a reflexive Gorenstein cone of index \( r \). Moreover, a Gorenstein cone is reflexive of index \( r \) if and only if its support is a Gorenstein polytope of index \( r \) (see [BN08, Prop. 1.11]).

**Definition 1.19.** Let \( P \subset M_\mathbb{R} \) be an \( n \)-dimensional Gorenstein polytope of index \( r \) and \( C_P \) be the corresponding \((n+1)\)-dimensional reflexive Gorenstein cone of index \( r \). We define the dual \( n \)-dimensional Gorenstein polytope \( P^* \) to be the support
of the dual reflexive Gorenstein cone $C_P^\vee$, i.e.,
\[ C_P^* := C_P^\vee. \]

It follows immediately from the definition that $(P^*)^* = P$.

**Remark 1.20.** It is easy to show that the duality for Gorenstein polytopes commutes with the pyramid construction, i.e., if $P$ and $P^*$ are dual to each other $n$-dimensional Gorenstein polytopes of index $r$, then $\Pi(P)$ and $\Pi(P^*)$ are dual to each other $(n+1)$-dimensional Gorenstein polytopes of index $r+1$. In particular, a Gorenstein polytope $P$ of degree $d$ is a pyramid if and only if the dual Gorenstein polytope $P^*$ is a pyramid.

**Theorem 1.21** [BN08, Th. 2.6]. Let $\Delta_1, \ldots, \Delta_r \subset M_\mathbb{R}$ be lattice polytopes. Assume that the Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_r$ is $n$-dimensional. Denote by $\tilde{\Delta}$ the Cayley polytope $\Delta_1^{\vee} \ast \cdots \ast \Delta_r^{\vee}$ associated with $\Delta_1, \ldots, \Delta_r$. Then the following statements are equivalent:

1. $\tilde{\Delta}$ is a $(n+r-1)$-dimensional Gorenstein polytope of degree $n$ (and index $r$),
2. The polytope $\Delta$ is reflexive,
3. The cone $C_{\tilde{\Delta}}$ associated with $\tilde{\Delta}$ is a reflexive Gorenstein cone of index $r$.

This theorem shows a way for constructing examples of Gorenstein polytopes using Minkowski sum decompositions of reflexive polytopes.

**Example 1.22.** Let $\Delta$ be the reflexive cube $[-1, 1]^n \subset \mathbb{R}^n$. Then $\Delta$ is a Minkowski sum of $2n$ 1-dimensional lattice polytopes $\Delta_i^{\pm} := [0, \pm e_i]$ ($i = 1, \ldots, n$), where $e_1, \ldots, e_n$ the standard basis of $\mathbb{R}^n$. The corresponding $(3n-1)$-dimensional Gorenstein polytope
\[ \Delta_1^{\ast} \ast \cdots \ast \Delta_n^{\ast} \ast \Delta_1^{-} \ast \cdots \ast \Delta_n^{-} \]
of degree $n$ will be denoted by $\Theta_n$.

**Remark 1.23.** In [ET06], I. Emiris and E. Tsigaridas computed all possible Minkowski decompositions of all reflexive lattice polytopes in dimension 2. Their results are listed up to isomorphism in Table 1 and Table 2.

**Remark 1.24.** Let $P \subset M_\mathbb{R}$ be an $n$-dimensional Gorenstein polytope of index $r$. By Remark 1.18, the dual polytope $P^*$ of a Gorenstein polytope $P$ of index $r$ is also a Gorenstein polytope of index $r$. Moreover, $P$ and $P^*$ are combinatorially dual to each other, i.e., there exists a natural bijection $F \leftrightarrow F^*$ between $k$-dimensional faces $F$ of $P$ and $(n-k-1)$-dimensional faces $F^*$ of $P^*$.

**Remark 1.25.** If $r > 1$ then the reflexive polytopes $rP$ and $rP^*$ are not dual to each other as reflexive polytopes (see Definition 1.9).

Denote by $M' \subset M_\mathbb{R}$ the $n$-dimensional lattice generated by $M$ and $m/r \in M_\mathbb{R}$, where $m \in rP$ is the unique interior point of the reflexive polytope $rP$. In order to get the dual Gorenstein polytope $P^*$ one has to dualize the reflexive polytope $rP$, i.e., to get the dual polytope $(rP)^*$ and replace the dual lattice $N$ by the sublattice $N' \subset N$, $|N/N'| = r$, which is dual to $M'$ (see [BN08, Prop. 1.15]).
Table 1. Minkowski sum decompositions of reflexive polygons

<table>
<thead>
<tr>
<th>Minkowski Sum Decompositions</th>
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<th>Minkowski Sum Decompositions</th>
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<tbody>
<tr>
<td><img src="image1" alt="Diagram 1" /></td>
<td><img src="image2" alt="Diagram 2" /></td>
<td><img src="image3" alt="Diagram 3" /></td>
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<tr>
<td><img src="image4" alt="Diagram 4" /></td>
<td><img src="image5" alt="Diagram 5" /></td>
<td><img src="image6" alt="Diagram 6" /></td>
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<tr>
<td><img src="image7" alt="Diagram 7" /></td>
<td><img src="image8" alt="Diagram 8" /></td>
<td><img src="image9" alt="Diagram 9" /></td>
</tr>
</tbody>
</table>

Remark 1.26. By [BN08, Prop. 1.16], one has

\[ \partial (rP)^* \cap N = \partial (rP)^* \cap N', \]

i.e., there exists a natural bijection between the set of \( N' \)-lattice points in \( P^* \) and the set of boundary \( N \)-lattice points in the dual reflexive polytope \( (rP)^* \). Moreover, every facet \( \Gamma \) of \( P^* \) considered as a lattice polytope with respect to \( N' \) is isomorphic to the facet \( \Gamma \) of \( \Delta^* \) considered as a lattice polytope with respect to \( N \).

Proposition 1.27. Let \( P \subset M_\mathbb{R} \) be an \( n \)-dimensional Gorenstein polytope of degree \( d \). Denote by \( \Gamma_1, \ldots, \Gamma_k \) all codimension-1 faces of \( P \). Then one has

\[ \sum_{i=1}^{k} \text{Vol}_S(\Gamma_i) = (n + 1 - d) \text{Vol}_S(P). \]

Proof. The polytope \( \Delta = (n + 1 - d)P \) is reflexive. Up to a shift by a lattice vector, we can assume that 0 is the unique interior lattice point of \( \Delta \). Then one can decompose \( \Delta \) into a union of \( n \)-dimensional pyramids with vertex 0 over \( (n - 1) \)-dimensional faces \( (n + 1 - d)\Gamma_1, \ldots, (n + 1 - d)\Gamma_k \) of \( \Delta \). Since heights of all these
Let \( N \) be a vertex of an \( n \)-dimensional Gorenstein polytope \( P \) of degree \( d \). We denote by \( C(v_i) \) the \( n \)-dimensional cone generated by vectors \( v - v_i \), where \( v \) runs over all lattice points of \( P \).

**Proposition 1.29.** Let \( v_i \in P \) be an arbitrary vertex of an \( n \)-dimensional Gorenstein polytope \( P \) of degree \( d \) and let \( \Gamma_i \subset P^* \) be the dual facet in the dual polytope.

}\[
\sum_{i=1}^{k} \text{Vol}_N((n+1-d)\Gamma_i) = \text{Vol}_N(\Delta).
\]

It remains to combine it with the equalities \( \text{Vol}_N(\Delta) = (n+1-d)^n \text{Vol}_N(P) \) and \( \text{Vol}_N((n+1-d)\Gamma_i) = (n+1-d)^{n-1} \text{Vol}_N(\Gamma_i), \quad 1 \leq i \leq k. \)
Then cone $C(v_i)$ is dual to the cone $C_{\Gamma_i}$ over the lattice polytope $\Gamma_i$ (see Example 1.30).

**Example 1.30.**

![Figure 2. Duality between $C(v_1)$ and $C_{\Gamma_1}$](image)

**Proof.** Let $\Delta = (n + 1 - d)P \subset M_\mathbb{R}$ be the reflexive polytope corresponding to $P$. By Remark 1.25, $P^*$ is obtained from the dual reflexive polytope $\Delta^* \subset N_\mathbb{R}$ by choosing a sublattice $N' \subset N$ of index $r = n + 1 - d$. By Remark 1.26, every facet $\Gamma$ of $P^*$ with respect to the lattice $N'$ is isomorphic to the facet $\Gamma$ of $\Delta^*$ (with respect to the lattice $N$). On the other hand, the cone $C(v_i)$ is equal to the cone generated by vectors $v - (n + 1 - d)v$, where $v$ runs over all lattice points of $\Delta = (n + 1 - d)P$. The duality between reflexive polytopes $\Delta$ and $\Delta^*$ shows that $C(v_i)$ in dual to cone $C_{\Gamma_i}$ over $\Gamma_i$ is a facet of $\Delta$. \qed

## 2. The Graded Gorenstein Algebra $S_P$

Let $K$ be an arbitrary field of characteristic 0 and let $P \subset M_\mathbb{R}$ be an arbitrary $n$-dimensional lattice polytope. We consider the $(n + 1)$-dimensional lattice $M_\mathbb{R} = M \oplus \mathbb{Z}$ and the $(n + 1)$-dimensional cone $C_P \subset M_\mathbb{R}$ over $P$. There exists a natural grading of lattice points $\mathbf{m} = (m, k) \in C_P \cap \overline{M}$ ($m \in M, k \in \mathbb{Z}_{\geq 0}$):

$$\deg \mathbf{m} := k,$$

so that $C_P \cap \overline{M}$ becomes a graded monoid.

