WEAK TOTAL RIGIDITY FOR POLYNOMIAL VECTOR FIELDS
OF ARBITRARY DEGREE

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To the memory of Vladimir Arnold, a teacher, a leader, a poet in mathematics

Abstract. We prove that in the space of the polynomial vector fields of arbitrary degree \( n \) with \( n + 1 \) different singular points at infinity the set of vector fields that are orbitally topologically equivalent to a generic vector field (modulo affine equivalence) is no more than countable.

This is the second one of two closely related papers. It was started after the first one, “Total rigidity of generic quadratic vector fields”, was completed. The present paper is motivated by the problem stated at the end of the first paper. The problem remains open. A slightly weaker problem is solved below.

This paper is independent on the first one. For the sake of convenience, it is published first.


Key words and phrases. Foliations, topological equivalence, rigidity.

1. Introduction

Consider the space of all polynomial vector fields of fixed degree \( n \) in a fixed affine chart in \( \mathbb{C} P^2 \). Denote it by \( A_n \). Generic vector fields of this class have an invariant line at infinity; this line with the singular points deleted is called an infinite leaf.

Let \( \hat{A}_n \) be the set of all vector fields from \( A_n \) with exactly \( n + 1 \) singular points at infinity, different from a chosen base point \( a_0 \) at the infinite leaf. Let \( \hat{A}_n \) be the universal cover over \( A_n \) with a base point \( v_0 \) to be chosen later, and \( p: \hat{A}_n \to \hat{A}_n \) be the corresponding projection. The affine group action may be lifted from \( \hat{A}_n \) to \( \hat{A}_n \).

Definition 1. Two vector fields \( v, w \in A_n \) are orbitally topologically equivalent provided that there exists a homeomorphism that conjugates the foliations with singularities defined by \( v \) and \( w \) on the projective plane. We also suppose that this conjugacy preserves the orientation on the leaves and in the phase space.

Received October 10, 2010.

The author was supported in part by the grants NSF 0700973, RFBR 10-01-00739-a, RFBR-CNRS 10-01-95115-НЦНИЛа.
In the sequel, we will skip for brevity the word orbitally.

**Definition 2.** A vector field \( v \in \mathcal{A}_n \) is weakly totally rigid provided that there exists no more than a countable set of vector fields \( w \in \mathcal{A}_n \) modulo affine equivalence, that are topologically equivalent to \( v \).

**Theorem 1.** Generic vector field in \( \mathcal{A}_n \) for \( n > 1 \) is weakly totally rigid. More precisely, there exists a finite union of proper real and complex analytic subsets of \( \tilde{\mathcal{A}}_n \) such that any vector field outside the projection \( p \) of this union into \( \hat{\mathcal{A}}_n \) is weakly totally rigid.

This result was obtained right after the paper [1] was completed. Main results of these two papers are parallel. Theorem 1 of [1] claims that a generic quadratic planar foliation is topologically equivalent to a finite number of foliations of the same class (modulo affine equivalence). Theorem 1 of this paper claims that a generic polynomial planar foliation of arbitrary degree is topologically equivalent to at most a countable number of foliations (modulo affine equivalence). The difference occurs for the following reason. In both cases, we construct moduli of topological equivalence that are of analytic origin.

In case of the quadratic foliations, the moduli are Baum-Bott indexes of singular points, that is, the quantities \( \frac{\lambda}{\mu} + \frac{\mu}{\lambda} \), where \( \lambda \) and \( \mu \) are the eigenvalues of the singular point. These moduli are well defined on the space of quadratic foliations, which is compact. Generic level sets of the corresponding moduli map are discrete; discrete subsets of compact spaces are finite.

In case of general polynomial foliations, the moduli are the tuples of higher order jets of generators of the holonomy group at infinity; their order depends on the degree of the foliation. These jets are multivalued vector functions on the space of foliations with different singular points at infinity. They are univalent on the universal cover of this space. Once again, generic level sets for these new moduli maps are discrete. But the domain of the moduli map is now non-compact, and discrete sets in such domain may be infinite, though countable.

## 2. Invariants of Topological Equivalence

Topological equivalence of polynomial vector fields implies topological equivalence of their monodromy groups at infinity, see [2, Proposition 28.2].

In more detail, let us choose the generators \( \gamma_1, \ldots, \gamma_n \) of the fundamental group of the infinite leaf of \( v \). Denote by \( f_{j,v} \) the germ of the holonomy transformation that corresponds to \( \gamma_j \) and \( v \). If two vector fields \( v \) and \( w \) are topologically equivalent, and \( H \) is the conjugating homeomorphism, then let \( g_{j,w} \) be the germ of the holonomy transformation for \( H(\gamma_j) \) and \( w \). The previous proposition claims that there exists a germ of a homeomorphism \( h: (\mathbb{C}, 0) \to (\mathbb{C}, 0) \) that conjugates the corresponding holonomy groups and moreover,

\[
h \circ f_{j,v} = g_{j,w} \circ h.
\]

An important theorem [4] claims
Theorem 2. Suppose that the holonomy group of the vector field \( v \) at infinity is unsolvable, and \( w \) is topologically equivalent to \( v \). Then the germ \( h \) in (1) is biholomorphic.

