NORMAL FORMS OF HYPERSURFACE SINGULARITIES IN
POSITIVE CHARACTERISTIC

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Dedicated to Sabir Medzhidovich Gusein-Zade at the occasion of his 60th birthday

Abstract. The main purpose of this article is to lay the foundations for a classification of isolated hypersurface singularities in positive characteristic. Although our article is in the spirit of Arnol’d who classified real and complex hypersurfaces in the 1970’s with respect to right equivalence, several new phenomena occur in positive characteristic. Already the notion of isolated singularity is different for right resp. contact equivalence over fields of characteristic other than zero. The heart of this paper consists of the study of different notions of non-degeneracy and the associated piecewise filtrations induced by the Newton diagram of a power series \( f \). We introduce the conditions AC and AAC, which modify and generalise the conditions A and AA of Arnol’d resp. Wall and which allow the classification with respect to contact equivalence in any characteristic. Using this we deduce normal forms and rather sharp determinacy bounds for \( f \) with respect to right and contact equivalence.

We apply this to classify hypersurface singularities of low modality in positive characteristic.

Key words and phrases. Hypersurface singularities, finite determinacy, Milnor number, Tjurina number, normal forms, semi-quasihomogeneous, inner Newton non-degenerate.

1. Introduction

Throughout this paper \( K \) denotes an algebraically closed field of arbitrary characteristic unless explicitly stated otherwise. By \( K[[x]] = K[[x_1, \ldots, x_n]], n \geq 1 \), we denote the formal power series ring over \( K \), and by \( \mathfrak{m} = (x_1, \ldots, x_n) \) the maximal ideal of \( K[[x]] \).

In the classification of power series \( f \in K[[x]] \) there are two natural equivalence relations, right equivalence and contact equivalence. Two power series \( f, g \in K[[x]] \) are right equivalent if there is an automorphism \( \varphi \in \text{Aut}(K[[x]]) \) such that \( f = \varphi(g) \), and we denote this by \( f \sim_r g \). We call \( f, g \in K[[x]] \) contact equivalent

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if there is an automorphism \( \varphi \in \text{Aut}(K[[x]]) \) and a unit \( u \in K[[x]]^* \) such that \( f = u \cdot \varphi(g) \), and we denote this by \( f \sim_c g \). Note that if \( f, g \in K[[x]] \) are contact equivalent if and only if their associated hypersurface singularities \( R_f = K[[x]]/(f) \) and \( R_g = K[[x]]/(g) \) are isomorphic as local analytic \( K \)-algebras.

For a power series \( f \in K[[x]] \) we denote by \( j(f) := \langle f_2, \ldots, f_{2n} \rangle \subseteq K[[x]] \) the Jacobian ideal of \( f \), generated by the partial derivatives of \( f \), and we call the associated algebra \( M_f := K[[x]]/j(f) \) the Milnor algebra of \( f \) and its dimension \( \mu(f) := \dim_K(M_f) \) the Milnor number of \( f \). We then call \( f \) an isolated singularity if \( \mu(f) < \infty \), which is equivalent to the existence of a non-negative integer \( k \) such that \( m^k \subseteq j(f) \).

Similarly we define the Tjurina ideal \( \tau(f) := \langle f, f_2, \ldots, f_{2n} \rangle = \langle f \rangle + j(f) \subseteq K[[x]] \) of \( f \), the associated Tjurina algebra \( T_f := K[[x]]/\tau(f) \) of \( f \) and its dimension \( \tau(f) := \dim_K(T_f) \), the Tjurina number of \( f \). We then call \( R_f \) an isolated hypersurface singularity if \( \tau(f) < \infty \), or equivalently \( m^k \subseteq \tau(f) \) for some \( k \geq 0 \).

It is straightforward to see that the Milnor number is invariant under right equivalence and the Tjurina number is invariant under contact equivalence.

Our principal interest is the classification of power series in positive characteristic with respect to right (respectively contact) equivalence, where the latter is the same as to say that we are interested in classifying hypersurface singularities up to isomorphism. In order to have good finiteness conditions at hand we restrict to the case that \( f \) is an isolated singularity for right equivalence (respectively that \( R_f \) is an isolated hypersurface singularity for contact equivalence). Note that these are two distinct conditions in positive characteristic (see also [BGM]).

A first important step in the attempt to classify singularities from a theoretical point of view as well as from a practical one is to know that the equivalence class is determined by a finite number of terms of the power series \( f \) and to find the corresponding degree bound. We say that \( f \) is right (respectively contact) \( k \)-determined if \( f \) is right (respectively contact) equivalent to every \( g \) which coincides with \( f \) up to order \( k \).

