ON POINT-LIKE INTERACTION BETWEEN \( n \) FERMIOSNS AND ANOTHER PARTICLE

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ABSTRACT. In this note the point-like interaction of \( n \) fermions with a particle of a different nature is considered in a framework of the theory of self-adjoint extensions of symmetric operators. It introduces the family of extensions of the original symmetric operator, most natural from the physical point of view (the so-called Ter-Martirosian–Skornyakov’ extensions). Here we prove that for \( n \leq 4 \) and large enough values of the mass of the separate particle these extensions are self-adjoint and bounded from below.

Key words and phrases. Symmetric operator, selfadjoint extension, boundedness below, Ter-Martirosian–Skornyakov extension.

1. Introduction

The problem of the point-like interaction for a system of several particles in considered in many papers (see [1]–[9]). In these works different approaches are applied to the study of a point-like interaction. In this paper we follow a method based on the theory of self-adjoint extensions of symmetric operators. Namely, the original symmetric operator is given as the sum of Laplace operators on the set of functions which vanish if positions of any two particles coincide with each other. The real self-adjoint Hamiltonian of our system is given as a self-adjoint extension of the original operator.

At first such method was applied in the work [5] for the system of two particles. Then the same method was used in [3], [4] for the system of three bosons, in [7] for the system of two fermions and a separate particle and finally in [8] for the general case of three particles.

The present paper continues the line of the paper [7]; here we consider the system of \( n \) fermions and a separate particle of another nature with the mass \( m \) (the mass of fermions equals 1). The original operator has the form

\[
H_0 = -\frac{1}{2} \left( \frac{1}{m} \Delta_y + \Delta_{x_1} + \cdots + \Delta_{x_n} \right),
\]

(1.1)
where \((x_1, \ldots, x_n)\) are the positions of fermions and \(y\) is the position of the separate particle. The operator (1.1) acts in the space \(L_2(\mathbb{R}^3) \otimes L_2^{\text{asym}}((\mathbb{R}^3)^n)\), where the space \(L_2^{\text{asym}}((\mathbb{R}^3)^n)\) consists of antisymmetric functions of \(n\) variables. The domain \(D(H_0)\) of \(H_0\) contains enough smooth functions vanishing on hyperplanes
\[
\Gamma_i = \{x_i = y\} \subset (\mathbb{R}^3)^{n+1}, \quad i = 1, 2, \ldots, n
\] (1.2)
(because of the anti-symmetry these functions vanish when \(x_i = x_j, i \neq j\)). This operator is a symmetric operator and we shall construct the more natural family of its self-adjoint extensions. In order to understand how these extensions are selected we consider the simplest example of the point-like interaction: a particle interacting with a force field at zero. The Hamiltonian of such system is the selfadjoint extension of symmetric operator
\[
h_0 = -\frac{1}{2} \Delta
\]
with the domain \(D(h_0) = \{\varphi \in L_2(\mathbb{R}^3) : \varphi(0) = 0\}\). After Fourier transform this operator becomes the operator
\[
(\tilde{h}_0 \tilde{\varphi})(p) = \frac{1}{2} p^2 \tilde{\varphi}(p)
\]
with the domain \(D(\tilde{h}_0) = \{\tilde{\varphi} : \int_{\mathbb{R}^3} \tilde{\varphi}(p) \, dp = 0\}\). The domain \(D(\tilde{h}_0^*)\) of the conjugate operator has the form
\[
D(\tilde{h}_0^*) = \left\{ \psi \in L_2(\mathbb{R}^3) : \int_{|p| < N} \psi(p) \, dp = 4\pi N c + b + o(1), \ N \to \infty \right\}, \quad (1.3)
\]
where \(c = c(\psi), b = b(\psi)\) are some constants.

Every selfadjoint extension \(\tilde{h}_2\) of the operator \(\tilde{h}_0\) is defined as restriction of \(\tilde{h}_0^*\) to the functions \(\psi \in D(\tilde{h}_0)\) for which
\[
\varepsilon c = b, \quad (1.4)
\]
where \(\varepsilon\) is a real parameter defining extension. A similar family of selfadjoint extensions \(H_\varepsilon\) will be constructed for the operator \(H_0\). As a preliminary we perform some reduction. After Fourier transform the operator \(H_0\) takes the form
\[
(\tilde{H}_0 f)(q, k_1, \ldots, k_n) = \frac{1}{2} \left( \frac{1}{m} q^2 + k_1^2 + \cdots + k_n^2 \right) f(q, k_1, \ldots, k_n).
\]
Then we perform the change of variables
\[
P = q + k_1 + \cdots + k_n,
\]
\[
p_i = \frac{P}{m + n} - k_i, \quad i = 1, \ldots, n.
\]
After it the operator \(H_0\) is represented in the form of tensor sum:
\[
\tilde{H}_0 = \tilde{H}_0^{(1)} + \tilde{H}_0^{(2)},
\]
(1.5)
where \(\tilde{H}_0^{(1)}\) is a selfadjoint operator in \(L_2(\mathbb{R}^3)\)
\[
(\tilde{H}_0^{(1)} f)(P) = \frac{1}{m + n} P^2 f(P), \quad f \in L_2(\mathbb{R}^3)
\]
and $\tilde{H}_0^{(2)}$ is a symmetric operator in $L^2_{\text{asym}}((\mathbb{R}^3)^n)$

$$(\tilde{H}_0^{(2)}) g(p_1, \ldots, p_n) = \frac{m + 1}{m} \left( \sum_{i=1}^{n} p_i^2 + \frac{2}{m + 1} \sum_{1 \leq i < j \leq n} (p_i, p_j) \right) g(p_1, \ldots, p_n)$$

with the domain

$$D(\tilde{H}_0^{(2)}) = \left\{ g \in L^2_{\text{asym}}((\mathbb{R}^3)^n) : \int_{\mathbb{R}^3} g(p_1, \ldots, p_n) \, dp_i = 0, \quad i = 1, \ldots, n \right\}. \tag{1.7}$$

Further we shall construct a family $\{\tilde{H}_\varepsilon^{(2)}, \varepsilon \in \mathbb{R}^1\}$ of extensions of the operator $\tilde{H}_0^{(2)}$ and prove for them the following assertion.

