METRICS AND SMOOTH UNIFORMISATION OF LEAVES OF HOLOMORPHIC FOLIATIONS

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ABSTRACT. We consider foliations of complex projective manifolds by analytic curves. In a generic case each leaf is hyperbolic and there exists unique Poincaré metric on the leaves. It is shown that in a generic case this metric smoothly depends on a leaf. The manifold of universal covering of the leaves passing through some transversal base has a natural complex structure. It is shown that this structure can be defined as a smooth almost complex structure on the product of the base and a fiber and there exists a natural pseudoconvex exhaustion.

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INTRODUCTION

Let $X$ be a complex manifold. We say that a foliation with singularities is defined on $X$ if there exists an analytic subset $\Sigma$ of codimension at least two and a foliation of its complement by analytic curves that cannot be extended to a neighborhood of any point of $\Sigma$. A foliation on $\mathbb{CP}^n$ can be locally defined by polynomial vector fields. We consider also foliations on $\mathbb{C}^n$ that are defined locally by polynomial vector fields and globally by rational fields of directions. In generic case the singular set $\Sigma$ consists of isolated points.

A covering manifold of leaves of a foliation was defined in [Il1], [Il2]. Let $\mathcal{F}$ be a foliation with singularities on a complex manifold $X$ and let $B$ be a transversal cross-section. Let $\varphi_p$ be a leaf passing through a point $p \in B$ and let $\tilde{\varphi}_p$ be the universal cover of this leaf with the marked point $p$. Define $M = \bigcup_{p \in B} \tilde{\varphi}_p$. It is shown in [Il1], [Il2] that at least in affine case or, in more general Stein case, a topology and a complex structure on this union can be defined so that it is a complex manifold with locally biholomorphic projection $\tilde{\pi}: M \to X$ and a holomorphic section $B \to M$ right inverse to the holomorphic retraction $\pi: M \to B$. For any leaf $\varphi_p$ the restriction of $\tilde{\pi}$ to $\tilde{\varphi}_p$ is the universal covering map over $\varphi_p$. For a foliation on $\mathbb{CP}^n$ the manifold of universal coverings can be non-Hausdorff but in a generic case it is Hausdorff (see [Br1], [Br2]). It is possible to define a Hausdorff
universal covering for general foliations on compact Kahler manifolds if we include the singular points in the leaves in some not generic cases [Br1], [Br3] but here we don’t consider such situations.

For generic foliation on $\mathbb{C}^n$ or $\mathbb{C}P^n$ each leaf is hyperbolic ([Gl1] or [LN]). The uniformising map of every leaf is unique modulo authomorphisms of the disk, and after some normalization (to get uniqueness) we may ask: how the uniformising map of $\varphi_p$ depends on the point $p$? Equivalently, we may put on every leaf its Poincaré metric, i.e., the unique complete hermitian metric of curvature $-1$ and ask about the dependence of this metric on the point $p$. The simultaneous uniformisation conjecture states that under a suitable choice of the uniformising map $\varphi_p$ onto an appropriate $p$-depending domain on the Riemann sphere, this map may become analytic in $p$. It is known that this conjecture is wrong for general foliation in dimension of more than two or even for foliations of general two-dimensional manifolds [Gl2]. It is not known is this conjecture true or not for generic foliations on $\mathbb{C}^2$ or $\mathbb{C}P^2$.

In the paper we prove that this uniformising map or the Poincaré metric smoothly depends on the point $p$. Everywhere below we consider only foliations on $\mathbb{C}P^n$ or foliations on $\mathbb{C}^n$ defined by polynomial vector fields but all results can be generalized to foliations on compact Kahler manifolds with negative divisor without any difficulties. More exactly we can formulate our result if we introduce almost complex structures. The manifold of universal coverings is topologically equivalent to a product of the base, which we can suppose to be a polydisk $D^{n-1}$ or a Stein submanifold $B$, and the disk $D$. Our main result is (Theorem 2 in Section 2):

**Theorem.** Suppose that an analytic foliation $\mathcal{F}$ is not tangent to a linear vector field. Suppose that the singular set $\Sigma$ consists of isolated points which are non-degenerate and the vector field locally defining $\mathcal{F}$ at each singular point is analytically linearizable and the linear part is diagonalizable. Let $M$ be a manifold of universal coverings with base $B$. Then $M$ is biholomorphic to the product $B \times D$ with an almost complex structure defined by the forms of type $(1, 0)$:

$$d\zeta = \{d\zeta_j\}, \quad j = 1, \ldots, n-1,$$

$$\omega = dw + \langle c, \overline{d\zeta} \rangle.$$

Here $\zeta_j$ are coordinates on the base $B$, $w$ is a holomorphic chart on the fibers of $M$, $c = \{c_j\}$ is a smooth ($C^\infty$) vector-function holomorphic with respect to $w$, $\langle c, d\zeta \rangle$ is the scalar product $c_1d\zeta_1 + \cdots + c_{n-1}d\zeta_{n-1}$.

Note that the equation $c_\bar{w} = 0$ is the condition of integrability of the complex structure according to the Newlender–Nirenberg theorem [H]. The leaves of the foliation are hyperbolic under the conditions of the theorem as we prove in the next section.

In the proof of Theorem 2 we use some metric of negative curvature on the leaves of the foliation. Existence of metrics with a curvature on the leaves uniformly bounded away from zero for generic foliations was proved by Glutsyuk [Gl1] and Lins Neto [LN]. In the construction of Lins Neto the metric was in addition complete on the leaves. Theorem 1 in Section 1 is some improvement of this result: the metric
in this theorem is complete and its curvature on the leaves is negative, uniformly bounded away from zero and uniformly bounded from below.

As a consequence of above results and some results of M. Brunella we prove that in our case the manifold of universal coverings $M$ is Stein. It is Theorem 4 in Section 3. In the case, when the manifold $X$ is Stein itself, this was proved in [B2], [BSh]. M. Brunella [Br1], [Br2], [Br3] proved that for foliations on compact Kahler manifolds the leafwise Poincaré metric changes in some “plurisubharmonic” way. (The accurate definition is given in Section 3). M. Brunella also pointed out that $M$ is Stein if the leaves are hyperbolic and the Poincaré metric is smooth, and this follows from his results. We give this proof in Section 3.

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1. Metrics on Foliations

We consider foliations on the affine space $\mathbb{C}^n$ and on the projective space $\mathbb{CP}^n$. In what follows $x_0, x_1, \ldots, x_n$ are homogeneous coordinates on $\mathbb{CP}^n$ and $z_1 = x_1/x_0, \ldots, z_n = x_n/x_0$ are coordinates on the affine domain $\{x_0 \neq 0\}$. A foliation of class $A_d$ on $\mathbb{C}^n$ is determined by a vector field of the form

$$X = P_1(z) \frac{\partial}{\partial z_1} + \cdots + P_n(z) \frac{\partial}{\partial z_n},$$

where $P_1, \ldots, P_n$ are polynomials of degree $d$. A foliation of class $B_d$ on $\mathbb{CP}^n$ on the domain $\{x_0 \neq 0\}$ is determined by a vector field

$$X = X_{(0)} + X_{(1)} + \cdots + X_{(d)} + X_{(d^*)} \left( z_1 \frac{\partial}{\partial z_1} + \cdots + z_n \frac{\partial}{\partial z_n} \right),$$

where $X_{(k)}$ is the $k$-th homogeneous component of $X$ and $X_{(d^*)}$ is a homogeneous polynomial of degree $d$. This foliation can be equivalently defined as the radial projection of a foliation on $\mathbb{C}^{n+1}$ determined by a degree $d$ homogeneous vector field

$$Y = H_0(x) \frac{\partial}{\partial x_0} + \cdots + H_n(x) \frac{\partial}{\partial x_n}.$$  

In the chart $z = x/x_0$ the coefficients of the vector field $X$ defined on $\mathbb{C}^n$ are

$$P_i(z) = H_i(1, z) - z_i H_0(1, z).$$

In what follows we will denote by $\langle x, y \rangle$ a scalar product $x_1 y_1 + \cdots + x_n y_n$ or $x_0 y_0 + \cdots + x_n y_n$ depending on a context. Let $\Sigma$ be a set of singular points of a foliation. We assume that $\Sigma$ is a discrete set of points and that all these points are non-degenerate. On $\mathbb{CP}^n \setminus \Sigma$ we define a metric which gives a metric of negative curvature bounded away from zero after restriction on a leaf. We will do it in several stages.