Vector fields from \( \tilde{A}_n \) with unsolvable holonomy group at infinity form a Zariski open set \([4], [3]\). Denote it by \( U \). Note that the non-solvability is a property that does not depend on the choice of the generators. Let us now define normal forms of the holonomy group at infinity. This normal form depends on the choice mentioned above. Consider a vector field \( v_0 \in U \) for which under some choice of generators in the fundamental group of the corresponding infinite leaf, the germ \( f_{1,v} \) is hyperbolic, and the two-jets of \( f_{1,v} \) and \( f_{2,v} \) do not commute. Let us take this \( v_0 \) as the base point of the universal cover \( \tilde{A}_n \); we shall identify the base point of the universal cover with its projection into \( U \) and denote both points by \( v_0 \). For this \( v_0 \) and the nearby fields \( v \), the following normal form of the holonomy group at infinity is well defined. Let us take a linearizing chart for \( f_{1,v} \). It exists by the renown Schr"{o}der Theorem, and is uniquely defined up to a multiplication by a constant, called a scaling parameter. In this coordinate, the map \( f_{1,v} \) is linear, and the two-jet of \( f_{2,v} \) is not, because the two-jets of these holonomy transformations do not commute. The scaling parameter may be chosen, and in the unique way, so that the latter jet has the quadratic coefficient equal to 1. This completes the choice of the canonical chart \( z \) for the holonomy group with marked generators. The normal form for the set of generators of this group is therefore:

\[
M(v) = (\nu_1 z, \nu_2 z + z^2 + z^3 F_{2,v}, \tilde{f}_{3,v}, \ldots, \tilde{f}_{n,v}),
\]

where \( \nu_2 z + z^2 + z^3 F_{2,v}, \tilde{f}_{3,v}, \ldots, \tilde{f}_{n,v} \) are the germs \( f_{j,v} \) written in the canonical chart \( z \).

This defines a germ of a holomorphic map \( M : (U, v_0) \to (\mathbb{C}^*)^2 \times \mathcal{O}(\mathbb{C}, 0)^{n-1} \). It may be analytically continued onto a complement of the universal cover \( \tilde{A}_n \) to a finite union of real and complex proper analytic subsets. First, let \( a(v) \) be the set of singular points at infinity for any vector field \( v \in \tilde{A}_n \). Denote by \( \tilde{C}_v \) the infinite leaf of \( v \), that is, the difference \( \tilde{C} \setminus a(v) \). Note that for all \( v \in \tilde{A}_n \), the same base point \( v_0 \) belongs to the infinite leaf. For any point \( \tilde{w} \in \tilde{A}_n \), let \( w = p \tilde{w} \). Then for any point \( \tilde{w} \in \tilde{A}_n \), an equivalence class of isotopies \( \tilde{C}_{v_0} \to \tilde{C}_w \) is well defined. Hence, an isomorphism of fundamental groups, induced by these isotopies, is defined as well. Therefore, if \( \gamma_j \) are the above chosen generators of \( \pi_1(\tilde{C}_{v_0}) \), then the corresponding generators of \( \pi_1(\tilde{C}_w) \) are well defined, as soon as \( \tilde{w} \) is chosen. Denote them \( \gamma_j(\tilde{w}) \). The germ of the holonomy map corresponding to this loop and the field \( w \) is denoted by \( f_{1,\tilde{w}} \). Now let \( \tilde{U} \) be the lift of \( U \) to the universal cover \( \tilde{A}_n \), and \( \tilde{V} \subseteq \tilde{U} \) be the set of all \( \tilde{w} \in \tilde{U} \) for which the germ \( f_{1,\tilde{w}} \) is hyperbolic, and the two jets of \( f_{1,\tilde{w}}, f_{2,\tilde{w}} \) do not commute. The difference \( \tilde{A}_n \setminus \tilde{V} \) is a union of real and complex proper analytic subsets of \( \tilde{A}_n \). The map \( M \) may be analytically extended to all of \( \tilde{V} \):

\[
M(\tilde{w}) = (\nu_1 z, \nu_2 z + z^2 + z^3 F_{2,\tilde{w}}, \tilde{f}_{3,\tilde{w}}, \ldots, \tilde{f}_{n,\tilde{w}}).
\]

Lemma 1. The set (2) is an invariant of the topological classification on the space \( p\tilde{V} \).
This means that if \( v, w \in p\hat{V} \) are topologically equivalent, then for some lifts \( \tilde{v}, \tilde{w} \), the invariants \( M(\tilde{v}), M(\tilde{w}) \) are the same.

Lemma 1 is proved in the next section. We will prove further that this invariant is even the modulus of the topological classification.

