RESONANCE-INDUCED SURFATRON ACCELERATION OF A RELATIVISTIC PARTICLE

A. I. NEISHTADT, A. A. VASILIEV, AND A. V. ARTEMYEV

Abstract. We study motion of a relativistic charged particle in a plane slow electromagnetic wave and background uniform magnetic field. The wave propagates normally to the background field. The motion of the particle can be described by a Hamiltonian system with two degrees of freedom. Parameters of the problem are such that in this system one can identify slow and fast variables: three variables are changing slowly and one angular variable (the phase of the wave) is rotating fast everywhere except for a neighborhood of a certain surface in the space of the slow variables called a resonant surface. Far from the resonant surface dynamics of the slow variables may be approximately described by the averaging method. In the process of evolution of the slow variables the particle approaches this surface and may be captured into resonance with the wave. Capture into this resonance results in acceleration of the particle along the wave front (surfatron acceleration). We study the phenomenon of capture and show that a captured particle never leaves the resonance and its energy infinitely grows. Passage through the resonant surface without capture leads to scattering at the resonance, i.e., a small phase-sensitive deviation of actual motion from the motion predicted by the averaging method. We find that repeated scatterings result in diffusive growth of the particle energy. The considered problem is a representative of a wide class of problems concerning passages through resonances in nonlinear systems with fast rotating phases. Estimates of accuracy of the averaging method in this class of problems were for the first time obtained by V. I. Arnold.

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1. Introduction

Motion of a charged particle in a background uniform magnetic field and a field of electromagnetic or electrostatic plane wave, propagating in plasma perpendicularly

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to the background magnetic field, is one of classical problems in plasma physics. This motion can be described by a Hamiltonian system with two degrees of freedom which does not allow for an exact analytic solution. Depending on relations between parameters of the problem, completely different dynamical phenomena take place, and different methods of the perturbation theory are used for description of these phenomena. We consider the case of a high-frequency wave of an amplitude comparable with the amplitude of the background field. In this case one can identify slow and fast variables in the system. Three variables are changing slowly and one angular variable (the wave phase) is rotating fast everywhere except for a neighbourhood of a certain surface in the space of the slow variables called a resonant surface. For an approximate description of motion far from the resonant surface the classical averaging method prescribes to average the rates of changing of the slow variables over the fast phase (see, e.g., [3]). The averaged system describes a slow Larmor rotation in the background magnetic field. In the process of this rotation the phase point of the system approaches the resonant surface: namely, the projection of the particle velocity onto the wave propagation direction becomes equal to value of the phase velocity of the wave. Here the particle may either be captured into the resonance or pass through the resonance without capture. In the case of capture the particle moves in such a way that the resonant condition is approximately kept; in the phase space the particle moves along the resonant surface. In the case of passage through the resonance without capture there is scattering at the resonance, i.e. a small phase-sensitive deviation of actual motion from the motion predicted by the averaging method. Capture into a resonance was first considered in [14], while scattering at resonances was first considered in [7]. General results about averaging in systems with slow and fast variables [1] imply that the averaging method approximately describes evolution of slow variables for majority of initial conditions at least on time intervals that are long enough to see a considerable evolution of these variables. The study of the accuracy of the averaging method in this class of problems was started in [2], where for the first time estimates were obtained of this accuracy in two-frequency systems which pass through states of resonance in the course of their evolution. Development of the theory of averaging in such systems is one of the problems stated by V.I. Arnold for his seminar ([4], problems 1972-9, 1972-10, 1976-10). In the current paper we use this theory in the form presented in [5] for description of motion of the relativistic charged particle in a plane slow electromagnetic wave and background uniform magnetic field. The capture into resonance in this problem leads to a remarkable phenomenon of acceleration along the wave front called surfatron acceleration.

The mechanism of the surfatron acceleration of charged particles is often considered for description of various plasma-physics phenomena [27], [19]. Originally this mechanism was suggested for description of charged particles acceleration along the front of a shock wave [27] and this application is still actual [20]. On the other hand, there are various astrophysical applications of the surfatron acceleration mechanism to problems of generation of high energy particles [10], [9], [31], [8]. Surfatron acceleration of relativistic particles was considered, for example, in [6], [17]. In all these papers authors consider a particle interaction with an electrostatic wave. The averaged system describes a slow Larmor rotation in the background magnetic field.
Surfatron acceleration of a particle by an electromagnetic wave is less studied. The analytical theory was constructed only for nonrelativistic [24] or ultrarelativistic [6, 16] particles. The effect of large particle velocity was estimated numerically in [29, 16]. Also, numerical calculations were carried out for the case when the wave amplitude is small compared to the background magnetic field [13, 18]; in this case, the particle is accelerated by multiple scatterings on the wave. In addition, several laboratory experiments with relativistic particles and large wave amplitudes were carried out for the investigation of surfatron acceleration of charged particles by electromagnetic waves (see [32] and references therein). Therefore, a complete analytic theory of relativistic charged particle captures by electromagnetic waves and the resulting acceleration is important.

