DYCK AND MOTZKIN TRIANGLES WITH MULTIPlicITIES

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Abstract. Exponential generating functions for the Dyck and Motzkin triangles are constructed for various assignments of multiplicities to the arrows of these triangles. The possibility to build such a function provided that the generating function for paths that end on the axis is a priori unknown is analyzed. Asymptotic estimates for the number of paths are obtained for large values of the path length.

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Introduction

By definition, the Dyck triangle (Fig. 1(a)) enumerates paths in the positive quadrant of the plane \((t, x)\) issuing from the origin and consisting of vectors \((1, 1)\) and \((1, -1)\). Paths that end on the \(x\)-axis are called Dyck paths [10].

The Motzkin triangle (Fig. 2(a)) enumerates paths in the positive quadrant of the plane \((t, x)\) issuing from the origin and consisting of the vectors \((1, 1)\), \((1, 0)\), and \((1, -1)\). Paths that end on the \(x\)-axis are called Motzkin paths [10].

Let \(d_{n,k}\), \(n = 0, 1, 2, \ldots, k \leq n\), be the elements of the Dyck triangle; they are nonzero only for even \(n + k\). For the numbers \(d_{n,k}\) we consider a generating function \(D(t, x)\) of the form

\[
D(t, x) = \sum_{k,n=0}^{\infty} d_{n,k}x^k t^n.
\]

By construction, the coefficients \(d_{n+1,k}\) for \(n \geq 2\), \(k = n - 2, n - 4, \ldots\), are related to the coefficients \(d_{n,k+1}\) and \(d_{n,k-1}\) by the following recurrence relation:

\[
d_{n+1,k} = d_{n,k+1} + d_{n,k-1}.
\]

With this relation, we can easily obtain an equation for the generating function \(D(t, x)\) (see, for example, [6]):

\[
\frac{D(t, x) - D(0, x)}{t} = \frac{D(t, x) - D(t, 0)}{x} + xD(t, x).
\]

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Since $D(0, x) = 1$ and $D(t, 0)$ is the generating function for the Catalan numbers (see, for example, [10]), which is equal to

$$D(t, 0) = \frac{1 - \sqrt{1 - 4t^2}}{2t^2},$$

for $D(t, x)$ we obtain

$$D(t, x) = \frac{1 - \sqrt{1 - 4t^2 - 2tx}}{2t(tx^2 + t - x)}.$$

Similar reasoning allows us to obtain the generating function $M(t, x)$ for the Motzkin triangle:

$$M(t, x) = \frac{1 - \sqrt{1 - 2t - 3t^2 - 2tx - t}}{2t(tx^2 + tx + t - x)}.$$
Let us slightly modify the Dyck and Motzkin triangles by assigning positive integers $\alpha_k$, $\beta_k$, and $\gamma_k$ to the arrows (Fig. 1(b) and Fig. 2(b)). The number assigned to the arrow will be interpreted as its multiplicity, i.e., as the number of various arrows going in the given direction. Triangles of this kind are called Dyck and Motzkin triangles with multiplicities [6]. It is evident that the conventional Dyck and Motzkin triangles provide special cases of the corresponding triangles with multiplicities, with $\alpha_k = \beta_k = \gamma_k = 1$.

The following result is known [6]:

**Theorem 1.** Let $\alpha_k$, $\beta_k$, and $\gamma_k$ be the respective multiplicities of the vectors $(1, 1)$, $(1, -1)$, and $(1, 0)$ at the $k$-th level of a Motzkin triangle with multiplicities. Then the generating function $F_k(t)$ enumerating paths that start and end at height $k$ and do not descend below this height can be represented as a continued fraction of the form

$$F_k(t) = \frac{1}{1 - \gamma_k t - \frac{\alpha_k \beta_k t^2}{1 - \gamma_{k+1} t - \frac{\alpha_{k+1} \beta_{k+1} t^2}{1 - \ldots}}}.$$

**Corollary 1.** The analogous generating function for the Dyck triangle with multiplicities $\alpha_k$, $\beta_k$ of the vectors $(1, 1)$ and $(1, -1)$ is

$$F_k(t) = \frac{1}{1 - \alpha_k \beta_k t^2 - \frac{\alpha_{k+1} \beta_{k+1} t^2}{1 - \ldots}}.$$

Despite the universality of this result, it has a number of limitations and weaknesses. First, the representation of $F_k(t)$ in the form of a continued fraction is not always convenient for the analysis of this function. Second, there is no way to use $F_k(t)$ to derive the generating function $w(t, x)$ for the number $w_{n,k}$ of paths that come to a point $(n, k)$. The purpose of this paper is to investigate the possibility of constructing an exponential generating function for the numbers $w_{n,k}$ provided that the coefficients $\alpha_k$, $\beta_k$, and $\gamma_k$ are linearly dependent on $k$.

1. **Examples of Constructing a Generating Function.**

**Heuristic Arguments**

Let us consider the Dyck triangle with multiplicities shown in Fig. 3. By the construction itself, the members of the number sequence $\{w_{n,k}\}$ in this triangle are connected by the relation

$$w_{n+1,k} = (k + 1)w_{n,k+1} + kw_{n,k-1} = (k + 1)w_{n,k+1} + (k - 1)w_{n,k-1} + w_{n,k-1}.$$  \(1\)

First we will seek the generating function for $w_{n,k}$ in the form

$$\tilde{w}(t, x) = \sum_{k,n=0}^{\infty} w_{n,k} x^k t^n.$$  \(2\)
Then, in view of (1), we obtain the following equation for \( \tilde{w}(t, x) \):
\[
\frac{\tilde{w}(t, x) - \tilde{w}(0, x)}{t} = \frac{\tilde{w}(t, x) - 1}{t} = \frac{\partial \tilde{w}}{\partial x} + x^2 \frac{\partial \tilde{w}}{\partial x} + x \tilde{w}.
\]

Obviously, we have an ordinary differential equation for \( \tilde{w}(t, x) \) in the variable \( x \) that depends on \( t \) as a parameter. To solve this equation, we must assign some value to the function at \( x = 0 \), i.e., find the generating function \( \tilde{W}_0(t) \) for the number of paths in the triangle under consideration that start and end on the \( x \)-axis. Since the function \( \tilde{W}_0(t) \) is a priori unknown, the above representation of the generating function for the sequence \( w_{n,k} \) appears to be inconvenient.