**Definition 2.1.** We denote by $S_P := K[C_P \cap \overline{M}]$ the graded $K$-algebra of the semigroup $C_P \cap \overline{M}$. For each lattice point $\mathbf{m} \in C_P \cap \overline{M}$ we denote by $x^\mathbf{m}$ the corresponding monomial in $S_P$. It is well-known that the $K$-algebra $S_P$ is always Cohen–Macaulay (see, e.g., [Sta78]).

**Definition 2.2.** A lattice point $m \in C_P \cap \overline{M}$ is called irreducible if it is not representable as a sum of two lattice points $\mathbf{m} = \mathbf{m}' + \mathbf{m}''$ ($\mathbf{m}', \mathbf{m}'' \in C_P \cap \overline{M}$).

Obviously every lattice point of degree 1 in $C_P \cap \overline{M}$ is irreducible. Denote by $I_P(k)$ the number of irreducible lattice gpoints of degree $k$ in $C_P \cap \overline{M}$. Since the semigroup $C_P \cap \overline{M}$ is finitely generated we obtain that

$$I(P, t) := \sum_{k \geq 0} I_P(k) t^k$$
is a polynomial, i.e., \( I_P(k) = 0 \) for \( k \gg 0 \). The number \( I_P(k) \) can be identified with the number of lattice points of degree \( k \) in a minimal generating set of the semigroup \( C_P \cap \overline{M} \).

**Definition 2.3.** Let \( S = \bigoplus_{i \geq 0} S_i \) be a finitely generated graded \( K \)-algebra, \( S_0 \cong K \). For any \( k, l \in \mathbb{N} \) we denote by \( S_k S_l \) the \( K \)-subspace in \( S_{k+l} \) generated by products \( xy \), where \( x \in S_k \), \( y \in S_l \). We define the numbers \( g_k(S) \) as follows: 

\[
g_0(S) = 1, \quad g_k(S) := \text{dim}_K S_{k}/\left( \sum_{j=1}^{k-1} S_j S_{k-j} \right), \quad \forall k > 0.
\]

It is clear that \( g_1(S) = \text{dim}_K S_1 \) and the sum 

\[
G_S(t) := \sum_{k \geq 0} g_k(S)t^k
\]

is a polynomial, i.e., \( g_k(S) = 0 \) for \( k \gg 0 \). The number \( g_k(S) \) \((k > 0)\) can be identified with the number of elements of degree \( k \) in the minimal generating system of the \( K \)-algebra \( S \). Here the minimality means that any proper subset of this system does not generate \( S \).

**Proposition 2.4.** Let \( S = S_P \) be the graded algebra of a lattice polytope \( P \). Then the polynomial \( G_S(t) \) is equal to \( I(P, t) \).

**Proof.** Since \( S_P = K[C_P \cap \overline{M}] \) is generated by monomials \( x^\mathbf{m} \) corresponding to lattice points \( \mathbf{m} \in C_P \cap \overline{M} \), a minimal generating system \( G = \{ \gamma_1, \ldots, \gamma_l \} \) of the \( K \)-algebra \( S_P \) can be chosen as a finite subset of these monomials. It remains to show that \( x^\mathbf{m} \in G \) if and only if \( \mathbf{m} \in C_P \cap \overline{M} \) is irreducible. Indeed, if \( x^\mathbf{m} \in G \) and \( \mathbf{m} = \mathbf{m}' + \mathbf{m}'' \mathbf{m}', \mathbf{m}'' \in C_P \cap \overline{M} \) and \( k = \text{deg} \mathbf{m}', l = \text{deg} \mathbf{m}'' \), then \( x^\mathbf{m} = x^\mathbf{m'} x^\mathbf{m''} \in S_{k} S_{l} \). This contradicts the minimality of \( G \). On the other hand, if \( \mathbf{m} \in C_P \cap \overline{M} \) is an irreducible lattice point of degree \( k \), then \( x^\mathbf{m} \) can not be a polynomial expression of elements of degree \( < k \). Hence \( x^\mathbf{m} \) must appear in \( G \). \( \square \)

**Proposition 2.5.** Let \( P \) be an \( n \)-dimensional lattice polytope. Denote by \( A_P \) an Artinian graded \( K \)-algebra obtained by quotient of \( S_P \) modulo a regular sequence of \( n + 1 \) elements \( y_0, y_1, \ldots, y_n \) of degree 1. Then 

\[
G_{A_P}(t) = I_P(t) - (n + 1)t.
\]

**Proof.** Obviously, we have 

\[
g_1(A_P) = \text{dim}_K(A_P)_1 = |P \cap M| - (n + 1) = I_P(1) - (n + 1).
\]

Denote by \( \psi \) the epimorphism \( S_P \to A_P \). If \( G \subseteq S_P \) is minimal generating system of monomials, then \( \psi(G) \) generate \( A_P \) as \( K \)-algebra. This implies \( g_k(A_P) \leq I_P(k) \forall k \geq 2 \). In fact, we have the equality \( g_k(A_P) = I_P(k) \forall k \geq 2 \), because the kernel of the epimorphism 

\[
\psi: (S_P)_k \to (A_P)_k
\]

is contained in \((S_P)_1(S_P)_{k-1} \). \( \square \)
Let \( P \) be an arbitrary \( n \)-dimensional lattice polytope and let \( I(P, 1) = \sum_{k \geq 0} I_P(k) \) be the number of irreducible lattice points \( C_P \cap \mathcal{M} \). Then we can describe the graded algebra \( S_P \) as a quotient of a polynomial ring \( R := K[x_1, \ldots, x_{I(P,1)}] \) such that exactly \( I_P(k) \) of variables \( x_i \) have weight \( k \) and \( J := \text{Ker}(R \to S_P) \) is an ideal generated by binomials
\[
B = x_i^{a_i} \cdots x_j^{a_j} - x_{i'}^{b_i} \cdots x_{j'}^{b_j}.
\]

We use the following two statements:

**Proposition 2.6** [Bat06, Prop. 3.5]. Let \( \{B_1, \ldots, B_p\} \) be a set of binomials which minimally generate the ideal \( J \). Then \( P \) is a pyramid over an \((n-1)\)-dimensional lattice polytope \( P' \), i.e., \( P = \Pi(P') \), if and only if the exists a variable \( x_i \in \{x_1, \ldots, x_{I(P,1)}\} \) which does not appear in any of binomials \( B_1, \ldots, B_p \).

**Proposition 2.7** [Bat06, Prop. 3.6]. Let \( \overline{R} := R/(y) \) be a polynomial ring in \( I(P, 1)-(n+1) \) variables obtained from the polynomial ring \( R := K[x_1, \ldots, x_{I(P,1)}] \) by quotient modulo a regular sequence \( y_0, y_1, \ldots, y_n \) of elements of degree 1. We denote by \( \overline{J} \) the ideal \( J/(y)J \) in \( \mathbb{R} \). Then the numbers and the degrees of minimal generators of ideals \( J \subset R \) and \( \overline{J} \subset \overline{R} \) are the same.

Now we apply the above statements to a Gorenstein polytope \( P \). In this case, the graded \( K \)-algebras \( S_P \) and \( A_P \) are known to be Gorenstein. If \( A := A_0 \oplus \cdots \oplus A_l \) is an Artinian graded Gorenstein \( K \)-algebra \((A_0 = K, A_l \neq 0)\), then the zero ideal in \( A \) is irreducible and the multiplication in \( A \) defines a perfect pairing
\[
A_l \times A_{l-i} \to A_l \cong K.
\]

For our purposes it will be sufficient to use the following more special fact:

**Proposition 2.8** [Hum07]. Let \( A := A_0 \oplus A_1 \oplus A_2 \) be an Artinian graded \( K \)-algebra where \( A_0 \cong A_2 \cong K \) is a field, and let \( \langle \cdot, \cdot \rangle \) be a symmetric bilinear form on \( A_1 \) given by the multiplication
\[
\langle \cdot, \cdot \rangle : A_1 \times A_1 \to A_2 \cong K, \quad (x, y) \mapsto \langle x, y \rangle := xy.
\]

Then \( A \) is a Gorenstein ring if and only if \( \langle \cdot, \cdot \rangle \) is a non-degenerate symmetric bilinear form on \( A_1 \).