3. SOME REMARKS ABOUT HOMEOMORPHISMS OF PUNCTURED SPHERES

**Definition 3.** Consider a punctured sphere \( \hat{C}_a, \ a = (a_1, \ldots, a_{n+1}) \), with a base point \( a_0 \notin b \). Consider another copy of a punctured sphere, \( \hat{C}_b, \ b = (b_1, \ldots, b_{n+1}) \), \( a_0 \notin b \), and a homeomorphism \( H \) of one copy into another that preserves the base point \( a_0 \). An isomorphism of the groups \( \pi_1(\hat{C}_a, a_0) \) and \( \pi_1(\hat{C}_b, a_0) \) is defined. It is called an isomorphism induced by the homeomorphism \( H \).

**Definition 4.** Consider the same punctured spheres \( \hat{C}_a, \hat{C}_b \) as above, and let \( i \) be an isotopy of \( \hat{C}_a \) to \( \hat{C}_b \) that preserves the base point \( a_0 \). Then it generates an isomorphism of the groups \( \pi_1(\hat{C}_a, a_0) \) and \( \pi_1(\hat{C}_b, a_0) \) which is called an isomorphism induced by the isotopy \( i \).

A well known folk theorem claims.

**Theorem 3.** Any isomorphism \( \pi_1(\hat{C}_a, a_0) \to \pi_1(\hat{C}_b, a_0) \) induced by a homeomorphism is at the same time induced by some isotopy.

Consider a vector field \( v \in p\hat{V} \subset A_n \). This means that there exists \( \tilde{v} \in \hat{V} \) such that \( v = p\tilde{v} \). Hence there exists a curve \( \lambda_\tilde{v} \) that connects \( v_0 \) and \( v \) and represents \( \tilde{v} \).

Suppose now that two vector fields \( v \) and \( w \) of class \( p\hat{V} \subset A_n \) are topologically equivalent. Let \( H \) be a restriction of the conjugating homeomorphism to the infinite leaf. Without loss of generality, we may assume that it preserves the base point. By the theorem above, it is induced by some isotopy. This isotopy may be determined by a curve \( \lambda \) in the space \( \hat{A}_n \) that goes from \( v \) to \( w \).

The curve \( \lambda_\tilde{v} \) defines the set of generators \( \gamma_j \) of \( \pi_1(\hat{C}_v) \). The curve \( \lambda \) defines the set of generators \( \tilde{\gamma}_j \) of \( \pi_1(\hat{C}_w) \) in such a way that \( \tilde{\gamma}_j = H(\gamma_j) \).

Then

\[ M(\tilde{v}) = M(\tilde{w}). \]

Hence, the normalized sets of generators of the monodromy group at infinity for \( \tilde{v} \) and \( \tilde{w} \) coincide. This proves Lemma 1.

4. MODULI OF TOPOLOGICAL EQUIVALENCE

The normal form (2) is well defined on \( \hat{A}_n \). It is the same for vector fields that are affine equivalent modulo a constant factor. Let \( B_n = \hat{A}_n/Aff(C^2) \times \mathbb{C}^* \). Consider a well defined map

\[ M_N : B_n \to \mathbb{C}^{(n-1)N}, \quad v \mapsto (\nu_1, \nu_2, j^{N-3}F_2, j^N f_3, \ldots, j^N f_n). \quad (3) \]

**Theorem 4.** There exists \( N \) for which the map \( M_N \) has the full rank.
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Proof. Suppose that the converse is true. Then for any \( N \) the rank of \( M_N \) at any point is smaller than the dimension of \( B_n \). Hence, the fiber of this map passing through any point will have a positive dimension. Fix any point \( v \in B_n \), and let \( V_N \) be the fiber of \( M_N \) passing through \( v \). This is a nested sequence of analytic sets. Hence it stabilizes at some point. Denote the resulting set by \( V_\infty \). For all the fields from \( V_\infty \) the normal form of the holonomy group is the same. So this set is a so called isoholonomy deformation of \( v \). In [2, 28E] it is proved that such a deformation consists of vectors fields that are affine equivalent to \( v \) modulo a constant factor. But different points of \( B_n \) correspond to vector fields that are not affine equivalent modulo \( C^* \). This contradiction proves the theorem. \( \square \)

5. WEAK TOTAL RIGIDITY

Theorem 4 implies Theorem 1. Indeed, take \( N \) for which the map \( M_N \) has the full rank at some point. By analicity, it has full rank everywhere outside some analytic subset \( \Sigma \) of \( \tilde{A}_n \). Recall that the domain \( \tilde{V} \) of the moduli map is a complement to a union of real and complex proper analytic subsets of \( \tilde{A}_n \). The projection \( p \) of the difference \( \tilde{V} \setminus \Sigma \) to \( A_n \) is the set mentioned in Theorem 1. The fibers of the moduli map restricted to this difference are discrete, hence at most countable. This proves Theorem 1.

Acknowledgments. The author is grateful to the participants of the workshop on holomorphic foliations, Mexico, August 9–20: Dmitry Novikov, Laura Ortiz, Natalya Pasyi, Ernesto Rosales and Sergei Voronin. Theorem 1 appeared as a result of our mutual discussions in the friendly environment of the Instituto de Matematico de UNAM. The author is also grateful to Jim West for fruitful discussions.

References


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