Now consider the generating function for \( w_{n,k} \) of the form
\[
w(t, x) = \sum_{k,n=0}^{\infty} w_{n,k} x^k t^n n!
\]
Using the properties of both ordinary and exponential generating functions, from (1) we obtain the following equation for the function \( w(t, x) \):
\[
\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} + x^2 \frac{\partial w}{\partial x} + x w.
\]

The general integral of this first-order partial differential equation has the following form:
\[
w(t, x) = \frac{\Phi(t + \arctan(x))}{\sqrt{1 + x^2}},
\]
where \( \Phi \) is an arbitrary function of its argument. Since at \( x = 0 \) the generating function satisfies the condition \( w(t, 0) = \Phi(t) \), the function \( \Phi(t) \) must coincide with the generating function for the number of paths that start and end on the \( x \)-axis.

This function is a priori unknown as well, but it can be found given the general integral (5) of equation (4) together with the specified value of this function at
Figure 4. Two more examples of Dyck triangles, the special case $\alpha_0 = 0$ (b)

$t = 0$. Indeed, as obvious from (3), the generating function $w(t, x)$ for this value of the argument is equal to $w_{0,0} = 1$. Substituting $t = 0$ into (5), we obtain

$$\Phi(\arctan(x)) = \sqrt{1 + x^2} \quad \Rightarrow \quad \Phi(y) = \sqrt{1 + \tan^2(y)} = \frac{1}{\cos(y)}.$$ 

Therefore, the final expression for the generating function has the form

$$w(t, x) = \frac{1}{\sqrt{1 + x^2 \cos(t + \arctan(x))}}.$$ 

Now consider the Dyck triangle shown in Fig. 4(a). We will seek the generating function for this triangle in the form (3). The coefficients $w_{n,k}$ in the expansion of this generating function satisfy the recurrence relation

$$w_{n+1,k} = (k + 2)w_{n,k+1} + kw_{n,k-1} = w_{n,k+1} + (k + 1)w_{n,k+1} + (k - 1)w_{n,k-1} + w_{n,k-1}. \quad (7)$$

Using (7), we can obtain the following equation for the generating function $w(t, x)$:

$$\frac{\partial w}{\partial t} = \frac{w - w(t, 0)}{x} + \frac{\partial w}{\partial x} + x^2 \frac{\partial w}{\partial x} + xw. \quad (8)$$

It is evident that equation (8), as distinguished from (4), involves an unknown function $W_0(t) = w(t, 0)$ which describes the number of paths in the triangle under consideration that start and end on the $x$-axis. Taking this into account, there is no way to construct the general integral of equation (8) and, consequently, an explicit representation for the required generating function $w(x, t)$.

From the above examples it follows that progress in constructing a generating function of the form (3) depends on whether or not the corresponding partial differential equation for the generating function $w(x, t)$ involves an a priori unknown function $W_0(t) = w(t, 0)$ that describes the number of paths in the triangle under
consideration that both start and end on the $x$-axis. The aim of the next two sections is to formulate the corresponding conditions in terms of the multiplicities $\alpha_k$, $\beta_k$, and $\gamma_k$ of the original triangle.

2. A Polynomial Approach to Construction of Generating Functions for the Dyck Triangle with Multiplicities

Rewrite the generating function $w(t, x)$ given by (3) in the form

$$w(t, x) = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}, \quad P_n(x) = \sum_{k=0}^{n} w_{n,k} x^k.$$  \hfill (9)

As the example analyzed in the previous section shows, equation (4) for the generating function $w(t, x)$ results in the following recurrence relation for the polynomials $P_{n+1}(x)$:

$$P_{n+1}(x) = (1 + x^2) P'_n(x) + x P_n(x).$$

Note that this relation can be immediately obtained from analyzing the triangle shown in Fig. 3. Indeed, the contributions of any term $w_{n,k}$ of the number sequence $\{w_{n,k}\}$ to the coefficients $w_{n+1,k-1}$ and $w_{n+1,k+1}$ are equal to $k w_{n,k}$ and $(k+1) w_{n,k}$, respectively. But $w_{n,k}$ is the coefficient of the monomial $x^k$ in the polynomial $P_n(x)$, while $w_{n+1,k-1}$ and $w_{n+1,k+1}$ are the coefficients of the monomials $x^{k-1}$ and $x^{k+1}$ in the polynomial $P_{n+1}(x)$. Therefore, for any $k = 0, \ldots, n$ we have the following diagram:

$$\begin{align*}
F(k, k+1) &\iff F(k, k) + W_{n,k}(x^k)
\end{align*}$$

Taking the sum of all these contributions of the monomials of $P_n(x)$ over $k$ from 0 to $n$, we obtain the polynomial $P_{n+1}(x)$, which, according to (10), is equal to

$$P_{n+1}(x) = \sum_{k=0}^{n+1} W_{n+1,k} x^k = \sum_{k=0}^{n} W_{n,k} (x^k) + \sum_{k=0}^{n} W_{n,k} ((x^k)' x^2 + x) = (1 + x^2) P'_n(x) + x P_n(x).$$