Particle capture and surfatron acceleration is possible if phase velocity of the wave is smaller than the absolute value of the particle velocity (and hence, smaller than the speed of light). There are several plasma modes that can support a wave with needed properties: the magnetosonic wave with frequency close to lower-hybrid [24], the plasma wave with frequency close to higher-hybrid [10] or various drift modes of plasma instability [33]. In addition, a relatively small population of trapped particles can decrease the phase velocity of a wave [21] and establish the condition of resonance interaction.

Resonant phenomena arise in a variety of problems of physics, including hydrodynamics, celestial mechanics, and plasma physics. For several examples of resonant phenomena, see, e.g., [28], [11], [12], [22], [15], [30], [26].

2. Main Equations

We consider motion of a relativistic charged particle of mass $m$ and charge $e$ in a uniform magnetic field $B_0 = (0, 0, B_0)$ and the field of a plane linearly polarized electromagnetic wave propagating in perpendicular direction to $B_0$. Thus, in the Cartesian coordinates $(\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3)$ the resulting magnetic field components are $B_1 = B_2 = 0, B_3 = -B_w \sin(\mathbf{k} \cdot \mathbf{q}_1 - \mathbf{\omega} t) + B_0$, where $B_w$ is the amplitude of the magnetic field of the wave, $k$ is the magnitude of the wave vector directed along the $\mathbf{q}_1$-axis, and $\mathbf{\omega}$ is the wave frequency. The corresponding vector potential can be chosen as

$$\mathbf{A} = \left(0, B_0 \mathbf{q}_1 + \frac{B_w}{k} \cos(k \mathbf{q}_1 - \mathbf{\omega} t), 0\right).$$

(1)

Let $\mathbf{p}_i, i = 1, 2, 3$, be components of the particle’s momentum. Introduce

$$\mathbf{p}_2 = \mathbf{p}_2 + \frac{e}{c} B_0 \mathbf{q}_1 + \frac{e B_w}{c k} \cos(k \mathbf{q}_1 - \mathbf{\omega} t).$$

(2)

The Hamiltonian of the system is:

$$\dot{H} = \sqrt{m^2 c^4 + c^2 \mathbf{p}_2^2 + \left(c \mathbf{p}_2 - e B_0 \mathbf{q}_1 - \frac{e B_w}{k} \cos(k \mathbf{q}_1 - \mathbf{\omega} t)\right)^2 + c^2 \mathbf{p}_3^2},$$

(3)

and pairs of canonically conjugate variables are $(\mathbf{p}_1, \mathbf{q}_1)$, $(\mathbf{p}_2, \mathbf{q}_2)$, $(\mathbf{p}_3, \mathbf{q}_3)$. The Hamiltonian does not contain variables $\mathbf{q}_2$ and $\mathbf{q}_3$. Thus canonically conjugate momenta $\mathbf{p}_2$ and $\mathbf{p}_3$ are constants of motion. We can put $\mathbf{p}_3 = 0$ (it always can be done by redefining the particle’s mass); one can also make $\mathbf{p}_2 = 0$ choosing properly...
the origin in \( \dot{q}_1 \). Introduce Larmor frequency \( \omega_L = eB_0/(mc) \) and dimensionless parameter \( \varepsilon = eB_w/(mkc^2) \). We assume that \( \varepsilon \) is small: \( 0 < \varepsilon < 1 \). Use the following rescaling to make the system dimensionless:

\[
\begin{align*}
p_i &= \frac{\hat{p}_i}{mc}, & q_i &= \frac{\omega_L \hat{q}_i}{\varepsilon c}, \\
t &= \frac{\omega_L t}{\varepsilon}, & H &= \frac{\hat{H}}{mc^2}, \\
k &= \frac{k}{\omega_L}, & \omega &= \frac{\omega \varepsilon}{\omega_L}.
\end{align*}
\]

The Hamiltonian in the new variables is:

\[
H = \sqrt{1 + p_i^2 + \left( \varepsilon q_i + \varepsilon \cos(kq_1 - \varepsilon t) \right)^2}.
\]

Introduce new variable \( U = \omega t \). Let \( P_U \) be the variable, canonically conjugate to \( U \). Thus we obtain a 2 d.o.f. Hamiltonian system. The Hamiltonian takes the form:

\[
\mathcal{H} = \omega P_U + \sqrt{1 + p_i^2 + (\varepsilon q_i + \varepsilon \cos(kq_1 - U))^2}.
\]