This characterisation allows to describe all relations in \( A \) if \( \dim_K(A_1) \leq 2 \):

**Proposition 2.9.** Let \( A = A_0 \oplus A_1 \oplus A_2 \) be an Artinian graded Gorenstein \( K \)-algebra and \( a = \dim_K(A_1) \in \{0, 1, 2\} \). Then we have:

(i) if \( a = 0 \), then \( A \cong K[X]/(X^2) \), \( \deg(X) = 2 \);

(ii) if \( a = 1 \), then \( A \cong K[X]/(X^3) \), \( \deg(X) = 1 \);

(iii) if \( a = 2 \), then \( A \cong K[X, Y]/(XY, X^2 - \lambda Y^2) \), \( \deg(X) = \deg(Y) = 1 \), \( \lambda \neq 0 \).

**Proof.** (i) Let \( x \in A_2 \) be a nonzero element. Consider the \( K \)-algebra homomorphism \( \psi : K[X] \to A \) defined by the condition \( \psi(X) = x \). Since \( A_2A_2 \subset A_4 = 0 \) we have \( x^2 = 0 \), i.e., \( X^2 \in \text{Ker} \psi \). So \( \psi \) induces an isomorphism \( A \cong K[X]/(X^2) \), because both rings \( A \) and \( K[X]/(X^2) \) are 2-dimensional \( K \)-vector spaces.
(ii) Let \( x \in A_1 \) be a nonzero element. Since the bilinear map \( A_1 \times A_1 \to A_2 \) is nondegenerate, we have \( x^2 \neq 0 \), i.e., \( \psi \) is surjective. However, \( x^3 = 0 \), because \( A_3 = 0 \). Consider the \( K \)-algebra homomorphism \( \psi : K[X] \to A \) defined by the condition \( \psi(X) = x \). Then \( X^3 \in \ker \psi \). So \( \psi \) induces an isomorphism \( R \cong K[X]/(X^2) \), because both rings \( R \) and \( K[X]/(X^2) \) are 3-dimensional \( K \)-vector spaces.

(iii) Since the bilinear map \( A_1 \times A_1 \to A_2 \cong K \) is nondegenerate there exist \( K \)-linearly independent elements \( x, y \in A_1 \) such that \( xy = 0 \) and \( x^2 \neq 0 \), \( y^2 \neq 0 \). Consider the \( K \)-algebra homomorphism \( \psi : K[X, Y] \to A \) defined by the conditions \( \psi(X) = x \), \( \psi(Y) = y \). Then \( XY, X^2 - \lambda Y^2 \in \ker \psi \) for some \( \lambda \in K \setminus \{0\} \). Comparing the dimensions of \( K[X, Y]/(XY, X^2 - \lambda Y^2) \) and \( A \) we get an isomorphism \( A \cong K[X, Y]/(XY, X^2 - \lambda Y^2) \).

**Theorem 2.10.** Let \( P \) be an \( n \)-dimensional Gorenstein polytope of degree 2 such that its \( h^* \)-vector is \((1, a, 1)\) where \( a \in \{0, 1, 2\} \), i.e., \( \text{Vol}_h(P) = a + 2 \leq 4 \). Assume that \( P \) is not a pyramid over an \((n-1)\)-dimensional Gorenstein polytope. We write the graded Gorenstein ring \( S_P \) as quotient of a polynomial ring \( K[x_1, \ldots, x_l] \) modulo an ideal \( J \) generated by binomials. Then \( n \leq 5 \) and there exist up to isomorphism exactly the following 15 possibilities for \( P \):

1. \( \dim P = 2 \); \( \Delta_n, 1 \leq i \leq 4 \);
2. \( \dim P = 3 \); \( P_i, 1 \leq i \leq 7 \);
3. \( \dim P = 4 \); \( Q_i, 1 \leq i \leq 3 \);
4. \( \dim P = 5 \); \( R_1 \).

The corresponding binomial relations in \( S_P \) are presented in the table below

| \( \Delta_1 \) | \( x_1x_2x_3 = x_1^4 \) | \( P_4 \) | \( x_1x_2 = x_3^2, x_1x_4 = x_5x_6 \) |
| \( \Delta_2 \) | \( x_1x_2 = x_3^2, x_3x_4 = x_5^2 \) | \( P_5 \) | \( x_1x_2 = x_3^2, x_3x_4 = x_5x_6 \) |
| \( \Delta_3 \) | \( x_1x_2 = x_3^2, x_3x_4 = x_4x_5 \) | \( P_6 \) | \( x_1x_2 = x_3x_4, x_1x_5 = x_2x_6 \) |
| \( \Delta_4 \) | \( x_1x_2 = x_3^2, x_3x_4 = x_4^2 \) | \( P_7 \) | \( x_1x_2 = x_3x_5, x_1x_2 = x_3x_4 \) |
| \( P_1 \) | \( x_1x_2x_3x_4 = x_5^2 \) | \( Q_1 \) | \( x_1x_2x_3 = x_4x_5x_6 \) |
| \( P_2 \) | \( x_1x_2x_3 = x_4x_5^2 \) | \( Q_2 \) | \( x_1x_2 = x_3x_4, x_5x_6 = x_7x_8 \) |
| \( P_3 \) | \( x_1x_2 = x_3^2, x_4x_5 = x_6^2 \) | \( Q_3 \) | \( x_1x_2 = x_3x_4, x_5x_6 = x_7x_8 \) |
| \( R_1 \) | \( x_1x_2 = x_3x_4, x_5x_6 = x_7x_8 \) |

**Proof.** Let us consider all three values of \( a \) separately.

Case 1: \( a = 0 \). By Proposition 2.5, the graded \( K \)-algebra \( S_P \) is minimally generated by \( n + 1 \) elements of degree 1 and by one element of degree 2. First we note that \( n \geq 3 \), because for any 2-dimensional lattice polytope \( \Delta \) the graded \( K \)-algebra \( S_\Delta \) is generated by elements of degree 1. So \( S_P \) has \( n + 2 \geq 5 \) generators. Since \( \dim(S_P) = 1 \) the polytope \( P \) contains exactly \( n + 1 \) lattice points. These lattice points must be vertices, because \( \dim P = n \). Thus, \( P \) is an \( n \)-dimensional lattice simplex. Let \( x_1, \ldots, x_{n+1} \) be variables in the polynomial ring \( R = K[x_1, \ldots, x_{n+1}, x_{n+2}] \) whose images in \( S_P \) are monomials \( x_1^{m_1} \) corresponding
to vertices \( v_1, \ldots, v_{n+1} \) of \( P \). Since \( \deg x_{n+2} = 2 \), the image of \( x_{n+2} \) in \( S_P \) is a monomial corresponding to an irreducible lattice point \( v_{n+1} \in 2P \cap M \) which is not an integral nonnegative linear combination of \( v_1, \ldots, v_{n+1} \). Since \( P \) is a convex hull of \( v_1, \ldots, v_{n+1} \), there exist nonnegative rational numbers \( c_1, \ldots, c_{n+1} \) such that

\[
\sum_{i=1}^{n+1} c_i (v_i, 1) = (v_{n+2}, 2).
\]

By Proposition 2.7, there is only one binomial relation among \( n + 2 \) generators of \( S_P \) and this relation has degree 4. This implies, \( 2c_i \in \mathbb{Z} \) (\( i = 1, \ldots, n + 1 \)) and

\[
\begin{align*}
P_1 & : x_1x_2x_3x_4 = x_3^2 \\
P_2 & : x_1x_2x_3 = x_4x_5^2 \\
P_3 & : x_1x_2 = x_3^2, \ x_4x_5 = x_6^2 \\
P_4 & : x_1x_2 = x_3^2, \ x_1x_4 = x_5x_6 \\
P_5 & : x_1x_2 = x_3^2, \ x_3x_4 = x_5x_6 \\
P_6 & : x_1x_2 = x_3x_4, \ x_1x_5 = x_2x_6 \\
P_7 & : x_1x_2 = x_5x_6, \ x_1x_2 = x_3x_4
\end{align*}
\]
the single binomial relation in \( S_p \) has form \( \prod_{i=1}^{n+1} x_i^{2c_i} = x_{n+2}^2 \). Since \( P \) is not a pyramid, by Proposition 2.6, \( 2c_i \geq 1 \) \((i = 1, \ldots, n + 1)\). On the other hand, \( \sum_{i=1}^{n+1} 2c_i = 4 = \deg x_{n+2}^2 \). This implies \( n + 1 \leq 4 \). Thus, 3 is the only possible value of \( n \) and \( 2c_2 = 2c_3 = 2c_4 = 1 \). So we come to the binomial relation \( x_1 x_2 x_3 x_4 - x_5^2 = 0 \). This relation defines a 3-dimensional Gorenstein simplex, which we denote by \( P_1 \).