A similar approach can be used for the analysis of a Dyck triangle with more general weights

$$\alpha_k = ak + \alpha_0, \quad \beta_k = bk + \beta_0, \quad \alpha_0, \beta_0 \in \mathbb{Z}_+, \quad a, b \in \mathbb{N},$$

linearly dependent on $k$ (Fig. 1(a)). Indeed, from Fig. 1(a) it follows that the contribution of an arbitrary element $w_{n,k}$ of the sequence $\{w_{n,k}\}$ to the coefficient $w_{n+1,k-1}$ of the monomial $x^{k-1}$ in the polynomial $P_{n+1}(x)$ is equal to $\beta_{k-1} w_{n,k}$, where $\beta_{k-1} = b(k-1) + \beta_0 = bk + \beta_0 - b$. As to the coefficient $w_{n+1,k+1}$ of the
monomial $x^{k+1}$, it appears in this coefficient with the multiplier $\alpha_k = ak + \alpha_0$:

$$w_{n,k} x^k$$

$$(bk + \beta_0 - b)w_{n,k}x^{k-1} = (ak + \alpha_0)w_{n,k}x^{k+1}.$$  

But

$$(bk + \beta_0 - b)w_{n,k}x^{k-1} = b(x^k)' + \frac{\beta_0 - b}{x} x^k, \quad (ak + \alpha_0)x^{k+1} = ax^2(x^k)' + \alpha_0 xx^k,$$

and, consequently,

$$P_{n+1}(x) = \sum_{k=0}^{n+1} w_{n+1,k} x^k = \sum_{k=0}^{n} \left\{ w_{n,k}(x^k)'(b + ax^2) + w_{n,k}x^k(\frac{\beta_0 - b}{x} + \alpha_0 x) \right\}.$$  

So that for $P_n(x)$ we obtain the following recurrence relation:

$$P_{n+1}(x) = (b + ax^2)P_n(x) + \left(\frac{\beta_0 - b}{x} + \alpha_0 x\right)P_n(x).$$  

This formula implies a first-order partial differential equation for the generating function $w(t, x)$:

$$\frac{\partial w}{\partial t} = (b + ax^2)\frac{\partial w}{\partial x} + \alpha_0 x w + (\beta_0 - b)\frac{w - w(t, 0)}{x}.$$  

As can be seen from this equation, constructing the general integral of this partial differential equation at $\beta_0 = b$ does not require a knowledge of any values of the function $w(t, 0)$ on the axis. Indeed, in this case the general integral of the equation

$$\frac{\partial w}{\partial t} = (b + ax^2)\frac{\partial w}{\partial x} + \alpha_0 xw$$  

has the form

$$w(x, t) = (1 + ab^{-1}x^2)^{-\alpha_0/2a}\Phi(\arctan \sqrt{ab^{-1}x} + \sqrt{ab} t),$$  

where $\Phi$ is an arbitrary function of its argument. In the case $\alpha_0 > 0$, in order to find an explicit form of this function, one should use the condition $w(0, x) = 1$, i.e.,

$$\left(1 + ab^{-1}x^2\right)^{-\alpha_0/2a} = \Phi(\arctan \sqrt{ab^{-1}x}) \quad \Rightarrow \quad \Phi(y) = (\cos y)^{-\alpha_0/a}.$$  

Hence it follows that for $\beta_0 = b$ and $\alpha_0 > 0$, the Dyck triangles with multiplicities are described by general functions of the form

$$w(t, x) = (1 + ab^{-1}x^2)^{-\alpha_0/2a}\left\{\cos(\arctan \sqrt{ab^{-1}x} + \sqrt{ab} t)\right\}^{-\alpha_0/a}.$$  

The case $\alpha_0 = 0$ is of special interest. The simplest example of a Dyck triangle with this value of $\alpha_0$, which corresponds to the case $a = 1, \beta_0 = b = 1$, is demonstrated in Fig. 4(b). For triangles of this kind, it is convenient to seek the generating function in the form

$$w(t, x) = \sum_{n=0}^{\infty} P_{n+1}(x) \frac{t^n}{n!}, \quad P_{n+1}(x) = \sum_{k=0}^{n+1} w_{n,k}x^k, \quad P_1(x) = x.$$
For such a representation of the generating function, the condition for finding an explicit form of the function $\Phi$ can be written as $w(0, x) = x$. Substituting this condition into (13) gives

$$x = \Phi(\arctan \sqrt{ab^{-1}} x) \implies \Phi(y) = \sqrt{ab^{-1}} \tan(y).$$

As a consequence, for $\beta_0 = b$, $\alpha_0 = 0$ the generating function is equal to

$$w(t, x) = \sqrt{ab^{-1}} \tan(\arctan \sqrt{ab^{-1}} x + \sqrt{ab} t).$$  \tag{15}$$

Thus the following result is established:

**Theorem 2.** The generating function that describes the Dyck triangle with multiplicities $\alpha_k$, $\beta_k$ is given by (14) for $\alpha_0 > 0$, $\beta_0 = b$ and by (15) for $\beta_0 = b$, $\alpha_0 = 0$.