**Theorem.** For any real $\varepsilon$, $n \leq 4$ and large enough values of $m$, the operator $\tilde{H}_\varepsilon^{(2)}$ is selfadjoint and bounded from below.

**2. Definition of $\tilde{H}_\varepsilon^{(2)}$ and its Selfadjointness**

We follow here the prescriptions of the general theory of selfadjoint extensions of symmetric operators bounded from below (see [10] and [12]).

**2.1.** Defect subspace $\mathcal{H}_{-1}$ of the operator $\tilde{H}_0 = \frac{m}{m+1} \tilde{H}_0^{(2)}$ (i.e., the set of functions which are orthogonal to subspace of form $\{ (\tilde{H}_0 + E) \psi, \psi \in D(\tilde{H}_0) \}$) consists of elements of the form

$$U_\varphi(p_1, \ldots, p_n) = \sum_{i=1}^{n} \varphi(p_1, \ldots, \hat{p}_i, \ldots, p_n)(-1)^{i-1} \frac{G(p_1, \ldots, p_n) + 1}{G(p_1, \ldots, p_n) + 1}. \tag{2.1}$$

Here the sign $\hat{}$ means that the corresponding variable is omitted,

$$G(p_1, \ldots, p_n) = \sum_{i=1}^{n} p_i^2 + \frac{2}{m + 1} \sum_{1 \leq i < j \leq n} (p_i, p_j) \tag{2.2}$$

and $\varphi$ is an antisymmetric function of $(n - 1)$ variables belonging to the Hilbert space $L^2_{\text{asym}} \subset L^2$, where $L^2$ is the Hilbert space of functions $\varphi(k_1, \ldots, k_{n-1})$ with inner product

$$\langle \varphi_1, \varphi_2 \rangle_L = (W \varphi_1, \varphi_2)_{L^2((\mathbb{R}^3)^{n-1})}, \tag{2.3a}$$

where $W$ is a positive operator with the kernel

$$\tilde{W}(k_1, \ldots, k_{n-1}; k'_1, \ldots, k'_{n-1}) = \frac{n \pi^2}{\sqrt{H(k_1, \ldots, k_{n-1}) + 1}} \prod_{i=1}^{n-1} \delta(k_i - k'_i)$$

$$+ \frac{n (n - 1)}{(G(k_1, k'_1, k_2, \ldots, k_{n-1}) + 1)^2} \prod_{i=2}^{n-1} \delta(k_i - k'_i). \tag{2.3b}$$

Here

$$H(k_1, \ldots, k_{n-1}) = \frac{m(m+2)}{(m+1)^2} \sum_{i=1}^{n-1} k_i^2 + \frac{2m}{m + 1} \sum_{1 \leq i < j \leq n-1} (k_i, k_j). \tag{2.4}$$
Note that for $\varphi_1, \varphi_2 \in L^{\text{asym}}$ one has
\begin{equation}
\langle \varphi_1, \varphi_2 \rangle = (U_{\varphi_1}, U_{\varphi_2})_{L_2^{\text{asym}}(\mathbb{R}^n)}.
\end{equation}

**Remarks.**
1. As the operator $W$ is positive, there exists an inverse operator $W^{-1}$. Every continuous linear functional on the space $L$ can be represented in the form
\begin{equation}
F(\psi) = (\psi, \chi)_{L_2(\mathbb{R}^n)}, \quad \psi \in L, \; \chi \in L',
\end{equation}
where $L'$ is the space of functions with inner product $(W^{-1} \chi_1, \chi_2)$.

As it follows from (2.3a) the operator $W$ maps $L$ to $L'$ and the inverse operator $W^{-1}$ maps $L'$ to $L$.

2. It is easy to see that the functions satisfying the condition
\begin{equation}
\int_{\mathbb{R}^{n-1}} |\varphi(k_1, \ldots, k_{n-1})|^2 / (H(k_1, \ldots, k_{n-1}) + 1)^{1/2} \prod_{i=1}^{n-1} dk_i < \infty
\end{equation}
belong to the space $L$.

**2.2.** Consider now the operator $(\tilde{H}_0)^*$ conjugate to the operator $\tilde{H}_0$. Its domain $D((\tilde{H}_0)^*)$ consists of the functions of form
\begin{equation}
g(p_1, \ldots, p_n) = f(p_1, \ldots, p_n) + U_{\varphi}(p_1, \ldots, p_n) + \frac{U_{\psi}(p_1, \ldots, p_n)}{G(p_1, \ldots, p_n) + 1},
\end{equation}
where $f \in D(\tilde{H}_0)$ and $\varphi, \psi \in L^{\text{asym}}$. The action of the operator $(\tilde{H}_0)^*$ is given by formula
\begin{equation}
((\tilde{H}_0)^* g)(p_1, \ldots, p_n) = G(p_1, \ldots, p_n) g(p_1, \ldots, p_n) - \sum_{i=1}^{n-1} \varphi(p_1, \ldots, \hat{p}_i, \ldots, p_n) (-1)^{i-1},
\end{equation}
where the $\varphi$ is defined in (2.7).