At first define on $\mathbb{CP}^n \setminus \Sigma$ a following metric (here $x$ are homogenous coordinates):

$$g = F(x) \frac{|x|^2 \langle dx, d\bar{x} \rangle - \langle \bar{x}, dx \rangle \langle x, d\bar{x} \rangle}{|x|^2 \langle Y, Y \rangle - \langle \bar{x}, Y \rangle \langle x, Y \rangle}.$$  

(1.4)
There $F$ is a real positive function on the complement of $\mathbb{C}^{n+1}$ to the projection preimage of the singular set $\Sigma$. This function is non-zero except for the origin and has a property

$$F(ax) = |a|^{2(d-1)}F(x).$$

(1.5)

In the chart $z$ the metric $g$ has a form:

$$g = f(z)^2 \frac{(1 + |z|^2)(dz, d\bar{z}) - \langle \bar{z}, dz \rangle \langle z, d\bar{z} \rangle}{(1 + |z|^2)(X, \bar{X}) - \langle \bar{z}, X \rangle \langle z, \bar{X} \rangle}.$$  

(1.6)

There $f(z) = F(1, z)$. If we consider a foliation of the class $A_d$ we also use a metric of form (1.6) with a function $f$ having an origin from some function $F(x)$ homogeneous in the sense of (1.5), that is, the function

$$F(x) = |x_0|^{2(d-1)}f(x_1/x_0, \ldots, x_n/x_0)$$

satisfies to property (1.5). If we define $h = (1 + |z|^2)(dz, d\bar{z}) - \langle \bar{z}, dz \rangle \langle z, d\bar{z} \rangle$ we can write metric (1.6) in the form:

$$g = f(z) \frac{h}{h(X)}.$$  

(1.7)

We assume the following condition on the function $F$:

a) for each affine chart $U_j$ the function $f_j(z) = F(z_1, \ldots, 1, \ldots, z_n)$, $z_k = x_k/x_j$ is such that $\ln(f_j)$ is strictly plurisubharmonic.

**Example.** Let’s take

$$F(x) = \sum_{j=0}^{n} |x_j|^2 \sum_{j=0}^{n} |x_j^{d-3}|^2.$$  

(1.8)

In the chart $z$ the corresponding function $f$ has a form:

$$f(z) = (1 + l_1(z))(1 + l_2(z)),$$

where $l_1(z) = z\bar{z}$, $l_2(z) = \sum_{j=1}^{n} |z_j^{d-3}|^2$. Let $L_\varphi$ be a Levi form of a function $\varphi$. We have:

$$L_{\ln f} = f^{-2}(\partial f \partial \bar{f} f^{-2} f \partial f - \partial \bar{f} f^{-2} f)$$

$$f^{-2} \{(1 + l_1)(1 + l_2)(1 + l_1)l_2 + 2 \Re(\partial l_1 \partial \bar{l}_2) + (1 + l_1) L_1 l_2 - |(1 + l_2) \partial l_1 + (1 + l_1) \partial \bar{l}_2|^2\}$$

$$= f^{-2} \{(1 + l_1)(1 + l_2)(1 + l_2) L_1 l_1 + (1 + l_1) L_2 l_2 - |(1 + l_2) \partial l_1|^2 - |(1 + l_1) \partial \bar{l}_2|^2\}$$

We have by the Schwarz inequality:

$$(1 + l_1(z))L_1(z)(\zeta) - |\partial L_1(z)(\zeta)|^2 = \zeta \bar{\zeta} + |z|^2|\zeta|^2 - |(\zeta \zeta)|^2 \geq \zeta \bar{\zeta}.$$  

Analogously

$$(1 + l_2(z)) L_2(z)(\zeta) - |\partial L_2(z)(\zeta)|^2$$

$$(d - 3)^2 \left[ \sum_{j=1}^{n} |z_j^{d-3}|^2 \right] + \sum_{j=1}^{n} |z_j^{d-3}|^2 \sum_{j=1}^{n} |z_j^{d-4}|^2 - \sum_{j=1}^{n} |z_j^{d-3} z_j^{d-4}|^2 \geq (d - 3)^2 \sum_{j=1}^{n} |z_j^{d-3}|^2 |\zeta_j|^2$$
also by the Schwarz inequality because the last term is scalar product of the vectors \((\bar{z}^{d-3}, \ldots, \bar{z}^{d-3})\) and \((\bar{z}^{d-k}z_1, \ldots, \bar{z}^{d-k}z_n)\).

The next theorem is some improving of the main result of [LN]. We will modify the metric in neighborhoods of singular points \(\Sigma\) in such a way that the curvature will be bounded away from zero. Besides, in the following theorem a metric is smooth and the curvature is uniformly bounded.

**Theorem 1.** Suppose we have a foliation of class \(A_d\) on \(\mathbb{C}^n\) or of class \(B_d\) on \(\mathbb{C}P^n\) (in both cases \(d \geq 2\)) with non-degenerate singular points \(\Sigma\) and a vector field \(X\) defining the foliation at each singular point is analytically linearizable, and the linear part is diagonalizable. Then there exists a \(C^\infty\)-smooth metric \(\mu\) on \(\mathbb{C}P^n \setminus \Sigma\), and for each singular point \(p\) there exists a neighborhood \(U_p\) such that

1. on \(\mathbb{C}P^n \setminus \bigcup_p U_p\) the metric \(\mu\) has form \((1.4)\) where \(F\) satisfies \((1.5)\) and condition a) above,
2. \(\mu\) is complete on the leaves,
3. there exist some constants \(C_0 < 0, C_1 > -\infty\) such that \(C_1 < K_\mu < C_0 < 0\), where \(K_\mu\) is the curvature on the leaves.

For example we can take for \(F\) function \((1.8)\).

**Proof.** In what follows we don’t write the index \(p\) at \(U_p\) or \(V_p\) when considering a neighborhood of a singular point \(p\). We need a definition [LN].

**Definition 1.** Let \(U\) be a neighborhood of a point \(p\) in a complex manifold and \(g\) be a continuous hermitian metric on \(U \setminus p\). We say that \(g\) is complete at \(p\) if any continuous path \(\gamma: [0, 1] \to U\) such that \(\gamma(0), \gamma(1) \in U \setminus p\) and \(\gamma(1) = p\) has infinite length

\[ l_g(\gamma) = \int_\gamma g^{1/2}. \]

Note that a continuous metric \(\mu\) on \(\mathbb{C}P^n \setminus \Sigma\) is complete if and only if it is complete at all points of \(\Sigma\).

We consider a local situation. Let \(X\) be a holomorphic vector field on \(B_r := \{ w \in \mathbb{C}^n : |w| < r \} \). Suppose that 0 \(\in\) \(B_r\) is the unique singularity of \(X\) in \(B_r\), this singularity is non-degenerate and the field \(X\) at 0 is analytically linearizable and the linear part is diagonalizable. Let \(a_1, \ldots, a_n\) be eigenvectors of this linear part. In what follows we suppose that the eigenvectors are directed along the coordinate axes and \((0, 0, \ldots, 1, \ldots, 0)\) (1 on the \(k\)-th place) is eigenvector with eigenvalue \(\lambda_k\), that is, \(X_k = \lambda_k w_k + o(|w|)\). Let \(h\) be a smooth hermitian metric on \(B_r\) and \(u\) be a pluriharmonic function on \(B_r\). We will consider metrics of the form:

\[ \mu = \exp(u) f(w) \frac{h}{h(X)}, \]

where \(f\) is some real function of \(w\). Let \(t\) be a holomorphic coordinate on a leaf, \(w(t)\) be a solution of the equation \(dw/dt = X[w(t)]\). The curvature of the leaves for metric \((1.9)\) is:

\[ K_\mu(w) = -\frac{2}{\exp(u)f(w)} \frac{\partial^2}{\partial t \partial \bar{t}} \ln f(w) \]

\[(1.10)\]
The curvature is negative if the function $\ln f$ is strictly plurisubharmonic.

**Lemma 1.** Let a vector field $X$ be analytically linearizable and the linear part be diagonalizable at the singular point, and let $w$ be a diagonalizing chart. Define metric (1.9) with the function $f_1$ of the form

$$f_1(w) = \frac{1}{\ln^4(|w|^2)}. \quad (1.11)$$

The function $\ln f_1$ is strictly plurisubharmonic. The punctured ball $0 < |w| < 1$ and the curvature for the corresponding metric is negative.

Define also the metric with the following function $f = f_2$:

$$f_2(w) = \frac{1}{\ln^2(|w|^2)}. \quad (1.12)$$

The function $\ln f_2$ is strictly plurisubharmonic at those $w \neq 0$ for which $|w_i| < 1$ for all $i$, and the curvature of the leaves for the corresponding metric $\mu$ is uniformly bounded away from zero and from $-\infty$ on any domain $0 < |w| \leq r, 0 < r < 1$. The metric is complete at zero.