**Remark 1.** The condition $\beta_0 = b$ is sufficient for constructing the generating function that describes the Dyck triangle with multiplicities $\alpha_k$, $\beta_k$ given by (11) without an a priori knowledge of the values $w(t, 0)$ of this function on the axis $x = 0$. However, the violation of the condition $\beta_0 = b$ does not imply that we cannot build the generating function $w(x, t)$ without specifying the value $w(t, 0)$ of this function on the axis $x = 0$. As an example, consider the triangle corresponding to the case $\alpha_0 = a = 1$, $\beta_0 = 2$, and $b = 1$ and shown in Fig. 4(a). Compare this triangle with that shown in Fig. 4(b). It is apparent that the number of paths in the triangle shown in Fig. 4(a) coming to the point with coordinates $(n, k)$, $k = 1, 2, \ldots$, agrees with the number of paths in the original Dyck triangle shown in Fig. 4(b) coming to the point with coordinates $(n, k)$, $k = 0, 1, \ldots$. Let us denote the generating function describing the Dyck triangle in Fig. 4(a) by $w(x, t)$, and that describing the triangle in Fig. 4(b), by $a(x, t)$. It is evident that

$$w(t, x) = a(t, x) - a(t, 0).$$

In compliance with (15), the generating function $a(x, t)$ is equal to

$$a(t, x) = \tan(t + \arctan x).$$

A similar technique applies to all Dyck triangles with $\alpha_0 = 1$ and $\beta_0 = b + 1$.

**Remark 2.** The results derived above are also available for the special case of so-called semi-linear Dyck triangles, for which the weights $\alpha_k$ are constant, while $\beta_k$ remain linearly dependent on $k$ (Fig. 5). Putting $a = 0$ in (11), instead of (12) we obtain, with $\beta_0 = b$, the following equation for the generating function $h(t, x)$ describing these triangles:

$$\frac{\partial h}{\partial t} = b \frac{\partial h}{\partial x} + \alpha_0 x h.$$

Solving this equation with the initial condition $h|_{t=0} = 1$ yields

$$h(t, x) = h(t, x; b, \alpha_0) = \exp(\alpha_0 (bh^2/2 + tx)).$$ \tag{16}$$

Note that $h(t, x; 1, 1)$ (see Fig. 5(a)) is the generating function for polynomials whose coefficients are equal in absolute value to the coefficients of modified Hermite polynomials, while $h(t, x; 1, 2)$ (see Fig. 5(b)) is the generating function for polynomials whose coefficients are equal in absolute value to the coefficients of ordinary Hermite polynomials (see [1], [8], and [9]).
3. Generating Functions for the Motzkin Triangle with Multiplicities

Now let us turn our attention to the more general case of Motzkin triangles with weights linearly dependent on $k$, i.e.,

$$\alpha_k = ak + \alpha_0, \quad \beta_k = bk + \beta_0, \quad \gamma_k = ck + \gamma_0, \quad \alpha_0, \beta_0, \gamma_0, a, b, c \in \mathbb{Z}^+.$$  \hspace{1cm} (17)

An argument similar to that in the previous section results in the following recurrence relation for the polynomials $P_n(x)$ that are the coefficients of the expansion of the generating function $w(t, x)$ (see (3)) in powers of $t$:

$$P_{n+1}(x) = (b + cx + ax^2)P'_n(x) + \left(\frac{\beta_0 - b}{x} + \gamma_0 + \alpha_0 x\right)P_n(x).$$

The corresponding equation for the generating function $w(t, x)$,

$$\frac{\partial w}{\partial t} = (b + cx + ax^2)\frac{\partial w}{\partial x} + \alpha_0 x w + \gamma_0 w + (\beta_0 - b)\frac{w - w(t, 0)}{x},$$

in the case $\beta_0 = b$ has the form

$$\frac{\partial w}{\partial t} = (ax^2 + cx + b)\frac{\partial w}{\partial x} + (\alpha_0 x + \gamma_0)w.$$  \hspace{1cm} (18)

Now let us write the characteristic system of ordinary differential equations corresponding to (18):

$$\frac{dt}{\Gamma} = -\frac{dx}{ax^2 + cx + b} = \frac{dw}{w(\alpha_0 x + \gamma_0)}.$$  \hspace{1cm} (19)

Notice that one of these equations rewritten in the form

$$\frac{d \ln w}{dx} = -\frac{\alpha_0 x + \gamma_0}{ax^2 + cx + b}$$  \hspace{1cm} (20)

is the so-called Pearson equation, well known both in mathematical statistics and the theory of classical orthogonal polynomials [3, 12, 11]. The nature of this equation, along with the original equation (18), depends on the coefficients $a, b, c,$
\( \alpha_0 \), and \( \gamma_0 \) appearing in it. Now let us consider all possible alternatives for solving this equation.

**Case I:** \( a = 0 \). In this situation, assuming that \( \alpha_0 > 0 \) and \( b > 0 \), we have the Motzkin triangle with the weights

\[
\alpha_k = \alpha_0, \quad \beta_k = b(k + 1), \quad \gamma_k = \alpha k + \gamma_0.
\]

The solution of equation (18) with the initial condition \( w|_{t=0} = 1 \) substantially depends on the coefficient \( c \). For \( c \neq 0 \), the generating function is

\[
w(x, t) = \exp\left( \frac{\alpha_0}{c^2}(b + cx)(\exp ct - 1) + t \frac{c\gamma_0 - b\alpha_0}{c^2} \right). \tag{21}\]

For \( c = 0 \), the generating function has the form

\[
w(x, t) = \exp(\alpha_0(bt^2/2 + tx) + \gamma_0 t). \tag{22}\]

The special case \( \gamma_0 = 0 \) in (22) corresponds to the generating function (16) for semi-linear Dyck triangles.