Case 2: \( a = 1 \). By Propositions 2.5 and 2.7, the graded \( K \)-algebra \( S_p \) is minimally generated by \( n + 2 \) elements of degree 1 satisfying a single cubic binomial relation. Such a binomial relation includes at most 6 variables from \( \{x_1, \ldots, x_{n+2}\} \subset R \). By Proposition 2.6, this implies \( n + 2 \leq 6 \), i.e., \( n \leq 4 \). Moreover, if \( n = 4 \), then up to renumeration of variables this binomial must be \( x_1 x_2 x_3 - x_4 x_5 x_6 \) (polytope \( Q_1 \)). If \( n = 2 \), then \( P \) is a reflexive triangle and the binomial has form \( x_1 x_2 x_3 - x_4^2 \) (polytope \( \Delta_1 \)). By Proposition 2.6, if \( n = 3 \) then at least one monomial in the binomial is product of 3 different variables. Without loss of generality, we can assume that this monomial is \( x_1 x_2 x_3 \). Then the second monomial must have form \( x_2^2 x_j \). Up to renumeration of variables we obtain the binomial relation \( x_1 x_2 x_3 - x_4 x_2^2 \), which defines a 3-dimensional Gorenstein polytope \( P_2 \).

Case 3: \( a = 2 \). By Propositions 2.5 and 2.7, the graded \( K \)-algebra \( S_p \) is minimally generated by \( n + 3 \) elements of degree 1 satisfying two quadratic binomial relations. These two quadratic relations include at most 8 variables from \( \{x_1, \ldots, x_{n+3}\} \subset R \). By Proposition 2.6, this implies \( n + 3 \leq 8 \), i.e., \( n \leq 5 \). The quadratic monomials appearing in the binomials are either \( x_i^2 \), or \( x_i x_j \). It is not difficult to enumerate all possibilities and get the corresponding Gorenstein polytopes

\[
\Delta_2, \Delta_3, \Delta_4, P_3, P_4, P_5, P_6, P_7, Q_2, Q_3, R_1.
\]

\( \square \)

3. GORENSTEIN POLYTOPES OF DEGREE 2

Let \( P \subset M_\mathbb{R} \) be an \( n \)-dimensional Gorenstein polytope of degree 2 with \( h^* \)-polynomial \( h^*_p(t) = 1 + at + t^2 \), \( a \geq 0 \). Then we have

\[
i(P, k) = \binom{n+k}{n} + \binom{n+k-1}{n} + \binom{n+k-2}{n} = \frac{1}{k!} (a+2)k^n + \text{terms of lower order},
\]

\[
|P \cap M| = i(P, 1) = \binom{n+1}{n} + \binom{n}{n} + \binom{n-1}{n} = n + a + 1.
\]

In particular,

\[
\text{Vol}_S(P) = a + 2 = |P \cap M| - n + 1.
\]

**Proposition 3.1.** Let \( P \subset M_\mathbb{R} \) be an \( n \)-dimensional Gorenstein polytope of degree 2. Then for any lattice subpolytope \( P' \subset P \) one has \( \text{Vol}_S(P') \leq |P' \cap M| - n + 1 \). Moreover, one has:

1. \( \text{Vol}_S(P') = |P' \cap M| - n + 1 \) if and only if \( P' \) a Gorenstein polytope of degree 2;
2. \( \text{Vol}_S(P') = |P' \cap M| - n \) if and only if \( \deg P' \leq 1 \).
Proof. We use the formula for the $h^*$-polynomial of $P$
\[ h^*_P(t) = 1 + (|P \cap M| - n - 1)t + t^2 \]
and the equality $\text{Vol}_N(P) = h^*_P(1) = |P \cap M| - n + 1$. By Remark 1.2, we obtain that either $\deg P' = 2$ and $P'$ is a Gorenstein polytope (see Corollary 1.15), or $\deg P' \leq 1$. In the first case, we obtain $\text{Vol}_N(P') = |P' \cap M| - n + 1$ (as for $P$ above). In the second case, the $h^*$-polynomial of $P'$ is linear and we obtain $\text{Vol}_N(P') = h^*_{P'}(1) = |P' \cap M| - n$.
\[ \square \]
Let us make some general remarks about pyramids:

**Proposition 3.2.** an $n$-dimensional Gorenstein polytope $P$ of degree 2 is a pyramid $\Pi(P')$ over an $(n - 1)$-dimensional Gorenstein polytope $P'$ of degree 2 if and only if there exists a proper face $Q \subset P$ such that $\deg Q = 2$.

Proof. One direction of this statement (i.e., if $P$ is a pyramid) is obvious.

Now we assume that a proper face $Q \subset P$ such that $\deg Q = 2$. Then $Q$ is contained in a facet $\Gamma \subset P$. By Remark 1.2, $\deg Q \leq \deg \Gamma \leq \deg P$ and we conclude that $\deg \Gamma = 2$. Thus, without loss of generality we can assume that $Q$ has codimension 1 (i.e., $Q = \Gamma$). Moreover, since $h^*$-polynomial is monotone, the leading coefficient of $h^*$-polynomial of $\Gamma$ must be 1. Thus, $\Gamma$ is a Gorenstein polytope of degree 2 (see Corollary 1.15). Let $p$ be the unique interior lattice point of $(n - 2)\Gamma$. First we show that there exists only one lattice point $q$ of $P$ which is not contained in $\Gamma$. If there were two different lattice points $q_1, q_2 \in P \setminus \Gamma$, then we would have two different lattice points $q_1 + p$ and $q_2 + p$ in the interior of $(n - 1)P$ (contradiction). Therefore, $q$ is a vertex of $P$ which is combinatorially equivalent to a pyramid over $\Gamma$. It remains to show that the integral height $h$ of the vertex $q$ in this pyramid is 1. One has $\text{Vol}_N(P) = h \text{Vol}_N(\Gamma)$. One the other hand, by Proposition 3.1, we have
\[ \text{Vol}_N(P) = |P \cap M| - n + 1, \quad \text{Vol}_N(\Gamma) = |\Gamma \cap M| - n. \]
We have already shown that $|P \cap M| = |\Gamma \cap M| + 1$. Therefore, $\text{Vol}_N(P) = \text{Vol}_N(\Gamma)$ and $h = 1$.
\[ \square \]

**Corollary 3.3.** A Gorenstein polytope $P$ of degree 2 is not a pyramid if and only if all proper faces of $P$ and $P^*$ have degree $\leq 1$.

Proof. It follows immediately from Remark 1.20 and Proposition 3.2.
\[ \square \]

From now until the end of this section we assume that $P \subset M_R$ is a 3-dimensional Gorenstein polytope of degree 2.

**Proposition 3.4.** If $P$ is a 3-dimensional Gorenstein polytope of degree 2, then
\[ \sum_{E \subset P, \dim E = 1} \text{Vol}_N(E) \cdot \text{Vol}_N(E^*) = 12. \]

Proof. Let $\Delta$ be a reflexive polytope of dimension 3. Then
\[ \sum_{F \subset \Delta, \dim F = 1} \text{Vol}_N(F) \cdot \text{Vol}_N(F^*) = 24, \]
where \( F^* \subset \Delta \) denote the dual to \( F \) 1-dimensional face of the dual reflexive polytope \( \Delta^* \). This equality follows from the formula for the Euler number of Calabi–Yau hypersurfaces in toric varieties [BD96, Cor. 7.10] and the fact that the Euler number of \( K3 \) surfaces is 24 (see also [Haa05]).

We apply this formula to the Gorenstein polytope \( \Delta = 2P \). Then \( F = 2E \) and hence \( \text{Vol}_N(F) = 2 \text{Vol}_N(E) \). On the other hand, by Remark 1.26, the dual 1-dimensional faces \( F^* \) and \( E^* \) are isomorphic as lattice polytopes, i.e., \( \text{Vol}_N(F^*) = \text{Vol}_N(E^*) \). So we get

\[
24 = \sum_{F \subset \Delta, \text{dim } F = 1} \text{Vol}_N(F) \cdot \text{Vol}_N(F^*) = \sum_{E \subset P, \text{dim } E = 1} 2 \text{Vol}_N(E) \cdot \text{Vol}_N(E^*).
\]

Dividing by 2, we obtain the required equality.

\[\square\]