Every selfadjoint extension $\tilde{H}_A$ of the operator $\tilde{H}_0$ is obtained by restriction of the operator $(\tilde{H}_0)^*$ to a domain $D(\tilde{H}_A) \subset D((\tilde{H}_0)^*)$ of the form
\begin{equation}
D(\tilde{H}_A) = \{ g \in D((\tilde{H}_0)^*): \psi = A \varphi \},
\end{equation}
where $\varphi$ and $\psi$ are defined in (2.7) and $A$ is an arbitrary selfadjoint operator acting in $L^{\text{asym}}$.

From the representation (2.7) we find that
\begin{equation}
\int_{|p_1| < N} g(p_1, \ldots, p_n) dp_1 = 4\pi N \varphi(p_2, \ldots, p_n) + b(p_2, \ldots, p_n) + o(1)
\end{equation}
as $N \to \infty$. Here
\begin{equation}
b(p_2, \ldots, p_n) = - (T \varphi)(p_2, \ldots, p_n) + ((VA) \varphi)(p_2, \ldots, p_n),
\end{equation}
where $T \varphi$ is given by expression
\begin{equation}
(T \varphi)(k_1, \ldots, k_{n-1}) = 2\pi^2 \int_{\mathbb{R}^{n-1}} \varphi(t, k_1, \ldots, \hat{k}_i, \ldots, k_{n-1}) (-1)^{i-1}
\end{equation}
\begin{equation}
+ \int_{\mathbb{R}^{n-1}} \varphi(t, k_1, \ldots, \hat{k}_i, \ldots, k_{n-1}) (-1)^{i-1}
\end{equation}
\begin{equation}
\frac{G(t, k_1, \ldots, k_{n-1}) + 1}{G(t, k_1, \ldots, k_{n-1}) + 1} dt.
\end{equation}
and $V$ is the operator acting on $\mathcal{L}^{\text{asym}}$ by formula

$$(V \varphi)(k_1, \ldots, k_{n-1}) = \pi^2 \frac{\varphi(k_1, \ldots, k_{n-1})}{\sqrt{H(k_1, \ldots, k_{n-1}) + 1}} + \int_{\mathbb{R}^3} \frac{\sum_{i=1}^{n-1} \varphi(t, k_1, \ldots, \hat{k}_i, \ldots, k_{n-1})(-1)^{i-1}}{(G(t, k_1, \ldots, k_{n-1}) + 1)^2} dt. \quad (2.12)$$

Note that the following relation is satisfied:

$$V = \frac{1}{n} P WP,$$  \quad (2.13)

where $P$ is the orthogonal projection in $\mathcal{L}$ on the subspace $\mathcal{L}^{\text{asym}}:

$$(P \psi)(k_1, \ldots, k_{n-1}) = \frac{1}{(n-1)!} \sum_{\sigma} \psi(k_{\sigma(1)}, \ldots, k_{\sigma(n-1)}) (-1)^{\Delta(\sigma)}. \quad (2.14a)$$

Here the summation is taken over all permutations of the indexes $(1, \ldots, n-1)$ and $\Delta(\sigma)$ is the parity of $\sigma$. From (2.13) it follows that $V$ is a positive operator in $L_2^{\text{asym}}(\mathbb{R}^3)^{n-1})$ and hence has a positive inverse $V^{-1}$. From (2.13) it follows that inner product in $\mathcal{L}^{\text{asym}}$ can be written in the form

$$\langle \varphi, \psi \rangle = n(V \varphi, \psi). \quad (2.14b)$$

### 2.3. The Ter-Martirosian–Skornyakov extension $H_\varepsilon$ of the operator $\tilde{H}_0$ is defined by the relation

$$b(k_1, \ldots, k_{n-1}) = \varepsilon \varphi(k_1, \ldots, k_{n-1}) \quad (2.15)$$

similar to (1.4). Here $b$ and $\varphi$ are the functions from (2.9), but $\varepsilon$ is a real parameter determining the extension. From (2.10) and (2.15) it follows that

$$(T + \varepsilon E) = VA \quad \text{or} \quad A = V^{-1}(T + \varepsilon E),$$

where $A = A_\varepsilon$ is the selfadjoint operator in $\mathcal{L}^{\text{asym}}$ corresponding to the extension $\tilde{H}_\varepsilon$. Note that the selfadjointness $A_\varepsilon$ in $\mathcal{L}^{\text{asym}}$ is equivalent, due to (2.14b), to the selfadjointness of $T$ in $L_2^{\text{asym}}(\mathbb{R}^3)^{n-1})$. Since the operator $T$ is unbounded in $L_2^{\text{asym}}(\mathbb{R}^3)^{n-1})$ it is necessary to define this operator more accurately so that it becomes selfadjoint. At first we define the operator $T_0$ on the set of all finite functions from $L_2^{\text{asym}}(\mathbb{R}^3)^{n-1})$. It is easy to prove that $T_0$ is a symmetric operator. Denote by $T = (T_0)^*$ the conjugate operator. It is defined on the set of all functions from $L_2^{\text{asym}}(\mathbb{R}^3)^{n-1})$ which the expression (2.11) leaves in this space.