**Case II:** \( a \neq 0 \), while the quadratic equation \( ax^2 + cx + b = 0 \) has two different real roots, so that

\[
ax^2 + cx + b = a(x + x_1)(x + x_2), \quad c = a(x_1 + x_2) > 0, \quad b = ax_1x_2 > 0, \quad x_1, x_2 > 0.
\]

For \( \alpha_0 > 0 \), the solution of equation (18) with the initial condition \( w|_{t=0} = 1 \) can be written as

\[
w(x, t) = \left( \frac{x + x_2}{x_2 - x_1} \exp(at(x_1 - x_2)) - \frac{x + x_1}{x_2 - x_1} \right)^{-\alpha_0/a} \exp(t(\gamma_0 - \alpha_0 x_2)). \tag{23}\]

As for the Dyck triangle, the case \( \alpha_0 = 0 \) is special, because for this value of \( \alpha_0 \) the solution of equation (18) with the initial condition \( w|_{t=0} = x \) has the following form:

\[
w(x, t) = \frac{x_2(x + x_1) - x_1(x + x_2) \exp(at(x_1 - x_2))}{(x + x_2) \exp(at(x_1 - x_2)) - (x + x_1)} \exp(\gamma_0 t). \tag{24}\]

**Case III:** \( a \neq 0 \), while the quadratic equation \( ax^2 + cx + b = 0 \) has only one real root \( x_0 \), so that

\[
ax^2 + cx + b = a(x + x_0)^2, \quad b = ax_0^2 > 0, \quad c = 2ax_0 > 0, \quad x_0 > 0.
\]

For \( \alpha_0 > 0 \), using the initial condition \( w|_{t=0} = 1 \), we have

\[
w(x, t) = (1 - at(x + x_0))^{-\alpha_0/a} \exp(t(\gamma_0 - \alpha_0 x_0)). \tag{25}\]

If \( \alpha_0 = 0 \), then the solution of equation (18) with the initial condition \( w|_{t=0} = x \) has the form

\[
w(x, t) = \frac{x + atx_0(x + x_0)}{1 - at(x + x_0)} \exp(\gamma_0 t). \tag{26}\]

**Case IV:** \( a \neq 0 \), while the quadratic equation \( ax^2 + cx + b = 0 \) has no real roots, so that

\[
ax^2 + cx + b = a((x + f)^2 + g^2), \quad b = a(f^2 + g^2) > 0, \quad c = 2af > 0, \quad f, g > 0.
\]
Then, for \( \alpha_0 > 0 \), \( w|_{t=0} = 1 \), the solution of equation (18) has the form
\[
w(x, t) = \left( \cos(gat) - \frac{x + f}{g} \sin(gat) \right)^{-\alpha_0/a} \exp(t(\gamma_0 - f\alpha_0)). 
\]
(27)

In the special case of \( c = \gamma_0 = 0 \), this formula reduces to representation (14) for the generating function describing a Dyck triangle.

Finally, if \( \alpha_0 = 0 \) and \( w|_{t=0} = x \), we have
\[
w(x, t) = \left( g \tan(gat + \arctan \frac{x + f}{g}) - f \right) \exp \gamma_0 t.
\]
(28)

The case \( c = \gamma_0 = 0 \) describes the corresponding Dyck triangle (see (15)).

Remark. Recall that when describing Motzkin triangles, one of the two characteristic ordinary differential equations (19) corresponding to the partial differential equation (18) for the generating function \( w(x, t) \) coincides with the Pearson equation (20). This raises an interesting question about the correlation between Motzkin triangles with multiplicities discussed above and some types of continuous distributions known in mathematical statistics (chi-square distribution, beta distribution, Fisher distribution, Student distribution, Pareto distribution, etc.) and, in particular, about the correlation between the generating functions (21)–(28) and basic Pearson-type density functions of continuous distributions [3].

4. Asymptotics of the Polynomials \( P_n(x) \) for Large \( n \)

Using the generating functions derived above, one can obtain asymptotic estimates of the polynomials \( P_n(x) \) appearing in (9). The simplest method for obtaining such estimates in the case under study is the method of generating functions, also known as Darboux’s method (see, for instance, [7]). This method is based on the analysis of singularities of the generating function \( w(x, t) \), regarded as a function of \( t \), on the boundary of the circle of convergence of the series (9).

As a typical example, we consider the generating function (15). For this function, the singularity nearest to the origin is a point on the positive real semi-axis,
\[
t_p = \frac{1}{\sqrt{ab}} \left( \frac{\pi}{2} - \arctan \sqrt{ab}^{-1}x \right),
\]
in which \( w(x, t) \) has a simple pole. The Laurent expansion of \( w(x, t) \) in the vicinity of this singularity is
\[
w(t, x) = -\frac{1}{a(t - t_p)} + b \frac{ax^2 - 3b - 3b^2}{3ax^2}(t - t_p) + \cdots.
\]

Any partial sum of this expansion in the vicinity of zero gives a desired asymptotics for the polynomials \( P_n(x) \) appearing in (9). Thus, taking only the leading term
\[
v(t, x) = -\frac{1}{a(t - t_p)}
\]
of the Laurent series and expanding it in powers of \( t \) in the vicinity of zero,
\[
v(t, x) = \sum_{n=0}^{\infty} Q_n(x) \frac{t^n}{n!}, \quad Q_n = \frac{n!}{a t_p(x)},
\]
we obtain the following asymptotic estimate for the polynomials \( P_n(x) \):
\[
P_n(x) \sim n! b^{(n+1)/2} a^{(n-1)/2} \left( \frac{\pi}{2} - \arctan \sqrt{ab^{-1}x} \right)^{-n-1}, \quad n \to \infty.
\]  
(29)

As a second example, we consider the Motzkin triangle with multiplicities \( a = 2 \), \( \alpha_0 = 1 \), \( \beta_0 = b = 1 \), \( c = 3 \), and \( \gamma_0 = 1 \). This triangle is described by the generating function \( w(t, x) \) given by equation (23) in which we put \( x_1 = 1/2 \) and \( x_2 = 1 \), namely,
\[
w(t, x) = \frac{1}{\sqrt{(2x + 2) \exp(-t) - (2x + 1)}} = \sum_{n=0}^{\infty} P_n(x) \frac{t^n}{n!}.
\]