*Proof.* Denote $v_i = |w_i|^2, v = |w|^2$. For a function $f(v_1, \ldots, v_n)$ the Levi form of $\ln f$ on a vector $Y$ is:

$$L_{\ln f}(w)(Y) = \frac{1}{f} \left( \sum_i f_i |Y_i|^2 + \sum_i f_{i,i} w_i Y_i^2 \right) + 2 \sum_{i \neq j} f_{i,j} \Re[(\bar{w}_i w_j Y_i Y_j)] - f^{-1} \sum_i f_i \bar{w}_i Y_i^2. \quad (1.13)$$

Analogously for function $f(v)$ the corresponding Levi form is:

$$L_{\ln f}(w)(Y) = \frac{1}{f} (f_i |Y_i|^2 + f_{i,j} |\bar{w}_i w_j Y_i Y_j|)^2 - f^{-1} f_i |\bar{w}_i Y_i|^2. \quad (1.14)$$

For function (1.11) and $0 < |w| < 1$ the expression in the brackets in (1.14) is:

$$- \frac{4}{\ln^4(|w|^2)} |\bar{Y}|^2 + \frac{4}{\ln^4(|w|^2)} |\bar{\bar{w}} Y|^2 + \frac{20}{\ln^4(|w|^2)} |\bar{\bar{w}} Y|^2 - \frac{16}{\ln^4(|w|^2)} |\bar{\bar{w}} Y|^2 \geq 0.$$  

(We use the inequality $|w| < 1$ in the last equality). This form is always positive because of Schwarz inequality $|w|^2 |Y|^2 - |\bar{w}, Y|^2 \geq 0$.

Now denote $x_i = 1/\ln(|w_i|^2)$. Calculating the form in the brackets in (1.13) for function (1.12) we obtain:

$$-2 \sum_{i=1}^n x_i \frac{|Y_i|^2}{|w_i|^2} + 2 \sum_{i=1}^n x_i^2 \frac{|\bar{w}_i Y_i|^2}{|w_i|^4} + 6 \sum_{i=1}^n x_i^4 \frac{|\bar{w}_i Y_i|^2}{|w_i|^4} - 4 \left( \sum_{i=1}^n x_i \right) (-1) \sum_{i=1}^n x_i \left| \frac{\bar{w}_i Y_i}{w_i} \right| \quad (1.15)$$

\[= 6 \sum_{i=1}^n x_i \frac{|Y_i|^2}{|w_i|^2} - 4 \left( \sum_{i=1}^n x_i \right) (-1) \sum_{i=1}^n x_i \left| \frac{\bar{w}_i Y_i}{w_i} \right|^2. \]
Schwarz inequality applied to the vectors \((x_1, \ldots, x_n)\) and \((x_1^2 Y_1/w_1, \ldots, x_n^2 Y_n/w_n)\) yields:

\[
\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} \frac{|Y_i|^2}{|w_i|^2} \geq \left| \sum_{i=1}^{n} x_i^2 \frac{Y_i}{w_i} \right|^2.
\]

Hence, the expression in the brackets in (1.13) is greater or equal to

\[
2 \sum_{i=1}^{n} x_i^4 \frac{|Y_i|^2}{|w_i|^2}.
\]

Thus function (1.12) is strictly plurisubharmonic at \(|w| > 0\) with \(|w_i| < 1\).

Now let us estimate the curvature of our metric (1.9) with \(f = f_2\). We calculate the Levi form on the vector \(X = (X_1, \ldots, X_n)\) with \(X_i = \lambda_i w_i\). Taking into account calculations (1.15) we get:

\[-\frac{12}{\exp(u)} \sum_{i=1}^{n} x_i^4 |\lambda_i|^2 \leq K_\mu(w) \leq -\frac{4}{\exp(u)} \sum_{i=1}^{n} x_i^4 |\lambda_i|^2.\]

Existence of uniform estimates for the curvature follows from the next obvious inequalities:

\[
\frac{\min |\lambda_i|^2}{2n} \leq \sum_{i=1}^{n} x_i^4 |\lambda_i|^2 \leq \max |\lambda_i|^2.
\]

Now prove the last assertion of the lemma. There exists a constant \(C < 1\) such that for each vector \(V\), \(C^{-1}|V|^2 < h_w(v) < C|v|^2\), where \(h_w\) is the metric \(h\) at the point \(w\). It follows that

\[\mu_w(V) = \exp(u) \sum_{i=1}^{n} \frac{1}{\ln^2(|w_i|^2) h_w(X(w))} \geq -b \frac{|V|^2}{|w|^2 \ln^2 |w|}\]

for some \(b > 0\). The metric \(|dw|^2/|w|^2 \ln^2 |w|\) is complete at 0. \(\square\)

Now consider some metric \(g\) on \(\mathbb{CP}^n \setminus \Sigma\) of the form (1.6) with strictly plurisubharmonic \(\ln f\) (condition a)). Consider some singular point \(p\). We chose a coordinate system \(w\) in its neighborhood so that \(p\) corresponds to the point \(|w| = 0\). For the function \(\ln f\) we can write:

\[\ln f(w) = v(w) + L(w) + R(w),\]

where \(v(w) = \ln f(0) + p(w) + \bar{p}(w)\), and \(p\) is some degree two complex polynomial, \(L\) is the positive Levi form, and \(R(w) = o(w^2)\). We can take the coordinate \(w\) such that in this chart \(L(w) = 2|w|^2\). Define another chart \(\tilde{w}\) in which the field \(X\) is diagonal and take:

\[\tilde{f}_\epsilon(w) = c \sum_{i=1}^{n} \frac{1}{\ln^2(|\tilde{w}_i|^2)}.\]

We will chose the constant \(c\) later.

We should glue the metric determined by the function \(\tilde{f}_\epsilon\) and global metric (1.4). This part follows [LN] with some modifications. Let

\[l_\epsilon(w) = \ln f(0) + \frac{1}{2} L(w) = \ln f(0) + \epsilon + 2 \Re|p(w)| + |w|^2.\]
For every \( \varepsilon > 0 \) small enough there exist \( 0 < r_1 < r_2 < r_3 < 1 \) such that

\[
\ln f(w) > l_\varepsilon(w) \text{ if } r_2 < |w| < r_3 \text{ and } \ln f(w) < l_\varepsilon(w) \text{ if } |w| < r_1.
\] (*)

These \( r_1, r_2, r_3 \) can be made arbitrarily small if \( \varepsilon \) tends to 0.

Proof. Since \( R(w) = o(|w|^2) \) there exists an \( 0 < r_3 < 1 \) such that \( |R(w)| < |w|^2 / 2 \) for \( |w| \leq r_3 \). If \( \delta(w) = \ln f(w) - l_\varepsilon(w) = |w|^2 + R(w) - \varepsilon \) and \( |w| < r_3 \) we have

\[
\frac{|w|^2}{2} - \varepsilon \leq |w|^2 - |R(w)| - \varepsilon \leq |w|^2 + |R(w)| - \varepsilon \leq \frac{3|w|^2}{2} - \varepsilon.
\]

If we take \( r_2 = r_3 / 2 \) and \( \varepsilon < r_3^2 / 8 \) we get for \( r_2 < |w| < r_3 \) that \( \delta(w) \geq |w|^2 / 2 - \varepsilon \geq r_3^2 / 8 - \varepsilon > 0 \). This means that \( \ln f(w) > l_\varepsilon(w) \) for \( r_2 < |w| < r_3 \). On the other hand if \( r_1 < (2\varepsilon / 3)^{1/2} < r_2 \) we get for \( |w| < r_1 \):

\[
\delta(w) \leq \frac{3|w|^2}{2} - \varepsilon < \frac{3r_1^2}{2} - \varepsilon < 0
\]

and thus \( \ln f(w) < l_\varepsilon(w) \) whenever \( |w| < r_1 \).

Define \( k(w) \) by

\[
k(w) = \begin{cases} 
\exp(l_\varepsilon(w)) & \text{if } |w| < r_1, \\
\max\{\exp(l_\varepsilon(w)), f(w)\} & \text{if } r_1 \leq |w| < r_2, \\
f(w) & \text{if } |w| > r_2.
\end{cases}
\]

The function \( k \) is continuous and \( \ln k \) is strictly plurisubharmonic.