### 2.4. Further we prove that $T$ is selfadjoint. Introduce the operator $\hat{T}$ in the space $L_2((\mathbb{R}^3)^{n-1})$

$$(\hat{T} \psi)(k_1, \ldots, k_{n-1}) = 2\pi^2(H(k_1, \ldots, k_{n-1}) + 1)^{1/2} \psi(k_1, \ldots, k_{n-1}) + (n-1) \int_{\mathbb{R}^3} \frac{\psi(t, k_1, \ldots, k_{n-1}) dt}{G(t, k_1, k_2, \ldots, k_{n-1}) + 1} \quad (2.16)$$

defined on the set of all functions from $L_2((\mathbb{R}^3)^{n-1})$ which the expression (2.16) leaves in $L_2((\mathbb{R}^3)^{n-1})$. The operator $\hat{T} = (\hat{T}_0)^*$, where $\hat{T}_0$ is given by formula (2.16)
on the set of finite functions from $L_2((\mathbb{R}^3)^{n-1})$. Note that $T$ is connected with $\hat{T}$ by the relation
\[ T = P \hat{T} P, \]
where $P$ is a projection (2.14a). Thus the following relation holds:
\[ (T\varphi, \varphi)_{L_2^{\text{asym}}((\mathbb{R}^3)^{n-1})} = (\hat{T}\varphi, \varphi)_{L_2^{\text{asym}}((\mathbb{R}^3)^{n-1})} \quad \text{for} \quad \varphi \in L_2^{\text{asym}}((\mathbb{R}^3)^{n-1}). \]

**Lemma 1.** For $n \leq 4$ and large enough values of $m$, operator $\hat{T}$ is selfadjoint and bounded from below.

**Corollary.** For the same values of $n$ and $m$ the operator $T$ is selfadjoint and bounded from below. Indeed, from (2.16b) and boundedness from below of $\hat{T}$ it follows that $T$ and $T_0$ are bounded from below. If $T$ is not selfadjoint, it has an eigenvector as conjugate operator to symmetric bounded from below operator $T_0$ with arbitrarily large negative eigenvalue which contradicts boundedness from below of $\hat{T}$.

**Proof of Lemma 1.** After change of variables
\[ t = t' - \frac{1}{m+2} \sum_{i=2}^{n-1} k_i, \quad k_1 = s - \frac{1}{m+2} \sum_{i=2}^{n-1} k_i \]
the action of the operator $\hat{T}$ is represented in the form
\[ (\hat{T}f)(s, k) = 2\pi^2 \sqrt{\frac{m(m+2)}{(m+1)^2}} s^2 + D(k) + V \]
\[ + (n-1) \int_{\mathbb{R}^3} \frac{f(t', k) dt'}{t'^2 + s^2 + \frac{2}{m+1}(t', s) + D(k) + 1}. \]

Here $k = (k_2, \ldots, k_{n-1})$ and
\[ D(k) = \frac{m(m+3)}{(m+1)(m+2)} \sum_{2<i<n-1} k_i^2 + \frac{2m}{(m+1)(m+2)} \sum_{2<i<j<n-1} (k_i, k_j) \]
(in the case $n = 2$ the term $D(k)$ in (2.17b) is absent). Then we write $\hat{T}$ in the form
\[ \hat{T} = R(2\pi^2 E + L)R, \]
where the operator $R$ acts as multiplication by the function:
\[ (Rf)(s, k) = \left( \frac{m(m+2)}{(m+1)^2} s^2 + D(k) + 1 \right)^{1/4} f(s, k), \]
and the operator $L$ is
\[ (Lf)(s, k) = \frac{n-1}{\left( \frac{m(m+2)}{(m+1)^2} s^2 + D(k) + 1 \right)^{1/4}} \int_{\mathbb{R}^3} \frac{f(t', k) dt'}{t'^2 + s^2 + \frac{2}{m+1}(t', s) + D(k) + 1} \]
\[ \times \frac{1}{\left( \frac{m(m+2)}{(m+1)^2} t'^2 + D(k) + 1 \right)^{1/4}} dt' = (L_0 f)(s, k) + (Qf)(s, k), \]
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where

$$
(L_0 f)(s, k) = \frac{n - 1}{\left(\frac{m(m+2)}{(m+1)^2}\right) s^2 + D(k) + 1} \int_{\mathbb{R}^3} \frac{f(t', k)}{\left(\frac{m(m+2)}{(m+1)^2} t'^2 + s^2 + \frac{2}{m+1}(t', s)\right)^{1/4}} dt'
$$

(2.20b)

and

$$
(Qf)(s, k) = -\frac{n - 1}{\left(\frac{m(m+2)}{(m+1)^2}\right) s^2 + D(k) + 1} \int_{\mathbb{R}^3} \frac{f(t', k)}{\left(\frac{m(m+2)}{(m+1)^2} t'^2 + D(k) + 1\right)^{1/4} (t'^2 + s^2 + \frac{2}{m+1}(t', s))} dt'.
$$

(2.20c)

Let us estimate the norm $\|Q\|$ of operator $Q$. A simple calculation with an application of Schwarz inequality shows that

$$
\|Qf\|^2 \leq (n - 1)^2 \int \int \int \int \frac{(D(k) + 1)^2 |f(u, k)|^2 du dk ds dt}{\left(\frac{m(m+2)}{(m+1)^2} s^2 + D(k) + 1\right)^{1/2} \left(\frac{m(m+2)}{(m+1)^2} t^2 + D(k) + 1\right)^{1/2}} \times \frac{1}{(t^2 + s^2 + \frac{2}{m+1}(t, s))^2 (\xi^2 + \eta^2 + \frac{2}{m+1}(\xi, \eta))^2} \, dt^2 ds^2 \frac{d\eta d\xi}{\xi^2 + \eta^2 + \frac{2}{m+1}(\xi, \eta)^2} \frac{1}{(\xi^2 + \eta^2 + \frac{2}{m+1}(\xi, \eta))^2}.
$$

(2.21)

where $I(m)$ is the integral with respect to the variables $\eta$ and $\xi$ arising from (2.21) after change of variables $s$ and $t$

$$
I(m) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d\eta d\xi}{\xi^2 + \eta^2 + \frac{2}{m+1}(\xi, \eta)^2} \frac{1}{(\xi^2 + \eta^2 + \frac{2}{m+1}(\xi, \eta))^2}.
$$