Now we introduce the wave phase \( \varphi = kq_1 - U \) as a new variable. To this end, we make canonical transform with generating function \( W = pq_1 - I(kq_1 - U) \), where \( I \) is a new variable canonically conjugate to \( \varphi \). For the new variables \( p, q, I \) we have

\[
q = q_1, \quad p = p_1 - Ik, \quad I = -P_U. \quad \text{Denote } \varepsilon q = \hat{q}. \quad \text{Omitting the tilde, we find for the}
\]

Hamiltonian in the new variables:

\[
\mathcal{H} = -\omega I + \sqrt{1 + (p + Ik)^2 + (q + \varepsilon \cos \varphi)^2},
\]

where the pairs of canonically conjugate variables are \( (I, \varphi) \) and \( (p, \varepsilon^{-1}q) \). This Hamiltonian can be represented in the form \( \mathcal{H} = H_0 + \varepsilon H_1 \), where

\[
H_0 = -\omega I + \sqrt{1 + (p + Ik)^2 + q^2}, \quad \varepsilon H_1 = \varepsilon \frac{q \cos \varphi}{\sqrt{1 + (p + Ik)^2 + q^2}} + O(\varepsilon^2).
\]

In the main approximation, the equations of motion are

\[
\begin{align*}
\dot{I} &= -\varepsilon \frac{\partial \mathcal{H}}{\partial \varphi} = \varepsilon \frac{q \sin \varphi}{\sqrt{1 + (p + Ik)^2 + q^2}}, \\
\dot{\varphi} &= \frac{\partial \mathcal{H}}{\partial I} = -\omega + \frac{k(p + Ik)}{\sqrt{1 + (p + Ik)^2 + q^2}}, \\
\dot{p} &= -\varepsilon \frac{\partial \mathcal{H}}{\partial q} = -\varepsilon \frac{q}{\sqrt{1 + (p + Ik)^2 + q^2}}, \\
\dot{q} &= \varepsilon \frac{\partial \mathcal{H}}{\partial p} = \varepsilon \frac{p + Ik}{\sqrt{1 + (p + Ik)^2 + q^2}}.
\end{align*}
\]

In this system, variable \( \varphi \) is fast (its time derivative is a value of order \( \varepsilon \)), and the other variables are slow (their time derivatives are of order 1). Thus, one can average over fast phase \( \varphi \) and obtain the averaged system. Motion in this system is just the Larmor rotation in the uniform magnetic field \( B_0 \). The averaging, however, does not describe the motion adequately near the resonance \( \dot{\varphi} = 0 \). At
the resonance, projection of particle’s velocity onto the $q_1$-axis equals the phase velocity of the wave. Resonance condition $\partial H_0/\partial I = 0$ defines a resonance surface in the $(p, q, I)$-space:

$$ (p + Ik)^2(k^2 - \omega^2) = (1 + q^2)\omega^2. \quad (10) $$

We denote the value of $I$ on this surface as $I_r$. This is a function of variables $p, q$:

$$ I_r(p, q) = \frac{1}{k} (\omega \sqrt{1 + q^2}/(k^2 - \omega^2) - p). \quad (11) $$

Intersection of surface (10) and isoenergetic surface $H_0 = h$ defines the resonance curve. Its projection onto the $(p, q)$-plane is a hyperbola given by equation

$$ (h - v\phi p)^2 = (1 + q^2)(1 - v^2\phi^2), \quad (12) $$

where we introduced the dimensionless phase velocity of the wave $v\phi = \omega/k$. Note that $v\phi$ is always smaller than 1.

Variable $I$ is an integral of motion of the averaged system (see (8)) and thus an adiabatic invariant of the exact system. Far from the resonance the value of $I$ is preserved with the accuracy of order $\varepsilon$ on time intervals of order $1/\varepsilon$ (see, e.g., [5]). The adiabatic invariance of $I$ breaks down near the resonance, where the averaging does not work properly. In a neighborhood of the resonance phenomena of capture and scattering can take place. We study dynamics of the system near the resonance in the following sections.

3. Motion Near the Resonance

To study the system near the resonance, we apply the approach formulated in [23] (see also [25]). Close to the surface (10) the Hamiltonian can be expanded into series in $(I - I_r)$:

$$ \mathcal{H} = \Lambda(p, q) + \frac{1}{2} g(p, q)(I - I_r(p, q))^2 + \varepsilon H_1|_{I = I_r} + O(|I - I_r|^3) + O(\varepsilon(I - I_r(p, q))). \quad (13) $$

Here $\Lambda(p, q) = H_0|_{I = I_r}$ is the unperturbed Hamiltonian $H_0$ restricted onto the resonant surface (10). Function $g(p, q)$ in (13) is $\partial^2 H_0/\partial I^2$ restricted onto the resonant surface. It is straightforward to find

$$ \Lambda(p, q) = pv\phi + \sqrt{(1 + q^2)(1 - v^2\phi^2)}, \quad g(p, q) = k^2(1 - v^2\phi^2)^{3/2}/\sqrt{1 + q^2}. \quad (14) $$