Now fix \( 0 < r_0 < r_1 \), and define \( \varphi(t) = (\ln t)^{-1} \), \( \varphi_{a,\alpha}(t) = a\varphi[(t/r_0^{2(\alpha-1)/\alpha})^\alpha] = a\varphi[t^{\alpha-1}/r_0^{2(\alpha-1)/\alpha}] \). We can choose \( a, \delta \) and \( 0 < \alpha < 1 \) such that

\[
\varphi_{a,\alpha}(r_0^3) = e^{\delta^2}, \varphi_{a,\alpha}(t) \leq e^t \text{ if } r_0^3 \leq t \leq r_0^3 + \delta, \varphi_{a,\alpha}(t) \geq e^t \text{ if } r_0^3 - \delta \leq t \leq r_0^3. \text{ (**)}
\]

Proof. Let \( a > 0 \) be such that \( a\varphi(r_0^3) = e^{\delta^2} \). Note that

\[
\varphi_{a,\alpha}(r_0^3) = a\varphi[(r_0^3/r_0^{2(\alpha-1)/\alpha})^\alpha] = a\varphi(r_0^3) = e^{\delta^2}.
\]

On the other hand

\[
\varphi'_{a,\alpha}(t) = \frac{a\alpha t^{\alpha-1}}{r_0^{2(\alpha-1)/\alpha}} \varphi'[(t^{\alpha}/r_0^{2(\alpha-1)/\alpha})^\alpha].
\]

Hence

\[
\varphi'_{a,\alpha}(r_0^3) = a\alpha e^{\delta^2} 4\alpha r_0^2 \ln(r_0^3)/[4\alpha r_0^2 \ln(r_0^3)] = a\alpha e^{\delta^2} r_0^2 \ln(r_0^3)/[4\alpha r_0^2 \ln(r_0^3)] = e^{\delta^2} 4\alpha r_0^2 \ln(r_0^3)/[4\alpha r_0^2 \ln(r_0^3)].
\]

This value is positive, and if \( \alpha \) is small enough then \( 4\alpha / (r_0^2 \ln(r_0^3)) \) < 1. Hence

\[
0 < \varphi'_{a,\alpha}(r_0^3) < e^{\delta^2} = \frac{de^t}{dt} \bigg|_{t=r_0^3}.
\]

This implies the assertion (**)
The function
\[ \varphi_{a,\alpha}(\|w\|^2) = \frac{a}{16\alpha^2 \ln^4(\|w\|/r_0^{(\alpha-1)/\alpha})} \]  
(1.17)
is the function \( f_1 \) from Lemma 1 up to a linear change of variables. Hence \( \ln \varphi_{a,\alpha} \) is strictly plurisubharmonic. Define \( m \) by
\[
\begin{cases}
  m(w) = \exp(\|w\|^2) & \text{for } r_0 \leq |w| \leq r_1, \\
  m(w) = \varphi_{a,\alpha}(\|w\|^2) & \text{for } |w| \leq r_0.
\end{cases}
\]
This function is continuous and
\[
m(w) = \max\{\exp(\|w\|^2), \varphi_{a,\alpha}(\|w\|^2)\} \quad \text{if } r_0^2 - \delta \leq |w|^2 \leq r_0^2 + \delta
\]
for some \( \delta > 0 \). Thus \( \ln m = \max\{|w|^2, \ln \varphi_{a,\alpha}(|w|^2)\} \) is also plurisubharmonic.

Now take some \( c > 0 \) and define
\[
q(w) = \max\{\tilde{f}_c(w), m(w)\}
\]
for \( |w| < r_1 \), where \( \tilde{f}_c \) is defined by (1.16). Function (1.17) tends to zero as \( |w| \to 0 \) more swiftly than \( \tilde{f}_c \). Hence \( q(w) = \tilde{f}_c(w) \) if \( |w| < \delta_1 \) for some \( \delta_1 > 0 \). If \( c \) is small enough, then in addition, there exists a \( \delta_2 > \delta_1 \) such that \( q(w) = m(w) \) whenever \( \delta_1 < \delta_2 < |w| < r_1 \). The constant \( \delta_1, \delta_2 \) can be made arbitrarily small if \( c \) is small enough. In particular, we can take \( \delta_2 < r_0 \). The function \( q \) is continuous and \( \ln q \) is plurisubharmonic. Define
\[
\begin{cases}
  n(w) = k(w) & \text{for } r_1 < |w| < r_3, \\
  n(w) = \exp(\varepsilon + v(w))q(w) & \text{for } |w| \leq r_1.
\end{cases}
\]
This function is continuous because \( k(w) = \exp[\varepsilon + v(w)]e^{\|w\|^2} = \exp[\varepsilon + v(w)]q(w) \) at \( |w| = r_1 \). It is easy to see also that \( \ln n(w) = \varepsilon + v(w) + \ln q(w) \) is plurisubharmonic since the function \( \varepsilon + v(w) \) is plurisubharmonic.

On some neighborhood of the point \( p \) containing the ball \( B_{r_3} \) we define the metric \( \mu \):
\[
\begin{cases}
  \mu = n \frac{h}{h(X)} & \text{on } B_{r_3}, \\
  \mu = g & \text{on } \mathbb{CP}^n \setminus B_{r_3}.
\end{cases}
\]
This metric is continuous because at \( |w| = r_2 \) one has \( n(w) = k(w) = f(w) \).

As it follows from Lemma 1 such a metric is complete in a neighborhood of zero and its curvature is negative and bounded from below and from above by some constant \( C > 0 \). Let us note that the metric is smooth in some neighborhood of zero: in fact, on the punctured ball \( 0 < |w| < \delta_1 \).

In \( B_{r_3} \), the function \( \tilde{n} \) is plurisubharmonic by our construction and continuous but not necessary smooth. It is piecewise smooth and strictly plurisubharmonic everywhere where it is smooth. We apply the following smoothing procedure. Let \( \eta \) be a smooth “cap” with support in the unit ball, \( \int \eta(\zeta)d\zeta = 1 \). Fix some \( 0 <
\( \rho_1 < \delta_1 < r_3 < \rho_2 \) and for \( \rho_1 < |w| < \rho_2 \) define a smoothed function

\[
\tilde{n}_\delta = \int \ln n(w - \delta \zeta) \eta(\zeta) d\zeta.
\]

For each \( w \) the integral depends only on values of the function \( \ln(n) \) for \( |w - w'| \leq \delta \). For \( \delta \) small enough the function \( \tilde{n}_\delta(w) \) is \( C^2 \)-close to \( \ln(n) \). We can smoothly glue \( \tilde{n}_\delta \) with \( \ln(n) \) inside the ball \( B_{\rho_2} \) so that the resulting function will coincide with \( \ln(n) \) for \( |w| < \rho_1 \) and for \( |w| > \rho_2 \). We obtain a function that will be smooth and plurisubharmonic (see [H]). We can approximate it by strictly plurisubharmonic function \( \tilde{f} \) coinciding with \( \ln(n) \) for \( |w| < \rho_1 \) and for \( |w| > \rho_2 \). We denote by \( \Phi \) the function \( \exp(\tilde{f}) \).

We do this construction for all singular points. This finishes the proof of Theorem 1. \( \square \)

The resulting metric in each domain \( U_j \) has a form

\[
G = \Phi h/h(X). \tag{1.18}
\]

Consider now restriction of our metric on a leaf of the foliation. Assume that we work in the chart \( z \) on \( U_0 \) and \( z_j \) is a holomorphic coordinate on leaves in some neighborhood of a point \( p \), that is, \( X_j(z) \neq 0 \) in this neighborhood. We can represent the leaves as integral curves of the system:

\[
dz_k dz_j = X_k(z) X_j(z), \quad k = 2, \ldots, n.
\]

Hence the metric \( G \) restricted on a curve is:

\[
G = \frac{\Phi(z)}{|X_j(z)|^2} dz_j dz_k. \tag{1.19}
\]

A factor of connectivity \( \Gamma = \Gamma_{11} \) (in usual designations) is

\[
\Gamma = \frac{\partial}{\partial z_j} [\ln(\Phi) - \ln(|X_j|^2)] = \frac{\partial \Phi(X)}{\Phi} - \frac{\partial X_j(X)}{X_j^2}.
\]

Here \( \partial \Phi(X), \partial X_j(X) \) are the values of the forms \( \partial \Phi, \partial X_j \) on the vector \( X \).

The curvature is:

\[
K = -2 |X_j|^2 \frac{\partial \Gamma}{\partial z_j} = -\frac{2}{\Phi^3} [\Phi L_\Phi(X, X) - |\partial \Phi(X)|^2],
\]

where \( \Phi L_\Phi \) is a Levy form of the function \( \Phi \).

The curvature is invariant and doesn’t depend on a chart. For use in the next section we give also the expression for the curvature in a neighborhood of a singular point which we can easy obtain from (1.11) and (1.15):

\[
K = -2 \sum_{i=1}^n x_i^2 \sum_{i=1}^n x_i^4 |\lambda_i|^2 - 4 \frac{\sum_{i=1}^n x_i^2 |\lambda_i|^2}{\left(\sum_{i=1}^n x_i^2 \right)^3} f(w), \tag{1.20}
\]

where \( w \) is analytic linearizing and diagonalizing chart, \( x_i = 1/\ln(|w_i|^2) \) and \( f(w) \) is bounded together with its derivatives.
2. Almost complex structure in geodesic coordinates

We consider a manifold $M$ of universal coverings of leaves with some transversal base $B$. Let $\zeta$ be a holomorphic chart on this transversal and $w = v + iu$ be a complex coordinate on the leaves (this coordinate is not necessary holomorphic). An almost complex structure on $M$ can be determined by $(1, 0)$-forms:

$$d\zeta = \{d\zeta_j\}, \quad j = 1, \ldots, n - 1,$$

$$\omega = dw + \mu d\bar{w} + \langle c, d\bar{\zeta} \rangle.$$  \hfill (2.1)

Suppose $t(u, v, \zeta)$ is a local holomorphic coordinate on the leaves. Then $dt$ is a holomorphic 1-form:

$$dt = t_u du + t_v dv + \langle t_{\bar{\zeta}}, d\bar{\zeta} \rangle + \langle t_{\zeta}, d\zeta \rangle.$$  \hfill (2.2)

For reduction to form (2.2) we should subtract the last term and normalize the coefficient at $dw$. We obtain:

$$\mu = \frac{t_v + it_u}{t_v - it_u} = \frac{t_w}{t_w},$$  \hfill (2.3)

$$c = \frac{2t_{\bar{\zeta}}}{t_v - it_u} = \frac{t_{\bar{\zeta}}}{t_w}.$$  \hfill (2.4)

It is easy to see that the functions $\mu$ and $c_j$ don’t depend on a choice of a holomorphic coordinate $t$.