This function, regarded as a function of a complex variable \( t \), has countably many branch points. The branch point nearest to the origin lies on the real axis and is determined by the formula
\[
t_0(x) = \ln \frac{2x + 2}{2x + 1}.
\]

Expand the function \( w(t, x) \) into a series in the vicinity of this point:
\[
w(t, x) = \frac{1}{\sqrt{2x + 1}} \frac{1}{\sqrt{t_0(x) - t}} \left( 1 + \frac{t - t_0(x)}{4} + \frac{(t - t_0)^2}{96} + \cdots \right).
\]

According to Darboux’s method, any partial sum of this series expanded in powers of \( t \) in the vicinity of zero allows us to estimate the polynomials \( P_n(x) \) for large \( n \).

As an example of such a sum, we take the expression
\[
v(t, x) = \frac{1}{\sqrt{2x + 1}} \frac{1}{\sqrt{t_0(x) - t}} = \sum_{n=0}^{\infty} S_n(x) \frac{t^n}{n!}.
\]

Then
\[
P_n(x) \sim S_n(x) = n! \left( \frac{2n - 1}{2n} \right)! \frac{1}{\sqrt{2x + 1}} \left( \ln \frac{2x + 2}{2x + 1} \right)^{-n-1} \left( 1 + \frac{t - t_0(x)}{4} + \frac{(t - t_0)^2}{96} + \cdots \right), \quad n \to \infty.
\]

5. Asymptotics of the Numbers \( w_{n,k} \) for Large \( n \)

Now we will study the asymptotics of the number of paths in the Dyck and Motzkin triangles with multiplicities for large values of the parameter \( n \). Since
\[
P_n(x) = \sum_{k=0}^{n} w_{n,k} x^k, \quad w_{n,k} = \frac{1}{2\pi i} \int_{\Gamma} \frac{P_n(x)}{x^{k+1}} dx,
\]
in order to obtain estimates of \( w_{n,k} \), we can use the asymptotics of the polynomials \( P_n(x) \) derived by the method described in the previous section. Using the asymptotic estimate (29) obtained above, the following asymptotics for the numbers \( w_{n,k} \) can be obtained:
\[
w_{n,k} \sim \frac{k!}{2\pi i} b^{(n+1)/2} a^{(n-1)/2} \int_{\Gamma} \left( \frac{\pi}{2} - \arctan \sqrt{ab^{-1}x} \right)^{-n-1} \frac{dx}{x^{k+1}}, \quad n \to \infty.
\]
(30)
Let us evaluate the integral in (30) by the saddle-point method. First, introduce a parameter \( \alpha \) such that \( k + 1 = \alpha (n + 1) \). It follows that \( \alpha = O(1) \) as \( n \to \infty \). Then

\[
I(\alpha) = \int_{\Gamma} \exp \left\{ -(n + 1) \left( \ln \left( \frac{\pi}{2} - \arctan \sqrt{ab^{-1}} x \right) + \alpha \ln(x) \right) \right\} dx.
\]

Denote

\[
f(x, \alpha) = \ln \left( \frac{\pi}{2} - \arctan \sqrt{ab^{-1}} x \right) + \alpha \ln(x).
\]

In keeping with the saddle-point method, it is essential to find a point \( x_0 \) in which \( f'(x_0, \alpha) = 0 \), and to choose a contour that runs along the line of steepest ascent and descent. It is straightforward to show (see, for example, \[5\]) that this point lies on the real axis and can be found from the relation

\[
\frac{\pi}{2} - \arctan \sqrt{ab^{-1}} x = \frac{\sqrt{ab} x_0}{\alpha(b + ax_0^2)},
\]

while the required contour passes through the point \( x_0 \) normally to the real axis. Then

\[
I(\alpha) \sim i \exp(-(n + 1)f(x_0, \alpha)) \sqrt{\frac{2\pi}{(n + 1)|f''_{xx}(x_0, \alpha)|}}
\]

and the corresponding estimate for the numbers (30) is

\[
w_{n,k} \sim \frac{n!}{a \sqrt{2\pi(n + 1)|f''_{xx}(x_0, \alpha)|}} \left( \frac{\pi x_0^{\alpha+1}}{\alpha(b + ax_0^2)} \right)^{(n + 1)} \quad x_0 \to 1.
\]

From Fig. 3, Fig. 4 it follows that for a fixed \( n \), the numbers \( w_{n,k} \) are not monotone in \( k \), reaching the maximum at a certain \( k_m \in [0, n] \). Using the estimates derived above, it can be proved that the value of the parameter \( \alpha = \alpha_m \) at which this maximum is achieved corresponds to \( x_0 = 1 \). Substituting this value of \( x_0 \) into (31), we see that the maximum of the distribution (32) occurs at

\[
\alpha_m = \frac{\sqrt{ab}}{b + a} \left( \frac{\pi}{2} - \arctan \sqrt{ab^{-1}} \right)^{-1}.
\]

6. An Example: the Enumeration of Cascade Diagrams

Let us suppose that a connected graph inscribed in the unit disk has \( n \) 4-valent vertices inside the disk and \( k \) 1-valent vertices on the boundary circle. Such a graph is called a 4-valent web graph (Fig. 6(a)). Hereafter, by the vertices of a web graph we mean only its \( n \) internal vertices. The 1-valent vertices together with their incident edges are called legs of a web graph. Note that the number \( k \) of legs is even for any 4-valent web graph.