**Example 3.5.** Let \( \Delta \) be a 2-dimensional reflexive polytope and \( \Delta^* \) its dual. By Remark 1.20, \( P := \Pi(\Delta) \) and \( P^* := \Pi(\Delta^*) \) are two dual to each other 3-dimensional Gorenstein polytopes of degree 2. A 1-dimensional face \( E \subset P \) is either an edge connecting the vertex of the pyramid \( \Pi(\Delta) \) to a vertex \( v \in \Delta \) (\( \text{Vol}_N(E) = 1 \)), or a 1-dimensional face of \( \Gamma_j \subset \Delta \). Then the dual 1-dimensional face \( E^* \subset P^* \) is respectively either 1-dimensional face \( v^*_i \subset \Delta^* \), or an edge connecting the vertex of the pyramid \( \Pi(\Delta^*) \) with dual vertex \( \Gamma^*_j \in \Delta^* \) (\( \text{Vol}_N(E^*) = 1 \)). Therefore

\[
\sum_{E \subset P, \text{dim } E = 1} \text{Vol}_N(E) \cdot \text{Vol}_N(E^*) = \sum_i 1 \cdot \text{Vol}_N(v^*_i) + \sum_j \text{Vol}_N(\Gamma_j) \cdot 1.
\]

By Proposition 1.27, we have

\[
\text{Vol}_N(\Delta) = \sum_j \text{Vol}_N(\Gamma_j), \quad \text{Vol}_N(\Delta^*) = \sum_i \text{Vol}_N(v^*_i).
\]

So for 3-dimensional Gorenstein polytopes \( P := \Pi(\Delta) \) and \( P^* := \Pi(\Delta^*) \) the equality in Proposition 3.4 is equivalent to the well-known identity for reflexive polygons:

\[
\text{Vol}_N(\Delta) + \text{Vol}_N(\Delta^*) = 12.
\]

**Example 3.6.** Another example is shown in Figure 3 below. The Gorenstein simplex \( P \) has 6 edges \( E_1, \ldots, E_6 \) (\( \text{Vol}_N(E_i) = 1 \)). Its dual polytope \( P^* \) has also 6

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{fig3.png}
\caption{Dual Gorenstein polytopes in dimension 3}
\end{figure}
edges $E_1^\ast, \ldots, E_6^\ast$ ($\text{Vol}_N(E_i^\ast) = 2$). Therefore, we have
\[
\sum_{E \subseteq P, \dim E = 1} \text{Vol}_N(E) \cdot \text{Vol}_N(E^\ast) = \sum_{i=1}^{6} \text{Vol}_N(E_i) \cdot \text{Vol}_N(E_i^\ast) = 6 \cdot (1 \cdot 2) = 12.
\]

For any face $F \subseteq P$ we denote by $\text{Int}(E)$ the number of lattice points in the relative interior of $F$.

**Proposition 3.7.** Let $P$ be a 3-dimensional Gorenstein polytope of degree 2 which is not a pyramid. Then
\[
|P \cap M| + |P^\ast \cap N| + \sum_{E \subseteq P} \text{Int}(E) \cdot \text{Int}(E^\ast) = 14.
\]

**Proof.** If $P$ is not a pyramid, then all lattice points of $P$ are contained in edges (see Corollary 3.3), i.e.,
\[
|P \cap M| = v + \sum_{E \subseteq P} \text{Int}(E),
\]
where $v$ denotes the number of vertices of $P$. Denote by $e$ and $f$ the numbers of edges and faces of $P$ respectively. By Euler formula, $e = v + f - 2$. Applying Proposition 3.4 and $\text{Vol}_N(E) = \text{Int}(E) + 1$, one has
\[
12 = \sum_{i=1}^{e} (\text{Int}(E_i) + 1)(\text{Int}(E_i^\ast) + 1)
\]
\[
= e + \sum_{i=1}^{e} (\text{Int}(E_i) + \text{Int}(E_i^\ast) + \text{Int}(E_i) \cdot \text{Int}(E_i^\ast))
\]
\[
= v + f - 2 + \sum_{i=1}^{e} \text{Int}(E_i) + \sum_{i=1}^{e} \text{Int}(E_i^\ast) + \sum_{i=1}^{e} \text{Int}(E_i) \cdot \text{Int}(E_i^\ast).
\]
Using $|P \cap M| = \sum_{i=1}^{e} \text{Int}(E_i) + v$ and $|P^\ast \cap N| = \sum_{i=1}^{f} \text{Int}(E_i^\ast) + f$, we obtain the statement. \qed

**Example 3.8.** The dual to each other polytopes $P$ and $P^\ast$ in Example 3.6 contain respectively 4 and 10 lattice points. Since all lattice points of $P$ are vertices, so we obtain $\sum_{E \subseteq P} \text{Int}(E) \cdot \text{Int}(E^\ast) = 0$, i.e.,
\[
|P \cap M| + |P^\ast \cap N| + \sum_{E \subseteq P} \text{Int}(E) \cdot \text{Int}(E^\ast) = 4 + 10 + 0 = 14.
\]

**Proposition 3.9.** If $P$ is a 3-dimensional Gorenstein polytope of degree 2 which is not a pyramid, then
\[
\text{Vol}_N(P) + \text{Vol}_N(P^\ast) \leq 10.
\]

**Proof.** Recall that $\text{Vol}_N(P) = |P \cap M| - 2$ and $\text{Vol}_N(P^\ast) = |P^\ast \cap M| - 2$. By Proposition 3.7 and by the obvious inequality
\[
\sum_{E \subseteq P} \text{Int}(E) \cdot \text{Int}(E^\ast) \geq 0,
\]
we obtain the statement. \qed
4. Minimal Gorenstein Polytopes and the Classification

Definition 4.1. Let $P \subset M_{\mathbb{R}}$ be an arbitrary $n$-dimensional Gorenstein polytope of degree 2 with vertices $\{v_1, \ldots, v_k\}$. We call $P$ minimal if for all $j \in \{1, \ldots, k\}$ the convex hull of the set $\{v_1, \ldots, v_k\} \setminus \{v_j\}$ is not a Gorenstein polytope of degree 2. Analogously, we call $P$ maximal, if for any lattice point $v_{k+1} \in M \setminus \{v_1, \ldots, v_k\}$ the convex hull of the lattice points $\{v_1, \ldots, v_k, v_{k+1}\}$ is not Gorenstein polytope of degree 2.

Remark 4.2. If an $n$-dimensional Gorenstein polytope $P$ of degree 2 is isomorphic to a subpolytope of another $n$-dimensional Gorenstein polytope $Q$ of degree 2, then we write $P \preceq Q$. If $P \subseteq Q$, then we immediately obtain that both reflexive polytopes $(n-1)P$ and $(n-1)Q$ contain a common unique interior lattice point $m \in M$. By Remark 1.25, we obtain that the dual Gorenstein polytope $Q^*$ is naturally contained in the dual polytope $P^*$, i.e., $Q^* \preceq P^*$. Therefore the duality $P \leftrightarrow P^*$ establishes a bijection between the set of minimal and the set of maximal Gorenstein polytopes:

$$\{\text{minimal Gorenstein polytopes}\} \leftrightarrow \{\text{maximal Gorenstein polytopes}\}.$$

Proposition 4.3. Let $v_i \in P$ be an arbitrary vertex of an $n$-dimensional Gorenstein polytope $P$ of degree 2 and let $\Gamma_i^* \subset P^*$ be the facet of the dual polytope $P^*$ which is dual to $v_i$. Then

$$P_i := \text{conv}(\{P \cap M\} \setminus \{v_i\})$$

is a Gorenstein polytope of degree 2 if and only if the facet $\Gamma_i^*$ is a standard $(n-1)$-dimensional basic simplex. In particular, $P$ is minimal if and only if $\text{Vol}_S(\Gamma^*) \geq 2$ for all facets $\Gamma^* \subset P^*$.

Proof. By Proposition 3.1, we obtain that $P_i$ is Gorenstein of degree 2 if and only if

$$\text{Vol}_S(P_i) = |P_i \cap M| - n + 1 = |P \cap M| - n - 2 = \text{Vol}_S(P) - 1.$$

The latter holds true if and only if $P_i$ is obtained from $P$ by cutting out an $n$-dimensional lattice polytope $S_i$ with $\text{Vol}_S(S_i) = 1$ (i.e., $S_i$ is a basic simplex). In this case, the simplex $S_i$ is a convex hull of $v_i \in P$ and a simplicial facet $\Gamma_i \subset P_i$, where $\text{Vol}_S(\Gamma_i) = 1$, i.e., $\Gamma_i$ is a standard $(n-1)$-dimensional basic simplex. $\square$

Corollary 4.4. If $P$ and $Q$ two $n$-dimensional Gorenstein polytopes of degree 2 such that $Q \preceq P$. Then

$$\text{Vol}_S(P) + \text{Vol}_S(P^*) = \text{Vol}_S(Q) + \text{Vol}_S(Q^*).$$

Proof. We use the induction on $k = |P \cap M| - |Q \cap M|$. If $k = 0$ then $P = Q$ and there is nothing to prove. If $k \geq 1$ then $P \neq Q$ and there exists a vertex $v_i \in P$ which is not contained in $Q$. Let $P_i := \text{conv}(\{P \cap M\} \setminus \{v_i\})$. Then $Q \preceq P_i$. Moreover, the proof of Proposition 4.3 shows that $P_i$ is obtained from $P$ by cutting out an $n$-dimensional lattice basic simplex $S_i = \text{conv}(\Gamma_i, v_i)$. Let $v_i^*$ be the dual vertex of $P_i$ corresponding to the facet $\Gamma_i \subset P_i$. Then

$$P^* := \text{conv}(\{P_i^* \cap M\} \setminus \{v_i^*\}).$$
As in the proof of Proposition 4.3, we obtain
\[ \text{Vol}_N(P^*) = \text{Vol}_N(P'_i) - 1. \]
Together with \( \text{Vol}_N(P'_i) = \text{Vol}_N(P) - 1 \) this implies
\[ \text{Vol}_N(P) + \text{Vol}_N(P^*) = \text{Vol}_N(P'_i) + \text{Vol}_N(P'_i). \]
Since \( Q \not\simeq P_i \) and \( |P_i \cap M| - |Q \cap M| < |P \cap M| - |Q \cap M| \) we can apply the induction hypothesis and get
\[ \text{Vol}_N(P_i) + \text{Vol}_N(P^*_i) = \text{Vol}_N(Q) + \text{Vol}_N(Q^*). \]
This proves the statement. \( \square \)

The main purpose of this section is to prove the following result:

**Theorem 4.5.** Let \( P \) be a minimal \( n \)-dimensional Gorenstein polytope of degree 2. Then \( \text{Vol}_N(P) \leq 4 \).