Now we consider a foliation with metric (1.18). By the Hadamard–Cartan theorem the universal covering of a leaf is diffeomorphic to $\mathbb{C}$ by the exponential mapping. We define $(u, v)$ as geodesic “polar” coordinates on a fiber: $v$ is a length of a geodesic with origin at zero, that is, at the point of intersection of the fiber with the transversal $B$, and $u$ is an argument of the unit vector tangent to the geodesic at origin. This choice of coordinates is in accordance with usual orientation: in this case the orientation preserving mapping $(v + iu) \mapsto (1 - e^{-v}) \exp(iu)$ maps the domain $L^+ = \{0 \leq u < 2\pi, 0 \geq v < \infty\}$ onto the open unit disk. We obtain an almost complex structure on the product $B \times L^+$. Our main goal is to prove the next theorem:

**Theorem 2.** In the assumptions of Theorem 1 let $B$ be a ball transverse to the foliation, $D \subset \mathbb{C}$ be the unit disk with coordinate $\zeta$. Then the complex structure on the corresponding manifold of universal coverings (over the leaves intersecting $B$) is biholomorphically equivalent to an almost complex structure on the product $B \times D$ defined by the forms:

$$d\zeta = \{d\zeta_j\}, \quad j = 1, \ldots, n - 1,$$

$$\bar{\omega} = d\xi + \langle \bar{c}, d\bar{\zeta} \rangle.$$  \hfill (2.5)

Here $\bar{c}$ is a smooth vector-function on $B \times D$ that is holomorphic on the fibers, that is, $\bar{c}_{\bar{\zeta}} = 0$.

Theorem 2 means that the uniformisation mapping and Poisson metric smoothly depend on the fibers. Note that if the chart $\xi$ is holomorphic on the fibers then the equation $\bar{c}_{\bar{\zeta}} = 0$ is the condition of integrability of the almost complex structure,
and hence holds automatically if we can transform the geodesic chart \( w \) to such a chart.

Before proving this theorem we determine the coefficients \( \mu \) and \( c_j \) in (2.3), (2.4) in geodesic coordinates and obtain some estimates. We can assume that the transversal \( B \) is transverse to the vector \( \partial / \partial z_1 \). At those points where \( z_1 \) is a local holomorphic coordinate on the leaves we have the equation for geodesics:

\[
\frac{d^2 z_1}{dv^2} + \Gamma \left( \frac{dz_1}{dv} \right)^2 = 0,
\]

(2.5)

\[
z_1(0) = 0, \quad \frac{dz_1}{dv}(0) = e^{iu}.
\]

(2.6)

In what follows we will use the designation:

\[
\nabla_t v = \frac{d}{dv} + \Gamma \frac{dt}{dv},
\]

where \( t(v) \) is a holomorphic coordinate “along” geodesics and \( t_v \) is the tangent vector. We can write the equation for the geodesics in the form:

\[
\nabla_t v t_u = 0.
\]

(2.7)

We will need also the operator \( \nabla^2_{t_v} \). Taking into account that \( t_v^2 + \Gamma(t_v)^2 = 0 \) we have:

\[
\nabla^2_{t_v} y = \frac{d^2 y}{dv^2} + 2\Gamma t_v \frac{dy}{dv} + \frac{d\Gamma}{dv} t_v y + [\Gamma t_v^2 + \Gamma^2(t_v)^2]y
\]

\[
= \frac{d^2 y}{dv^2} + 2\Gamma t_v \frac{dy}{dv} + (\Gamma t_v + \Gamma^2(t_v))t_v y.
\]

(2.8)

Now consider \( z_{1,u} \) (the derivative of the coordinate \( z_1 \) with respect to initial conditions) or \( t_u \) in a more general case. We obtain the “equation in variations” by differentiating (2.7) with respect to \( u \) and taking into account (2.8). It is exactly the Jakoby equation for variation of geodesics:

\[
\nabla^2_{t_v} t_u + \Gamma t_v (t_v t_u - t_u t_v) = 0.
\]

(2.9)

The vectors \( t_v \) and \( t_u \) are orthogonal in the metric \( G \). This metric is conformal on the leaves and hence \( t_v t_u + t_u t_v = 0 \). Hence we can rewrite (2.9) as:

\[
\nabla^2_{t_v} t_u + \Gamma t_v (t_v t_u - t_u t_v) = \nabla^2_{t_v} t_u - 2\Gamma |t_v|^2 t_u = 0.
\]

(2.10)

The vector \( t_v \) has unit length in our metric. If we write the metric in the form \( G = g(t) dt dt \) we obtain:

\[
2\Gamma |t_v|^2 = 2\Gamma / g = -K,
\]

where \( K \) is a curvature. Thus we can rewrite the Jakoby equation (2.10) in the form:

\[
\nabla^2_{t_v} t_u + K t_u = 0.
\]

(2.11)

Now we represent \( t_u \) in the form:

\[
t_u = i \varphi t_v.
\]

(2.12)
Since \( t_u \) is orthogonal to \( t_v \), \( \varphi \) is a real scalar function. Later we will show that \( \varphi(v) \) is positive for \( v > 0 \). Equation (2.11) gives:

\[
\frac{d^2 \varphi}{dv^2} - k^2 \varphi = 0,
\]

(2.13)

where \( k = |K|^{1/2} \). From (2.5) we obtain the initial conditions for \( z_{1,u} \):

\[
z_{1,u}(0) = 0, \quad dz_{1,u}/dv(0) = ie^{iu}.
\]

It means that the initial conditions for \( \varphi \) are

\[
\varphi(0) = 0, \quad d\varphi/dv(0) = 1.
\]

(2.14)

We estimate the solution of (2.13) with initial conditions (2.14). The next proposition is a statement of elementary theory of ordinary differential equations.

**Proposition 1.** Let in an equation of form (2.13) the function \( k(v) \) be uniformly bounded and uniformly bounded away from zero. If \( \varphi(v) \) is the solution of (2.13) with initial conditions (2.14) then it can be represented in the form

\[
\varphi(v) = e^{s(v)} - 1,
\]

(2.15)

and for \( s \) we have the estimates:

\[
0 < c_1 < ds/dv < c_2 < \infty
\]

(2.16)

for some \( c_1 < c_2 \) if \( v > 0 \). These estimates depend only on the upper and lower bounds for the function \( k \).

**Proof.** Let’s note that \( \varphi(v) > 0 \) at \( v > 0 \). Indeed, the set \( \{v > 0: \varphi(v) > 0\} \) is open and non-empty by (2.14). It follows from (2.13) that \( d^2 \varphi/dv^2 > 0 \) if \( \varphi(v) > 0 \) for \( v < v_0 \) (since \( d\varphi/dv(0) = 1 \) by (2.14)). Hence also \( d\varphi/dv > 0 \) for \( v < v_0 \), the function \( \varphi \) is increasing and \( \varphi(v_0) > 0 \). Thus, the set \( \{v > 0: \varphi(v) > 0\} \) is closed also. Now it follows from (2.13) that \( d^2 \varphi/dv^2 > 0 \) and \( d\varphi/dv > 0 \) for \( v > 0 \). Hence also \( d(ln \varphi)/dv = \varphi^{-1}d\varphi/dv > 0 \). Thus we can represent the solution \( \varphi(v) \) in form (2.15), and for \( s \) we have the following initial conditions:

\[
s(0) = 0, \quad ds/dv(0) = d\varphi/dv(0) = 1.
\]

(2.17)

The inequality \( d\varphi/dv > 0 \) at \( v > 0 \) yields

\[
\frac{ds(v)}{dv} > 0
\]

at \( v > 0 \).

Now let us prove estimate (2.16). Denote \( \psi = ds/dv - k \). Taking into account initial conditions (2.17) we have the following equation for \( \psi \):

\[
\frac{d(k + \psi)}{dv} + \psi(k + \psi) + k\psi + k^2 e^{-\int_0^v (k(t) + \psi(t))dt} = 0
\]

(2.18)

Let us prove the right estimate in (2.16). We know that \( k < k_m \) for some \( k_m > 0 \). Suppose \( (k + \psi)(v_0) > \max\{1, k_m\} \). Then \( \psi(v_0) > 0 \) and, taking into account the second initial condition (2.17), we see that the derivative \( d(k + \psi)/dv \) must be positive at some \( v_1 \leq v_0 \) such that \( \psi(v_1) > 0 \). All terms in (2.18) except the first one are positive at the point \( v_1 \). It means the first term must be negative,
which is impossible. Hence the right inequality in (2.16) is proved for arbitrary $c_2 > \max\{1, k_m\}$.