Introduce notation $d = q\sqrt{1 - v^2\phi}/\sqrt{1 + q^2}$. Then

$$ \varepsilon H_1|_{I = I_r} = \varepsilon d \cos \varphi. \quad (15) $$

Now we introduce new momentum $K = I - I_r(p, q) + O(\varepsilon)$. To this end, we make a canonical transformation of variables $(I, \varphi, p, q) \rightarrow (K, \varphi, \bar{p}, \bar{q})$ with generating function $W_1 = \bar{p}\varepsilon^{-1}q + (K + I_r(p, q))\varphi$. Omitting bars, we find for the Hamiltonian in the new variables (we neglect terms of higher orders):

$$ \mathcal{H} = \Lambda(p, q) + \frac{1}{2} g(p, q)K^2 + \varepsilon d \cos \varphi + \varepsilon b(p, q)\varphi, \quad (16) $$
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Figure 1. Schematic view of the phase portrait of the system (18) for \( d/b > 1 \) (a) and for \( d/b < 1 \) (b)

where

\[ b(p, q) = \{I_r, \Lambda\} = \frac{\partial I_r}{\partial q} \frac{\partial \Lambda}{\partial p} - \frac{\partial I_r}{\partial p} \frac{\partial \Lambda}{\partial q} = \frac{q}{k \sqrt{1 + q^2}} \frac{1}{\sqrt{1 - v_\phi^2}}. \]  

(17)

Introduce \( P = K/\sqrt{\varepsilon}, \theta = t\sqrt{\varepsilon}, \) and rescaled Hamiltonian \( F = \mathcal{H}/\varepsilon \). The rescaled system is Hamiltonian, and pairs of canonically conjugate variables are \((P, \phi)\) and \((p, \varepsilon^{-3/2}q)\). In the main approximation, the Hamiltonian is

\[ F = \varepsilon^{-1} \Lambda(p, q) + F_0(P, \phi, p, q), \quad F_0 = \frac{1}{2} g P^2 + d \cos \phi + b \phi, \]  

(18)

and the equations of motion are

\[ p' = -\sqrt{\varepsilon} \frac{\partial \Lambda}{\partial q}, \quad q' = \sqrt{\varepsilon} \frac{\partial \Lambda}{\partial p}, \]

\[ P' = -\frac{\partial F_0}{\partial \phi}, \quad \phi' = \frac{\partial F_0}{\partial P}, \]  

(19)

where prime denotes derivative over \( \theta \). One can see that variables \((P, \phi)\) are fast, and variables \((p, q)\) are slow. Thus, as the first step to study this system, one can consider variation of the fast variables at fixed values of \( p \) and \( q \). Dynamics of the fast variables is defined by Hamiltonian \( F_0 \), which contains \( q \) as a parameter. This is a Hamiltonian of a pendulum under the action of external torque. Consider phase portrait of this system (fast subsystem). If \( d/b > 1 \), there is a separatrix on the phase portrait, and, correspondingly, domain of oscillatory motion (see Fig. 1). In the opposite case \( d/b < 1 \), there is no separatrix. Note that ratio \( d/b = k(1 - \hat{v}_\phi^2) \) is independent of \( q \). In original dimensional units \( d/b = (Bw/B_0)(1 - \hat{v}_\phi^2/c^2) \), where \( \hat{v}_\phi = \hat{\omega}/k \). We consider first the case \( d/b > 1 \).

It is straightforward to obtain for the area \( S \) inside the separatrix on the plane \((\phi, P)\)

\[ S = 2 \int_{\phi_m}^{\phi_m} \left( \frac{d}{2} \left[ \cos \phi_s + b \frac{d}{d} \phi_s - \cos \phi - b \frac{d}{d} \phi \right] \right)^{1/2} d\phi, \]  

(20)
where $\varphi_s = \arcsin(b/d)$ and $\varphi_m$ is the root of equation $\cos \varphi + (b/d)\varphi = \cos \varphi_s + (b/d)\varphi_s$, different from $\varphi_s$. One can see that both integration limits and the expression in square brackets in (20) do not depend on $q$. Thus,

$$S = \sqrt{\frac{d}{g}} A = \sqrt{\frac{q}{k^2 - \omega^2}} A,$$

where $A$ is a constant independent of $q$ (and $p$).