A web graph with \( n \) vertices and \( k \) legs can be identified with a map on the sphere with \( n \) 4-valent vertices and one \( k \)-valent vertex. It can also be regarded as a projection with \( n \) crossings of a certain tangle \[2, \[4\].
Let us divide the disk into a system of imbedded, simply connected domains so that only one vertex lies inside each annular layer (Fig. 6(b)). Let us cut the disk from the central vertex (vertex 1 in Fig. 6(b)) to the boundary circle so that the cut intersects the boundary of each region just once (dashed line in Fig. 6(b)), while it may meet the web graph only at crossings (dashed line in Fig. 6(b)). By opening the cut, we obtain a representation of the web graph in the form of a falling cascade (Fig. 6(c)). This representation is called a cascade diagram of the web graph.

There is only one vertex of the web graph at each level of the cascade diagram. Edges that leave this vertex intersect the upper and/or lower boundaries of the level that contains the vertex. Only five edge configurations are possible, and they correspond to the patterns \( \text{X}, \text{X}, \), \( \text{X}, \), and \( \text{X} \).

The \( \text{X} \) pattern, which is always the starting pattern of a cascade diagram, cannot lie in other levels, in view of the connectivity of the web graph and by its very construction. It can also be shown that for any web graph there is a cascade diagram that does not have the pattern \( \text{X} \). Moreover, any vertex of the web graph can be chosen as the starting vertex of such a diagram.

Thus any web graph can be represented by a cascade diagram having (except for the starting pattern \( \text{X} \)) only the patterns \( \text{X}, \), \( \text{X} \), and \( \text{X} \). The primary goal of this section is to evaluate the number \( a_{n,k} \) of different cascade diagrams with \( n \) crossings and \( k \) legs.

In the case \( n = 1 \), there is only one cascade diagram, corresponding to the pattern \( \text{X} \) and having four legs. In order to construct all cascade diagrams with two vertices, we should connect the patterns \( \text{X}, \), \( \text{X}, \), and \( \text{X} \) to the \( \text{X} \) pattern in all possible ways. In this case, there will obviously be four cascade diagrams for each of the types \( \text{X}, \), \( \text{X}, \), and \( \text{X} \). Every \( \text{X} \) diagram has \( k = 6 \) legs; therefore it is possible to connect the patterns \( \text{X}, \), \( \text{X}, \), and \( \text{X} \) to this diagram in six ways. Each of these patterns can be connected to any \( \text{X} \) diagram with four legs in four different ways. Finally, there exist two ways to connect these two patterns \( \text{X} \) and \( \text{X} \) to any diagram \( \text{X} \). As a result, the sequence
The enumerating triangle for cascade diagrams (a), the auxiliary triangle (b) yields 24 diagrams with \( n = 3 \) vertices and \( k = 8 \) legs, whereas the sequences \( \cdots \) and \( \cdots \) have 40 corresponding diagrams with \( n = 3 \) and \( k = 6 \). Then 48 diagrams with \( n = 3 \) and \( k = 4 \) arise from the combinations \( \cdots \), \( \cdots \), and \( \cdots \); lastly, the sequences \( \cdots \) and \( \cdots \) yield 24 diagrams with \( n = 3 \) and \( k = 2 \).

It is convenient to represent the number of various cascade diagrams \( a_{n,k} \) graphically in the plane (see Fig. 7(a)). In this figure, the numbers \( a_{n,k} \) are indicated at points with coordinates \((n, k)\), \( n = 1, 2, \ldots, k = 2, 4, \ldots, 2n + 2 \). These numbers for \( k = 4, \ldots, 2n - 2, n > 2 \), are constructed from the numbers \( a_{n-1,k+2} \), \( a_{n-1,k} \), and \( a_{n-1,k-2} \) of the \((n-1)\)-th level multiplied by \( k+2, k \), and \( k-2 \), respectively:

\[
a_{n,k} = (k+2)a_{n-1,k+2} + ka_{n-1,k} + (k-2)a_{n-1,k-2}.
\]

As seen from Fig. 7(a), the corresponding numbers \( a_{n-1,i} \) and \( a_{n,k} \) are connected by arrows. The numbers above the arrows indicate the weights (multiplicities) with which the numbers \( a_{n-1,i} \) occur in the corresponding coefficient \( a_{n,k} \).

**Theorem 3.** The enumeration of various cascade diagrams with \( n \) vertices and \( k \) legs is reduced to determining the number of paths in the Motzkin triangle with multiplicities shown in Fig. 7(b).

**Proof.** Denote by \( w_{n,k} \) the number of Motzkin paths from \((0, 0)\) to \((n, k)\) in the triangle shown in Fig. 7(b). Let \( w(t, x) \) be the generating function for the numbers \( w_{n,k} \):

\[
w(t, x) = \sum_{n=0}^{\infty} \sum_{k=1}^{n+1} w_{n,k} x^k \frac{t^n}{n!} = x + (x + x^2) t + (9x + 15x^2 + 12x^3 + 6x^4) \frac{t^3}{3!} + \cdots.
\]

Along with this triangle, let us consider the triangle with multiplicities multiplied by 2. The generating function \( \tilde{w}(t, x) \) for this triangle is expressed in terms of
Now suppose that \( \tilde{w}(t, x) \) is the generating function for the number of cascade diagrams \( a_{n,k} \) with \( n \) vertices and \( k \) legs:

\[
\tilde{w}(t, x) = \sum_{n=0}^{\infty} \sum_{k=1}^{n+1} \tilde{w}_{n,k} x^k t^n = \sum_{n=0}^{\infty} \sum_{k=1}^{n+1} w_{n,k} 2^n x^k t^n = w(2t, x).
\]