Now let us prove the left estimate.

**Claim 1.** Let $a = \frac{1}{\alpha(1+k_m)^2}$, $b = \exp\left[-\frac{1}{\alpha(1+k_m)^2}\right]$. Then $k + \psi > 1/2$ on the segment $[0, a]$ and

$$e^{-\int_a^b [k+\psi(t)] dt} < b$$

for every $v \geq a$.

**Proof.** Since $k + \psi > 0$ and taking into account that $k + \psi \leq \max\{1, k_m\}$, $k_m = \sup k$, we have: $|\psi| \leq 1 + k_m$, $|k + \psi| \leq 1 + k_m$. Hence from (2.18) we get:

$$|d(k + \psi)/dv| \leq 3(1 + k_m)^2$$

Since $(k + \psi)(0) = 1$ we get that $k + \psi > 1/2$ at least on the segment $[0, \frac{1}{\alpha(1+k_m)^2}]$. Hence $\int_0^v (k+\psi(t)) dt$ along this segment is greater than $\frac{1}{2(1+k_m)^2}$. Since $(k+\psi)(v) > 0$ for $v > 0$ we obtain Claim 1. \( \Box \)

**Claim 2.** Let $a$ and $b$ be as above and $(1 + b)/2 < \alpha < 1$. Suppose $v_0 < a$, $\psi(v_0) < -\alpha k v_0)$. Then $d(k + \psi)/dv > 0$ at $v = v_0$.

**Proof.** The inequalities $\psi < -\alpha k$, $k + \psi > 0$ imply $\alpha k < |\psi| < k$. Therefore, by (2.18)

$$\frac{d(k + \psi)}{dv} = 2k|\psi| - \psi^2 - k^2 e^{-\int_a^b [k+\psi(t)] dt} > 2\alpha k^2 - k^2(1+b) > 0. \quad \Box$$

Now we finish the proof of the proposition. Let us fix some $\alpha$ as in Claim 2. We claim that the left estimate (2.16) holds for arbitrary $c_1 < \min\{1/2, (1 - \alpha) \inf k\} < \min\{1/2, 1/2(1 - b) \inf k\}$. Indeed, let $k + \psi$ be monotonous on each interval $(a, v_1)$, $(v_1, v_2)$, $\ldots$, $(v_k, v_{k+1})$, $\ldots$. If $k + \psi$ increases on the interval $(a, v_1)$ then $k + \psi > 1/2$ on this interval by Claim 1. If $k + \psi$ decreases on some interval $(v_k, v_{k+1})$ then $k + \psi > (1 - \alpha) k$ on this interval by Claim 2. Hence the estimate $k + \psi > (1 - \alpha) k$ also holds on the adjacent interval $(v_{k+1}, v_{k+2})$, where $\psi$ increases. The number of monotonicity intervals may be infinite. In this case we obtain the required estimate by induction. \( \Box \)

We need also an estimate for $ds/du$.

**Proposition 2.** If $\varphi = e^s - 1$ is the solution of (2.13) with initial conditions (2.14) then

$$\left|\frac{ds}{du}\right| < Me^s, \quad 0 < M < \infty. \quad (2.19)$$

**Proof.** It will be convenient to introduce a new function $\tilde{s}$ such that $e^{\tilde{s}} = e^s - 1$. If we will prove an estimate analogous to (2.19) for $\tilde{s}$ we will prove it also for $s$ because $s_u/\tilde{s}_u = e^s/e^{\tilde{s}} = 1 - e^{-\tilde{s}}$. For $\tilde{s}_u$ we have the equation in variations:

$$\frac{d^2 \tilde{s}_u}{dv^2} + 2 \tilde{s}_u \frac{d\tilde{s}_u}{dv} - (k^2)_u = 0.$$
We can take the following initial conditions at \( v = 1 \):
\[
\tilde{s}_u(1) = a, \quad d\tilde{s}_u/dv(1) = b.
\] (2.20)

Here \( a \) and \( b \) depends on \( u \) and \( \zeta \) but are uniformly bounded. Denoting \( \theta = d\tilde{s}_u/dv \) we get
\[
\frac{d\theta}{dv} + 2\tilde{s}_v \theta = -K_u
\]
with initial condition \( \theta(1) = b \). The solution of this equation is:
\[
\theta(v) = e^{-2\tilde{s}(v)} \left[ b - \int_1^v K_u(\tau)e^{2\tilde{s}(\tau)} d\tau \right]
\] (2.21)
and
\[
\tilde{s}_u(v) = a + \int_1^v \theta(\tau) d\tau.
\] (2.22)

Now, reminding (2.12), we have
\[
K_u = K_t t_u + K_{\bar{t}} \bar{t}_u = i(K_t t_v - K_{\bar{t}} \bar{t}_v)e^\xi.
\] (2.23)

Here \( t \) is a holomorphic coordinate on a leaf. Note that the value \( K_t t_v \) doesn’t depend on the choice of coordinates: it is the value of the \( C \)-linear part of the differential of \( K \) on the unit vector tangent to geodesics. In what follows \( t \) will be one of affine coordinates \( z_j \) or one of analytic linearizing coordinates \( w_j \) in a neighborhood of a singular point and \( X = (X_1, \ldots, X_n) \) will be the coordinate representation of the vector field \( X \) in the chart \( z \) or \( w \). Since \( t_v \) has unit length in metric (1.19) we have
\[
|t_v| = \Phi^{-1/2}|X_j|.
\]

The value \( |K_t t_v| \) is uniformly bounded on any compact set in \( \mathbb{CP}^n \setminus \Sigma \) and we must estimate it only in neighborhoods of the singular points. In a neighborhood of such a point the metric is described by (1.12) in Lemma 1:
\[
G(w) = \sum_{i=1}^n \frac{1}{\ln^2(|w_i|^2)} a(w)h/h(X)
\]
in the chart \( w \) such that \( X_i(w) = \lambda_i w_i \), and the function \( a \) is non-zero at the singular point and has bounded derivatives. The curvature is given by (1.20).

Hence to estimate \( K_t t_v \) we must estimate the values \( q_t t_v \) and \( q_{\bar{t}} t_v \), where
\[
q(w) = 6 \sum_{i=1}^n x_i^2 \sum_{i=1}^n x_i^4 |\lambda_i|^2 - 4 \sum_{i=1}^n x_i^2 |\lambda_i|^2
\]
\[
\left( \sum_{i=1}^n x_i^2 \right)^3
\]
Remind that \( x_j = 1/\ln(|w_j|^2) \) and \( \Phi = \sum_{i=1}^n x_j^2 \). Recall that \( t = w_j \) for some \( j \) and \( X_j = \lambda_j w_j \). Hence for \( t_v \) we have the estimate:
\[
|t_v| = \left( \sum_{i=1}^n x_i^2 \right)^{-1/2} O(|w_j|).
\] (2.24)

We have:
\[
q_t = \frac{1}{\left( \sum_{i=1}^n x_i^2 \right)^2} \sum_{i=1}^n p_i(x) x_{i,t},
\]
where \( p_i(x) \) are homogeneous polynomials with respect to \( x_i \) of degree 7. For \( x_{i,t} \) we have
\[
x_{i,t} = \frac{\partial x_i}{\partial w_j} = -x_i^2 \frac{1}{w_i} \frac{dw_i}{dw_j} = -x_i^2 \frac{1}{w_j} \lambda_j.
\]
Thus:
\[
q_t = O\left(|w_j|^{-1} \sum_{i=1}^{n} |x_i| \right).
\]
Taking into account (2.24) we obtain
\[
q_t t_v = O(1)
\]
Analogously
\[
q_f t_v = O\left( \left( \sum_{i=1}^{n} x_i^2 \right)^{-1/2} |w| \right) = o(1).
\]
Thus from (2.23) \( K_{t,v} = O(1) e^{\tilde{s}} \). Now by (2.21), and taking into account that the estimates of Proposition 1 hold for \( \tilde{s} \) at large \( v \) we obtain:
\[
\theta(v) \leq M_0 e^{-2\tilde{s}(v)} \int_0^v e^{3\tilde{s}(\tau)} d\tau
\]
for some \( M_0 < \infty \). Taking into account estimate (2.16) we have:
\[
\int_0^v e^{3\tilde{s}(\tau)} d\tau \leq \frac{1}{3 \inf s'} \int_0^v 3 s'(\tau) e^{3\tilde{s}(\tau)} d\tau \leq \frac{1}{3c_1} (e^{3s(v)} - 1).
\]
Thus \( \theta(v) < Me^{s(v)} \) for some \( M \), and from (2.22) we obtain the required estimate for \( \tilde{s} \). \( \square \)

**Remark.** Note that it follows from the proof that the conclusion of the proposition holds if \( K_{t,v} - K_{t,v} \), that is, the derivative of the curvature along the geodesics is uniformly bounded, and if we can use the Proposition 1, that is, if the curvature is negative, uniformly bounded and uniformly bounded away from zero.