Now take into account slow variation of $p$ and $q$ according to the first two equations in (19). It follows from the expression (14) for $\Lambda(p, q)$ that if $v_{\phi} > 0$, variable $q$ grows with time. This means that area $S$ also grows in the process of motion. Therefore, phase points on the phase portrait of the fast subsystem can cross the separatrix and enter the domain of oscillations. This corresponds to a capture into resonance. The area $J$ encircled by a trajectory in the domain of oscillations on this portrait is an adiabatic invariant (it is called the \textit{inner adiabatic invariant}). Thus, as $S$ monotonously grows with time, a captured particle goes deeper and deeper inside the oscillation domain and cannot leave it. This means that a particle captured into the resonance is captured forever.

Motion of a captured particle can be described as follows. In the main approximation, it moves with the \textit{resonant flow} defined by Hamiltonian $\Lambda(p, q)$. The corresponding equations of motion are:

$$\dot{p} = -\varepsilon \frac{\partial \Lambda}{\partial q} = \varepsilon q \sqrt{1 - \frac{v_{\phi}^2}{1 + q^2}},$$

$$\dot{q} = \varepsilon \frac{\partial \Lambda}{\partial p} = \varepsilon v_{\phi}. \quad (22)$$
It means that a captured particle moves in $\hat{q}_1$-direction with the wave at a speed of the wave’s phase velocity. Since $P_2 = 0$ it follows from (2) that $p_2 \approx -q$, whence $p_2 \sim -\varepsilon v_\phi t$. Therefore $p_2$-component of the particle’s momentum varies (on average) linearly in time:

$$\dot{p}_2 \sim -\frac{e}{c} B_0 \hat{v}_\phi \hat{t} \sim mc(\omega_L \hat{v}_\phi \hat{t}/c)$$

Thus, the particle is accelerated along the wave front. This acceleration is called surfatron one. To find variation of $p_1$ in this motion, we use that $p_1 = p + Ik$ and expression (11) for $I$ on the resonant surface. Thus we obtain $p_1 = v_\phi \sqrt{1 + q^2}/(1 - v_\phi^2)$ and

$$\dot{p}_1 = \frac{\varepsilon v_\phi^2}{\sqrt{1 - v_\phi^2}} \frac{q}{\sqrt{1 + q^2}},$$

(23)

where we used the second equation in (22). At $q \gg 1$ we find that $p_1$ grows with time as $\varepsilon v_\phi^2 t/\sqrt{1 - v_\phi^2}$. In dimensional variables, we find that

$$\dot{p}_1 \sim \frac{eB_0 \hat{v}_\phi^2}{c^2 \sqrt{1 - \hat{v}_\phi^2/c^2}} \hat{t} \sim mc \frac{\hat{v}_\phi/c}{\sqrt{1 - \hat{v}_\phi^2/c^2}} (\omega_L \hat{v}_\phi \hat{t}/c).$$

For the energy of a captured particle $E = \sqrt{1 + p_1^2 + p_2^2}$ we find that it also grows linearly with time at large enough values of $q$. Namely, we have $E \sim \varepsilon v_\phi t/\sqrt{1 - v_\phi^2}$ and, in dimensional variables, $\dot{E} = mc^2 E \sim mc \hat{v}_\phi \omega_L \hat{t}/\sqrt{1 - \hat{v}_\phi^2/c^2}$.

The captured particle also oscillates in the potential well of the wave. These oscillations correspond to motion in the oscillatory domain in Fig. 1. One can evaluate the amplitude and the frequency of the oscillations using conservation of the inner adiabatic invariant $J$. For a captured particle $J$ equals the area inside the separatrix at the time when the particle crossed the separatrix. It follows from the expression for $F_0$ in (18) that for a captured particle

$$J = \frac{2^{3/2}}{k} \sqrt{\frac{q}{1 - v_\phi^2}} \int_{\varphi_1}^{\varphi_2} \sqrt{f_0 \cos \varphi - \frac{\varphi}{k(1 - v_\phi^2)}} d\varphi,$$

(24)
Figure 4. Oscillation frequency of $\dot{\varphi}$ as a function of time for two particles with different values of initial momentum. Grey dashed lines show the theoretical dependence $\omega_0 \sim t^{-1/2}$.

where $\tilde{f}_0 = F_0/d$ does not depend on $\varphi$; $\varphi_1$ and $\varphi_2$ are the minimal and the maximal values of $\varphi$ on a trajectory with fixed values of $F_0$ and $q$. Thus, growth of $q$ results in decreasing of the amplitude of the $\varphi$-oscillations. When the amplitude of these oscillations is sufficiently small, one can expand the Hamiltonian $F_0$ to obtain

$$2(F_0 - \bar{F}_0) \approx g P^2 + d |\cos \varphi_0| \cdot (\varphi - \varphi_0)^2,$$

where $F_0$ and $\varphi_0$ are values of $F_0$ and $\varphi$ at the bottom of the potential well inside the separatrix (see Fig. 1a). The frequency of oscillations (in terms of rescaled time $\theta$) is approximately $\sqrt{gd |\cos \varphi_0|}$, and in this approximation $2\pi J \sim (F_0 - \bar{F}_0)/\sqrt{gd}$. When the particle is captured, $J = J_0$.