**Remark 4.6.** If \( P \) is a pyramid \( \Pi(P') \) over an \((n-1)\)-dimensional Gorenstein polytope \( P' \) of degree 2, then \( \text{Vol}_N(P) = \text{Vol}_N(P') \) and minimality of \( P \) is equivalent to minimality of \( P' \). Therefore it is sufficient to prove Theorem 4.5 for Gorenstein polytopes of degree 2 which are not pyramids.

**Lemma 4.7.** Let \( P \) be an \( n \)-dimensional Gorenstein polytope of degree 2 and \( P^* \) its dual. Assume that
\[ \text{Vol}_N(P^*) \leq 4. \]
Then either \( P \) is not minimal, or \( \text{Vol}_N(P) \leq 4 \).

**Proof.** We can apply the classification of all \( n \)-dimensional Gorenstein polytopes \( P^* \) of degree 2 satisfying the condition \( \text{Vol}_N(P^*) \leq 4 \) (see Theorem 2.10).

If \( \text{Vol}_N(P^*) \in \{2, 3\} \) a direct check via this classification shows that all facets of \( P^* \) are standard simplices. Thus, by Proposition 4.3, we obtain that \( P = (P^*)^* \) is not not minimal.

If \( \text{Vol}_N(P^*) = 4 \), then \( P^* \) may contain facets which are not standard simplices, but if \( \text{Vol}_N(P) \geq 5 \) there always exists at least one facet which is a standard simplex. By Proposition 4.3, this implies that \( P \) is not minimal. \( \square \)

By Lemma 4.7, the statement of Theorem 4.5 follows from:

**Theorem 4.8.** Let \( P \) be a minimal \( n \)-dimensional Gorenstein polytope of degree 2. Then at least one of two inequalities \( \text{Vol}_N(P) \leq 4 \) and \( \text{Vol}_N(P^*) \leq 4 \) holds.

We prove Theorem 4.8 in several steps:

Let \( P \) be a minimal \( n \)-dimensional Gorenstein polytope of degree 2. We fix a vertex \( v_i \in P \) and the facet \( \Gamma_i^* \subset P^* \) which is dual to \( v_i \). Without loss of generality we assume that \( v_i = 0 \), i.e., \( C(v_i) \supset P \) is a cone with vertex 0.

Since \( P^* \) is not a pyramid, by Corollary 3.3, we obtain that \( \Gamma_i^* \) is an \((n-1)\)-dimensional Gorenstein polytope of degree \( \leq 1 \). By Proposition 4.3, the facet \( \Gamma_i^* \) can not have degree 0, i.e., \( \Gamma_i^* \) is not an \((n-1)\)-dimensional basic lattice simplex. Therefore, we have \( \deg \Gamma_i^* = 1 \). By the complete classification of lattice polytopes of
degree 1 (see Theorem 1.5), \( \Gamma^*_1 \) is either an \((n-1)\)-dimensional exceptional simplex, or a Lawrence prism with \( n \) vertices.

**Lemma 4.9.** If \( \Gamma^*_1 \) is an \((n-1)\)-dimensional exceptional simplex, then \( \text{Vol}_3(\Gamma^*_1) = 2 \).

**Proof.** Using Proposition 1.29, one obtains that the cone \( C(v_i) \supset P \) is isomorphic to the simplicial cone generated by \( 2e_1 - e_2 - e_3, e_2, e_3, \ldots, e_n \) with \( v_i \) and \( P \). By minimality, we have \( P = P'_i \). Therefore \( \text{Vol}_3(P) = \text{Vol}_3(P'_i) = 2 \).

**Lemma 4.10.** If \( \Gamma^*_1 \) is an \((n-1)\)-dimensional Lawrence prism with \( h_1, \ldots, h_{n-1} \), then \( \text{Vol}_3(\Gamma^*_1) = 3 \), then \( \text{Vol}_3(P) \in \{3, 4\} \).

**Proof.** Using Proposition 1.29, one obtains that the cone \( C(v_i) \supset P \) is isomorphic to the cone generated by \( n+1 \) vectors

\[
e_1, \ldots, e_n, -e_n + \sum_{i=1}^{n-1} h_i e_i,
\]

where \( e_1, \ldots, e_n \) is a basis of \( M \). We denote by \( P'_i \) the convex hull of 0 and \( n+1 \) lattice points \( e_1, \ldots, e_n, -e_n + \sum_{i=1}^{n-1} h_i e_i \). Then a direct calculation shows that \( \text{Vol}_3(P'_i) = \sum_{i=1}^{n-1} h_i \). Since \( P'_i \) lies between two parallel hyperplanes \( z_n = \pm 1 \), the intersections of \( P'_i \) with these hyperplanes are exactly two vertices \( p := e_n \) and \( q := -e_n + \sum_{i=1}^{n-1} h_i e_i \), all remaining lattice points of \( P'_i \) are contained in the hyperplane \( \{ z_n = 0 \} \). We want to show that the set \( M \cap P' \cap \{ z_n = 0 \} \) is either \( \{0, e_1, \ldots e_{n-1}\} \), or it possibly has one more lattice point \((p+q)/2 \) if \((p+q)/2 \in M \). Indeed, by Corollary 3.3 all lattice points of \( P \) are contained in \( 1 \)-dimensional faces of \( P \). Since \( P'_i \subset P \), the same is true also for \( P'_i \), i.e., all lattice points of \( P'_i \) are contained in \( 1 \)-dimensional faces of \( P'_i \). But \( P'_i \) is a convex hull of \( n+2 \) lattice points and all segments connecting two of them (except maybe \((p, q)\)) do not have interior lattice points, because the convex hull of \( \{0, e_1, \ldots e_{n-1}\} \) is a basic \((n-1)\)-dimensional simplex in the hyperplane \( \{ z_n = 0 \} \) and the \( z_n \)-coordinates of \( p \) and \( q \) are 1 and \(-1 \) respectively. We note that \( \{ z_n = 0 \} \cap \{ p, q \} = (p+q)/2 = (\sum_{i=1}^{n-1} h_i e_i)/2 \). Thus, we obtain \( |P'_i \cap M| \leq n+3 \). By Proposition 3.1, the latter implies \( \text{Vol}_3(P'_i) \leq |P'_i \cap M| - n + 1 \leq 4 \), i.e., \( \sum_{i=1}^{n-1} h_i \in \{3, 4\} \). If \( \sum_{i=1}^{n-1} h_i = 3 \), then \((p+q)/2 \notin M \) can not be a lattice point of \( P'_i \) and \( |P'_i \cap M| = n+2 \). Together with \( \text{Vol}_3(P'_i) = 3 \) this implies that \( P'_i \) is a Gorenstein polytope with \( h^\ast \)-polynomial \( 1+t+t^2 \). If \( \sum_{i=1}^{n-1} h_i = 4 \), then by Proposition 3.1, \( P'_i \) must be a Gorenstein polytope of degree 2, i.e., \( |P'_i \cap M| = n+3 \), \((p+q)/2 \) is a lattice point of \( P'_i \), and \( \text{Vol}_3(P'_i) = 4 \). By minimality, we have in both cases \( n \leq \text{Vol}_3(P) \), i.e., \( \text{Vol}_3(P) \in \{3, 4\} \). \( \square \)

The following statement completes the proof of Theorem 4.8 and Theorem 4.5:

**Lemma 4.11.** Assume that \( P \) is a minimal Gorenstein polytope of degree 2 such that every codimension-1 face \( \Gamma^*_1 \subset P^* \) is a Lawrence prism with \( \text{Vol}_3(\Gamma^*_1) = 2 \). Then \( \text{Vol}_3(P) \leq 4 \), or \( \text{Vol}_3(P^*) \leq 4 \).
Proof. In this situation, all generators of the cone $C(v_i)$ are contained in an affine hyperplane $H_i$ which divides $P$ into two lattice polytopes $P_i = \text{conv}(\{P \cap M \setminus \{v_i\}\})$ and a Lawrence prism $P'_i$ containing the vertex $v_i$ such that $\text{Vol}_3(P'_i) = 2$. Moreover, $P'_i$ is a pyramid with vertex $v_i$ over the $(n - 1)$-dimensional Laurence prism $P'_i = H_i \cap P$.