Now consider the almost complex structure (2.1), (2.2) on the universal covering in geodesic coordinates \( w = v + iu \) on the fibers.

**Lemma 2.** There exists a smooth change of variables \( (r, u, \zeta) \mapsto (v, u, \zeta) \), \( 0 \leq r < 1 \) with the following properties. Consider the pullback of the almost complex structure (2.1), (2.2) on the product \( B \times L^+ \) (\( B \) is the base, \( L^+ \) is the half-strip \( \{0 \leq u < 2\pi, 0 \leq v\} \)) induced by the complex structure on the universal covering \( M \). This pullback almost complex structure is defined on the product \( B \times D \) (\( D \) is the unit disk) by forms (2.1) and the form
\[
\omega_1 = dw_1 + \mu_1 d\bar{w}_1 + \langle c^1, \bar{d}\zeta \rangle.
\]
There \( w_1 = re^{iu} \) and the chart \( w_1 \) is “quasiconformal”, that is, \( |\mu_1| \leq b < 1 \) on \( D \). The vector-function \( c^1 \) has the form
\[
c^1 = \nu [c + 2(1 - r)g_1^2],
\]
where \( \nu \) is an uniformly bounded function, \( q_t r, u, \zeta = \eta [ - \ln (1 - r) ] \), \( u, \zeta \), and \( \eta \) is the inverse function to \( s(v) \), that is, \( v = \eta(s, u, \zeta) \).
Proof. From (2.3), (2.12), (2.15) we have:

\[ \mu = \frac{t_v + it_u}{t_v - it_u} = \frac{1 - \varphi}{1 + \varphi} = -1 + 2e^{-s}. \]

If \( v = g(r, u, \zeta) \) we can represent form (2.2) as:

\[ \omega = (1 + \mu)(g_r dr + (g_\zeta, d\zeta) + (g_{\bar{\zeta}}, d\bar{\zeta})) + [(1 + \mu)g_u + i(1 - \mu)]du + \langle c, d\zeta \rangle \]

\[ = \frac{1}{2}[(1 + \mu)g_r + r^{-1}(1 - \mu) - i\nu^{-1}(1 + \mu)g_u](dr + i\nu du) \]

\[ + \frac{1}{2}(1 + \mu)g_r - r^{-1}(1 - \mu) + i\nu^{-1}(1 + \mu)g_u](dr - i\nu du) + (1 + \mu)\langle g_\zeta, d\zeta \rangle \]

\[ + \langle c + (1 + \mu)g_\zeta, d\zeta \rangle. \]

If \( w_1 = re^{iu} \) then \( dw_1 = e^{iu}(dr + i\nu du) \), \( d\bar{w}_1 = e^{-iu}(dr - i\nu du) \). Omitting the term with \( d\zeta \) we see that the form \( \omega \) defining our almost complex structure in the chart \( w_1 \) will be replaced by the form:

\[ \omega_1 = dw_1 + \mu_1 d\bar{w}_1 + \langle c^1, d\zeta \rangle, \]

where

\[ \mu_1 = e^{2iu}r(1 + \mu)g_r - (1 - \mu) + i(1 + \mu)g_u \]

\[ = e^{2iu}re^{-s}g_r - 1 + e^{-s} + ie^{-s}g_u = e^{2iu}r g_r + 1 - e^{s} + ig_u \]

\[ e^{1} = \frac{2re^{iu}}{r(1 + \mu)g_r + 1 - \mu - i(1 + \mu)g_u}[c + (1 + \mu)g_\zeta] \] (2.27)

In the following calculations we will omit dependence on \( \zeta \). It follows from estimates (2.16) and initial conditions (2.17) that there exists an inverse function \( v = \eta(s, u) \), \( \eta(0, u) = 0 \). We take \( s = -\ln(1 - r) \), that is, \( v = g(r, u) = \eta(-\ln(1 - r), u) \). If we denote the derivative of \( \eta \) with respect to the first argument by \( \eta' \) we obtain the following formula for the coefficient \( \mu_1 \):

\[ \mu_1 = e^{2iu}r \eta' - r + i(1 - r)\eta_u \]

\[ = e^{2iu}r \eta' + r - i(1 - r)\eta_u \] (2.28)

It follows from Proposition 1 that \( \eta' \) is positive and uniformly bounded. Since \( \eta_u = -\eta' s_u \), \( \eta' s_u + \eta_u = d\eta'/du = 0 \), \( 1 - r = e^{-s} \) and for \( s_u \) we have the estimate of Proposition 2, we obtain:

\[ \eta_u(1 - r) \leq M \]

for some \( M < \infty \) and this estimate also is uniform.

It follows from (2.28) that \( |\mu_1| \leq 1 \) and \( |\mu_1| \leq c(\delta) \) for some \( c < 1 \) outside any disk \( r \leq \delta \). We should only consider \( \mu_1 \) at \( r \rightarrow 0 \). From (2.17) we have the initial conditions for \( s_u \):

\[ s_u(0) = 0, \quad ds_u/du(0) = 0. \] (2.29)
We have: \( \eta'(0) = 1/s_u(0) = 1, \eta_u(0) = -\eta'(0)s_u(0) = 0 \). Using (2.29) we have also:

\[
\frac{ds_u}{dr} = \frac{ds_u}{dr} g_r = \frac{ds_u}{dr} (1 - r)^{-1} \eta'.
\]

Hence \( \frac{ds_u}{dr}(0) = 0 \) and we obtain from (2.28):

\[
\mu_1 = \frac{o(r)}{2r + o(r)}.
\]

Hence \( \mu_1 \to 0 \) if \( r \to 0 \) and we proved the estimate on \( \mu_1 \).

Now by the same considerations we see that the fraction in (2.27) is bounded at zero. Also it is uniformly bounded and non-zero at \( r > 0 \). Hence we obtain (2.26).

**End of the proof of Theorem 2.** We must find a change of the chart \( w_1 \mapsto \xi \) transforming the form \( \omega_1 \) of the last lemma to a form proportional to \( \tilde{\omega} = d\xi + \bar{\zeta} \) after subtraction of a possible term proportional to \( d\zeta \). Such change of variable \( \xi = q(w_1, \zeta) \) must satisfy the Beltramy equation:

\[
q \bar{w}_1 = \mu_1 q w_1.
\] (2.30)

The existence of required solution follows from the theory of the Beltramy equation (see, e.g., [Ah]). We extend the Beltramy coefficient \( \mu_1 \) to the exterior of the unit disk \( D \) in the Riemann sphere by the symmetry (inversion) with respect to \( \partial D \). Then there exists a unique quasiconformal homeomorphism \( q : \mathbb{C} \to \mathbb{C} \) satisfying equation (2.30) and having 0, 1 and \( \infty \) as fixed points [Ah]. It conserves the unit circle, and hence, maps the unit disk onto itself. It is \( C^k \)-smooth for \( C^k \)-smooth \( \mu_1 \) since it is a solution of an elliptic system with \( C^k \)-smooth coefficients (see, e.g., [Tr]). If \( \mu_1 \) \( C^k \)-smoothly depends on parameters (in our case it depends on \( \zeta \) \( C^\infty \)-smoothly), then \( q \) also \( C^k \)-smoothly depends on the parameters [AhB].

Note that Theorem 2 follows from Lemma 2. The prove of this lemma uses only the estimates of Propositions 1 and 2. It is easy to see also that in all considerations we can take a metric of \( C^k \)-class for \( k \geq 2 \). Taking into account the remark after the proof of Proposition 2 we see that we have the following result:

**Theorem 3.** Suppose we have a holomorphic foliation with a singular set \( \Sigma \) on a complex manifold \( M \). If there exists a \( C^k \)-smooth \( (k \geq 2) \) leafwise hermitian metric complete on the leaves, and such that its curvature is negative, uniformly bounded and uniformly bounded away from zero and the derivative of the curvature along geodesics is also uniformly bounded, then the leaves are hyperbolic and the Poincaré metric \( C^k \)-smoothly depends on the leaves.

### 3. Pseudoconvex Exhaustion

In this section we prove that in our case the manifold of universal coverings of leaves of foliation on \( \mathbb{C}P^n \) is Stein. The result must be attributed to M. Brunella, who sent to me the outlines of the proof of Theorem 4 below in a letter.