(2.22)

Thus

$$
\|Q\| \leq (I(m))^{1/2}(n - 1).
$$
Note that for \( m \to \infty \) the integral \( I(m) \) tends to \( I(\infty) \), where

\[
I(\infty) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{d\eta d\xi}{(\xi^2 + 1)(\eta^2 + 1)(\xi^2 + \eta^2)(\xi^2 + \eta^2 + 1)^2}
\]

\[
= 16\pi^2 \int_0^\infty \int_0^\infty \frac{r_1^2 r_2^2 dr_1 dr_2}{((r_1^2 + 1)(r_2^2 + 1))(r_1^2 + r_2^2)^2(r_1^2 + r_2^2 + 1)^2}
\]

\[
= 16\pi^2 \int_0^\infty d\rho \int_0^{\pi/2} d\theta \frac{\rho^5 \cos^2 \theta \sin^2 \theta}{(\rho^2 \cos^2 \theta + 1)^{1/2}(\rho^2 \sin^2 \theta + 1)^{1/2}\rho^4(\rho^2 + 1)^2}
\]

(here \( r_1 = \rho \sin \theta, \ r_2 = \rho \cos \theta \))

\[
= 4\pi^2 \int_0^\infty du \int_0^{\pi/2} d\varphi \frac{\sin^2 \varphi}{(u^2 \sin^2 \varphi + 4u + 4)^{1/2}(u + 1)^2}
\]

\[
\leq 4\pi^2 \int_0^\infty du \int_0^{\pi/2} d\varphi \frac{\sin^2 \varphi}{(u \sin \varphi + 2)(u + 1)^2}
\]

From the representation

\[
\frac{1}{(u \sin \varphi + 2)(u + 1)^2} = \frac{\sin \varphi}{(2 - \sin \varphi)^2} \left( \frac{1}{u + 2/\sin \varphi} - \frac{1}{u + 1} \right) + \frac{1}{(2 - \sin \varphi)(u + 1)^2}
\]

we find that the last integral doesn’t exceed

\[
4\pi^2 \int_0^{\pi/2} \frac{\sin^2 \varphi d\varphi}{2 - \sin \varphi} < \pi^3.
\]

Thus for \( n \leq 4 \) and large enough \( m \) the norm \( \|Q\| \) doesn’t exceed

\[
\|Q\| < 3\frac{11}{10} \pi^{3/2} < 2\pi^2. \quad (2.23)
\]

Hence

\[
(Q \varphi, \varphi) \geq -\frac{33}{10} \pi^{3/2}(\varphi, \varphi), \quad \varphi \in L_2((\mathbb{R}^3)^{n-1}) \quad (2.24a)
\]

and

\[
((2\pi^2 E + Q) \varphi, \varphi) > \Delta(\varphi, \varphi), \quad (2.24b)
\]

where

\[
\Delta = 2\pi^2 - \frac{33}{10} \pi^{3/2} > 0. \quad (2.24c)
\]

Let us study now the operator \( L_0 \) which is written in the form

\[
L_0 = SN S,
\]

where

\[
(Sf)(s, k) = \frac{|s|^{1/2}f(s, k)}{(m(m+2)(m+1))^{1/2}} s^2 + D(k) + 1)^{1/4}
\]

and

\[
(Nf)(s, k) = \frac{n - 1}{|s|^{1/2}} \int_{\mathbb{R}^3} \frac{f(t, k) dt}{(s^2 + t^2 + \frac{2}{m+1}(t, s))^{1/2}}. \quad (2.26b)
\]
We represent the space $L_2((\mathbb{R}^3)^{n-1})$ as tensor product of two spaces
\[ L_2((\mathbb{R}^3)^{n-1}) = L_2(\mathbb{R}^3) \otimes L_2((\mathbb{R}^3)^{n-2}) \]
and write $N$ as tensor product of two operators
\[ N = M \otimes E, \]
where $M$ is operator in $L_2(\mathbb{R}^3)$ with the kernel
\[ M(s, t) = \frac{n-1}{|st|^{1/2} \times (s^2 + t^2 + \frac{2}{m+1}(t, s))}. \]
The space $L_2(\mathbb{R}^3)$ is represented also as tensor product
\[ L_2(\mathbb{R}^3) = L_2^1(\mathbb{R}^3) \otimes L^2(S), \quad (2.27) \]
where $L^2(S)$ is the space of functions square integrable on the unit sphere $S \subset \mathbb{R}^3$ with measure $d\sigma = \sin \theta \, d\theta \, d\varphi$, where $(\theta, \varphi)$ are spherical coordinates on $S$. The space $L^2(S)$ is the orthogonal sum of the spaces $L^l_2$,
\[ L_2(S) = \bigoplus_{l=0}^\infty L^l_2, \]
where $L^l_2 \subset L_2(S)$ is a subspace of the functions on $S$, which are transformed under representation of the group $O_3$ of rotations in $L^2(S)$
\[ g : f(\sigma) \to f(g^{-1} \sigma), \quad \sigma \in S, \ g \in O_3 \quad (2.28) \]
with the irreducible representation of weight $l$ (see [11]). There is an orthogonal basis in $L^l_2$ consisting of the spherical functions:
\[ Y^m_l(\theta, \varphi) = P^m_l(\cos \theta)e^{im\varphi}, \quad m = -l, -l+1, \ldots, l-1, l, \]
where $P^m_l(x)$ are polynomials of $x \in [-1, 1]$ (see [11]).