By minimality, $P_i$ must have degree 1 and by Theorem 1.5 we have the following two cases:

Case 1: The polytope $P_i$ is an exceptional simplex, i.e., $\text{Vol}_3(P_i) = 4$. Then its facet $P_i'$ is also a simplex. Moreover, $P_i'$ must be also a simplex, since $P_i'$ is a pyramid over $P_i$. Let $k$ be the number of vertices of $P$ (this is also the number of facets in $P'$). The whole polytope $P$ is a union of simplices $P_i$ and $P'_i$. Therefore $k \leq n + 2$. By Proposition 1.27, we have

$$\text{(n - 1) Vol}_3(P^*) = \sum_{i=1}^{k} \text{Vol}_3(\Gamma_i^*) \leq 2(n + 2).$$

If $n \geq 4$, then we obtain $\text{Vol}_3(P^*) \leq 2 + 6/(n - 1) \leq 4$. If $n = 3$, then $\text{Vol}_3(P) = \text{Vol}_3(P') + \text{Vol}_3(P_i) = 2 + 4 = 6$. By Proposition 3.9, we have $\text{Vol}_3(P^*) \leq 4$.

Case 2: The polytope $P_i$ is a Lawrence prism. Since $P_i'$ and $P_i$ are both Lawrence prisms and $\text{Vol}_3(P_i') = \text{Vol}_3(P_i) = 2$ the number of vertices of $P_i'$ is at most $n + 1$. On the other hand, $P_i'$ is a facet of $P_i$ which is not a basic $(n - 1)$-dimensional simplex. Therefore $P_i$ contains at most 2 more vertices than in $P_i'$. Together with $v_i \in P_i'$ we see that $P = P_i' \cup P_i$ contains $k \leq n + 4$ vertices. By Proposition 1.27, we have

$$\text{(n - 1) Vol}_3(P^*) = \sum_{i=1}^{k} \text{Vol}_3(\Gamma_i^*) \leq 2(n + 4).$$

If $n \geq 5$, then we obtain $\text{Vol}_3(P^*) \leq 2 + 10/(n - 1) < 5$, i.e. $\text{Vol}_3(P^*) \leq 4$.

If $n = 4$, then $(n - 1) \text{Vol}_3(P^*) = \sum_{i=1}^{k} \text{Vol}_3(\Gamma_i^*)$ together with $\text{Vol}_3(\Gamma_i^*) = 2$ implies that the number $k$ of vertices of $P$ must be divisible by 3. Since $k \leq n + 4 = 8$, we obtain $k = 6$ and $\text{Vol}_3(P^*) = 4$.

If $n = 3$, then it remains to exclude only the case $\text{Vol}_3(P) = \text{Vol}_3(P^*) = 5$ (all other cases follow from Proposition 3.9). By Proposition 1.27, in the last case both polytopes $P$ and $P^*$ must have 5 vertices. There exist exactly two different combinatorial types of 3-dimensional polytopes $P$ with 5 vertices, but only for one of these types the dual polytope $P^*$ has also 5 vertices. Therefore, both polytopes $P$ and $P^*$ must be combinatorially equivalent to a pyramid over a lattice 4-gon $S$ such that $\text{Vol}_3(S) = 2$. This implies that $\text{Vol}_3(P) = \text{Vol}_3(P^*) = 5$ is divisible by $\text{Vol}_3(S) = 2$. Contradiction. □

Proposition 4.12. Every $n$-dimensional Gorenstein polytope $P \neq 2S_3$ of degree 2 is a Cayley polytope $P = \Delta_1 + \cdots + \Delta_r$, where $\Delta_1, \ldots, \Delta_r$ are plane lattice polytopes such that the Minkowski sum $\Delta = \Delta_1 + \cdots + \Delta_r$ is a reflexive polygon.

Proof. It is enough to assume that $P$ is not a pyramid, because a pyramid over a Cayley polytope is again a Cayley polytope. On the other hand, every $n$-dimensional subpolytope of a Cayley polytope $\Delta_1 + \cdots + \Delta_r$ is again a Cayley polytope.
$\Delta'_i \cdots \Delta'_r$, where $\Delta'_i$ is a lattice subpolytope of $\Delta_i$ ($i = 1, \ldots, r$). By Theorem 4.5, if $P$ is a maximal Gorenstein polytope of degree 2, then $\text{Vol}_\mathbb{S}(P^*) \leq 4$, i.e., $P^*$ is one of polytopes classified in Theorem 2.10. One can check the whole list of these Gorenstein polytopes $P^*$ and obtain that among potential maximal polytopes $P$ only the 3-dimensional Gorenstein polytope $2S_3$ cannot be described as a Cayley polytope. Moreover, if $v_i$ is a vertex of $2S_3$ and $P_i$ is the convex hull of $(2S_3 \cap M) \setminus \{v_i\}$ then $P_i$ is again a Cayley polytope. This shows that any proper Gorenstein subpolytope of $2S_3$ is also a Cayley polytope. □

**Theorem 4.13.** There exist exactly 37 $n$-dimensional Gorenstein polytopes of degree 2 which are not pyramids:

1. 16 reflexive polygons;
2. 3-dimensional Gorenstein polytopes $P_1, \ldots, P_{15}$;
3. 4-dimensional Gorenstein polytopes $Q_1, \ldots, Q_5$;
4. 5-dimensional Gorenstein polytope $R_1$.

*Proof.* There are two ways to prove this statement.

The first one is to use Proposition 4.12 and the classification of all possible Minkowski sum decompositions of reflexive polygons due to Emiris and Tsigaridas [ET06].

The second one is to enumerate all Gorenstein subpolytopes of degree 2 of maximal Gorenstein polytopes of degree 2. The maximal Gorenstein polytopes are contained in the list of all $P$ such that $a_{P^*} = \text{Vol}_\mathbb{S}(P^*) - 2 \leq 2$ (see Theorem 2.10).

Both ways do not need any computer and can be easily realized. □

Below we give a complete list of $n$-dimensional ($n \geq 3$) Gorenstein polytopes $P$ of degree 2 which are not pyramids. This list includes some of combinatorial invariants of $P$ such as its $f$-vector $f = (f_0, \ldots, f_n)$ (where $f_i$ denotes the number of $i$-dimensional faces of $P$), the dual partner $P^*$, the lattice normalized volume $\text{Vol}_\mathbb{S}(P)$ as well as all possible descriptions of $P$ as a Cayley polytope.
\[
P_4 \quad f = (1, 5, 8, 5, 1) \quad P_4^* = P_{13}^* \quad \text{Vol}_8(P_4) = 4
\]

\[
P_5 \quad f = (1, 5, 9, 6, 1) \quad P_5^* = P_{10}^* \quad \text{Vol}_8(P_5) = 4
\]

\[
P_6 \quad f = (1, 6, 11, 7, 1) \quad P_6^* = P_{11}^* \quad \text{Vol}_8(P_6) = 4
\]

\[
P_7 \quad f = (1, 6, 12, 8, 1) \quad P_7^* = P_{12}^* \quad \text{Vol}_8(P_7) = 4
\]

\[
P_8 \quad f = (1, 7, 12, 7, 1) \quad P_8^* = P_8^* \quad \text{Vol}_8(P_8) = 5
\]

\[
P_9 \quad f = (1, 6, 10, 6, 1) \quad P_9^* = P_9^* \quad \text{Vol}_8(P_9) = 5
\]

\[
P_{10} \quad f = (1, 6, 9, 5, 1) \quad P_{10}^* = P_5^* \quad \text{Vol}_8(P_{10}) = 6
\]

\[
P_{11} \quad f = (1, 7, 11, 6, 1) \quad P_{11}^* = P_6^* \quad \text{Vol}_8(P_{11}) = 6
\]

\[
P_{12} \quad f = (1, 8, 12, 6, 1) \quad P_{12}^* = P_7^* \quad \text{Vol}_8(P_{12}) = 6
\]
Our classification of Gorenstein polytopes $P$ is based on the anticanonical class $-\mathcal{K}_P$.