**Definition 2.** Let \( \mathcal{F} \) be a foliation by holomorphic curves on a complex manifold \( X \) and let \( g \) be a leafwise hermitian metric on \( \mathcal{F} \). Let \( v \) be a holomorphic vector field, defined on some open subset \( U \subset X \) and tangent to the leaves of \( \mathcal{F} \). Then the function \( \ln \|v\|_g \) is defined on the domain \( U \), where, for every \( q \in U \), \( \|v(q)\|_g \) is
the $g$-norm of $v(q)$. We say that $g$ has a plurisubharmonic variation if this function is plurisubharmonic on $U$.

**Theorem** (M. Brunella [Br1], [Br2]). *If $F$ is a foliation by holomorphic curves on a compact Kahler manifold $X$ then the leafwise Poincaré metric has a plurisubharmonic variation.*

Note that in this theorem the leaves may include singular points in some exceptional cases but for generic hyperbolic foliation, as it is in our case, the definition of leaves coincides with the usual one.

**Theorem 4.** Suppose a foliation $F$ on a complex manifold $X$ has a continuous leafwise Poincaré metric on the leaves, and suppose this metric has a plurisubharmonic variation. Let $M$ be a manifold of universal coverings with a Stein base $B$. Take the function $d(q) =$ the distance in our metric from $q \in M$ to the base point $\pi(q) \in B \subset M$. Then this function is plurisubharmonic. If the fibers are hyperbolic and the Poincaré metric is $C^k$-smooth ($k \geq 2$) then there exists a strictly plurisubharmonic exhaustion of $M$, and the manifold of universal coverings is Stein.

**Proof.** Take a point $q$ in $M$ and consider a local holomorphic chart $(z, w)$ at $q$. It is enough to consider only two-dimensional case. We should prove the submean inequality for any small holomorphically embedded disk $D$ centered at $q$. That is, we should prove that

$$d(q) \leq \frac{1}{l(\partial D)} \int_{\partial D} d(p)ds,$$

where $l(\partial D)$ is the length of the curve $\partial D$ in some local hermitian metric. We can take the metric induced by the euclidean metric in $\mathbb{C}^2$ in our local chart. If the disk $D$ is contained in the fiber then inequality (3.1) holds since the distance in the Poincaré metric is a subharmonic function. Thus we should consider the disk $D$ transversal to the fibers.

In local holomorphic chart a metric having a plurisubharmonic variation has the form

$$g(p) = \exp(F(p))dwd\bar{w},$$

where $F$ is plurisubharmonic. We can approximate it by strictly plurisubharmonic function, and we can approximate in $C^k$ for any $k$ a smooth strictly plurisubharmonic function by real analytic plurisubharmonic functions. Thus we can assume that our metric is real analytic and has a plurisubharmonic variation.

For every $p \in D$ let $\gamma(p)$ be the geodesic in the fiber joining $p$ with the basepoint $\pi(p)$. The union of these geodesics is a three-dimensional filled cylinder which we denote $K$. The union of the geodesics passing through points of the circle $S_1 = \partial D$ is a real analytic totally real cylinder $C$. The upper boundary of this cylinder is the circle $S_1$, and the lower boundary is another circle $S_0$ contained in $B$ and bounding some disk $D_0$ in $B$. Now we show that there exists a real analytic family of holomorphic disks $D_t$, $t \in [0, d(q)]$ interpolating $D_0$ and $D_1 = D$ with boundaries contained in $C$. We use the next lemma from [BG]:

**Suppose** $K$ **is an embedded holomorphic disk in a complex surface $Y$ with boundary in a real analytic totally real surface $\Gamma \subset Y$. If the winding number of $\Gamma$ along**
\( \partial K \) is zero, then \( K \) belongs to a unique embedded family of holomorphic disks \( K^s \), \( s \in (-\varepsilon, \varepsilon) \), \( K^0 = K \) with boundaries in \( \Gamma \).

For our cylinder the condition of the lemma means that the boundary of the disk is a simple curve making one turnover around the cylinder.

Introduce a holomorphic coordinate \( w \) on the \( d(q) + 1 \)-neighborhood (in the Poincaré metric) of the point \( \pi(q) \) in the fiber containing \( q \). We can consider it as a holomorphic chart on the fibers in some neighborhood of the interior of the cylinder \( C \) if the disk \( D_1 \) is small enough. If \( z \) is the chart on the base we have the holomorphic coordinates \( (z, w) \) in a neighborhood of \( C \). Let \( t \) be a parameter on the geodesic joining \( \pi(q) \) and \( q \), \( t \in [0, d(q)] \). We will define a family of holomorphic functions \( w = h_t(z) \) with uniformly bounded derivatives such that their graphs interpolate the disks \( D_0 \) and \( D_1 \). Let \( T \) be the set \( \{ t \in [0, d(q)], t : h_\tau \text{ for } \tau \leq t \text{ exists and the point } (0, h_\tau(0)) \text{ belongs to the closed cylinder } K \} \). By the lemma above this set is open. Suppose \( t_n \) is a sequence contained in \( T \) and converging to some \( t_0 \). Then all functions \( h_{t_n} \) are uniformly bounded. Indeed, their graphs \( D^n \) cannot intersect \( D_1 \) until the point \( (0, h_{t_n}(0)) \) isn’t contained in \( D_1 \). Indeed, each graph \( D^n \) must be contained in the interior of a real three-dimensional cylinder, and its intersection with \( D_1 \) must be along a real curve. In this case such a graph coincides with \( D_1 \). We can always suppose that the functions \( h_{t_n} \) are defined on some domain including \( D_0 \) and independent on \( n \). Indeed, the boundaries of the disks \( D^n \) belong to the real analytic totally real cylinder \( C \). We can extend one-dimensional analytic set through such submanifold by local reflection to some domain which depends only on a local properties of the cylinder \( C \) (see, for example [Chi, 20.5]). Thus this extension doesn’t depend on \( n \).

Hence passing to subsequence, one can achieve that the sequence \( h_{t_n} \) converges uniformly by the Montel theorem. Thus the set \( T \) is closed. If for some \( t \) the point \( (0, h_t(0)) \) coincides with \( q \) then the corresponding disk coincides with \( D_1 \) by the above arguments. Thus we obtained the interpolation. We normalize the parameter \( t \) such that it belongs to the segment \( [0, 1] \).

Now we complexify the real-analytic family \( h_t \). We obtain a holomorphic family of functions \( h_{w_1} \), where \( w_1 \) belongs to some neighborhood of the segment \( [0, 1] \). The variables \( z, w_1 \) define a holomorphic coordinate system in a neighborhood of the cylinder \( C \). This system is such that

1. The fibers are given by the equation \( z = \text{const} \).
2. The geodesics \( \gamma(p), p \in \partial D \) are given by the conditions: \( z = \text{const} (|z| = 1), w_1 \in [0, 1] \).
3. \( D_0 = \{ w_1 = 0 \}, D = D_1 = \{ w_1 = 1 \} \).

The curve \( z = 0, w_1 \in [0, 1] \) is not necessary a geodesic but its length is not less than the geodesic distance between \( (0, 0) \) and \( q \).

In these coordinates the fiberwise metric has form (3.2): \( g(p) = \exp(F(p))dw_1d\bar{w}_1 \) with plurisubharmonic \( F \). The length \( d(p), p \in \partial D \) is:

\[
d(p) = \int_{[0,1]} \exp[F(z(p), w_1)/2]dw_1
\]
The mean of these values over the circle is greater or equal to the integral
\[ \int_{[0,1]} \exp[F(0, w_1)/2] |d|w_1| \]
since \( F \) is plurisubharmonic. But this integral is greater or equal to the distance \( d(q) \). We obtained submean inequality (3.1).

To prove that \( M \) is Stein we should define a strictly plurisubharmonic exhausting function on \( M \). Take some strictly plurisubharmonic exhausting function \( \psi \) on the base \( B \) and define \( H(q) = d(q) + \psi \circ \pi(q) \). This function is exhausting if the fibers are hyperbolic. It is easy to see that \( H \) is also strictly plurisubharmonic. Indeed, \( d \) is strictly plurisubharmonic along the fibers, and the value of the Levi form of \( H \) at any vector transversal to the fibers is positive because the Levi form of the function \( \psi \circ \pi \) is positive on such vectors. \( \square \)

**Corollary.** In the conditions of Theorem 1 the manifold \( M \) of universal covering of the foliation \( F \) is Stein if the base \( B \) is Stein.

In fact, smoothness of the Poincaré metric seems not necessary for proving that the manifold of universal coverings is Stein. According Ohsava [Oh] and Tomassini and Vajiti [TV] a complex surface is holomorhically convex if there exists a continuous plurisubharmonic exhaustion function and a non-constant holomorphic function on it. It covers the two-dimensional case. Due to Narasimhan [Nar] a complex manifold \( M \) is Stein iff there exists an exhausting continuous plurisubharmonic function and some \( C^2 \)-smooth strictly plurisubharmonic function. It isn’t obvious that such function exists in our case. However, this difficulty seems rather technical and not insurmountable.

**References**


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