Since the operator $M$ commutes with the operators (2.28), the spaces of the form
\[ L_2(R^l_+, r^2 \, dr) \otimes L^l_2 \subset L_2(\mathbb{R}^3) \quad (2.29) \]
are invariant with respect to $M$ and it acts in each subspace (2.29) as
\[ \hat{M}_l \otimes E_l, \]
where $E_l$ is unit operator in $L^l_2$ and $\hat{M}_l$ is an operator in $L_2(R^l_+, r^2 \, dr)$ acting by formula
\[ (\hat{M}_l f)(r) = (n-1) \int_0^\infty r'^2 a_l(r, r') f(r') \, dr', \quad r > 0. \]
The kernel $a_l(r, r')$ is found from the relation
\[ Y^m_l(\theta, \varphi) a_l(r, r') = \int_0^\pi d\theta' \int_0^{2\pi} d\varphi' Y^m_l(\theta', \varphi') \sin \theta' \]
\[ \times r'^{1/2}(r')^{1/2}(r^2 + r'^2 + \frac{2}{m+1}rr'\cos \theta \cos \theta' + \sin \theta \sin \theta'(\sin \varphi \sin \varphi' + \cos \varphi \cos \varphi')) \]
\[ = \frac{1}{r^1/2(r')^{1/2}(r^2 + r'^2 + \frac{2}{m+1}rr'\cos \theta \cos \theta' + \sin \theta \sin \theta'(\sin \varphi \sin \varphi' + \cos \varphi \cos \varphi'))}. \quad (2.30) \]
which is true for any \( m, \theta \) and \( \varphi \). Choosing \( m = 0 \) and \( \theta = 0 \) we get

\[
a_l(r, r') = (P_0^l(1))^{-1} 2\pi \int_{-1}^{1} \frac{P_0^l(x)}{(rr'x)^{1/2}(r^2 + r'^2 + 2m + 1)} dx.
\]

The polynomial \( P_0^l(x) \) is equal to

\[
P_0^l(x) = \sqrt{\frac{2l + 1}{2}} \frac{d^l}{dx^l} (x^2 - 1)^l,
\]

whence

\[
P_0^l(1) = \sqrt{\frac{2l + 1}{2}}.
\]

Thus the operator \( \hat{M}_l \) acts by formula

\[
(\hat{M}_l f)(r) = (n - 1) \int_{-1}^{1} \frac{P_0^l(x) r'^2 f(r') dr'}{(rr'x)^{1/2}(r^2 + r'^2 + 2m + 1)} dx.
\]

After unitary Mellin transform

\[
U: L_2(\mathbb{R}_+^l, r^2 dr) \to L_2(\mathbb{R}_+, ds):
\]

\[
(U \psi)(s) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} r^{-is+1/2} \psi(r) dr = g(s)
\]

(with inverse transform

\[
(U^{-1}g)(r) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} r^{is-3/2} g(s) ds
\]

and some calculations (see Appendix) we find that the operator \( \hat{M}_l \) in \( L_2(\mathbb{R}_+^l, r^2 dr) \) is unitary equivalent to the operator of multiplication by the function \( \Lambda_l(s) \) in \( L_2(\mathbb{R}, ds) \), where

\[
\Lambda_l(s) = (n - 1) \sqrt{\frac{2}{2l + 1}} \left\{ \begin{array}{ll}
\int_{0}^{1} P_0^l(x) \frac{\ch(s \vartheta(x)) dx}{\ch \frac{\pi}{2} \cos(\vartheta(x))} & \text{for even } l, \\
\int_{0}^{1} P_0^l(x) \frac{\sh(s \vartheta(x)) dx}{\sh \frac{\pi}{2} \cos(\vartheta(x))} & \text{for odd } l
\end{array} \right.
\]

with \( \vartheta(x) = \arcsin \frac{x}{m+1}, \ x \in (0, 1) \). From the formula (2.32) together with the inequality

\[
\cos^2 \vartheta(x) > 1 - \left( \frac{1}{m + 1} \right)^2
\]

we see that

\[
|\Lambda_l(s)| < (n - 1) \frac{\pi}{\sqrt{2l + 1}} \frac{m + 1}{\sqrt{m(m + 2)}}.
\]

Thus the operator \( M \) is bounded

\[
\|M\| < \max_{l,s} |\Lambda_l(s)| < (n - 1) \pi \frac{m + 1}{\sqrt{m(m + 2)}}.
\]

Besides, as \( m \to \infty \), all \( \Lambda_l(s) \) tend to zero except \( \Lambda_0(s) \): \( \Lambda_0(s) \to (n - 1) \pi \frac{1}{\ch \frac{\pi}{2}} > 0 \).
From here it follows that for $n \leq 4$ and large enough $m$ the quadratic form

$$
(M \varphi, \varphi) > -\frac{1}{2} \Delta(\varphi, \varphi), \quad \varphi \in L_2(\mathbb{R}^3)
$$

(2.33)

(the number $\Delta$ is defined in (2.24c)). Since for $m \to \infty$ the operator $S$ converges in norm to the operator of multiplication by the function

$$
\frac{|s|^{1/2}}{(s^2 + D(k) + 1)^{1/4}}
$$

and since this operator has norm 1, the quadratic form $L_0(\Phi, \Phi)$ satisfies the inequality

$$
(L_0 \Phi, \Phi) > -\frac{2}{3} \Delta(\Phi, \Phi), \quad \Phi \in L_2((\mathbb{R}^3)^{n-1})
$$

(2.34a)

for $n \leq 4$ and large enough $m$. Gathering the estimates (2.24b) and (2.34a) we get that quadratic form of the bounded selfadjoint operator

$$
((2\pi^2 E + L) \Phi, \Phi) > \frac{1}{3} \Delta(\Phi, \Phi).
$$

(2.34b)