It can easily be shown that the generating functions \( \tilde{w}(t, x) \) and \( w(t, x) \) for the triangles shown in Fig. 7(a) and Fig. 7(b) are connected by the relation

\[
\frac{\partial \tilde{w}(t, x)}{\partial t} = 2 \tilde{w}(t, x) + 2a(t, x).
\]

Consequently, the generating function \( a(t, x) \) for the number of various cascade diagrams is expressed in terms of \( w(t, x) \) as

\[
a(t, x) = \frac{1}{2} \frac{\partial w(2t, x)}{\partial t} - w(2t, x).
\]

**Theorem 4.** The generating function \( w(t, x) \) given by (33) for the Motzkin triangle with multiplicities shown in Fig. 7(b) is equal to

\[
w(t, x) = \frac{\sqrt{3}}{2} \tan \left( \frac{\sqrt{3}}{2} t + \arctan \left( \frac{2x+1}{\sqrt{3}} \right) \right) - \frac{1}{2} - W_0(t),
\]

where

\[
W_0(t) = \sum_{n=1}^{\infty} \frac{C_n t^n}{n!} = \frac{t^1}{1!} + \frac{t^2}{2!} + \frac{3t^3}{3!} + \frac{9t^4}{4!} + \frac{39t^5}{5!} + \cdots = \frac{\sin \left( \frac{\sqrt{3}}{2} t \right)}{\cos \left( \frac{\pi}{6} + \frac{\sqrt{3}}{2} t \right)}.
\]

**Proof.** The triangle shown in Fig. 7(b) is the Motzkin triangle with multiplicities \( \alpha_k = k + 1, \beta_k = k + 2, \gamma_k = k + 1 \implies a = b = c = \alpha_0 = \gamma_0 = 1, \beta_0 = 2 \).

Since \( \beta_0 \neq b \), we cannot take advantage of the above results directly, because the sufficient condition for constructing the generating function \( w(x, t) \) is violated. However, as it was mentioned in Section 2, in the case of \( \beta_0 = b + 1 \), we can construct the generation function.

Indeed, rewrite the generating function \( w(t, x) \) as

\[
w(t, x) = \sum_{n=0}^{\infty} P_{n+1}(x) \frac{t^n}{n!}, \quad P_{n+1}(x) = \sum_{k=1}^{n+1} w_{n,k} x^k.
\]

Note that for each \( k = 2, \ldots, n+1 \) any summand of the form \( w_{n-1,k} x^k \) contributes to the coefficients \( w_{n,k-1}, w_{n,k}, \) and \( w_{n,k+1} \) of the monomials \( x^{k-1}, x^k, \) and \( x^{k+1} \).
of the polynomial $P_{n+1}(x)$, this contribution being equal to the coefficient $w_{n-1,k}$ multiplied by $k$:

$$w_{n-1,k}x^k$$

$$kw_{n-1,k}x^{k-1}$$

$$kw_{n-1,k}x^{k+1}$$

If we turn from the polynomial $P_{n+1}(x)$ to the polynomial

$$\hat{P}_{n+1}(x) = \sum_{k=0}^{n+1} \hat{w}_{n,k} x^k,$$

then the diagram (38) will be true for all $k \in \{0, n+1\}$. Therefore, the polynomials $\hat{P}_n(x)$ and $\hat{P}_{n+1}(x)$ are related by the following recurrence formula:

$$\hat{P}_{n+1}(x) = \hat{P}'_n(x)(1 + x + x^2).$$

Let

$$\hat{w}(t, x) = \sum_{n=0}^{\infty} \hat{P}_n(x) \frac{t^n}{n!}$$

be the generating function for $\hat{P}_{n+1}(x)$; then

$$\hat{w}(t, x) = \hat{w}(t, x) - \hat{w}(t, 0).$$

Now we can derive an equation for the function $\hat{w}(t, x)$. Differentiating $\hat{w}(t, x)$ in $x$ and in $t$, we get

$$\frac{\partial \hat{w}}{\partial x} = \sum_{n=0}^{\infty} \hat{P}_n(x) \frac{t^n}{n!}, \quad \frac{\partial \hat{w}}{\partial t} = \sum_{n=0}^{\infty} \hat{P}_n(x) \frac{t^n}{n!}.$$ 

Taking into account (39), we obtain

$$\frac{\partial \hat{w}}{\partial t} = (1 + x + x^2) \frac{\partial \hat{w}}{\partial x}.$$ 

The solution of this equation with the initial condition $\hat{w}(0, x) = x$ is

$$\hat{w}(t, x) = \sqrt{3} \tan \left( \sqrt{3} \frac{t}{2} + \arctan \frac{2x + 1}{\sqrt{3}} \right) - \frac{1}{2}.$$ 

Substituting $x = 0$ into (41) and performing simple trigonometric transformations, we get the function $\hat{w}(t, 0)$ in the form $\hat{w}(t, 0) = W_0(t)$, which is similar to (37). Now formula (36) follows from (40).

**Conclusion**

In the paper, a detailed analysis of Dyck and Motzkin triangles with multiplicities $\alpha_k, \beta_k,$ and $\gamma_k$ is carried out for the case where these multiplicities are linearly dependent on $k$. Some sufficient conditions are found under which the generating function describing the corresponding triangles can be constructed. It is noted that there exists a certain link between the equations describing the generating functions and the Pearson equation. The classification of all possible types of
generating functions is developed depending on the coefficients occurring in the Pearson equation. Asymptotic formulas are derived that describe the distribution of the number \( w_{n,k} \) of paths in Dyck and Motzkin triangles with multiplicities for large values of the parameter \( n \). As an example of applying the results obtained in the paper, the problem of enumerating cascade diagrams for a 4-valent web graph is solved.

References


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