Using conservation of $J$ along the trajectory of the captured particle, one obtains the scalings $\Delta \varphi \sim q^{-1/4}(k^2 - \omega^2)^{1/4}$ and $\Delta P \sim q^{1/4}(k^2 - \omega^2)^{-1/4}$, where $\Delta \varphi$ and $\Delta P$ are amplitudes of $\varphi$-oscillations and $P$-oscillations accordingly. Thus, the amplitude of oscillations in $q$ decreases with time proportionally to $t^{-1/4}$, while amplitude of oscillations in $p_1$ grows with time proportionally to $t^{1/4}$. Accordingly, amplitude of oscillations in $p_2$ decreases with time as $t^{-1/4}$. In dimensional variables we have

$$\Delta \dot{q}_1 \sim \varepsilon (\dot{\varphi}_0 \dot{t})^{-1/4} \quad \text{and} \quad \Delta \dot{p}_1 \sim \sqrt{\varepsilon} (\dot{\varphi}_0 \dot{t})^{-1/4}.$$  

The frequency of these oscillations is $\omega_0 \approx \sqrt{\varepsilon gd}$ (we recall that we made time rescaling to obtain (18)). Thus,

$$\omega_0 \approx k(1 - \nu_0^2) \sqrt{\frac{\varepsilon q}{1 + q^2}}.$$  

(25)

At $q \gg 1$ we find that this frequency decreases as $\sqrt{\varepsilon/q}$. In dimensional variables we find that $\omega_0 \approx (\varepsilon \dot{t})^{-1/2}$. 

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Thus, the amplitude of oscillations in $q$ decreases with time proportionally to $t^{-1/4}$, while amplitude of oscillations in $p_1$ grows with time proportionally to $t^{1/4}$. Accordingly, amplitude of oscillations in $p_2$ decreases with time as $t^{-1/4}$. In dimensional variables we have

$$\Delta \dot{q}_1 \sim \varepsilon (\dot{\varphi}_0 \dot{t})^{-1/4} \quad \text{and} \quad \Delta \dot{p}_1 \sim \sqrt{\varepsilon} (\dot{\varphi}_0 \dot{t})^{-1/4}.$$  

The frequency of these oscillations is $\omega_0 \approx \sqrt{\varepsilon gd}$ (we recall that we made time rescaling to obtain (18)). Thus,

$$\omega_0 \approx k(1 - \nu_0^2) \sqrt{\frac{\varepsilon q}{1 + q^2}}.$$  

(25)

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The frequency of these oscillations is $\omega_0 \approx \sqrt{\varepsilon gd}$ (we recall that we made time rescaling to obtain (18)). Thus,
Here are more complete formulas in dimensional variables for the amplitudes of the oscillations at large enough values of $\hat{q}$ such that we can put $\hat{q} = \hat{v}_\phi \hat{t}$. One obtains:

$$\Delta \hat{q}_1 \sim \varepsilon c/\omega L \sqrt{B_0/B_w} \left(1 - \frac{\hat{v}^2\phi c^2}{c^2}\right)^{1/4} \left(\omega_L \hat{v}_\phi \hat{t}/c\right)^{-1/4},$$

$$\Delta \hat{p}_1 \sim \varepsilon^{1/2} \sqrt{B_w/B_0} \left(1 - \frac{\hat{v}^2\phi c^2}{c^2}\right)^{1/4} \left(\omega_L \hat{v}_\phi \hat{t}/c\right)^{1/4},$$

$$\hat{\omega}_0 \sim \omega_L \varepsilon^{-1/2} \sqrt{B_w/B_0} \left(1 - \frac{\hat{v}^2\phi c^2}{c^2}\right) \left(\omega_L \hat{v}_\phi \hat{t}/c\right)^{-1/2}.$$

Capture into the resonance is a probabilistic phenomenon (see, e.g., [23], [25]). Its probability is a small value of order $\sqrt{\varepsilon}$. However, the geometry of the system makes the particle pass through the resonance repeatedly, at each Larmor turn. The probability of capture after $\sim \varepsilon^{-1/2}$ Larmor turns is a value of order one (provided that $d/b > 1$).