Hence there exists a bounded inverse operator

$$
(2\pi^2 E + L)^{-1}.
$$

If the operator $\tilde{T}$ as conjugate to symmetric operator $\tilde{T}_0$ is not selfadjoint, i.e., has nonzero deficiency indexes (equal to each other) for any complex number $\lambda$, $\text{Im} \lambda \neq 0$, there is an eigenvector $\psi_\lambda$ of $\tilde{T}$ with eigenvalue $\lambda$:

$$
\tilde{T} \psi_\lambda = \lambda \psi_\lambda.
$$

(2.35)

Using now the representation (2.18) for $\tilde{T}$ the equation (2.35) is written in the form

$$
\psi_\lambda = \lambda R^{-1} (2\pi^2 E + L)^{-1} R^{-1} \psi_\lambda.
$$

Since the operator $R^{-1}$ is bounded, the norm

$$
\|\lambda R^{-1} (2\pi^2 E + L)^{-1} R^{-1}\| < 1
$$

for small enough $\lambda$, whence the equation (2.35) has no nonzero solutions. From this, selfadjointness of $\tilde{T}$ follows. Its boundedness from below follows from the inequality

$$
(\tilde{T} \Phi, \Phi) = ((2\pi^2 E + L) R \Phi, R \Phi) > 0 \quad \text{for } \Phi \in D(R),
$$

where $D(R)$ is the domain of $R$ which is everywhere dense in $L_2((\mathbb{R}^3)^{n-1})$.

Lemma 1 is proved and by the same token it is established that all extensions $\tilde{H}_\varepsilon$ of the operator $\tilde{H}_0$ are selfadjoint. □

3. BOUNDEDNESS FROM BELOW OF THE OPERATOR $\tilde{H}_\varepsilon$

We shall establish boundedness from below of $\tilde{H}_\varepsilon$ by proving the existence of resolvent $R_{\tilde{H}_\varepsilon}(z)$ of $\tilde{H}_\varepsilon$ for enough large negative $z$. Thus we begin to construct the resolvent $R_{\tilde{H}_\varepsilon}(z)$. Suppose the equation with respect to $g$ is given

$$
\tilde{H}_\varepsilon g - z g = f, \quad f \in L_2^{\text{sym}}((\mathbb{R}^3)^n).
$$
Using the formula (2.8) we find that
\[ g(p_1, \ldots, p_n) = \sum_{i=1}^{n} \varphi(p_1, \ldots, \hat{p}_i, \ldots, p_n) \frac{(-1)^{i-1} + f(p_1, \ldots, p_n)}{G(p_1, \ldots, p_n) - z}, \] (3.1)

where \( z \in \mathbb{R}^3, z < 0 \), and \( \varphi \in L^{\text{asym}} \) is the function occurring in (2.7). If we integrate both sides of the equality (3.1) we get
\[ \int_{|p_1|<N} g(p_1, \ldots, p_n) \, dp_1 = 4\pi N \varphi(p_2, \ldots, p_n) - (T(-z)\varphi)(p_2, \ldots, p_n) + \chi_z \varphi(p_2, \ldots, p_n) + o(1) \quad \text{for } N \to \infty. \] (3.2)

Here \( T(-z)\varphi \) is defined by a formula similar to (2.11):
\[ (T(-z)\varphi)(p_2, \ldots, p_n) = 2\pi^2 (H(p_2, \ldots, p_n) - z)^{1/2} \varphi(p_2, \ldots, p_n) + \int_{\mathbb{R}^2} \frac{\sum \varphi(t, p_2, \ldots, \hat{p}_i, \ldots, p_n) (-1)^i \, dt}{G(t, p_2, \ldots, p_n) - z} \] (3.3a)

and
\[ \chi_z(p_2, \ldots, p_n) = \int_{\mathbb{R}^3} \frac{f(p_1, \ldots, p_n) \, dp_1}{G(p_1, p_2, \ldots, p_n) - z}. \] (3.3b)

Using the relation (2.15) we get the equation with respect to \( \varphi \)
\[ (T(-z) + \varepsilon E)\varphi = \chi_z. \] (3.4a)

Note that as it follows from (3.3b) the function \( \chi_z(k_1, \ldots, k_{n-1}) \) is representable in the form
\[ \chi_z = R^{-2}(z)\psi, \]
where \( \psi \in L_2^{\text{asym}}(\mathbb{R}^{3n-1}) \) and \( R(z) \) is the operator
\[ (R(z)f)(k_1, \ldots, k_{n-1}) = (H(k_1, \ldots, k_{n-1}) - z)^{1/4} f(k_1, \ldots, k_{n-1}). \]

The function \( \varphi \) is found in the form
\[ \varphi = R^2(z)h, \]
where \( h \in L_2^{\text{asym}}(\mathbb{R}^{3n-1}) \) (see Remark 2).