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The list of all 4-dimensional Gorenstein non-pyramids $Q_i$ of index 2:

- $Q_1$: $f = (1, 6, 15, 18, 9, 1)$
  \[ Q_1^* = Q_4 \]
  \[ \text{Vol}_h(Q_1) = 3 \]

- $Q_2$: $f = (1, 7, 17, 18, 8, 1)$
  \[ Q_2^* = Q_5 \]
  \[ \text{Vol}_h(Q_2) = 4 \]

- $Q_3$: $f = (1, 6, 13, 13, 6, 1)$
  \[ Q_3^* = Q_3 \]
  \[ \text{Vol}_h(Q_3) = 4 \]

- $Q_4$: $f = (1, 9, 18, 15, 6, 1)$
  \[ Q_4^* = Q_1 \]
  \[ \text{Vol}_h(Q_4) = 6 \]

- $Q_5$: $f = (1, 8, 18, 17, 7, 1)$
  \[ Q_5^* = Q_2 \]
  \[ \text{Vol}_h(Q_5) = 5 \]

There exists a single 5-dimensional Gorenstein (selfdual) non-pyramid $R_1$ of index 3:

- $R_1$: $f = (1, 8, 24, 34, 8, 1)$
  \[ R_1^* = R_1 \]
  \[ \text{Vol}_h(R_1) = 4 \]

5. Toric almost Del Pezzo Manifolds

A smooth $n$-dimensional toric variety $X$ is called almost Del Pezzo manifold if the anticanonical class $-K_X$ is a semiample Cartier divisor which defines a $K_X$-trivial birational morphism $\varphi: X \to X'$, where $X'$ is a Gorenstein toric Del Pezzo variety. Our classification of Gorenstein polytopes $P$ of degree 2 in Theorem 4.13
is equivalent to a birational classification of all $n$-dimensional Gorenstein toric Del Pezzo varieties, i.e., of all $X'$. Let $P$ be an $n$-dimensional Gorenstein polytope of degree 2. We denote by $X_P$ the corresponding toric variety. First we need to understand when a given $n$-dimensional Gorenstein toric Del Pezzo variety admits a smooth crepant resolution.

**Theorem 5.1.** Let $P$ an $n$-dimensional Gorenstein polytope of degree 2 which is not a pyramid over an $(n-1)$-dimensional Gorenstein polytope. Then the Gorenstein toric Del Pezzo variety $X_P$ always admits a crepant desingularization $\hat{X}_P$.

**Proof.** Since a projective toric variety is smooth if and only if all its 0-dimensional torus orbits are nonsingular points, it is enough to analyse the singularities of 0-dimensional torus orbits in $X_P$, which correspond to vertices of the Gorenstein polytope $P$. If $v_i \in P$ such a vertex, then, by Proposition 1.29, the cone $C(v_i)$ generated by $v - v_i$ ($v \in P$) is dual to the cone $C_{\Gamma_i}$ over the dual facet $\Gamma_i \subset P^*$. By Corollary 3.3, all facets of $P^*$ are lattice polytopes of degree $\leq 1$. In [BN07, Prop. 5.1] it was shown that every lattice polytope of degree $\leq 1$ admits a unimodular triangulation. This implies existence of smooth crepant resolutions of all 0-dimensional orbits in $X_P$. □

**Example 5.2.** The statement in Theorem 5.1 is not true if $P$ is a pyramid. Let $X_{11(13)}$ be 4-dimensional Gorenstein toric Del Pezzo variety which is a cone over the 3-dimensional Gorenstein toric Del Pezzo variety $X_{P_{13}}$. Then the vertex of the cone is a terminal quotient singularity $\mathbb{A}^4/\pm \text{id}$, which does not admit a crepant resolution.

**Remark 5.3.** A smooth crepant desingularization of a Gorenstein toric Del Pezzo variety is not uniquely determined, because there might be many different unimodular triangulations of lattice polytopes of degree 1. For example, 3-dimensional Gorenstein toric Del Pezzo variety $X_{P_{13}}$ admits exactly two different crepant desingularizations related by a flop, because all except one facets of the dual polytope $P_3 = P_{13}^*$ admit a unique unimodular triangulation and the exceptional face defines an isolated conifold singularity in $X_{P_{13}}$, which has exactly two different crepant resolutions. Another example of a 3-dimensional Gorenstein toric Del Pezzo variety with many different crepant resolutions is a 3-dimensional singular cubic $X_{P_2} = \{x_1x_2x_3 - x_4x_5^2 = 0\} \subset \mathbb{P}_4$.

**Theorem 5.4.** Let $P$ be an $n$-dimensional Gorenstein polytope of degree 2 and let $(1, a_P, 1)$ be the $h^*$-vector of the dual Gorenstein polytope $P^*$. Then the Picard number of the crepant resolution $\hat{X}_P$ is equal to $a_P + 1$. In particular, if $X_P$ corresponds to a maximal Gorenstein polytope $P$, then $\text{rk} \text{Pic}(\hat{X}_P) \leq 3$.

**Proof.** Let $V$ be a smooth projective $n$-dimensional toric variety defined by a fan $\Sigma$. Denote by $\Sigma^{(1)}$ the set of all 1-dimensional cones in $\Sigma$. The Picard number of $V$ equals $|\Sigma^{(1)}| - n$ [Oda88]. Consider now the case $V = \hat{X}_P$. Then the fan $\Sigma$ defining $V$ is a simplicial subdivision of the fan defining the Gorenstein toric Del Pezzo variety $X_P$. By [Bat94], the latter is the fan of cones over proper faces of the dual reflexive polytope $((n-1)P)^*$, i.e., $|\Sigma^{(1)}| = \partial((n-1)P)^* \cap N$. By Remark 1.26, the number of $N$-lattice points in $\partial((n-1)P)^*$ equals the number of $N'$-lattice
points in the dual Gorenstein polytope $P^*$. The latter is equal to $a_{P^*} + n + 1$ (see Section 3). So we obtain $\text{rk Pic}(\hat{X}_P) = a_{P^*} + 1$. The last statement follows from the duality of Gorenstein polytopes and from the inequality $\text{Vol}_n(P^*) = 2 + a_{P^*} \leq 4$ (see Theorem 4.5).

**Theorem 5.5.** Let $P$ and $Q$ be two $n$-dimensional Gorenstein polytopes of degree 2 which are not pyramids such that $Q \nleq P$. Then, up to finitely many flops, the almost Del Pezzo manifold $\hat{X}_Q$ is obtained from the almost Del Pezzo manifold $\hat{X}_P$ by blow ups of $\text{Vol}_n(P) - \text{Vol}_n(Q)$ smooth points.

**Proof.** Consider $Q$ as a lattice subpolytope of $P$. Let $v_i \in P$ be a vertex of $P$ which is not contained in $Q$. In the proof of Proposition 4.3 it was shown that $P_i := \text{conv}(\{M \cap P\} \setminus \{v_i\})$ is another Gorenstein polytope of degree 2 containing $Q$. Moreover, $P_i$ is obtained from $P$ by cutting out an $n$-dimensional basic simplex $S_i$ ($\text{Vol}_n(S_i) = 1$) containing $v_i$ as one of its vertices, i.e., we have $\text{Vol}_n(P) - \text{Vol}_n(P_i) = 1$ and the vertex $v$ corresponds to a smooth fixed point $p(v_i) \in X_P$. Let $\Gamma_i$ be the facet of $S$ which does not contain $v_i$. Then $\Gamma_i$ is also a facet of $P_i$ and it defines a torus invariant divisor in $X_P$ which is isomorphic to $\mathbb{P}^{n-1}$ because $\Gamma_i \subset S_i$ is an $(n-1)$-dimensional basic simplex. By Corollary 4.4, we have

$$1 = \text{Vol}_n(P) - \text{Vol}_n(P_i) = \text{Vol}_n(P_i^*) - \text{Vol}_n(P^*) = a_{P^*} - a_{P^*}.$$  

Using Theorem 5.4, we obtain

$$\text{rk Pic}(\hat{X}_P) - \text{rk Pic}(\hat{X}_P) = 1.$$

Choose an unimodular triangulation of facets of $P^*$ defining a crepant birational morphism $\hat{X}_P \to X_P$. We can extend this triangulation to an unimodular triangulation of faces of $P_i^* \supset P^*$ which defines a crepant birational morphism $\hat{X}_{P_i} \to X_{P_i}$. The compatibility of triangulations defines a birational morphism $\phi: \hat{X}_{P_i} \to \hat{X}_P$ which contracts a divisor $D \cong \mathbb{P}^{n-1} \subset X_{P_i}$ to a smooth point $p(v_i)$. Thus, $\hat{X}_{P_i}$ is a blow up of a smooth point in $\hat{X}_P$. We can repeat the same procedure $\text{Vol}_n(P) - \text{Vol}_n(Q)$ times and obtain the statement. $\square$

Theorems 5.1, 5.4, 5.5 show that our combinatorial method in the biregular classification of all Gorenstein toric Del Pezzo varieties is parallel to the birational method in the classification of almost Del Pezzo manifolds due to Jahnke and Peternell [JP08].

**References**