For comparison with theoretical results we present the numerical solution of the system with Hamiltonian (4), parameters $\varepsilon = 0.1$, $\hat{\omega}/(\hat{k}c) = 0.25$ and initial value of the $p_1 = 0.1$. The particle trajectory in momentum space $(p_1, p_2)$ is shown in Fig. 2. At the first stage of modelling the particle rotates in the constant background magnetic field (Larmor rotation). This motion is slightly perturbed by influence of the wave: the Larmor circles in $(p_1, p_2)$ plane are “scattered”. Then after certain time interval the particle is captured by the wave and the magnitude of momentum $p_2$ grows with time while momentum $p_1$ oscillates around the resonant value (it is also increasing, yet much slower). To compare the scale of growth of momenta $p_1$ and $p_2$ we plot the same picture on a longer time range (see Fig. 2). The ratio of $p_1$ and $p_2$ after initial time interval is $1/40 \sim \varepsilon \hat{\omega}/(\hat{k}c)$, in agreement with the theory (see equation (23)). The particle energy $E = \sqrt{1 + p_1^2 + p_2^2}$ is shown in Fig. 3. The energy is almost constant before capture (if we neglect small scatterings due to wave impacts) and after the capture the energy grows linearly with time. In addition we examine the theoretical equation for frequency of oscillation of the captured particle — equation (25). For this purpose we plot the oscillation frequency of $\dot{\varphi}$ around the null value — Fig. 4.

4. SCATTERINGS ON THE RESONANCE

Capture into the resonance is impossible in the case $d/b < 1$, when there is no oscillatory domain on the phase portrait of the pendulum-like system (see Fig. 1b). However, in this case the particle energy also changes at the resonance crossing. This happens due to scatterings on the resonance. If $d/b < 1$, the average value of the jump in the energy is zero (see [23]), but the dispersion is non-zero, and thus diffusive variation of the particle energy may be possible. Here we study this topic more attentively.

When the particle is far from the resonance, its energy is approximately constant, because the impact of the wave can be averaged. Thus, to study variation of the particle energy we find its time derivative according to equations of motion (19)
and integrate it near the resonance. We have
\[ E = \sqrt{1 + p_1^2 + q^2} = \sqrt{1 + (p + k I)^2 + q^2}. \] (26)

Using (6) we can write
\[ \dot{E} = \frac{d}{dt}(H + \omega I) = \omega \dot{I} = -\omega \frac{\partial H}{\partial \varphi}. \] (27)

From (13) and (15) we find that near the resonance
\[ \dot{E} = \varepsilon \omega d \sin \varphi, \] (28)

To integrate (28) we change the integration variable from time \( t \) to phase \( \varphi \) according to
\[ \dot{\varphi} = \sqrt{\varepsilon \frac{\partial F_0}{\partial P}}. \] Thus we find for variation (jump) of the particle energy on one resonance crossing
\[ j_E = 2 \sqrt{\varepsilon} \frac{v_0}{\sqrt{1 - v_0^2}} \sqrt{q} \int_{-\infty}^{\varphi^*} \sin \varphi \, d\varphi \frac{\sin \varphi}{\sqrt{2[\cos \varphi^* + \frac{q}{2\varepsilon} \varphi - \cos \varphi - \frac{q}{2\varepsilon} \varphi^*]}}, \] (29)

where \( \varphi^* \) is the wave’s phase at the resonance crossing, and \( q \) is taken also at the crossing of the unperturbed trajectory with energy \( E \) and the resonant surface. The value of \( \varphi^* \) strongly depends on initial conditions and should be treated as random. Therefore, change in the particle’s energy on the resonance is also a random variable.

If there is no separatrix on the phase portrait in Fig. 1 (i.e., if \( d/b < 1 \)), the average value of this latter random value is zero. An important question is whether these values at successive crossings are statistically independent. Expressing \( q \) in (29) on the resonance via \( E \) (we use that at the resonance \( E = \sqrt{(1 + q^2)/(1 - v_0^2)} \)) we find that at \( E \gg 1 \) the variance of change in energy at the resonance scattering is estimated as
\[ \langle (j_E)^2 \rangle \sim \varepsilon E. \] (30)

On the plane \((q, p_1)\) unperturbed motion of the particle is rotation along the circle \( p_1^2 + q^2 = E^2 - 1 \). Using Hamiltonian equations of the unperturbed motion, one immediately finds that the frequency of this rotation is \( \omega_0 = \varepsilon/E \). The resonant curve on the plane \((q, p_1)\) is a branch of hyperbola \( p_1^2(1 - v_0^2) - q^2v_0^2 = v_0^2 \) with \( p_1 > 0 \). At large enough values of \( E \) the trajectory of the unperturbed motion crosses the resonant curve at two points. It is straightforward to find that the time of motion between these two points is a value of order \( E/\varepsilon \). Consider two successive resonance crossings. Let the values of \( \varphi \) at the first and the second crossings be \( \varphi^{(1)} \) and \( \varphi^{(2)} \) accordingly. To find \( \varphi^{(2)} - \varphi^{(1)} \) one can integrate equation of motion for \( \varphi \) in (8). Thus, one obtains \( \varphi^{(2)} - \varphi^{(1)} \sim E/\varepsilon \). A small variation \( \delta \varphi^{(1)} \) of the phase \( \varphi^{(1)} \) results in variation of the energy jump \( \delta j_E \sim \sqrt{\varepsilon E} \delta \varphi^{(1)} \) at the first resonance crossing. The resulting variation \( \delta \varphi^{(2)} \) of the phase at the second crossing can be found as \( \delta j_E \partial(\varphi^{(2)} - \varphi^{(1)})/\partial E \sim \delta j_E / \varepsilon \sim \delta \varphi^{(2)} \sqrt{E} / \sqrt{\varepsilon} \).