As a result the equation (3.4a) is represented in the form
\[ \mathbb{R}^2(z) (T(-z) + \varepsilon E) \mathbb{R}^2(z) h = \psi, \] (3.4b)
where the operator \( T(-z) \) for real \( z < 0 \) means the selfadjoint operator similar to the operator \( T \) from the previous section (for \( z = -1 \), \( T(1) = T \)). As above the following equality is true
\[ T(-z) = P \tilde{T}(-z) P, \]
where \( P \) is projection (2.14a) and \( \tilde{T}(-z) \) is operator in \( L_2((\mathbb{R}^3)^{n-1}) \) similar to \( \tilde{T} \) and has the form
\[ (\tilde{T}(-z) \varphi)(k_1, \ldots, k_{n-1}) = 2\pi^2 (H(k_1, \ldots, k_{n-1}) - z)^{1/2} \varphi(k_1, \ldots, k_{n-1}) \]
\[ + (n - 1) \int_{\mathbb{R}^3} \frac{\varphi(t, k_2, \ldots, k_{n-1}) \, dt}{G(t, k_1, \ldots, k_{n-1}) - z}. \] (3.5)
For the operator \((T(-z) + \varepsilon E)\) one has
\[
((T(-z) + \varepsilon E)\varphi, \varphi)_{L^2_{\text{sym}}(\mathbb{R}^3)^{n-1}} = ((\tilde{T}(-z) + \varepsilon \tilde{E})\varphi, \varphi)_{L^2(\mathbb{R}^3)^{n-1}} \tag{3.6}
\]
for \(\varphi \in L^2_{\text{sym}}(\mathbb{R}^3)^{n-1}\). Here \(\tilde{E}\) is a unity operator in \(L^2(\mathbb{R}^3)^{n-1}\).

Then we write the operator \(\tilde{T}(-z) + \varepsilon \tilde{E}\) in the form
\[
(\tilde{T}(-z) + \varepsilon \tilde{E}) = R(-z) \left(2\pi^2 \tilde{E} + \varepsilon R^{-2} + L(-z)\right) R(-z),
\]
where the operator \(L(-z)\) has the form (after the change of variables (2.17a))
\[
(L(-z)\varphi)(s, k) = \frac{n-1}{\left(\frac{m+1}{2}\right)^2} s^2 + D(k) - z \right)^{1/4} \\
\times \int_{\mathbb{R}^3} \varphi(t, k) dt \\
= (L_0(-z)\varphi)(s, k) + (Q(-z)\varphi)(s, k). \tag{3.7}
\]
Here the operators \(L_0(-z)\) and \(Q(-z)\) are similar to the operators \(L_0 = L_0(1)\) and \(Q = Q(1)\) correspondingly.

**Lemma 2.** For \(n \leq 4\) and large enough \(m\) and \(z < 0\) the operator \(2\pi^2 \tilde{E} + \varepsilon R^{-2}(-z) + L(-z)\) is strictly positive.

From this lemma the existence of the resolvent \(R_{\tilde{H}_s}(z)\) follows immediately. Indeed in this case the operator \(\tilde{T}(-z) + \varepsilon \tilde{E}\) is strictly positive and has a bounded inverse operator. From here with the help of (3.6) it follows that the operator \(T(-z) + \varepsilon E\) has the inverse operator and therefore the equation (3.4b) has a solution for any \(\psi\).

Thus the resolvent \(R_{\tilde{H}_s}(z)\) exists for large \(z < 0\). The proof of Lemma 2 almost literally follows the arguments of previous section together with the remark that operator \(\varepsilon R^{-2}(-z)\) has small enough norm for large negative \(z\).

Thus Theorem 1 is proved.

**Remark.** Unfortunately the lower bound on the quadratic form \((L\psi, \psi)\) (see (2.34b)) is too rough. More fine estimate permits to consider the systems with more than 4 fermions and also for each \(n\) to establish the boundary for values \(m\) for which the theorem 1 is true.

**Appendix. Calculating the Functions \(\Lambda_l(s)\)**

Here we give a short explanation of how the functions \(\Lambda_l(s)\) are calculated. After the transformation of the operator \(\tilde{M}_l\)
\[
U \tilde{M}_l U^{-1} = \tilde{N}_l,
\]
where \(U\) and \(U^{-1}\) are Mellin transforms (2.31c) it becomes the operator of multiplication by the function \(\Lambda_l(s)\) in \(L^2(\mathbb{R}, ds)\), where
\[
\Lambda_l(s) = (n-1) \sqrt{\frac{1}{2\pi}} 2\pi \int_{-\infty}^{1} d\xi \int_{-1}^{1} dx \frac{\xi^{-is}}{\xi^2 + \frac{2}{m+1} \xi x + 1} P^0_l(x). \tag{A.1}
\]
The function (for fixed $s \in \mathbb{R}$)
\[
\frac{\xi^{-is}}{\xi^2 + \frac{2}{m+1} \xi x + 1} = q(\xi; x)
\]
is a meromorphic function of $\xi$ on the complex plane with the cut $(0, \infty)$ along real axis which goes to zero for $\xi \to \infty$ as $|\xi|^{-2}$. It is easy to see that
\[
\int_0^\infty q(\xi; x) \, d\xi = \frac{1}{1 - e^{2\pi s}} \int_\gamma q(\xi; x) \, d\xi,
\]
where $\gamma$ is the contour consisting of the both sides of the cut.

For even $l$ the integral in (A.1) is equal to
\[
\frac{1}{1 - e^{2\pi s}} \int_0^1 dx P_0^l(x) \int_\gamma (q(\xi; x) + q(\xi; -x)) \, d\xi = \frac{1}{1 - e^{2\pi s}} \int_0^1 dx P_0^l(x) \left[ \sum \text{res} (q(\xi; x) + q(\xi; -x)) \right],
\]
where the sum is taken over four poles of the function $q(\xi; x) + q(\xi; -x)$. These poles are in the points
\[
z_{1,2} = e^{i(\frac{\pi}{2} + \vartheta(x))} \quad \text{and} \quad z_{3,4} = e^{i(\frac{3\pi}{2} + \vartheta(x))},
\]
where
\[
\vartheta(x) = \arcsin \frac{x}{m+1}.
\]
Calculating the residues in these points we get the expressions (2.32) for even $l$. The calculations for odd $l$ are similar.

References


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