Thus, the resulting variation in the phase at the second resonance crossing is much larger than \( \delta \varphi^{(1)} \). Therefore, the values of phase at successive resonance crossings should be considered as statistically independent. Hence, the jumps in the particle’s energy at the resonance produce diffusive variation of the energy and its unlimited stochastic growth. Note that in [24] the diffusive growth of energy was studied
Figure 5. The Poincaré sections for several cases with different initial energy ($\varepsilon = 0.1$). Number of resonance crossings in both cases is $10^6$. The energy diffusion is clearly seen.

in nonrelativistic case. It was found that, unlike in the relativistic case, for a nonrelativistic particle the energy diffusion slows down and finally stops at large enough energies.

One can estimate the rate of the energy diffusion as follows. Consider an ensemble of long phase trajectories with initial energy $E_0 \gg 1$. Introduce new variable $\kappa = \sqrt{E}$. It follows from (30) that at every resonant crossing $\kappa$ changes by a value of order $\sqrt{\varepsilon}$. If successive jumps in energy are not correlated, after $N$ resonance crossings we have $\langle (\Delta \kappa)^2 \rangle \sim \varepsilon N$, where angle brackets denote the ensemble averaging and $\Delta \kappa$ is a displacement of $\kappa$. Hence, we find that the ensemble average change in energy after $N$ jumps is proportional to $N$:

$$\langle E \rangle - E_0 \sim \varepsilon N. \quad (31)$$

Time interval between successive jumps is a value of order of the Larmor period. Hence, the time interval corresponding to $N$ resonance crossings is $t \sim NE/\varepsilon$. Combining this with (31), we find that the energy typically grows with time as

$$\langle E \rangle - E_0 \sim \varepsilon \sqrt{t}. \quad (32)$$

We also obtain that the number of jumps (resonance crossings) grows with time as $N \sim \sqrt{t}$.

These results on the energy diffusion of a relativistic particle can be examined numerically. For this purpose we construct the Poincaré section of a particle trajectory in $(p_1, p_2)$ plane. Points on this plane are plotted with time period $2\pi/\omega$. One can see that the diffusion in $(p_1, p_2)$ space becomes stronger as the initial energy of a particle grows (Fig. 5). In Fig. 6, we present results of numerics illustrating estimates (31) and (32).

5. Conclusions

In this paper we considered dynamics of a relativistic charged particle in the field of an electromagnetic wave in the presence of a background magnetic field. We have described the particle capture into resonance with the wave and consequent acceleration using approach of the adiabatic theory of motion. During the
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Figure 6. The energy of particle ensemble (number of particles is $10^3$) as a function of number of resonance crossings in the system with $\varepsilon = 0.1$. In subpanel: the number of resonance crossings and averaged energy as a function of time.

acceleration the particle’s momentum in the direction of the wave vector $\hat{p}_1$ and along the wave front $\hat{p}_2$ change with time linearly ($\hat{p}_1 \sim mc\frac{\hat{v}_L/\hat{v}_m}{\sqrt{1-\hat{v}^2/\hat{c}^2}}(\omega_L\hat{v}_L\hat{t}/\hat{c})$, $\hat{p}_2 \sim -mc(\omega_L\hat{v}_L\hat{t}/\hat{c})$). As a result the particle energy $\hat{E} \sim \sqrt{m^2c^4 + \hat{p}_1^2 + \hat{p}_2^2}$ grows with time as $\hat{E} \sim mc^2(\omega_L\hat{v}_L\hat{t}/\hat{c})$. If the condition of capture into the resonance is not satisfied (the magnitude of the wave is less than a certain value), particle can nevertheless gain energy by scatterings on the resonance. In this case $\hat{E} \sim N$, where $N$ is a number of resonance crossings.

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A.N.: Department of Mathematical Sciences, Loughborough University, LE11 3TU Loughborough, United Kingdom
E-mail address: a.neishtadt@lboro.ac.uk

A.N., A.V., A.A.: Space Research Institute, Profsoyuznaya st., 84/32, Moscow, 117997, Russia
E-mail address: valex@iki.rssi.ru
E-mail address: ante0226@yandex.ru