In [BGM] we have shown that \( f \) is finitely right (respectively contact) determined if and only if \( \mu(f) \) respectively \( \tau(f) \) is finite, and we have shown that \( 2 \cdot \mu(f) - \text{ord}(f) + 2 \) (respectively \( 2 \cdot \tau(f) - \text{ord}(f) + 2 \)) is an upper bound for the determinacy. Here \( \text{ord}(f) \) denotes the order. In Corollaries 4.8 and 4.9 we show how this degree bound can be considerably improved when the singularities satisfy the conditions AA respectively AAC introduced in Section 4 (see also the examples in Section 5).

Once we know that a finite number of terms of \( f \) suffices to determine its equivalence class then we would like to determine a normal form for \( f \), i.e. an “efficient” representative for the equivalence class. This is in general a difficult task. The first classes of singularities one comes across are those which have a quasihomogeneous representative. In characteristic zero they are determined by the fact that the Milnor number and the Tjurina number coincide (see Theorem 2.1). The next more complicated classes of singularities are those which have a representative with a quasihomogeneous principal part (that governs its topology over the complex numbers), i.e. the right semi-quasihomogeneous rSQH (respectively contact semi-quasihomogeneous) cSQH singularities. These are considered in Section 2, and among others we show that they are indeed isolated (see Proposition 2.4).
When obtaining normal forms of power series which are not right semi-quasi-
homogeneous, the only known classification method is due to Arnol’d, introduced
in [Arn75] (see also [AGZV85, Sec. 12.7]) over the complex numbers and slightly
generalised by Wall in [Wal99]. The method generalises semi-quasihomogeneity
and requires the principal part in\(\mathcal{P}(f)\) of the power series (with respect to some
\(C\)-polytope \(P\)) to be an isolated singularity and its Milnor algebra to have a fin-
ite regular basis — see Section 3 and 4. At the heart lies the study of piecewise
filtrations as introduced by Arnol’d [Arn75] and used by Kouchnirenko to study
non-degeneracy conditions [Kou76]. Section 3 is devoted to these. Arnol’d intro-
duced the important condition A and Wall noticed that the weaker con-
dition AA is sufficient to get normal forms. In Section 4 we generalise these condi-
tions to the situation of contact equivalence, calling them AC and AAC respective-
ly, and derive normal forms for right as well as for contact equivalence in arbitrary characteristic.

The results on normal forms and degree bounds apply to large classes of exam-
pies. In Corollary 4.11 we show that all rSQH singularities satisfy AA and all cSQH
singularities satisfy AAC. Moreover, we show that all inner Newton non-degenerate
singularities (for the definition see Remark 4.13) satisfy both AA and AAC (see
Theorem 4.14), which generalises a result of Wall. In Section 5 we then use our
results to derive normal forms for singularities of type \(T_{pq}, Q_{10}, W_{1,1}\) and \(E_7\) in
Arnol’d’s notation in positive characteristic.

2. Quasi- and Semi-Quasihomogeneous Singularities

A polynomial \(f \in K[x], x = (x_1, \ldots, x_n), n \geq 1,\) is called quasihomogeneous
with respect to the weight vector \(w \in \mathbb{Z}_{>0}^n\) if all monomials \(x^\alpha\) of \(f\) have the same
weighted degree \(d := \deg_w(x^\alpha) = w \cdot \alpha = \sum_{i=1}^n w_i \cdot \alpha_i.\) We say for short that \(f\)
is QH of type \((w; d)\). By the Euler formula a quasihomogeneous polynomial \(f\) of
weighted degree \(\deg_w(f) := d\) satisfies
\[d \cdot f = w_1 \cdot x_1 \cdot f_{x_1} + \cdots + w_n \cdot x_n \cdot f_{x_n},\]
so that
\[j(f) = t_j(f) \quad \text{if} \quad \text{char}(K) \nmid d.\]

In particular, for a quasihomogeneous polynomial \(f\) in characteristic zero the Milnor
number and the Tjurina number coincide. A famous result of K. Saito states that
over the complex numbers the converse is true as well (up to equivalence) (see
[Sai71]). One can check that his proof generalises to any algebraically closed field
of characteristic zero, a statement which we could not find in the literature.

Theorem 2.1 (Saito, [Sai71]). Let \(K\) be an algebraically closed field of characteris-
tic zero, and suppose that \(f \in K[[x]]\) is an isolated singularity. Then the following
are equivalent:

(a) \(f\) is right equivalent to a quasihomogeneous polynomial.
(b) \(f\) is contact equivalent to a quasihomogeneous polynomial.
(c) \(\mu(f) = \tau(f)\).
(d) \(f \in j(f)\).
The Milnor and the Tjurina numbers are important invariants, which even characterise the singularities for values 0 and 1 in any characteristic. In fact, let \( \text{ord}(f) := \min\{k \mid f \in m^k\} \), then, by the implicit function theorem we see easily for \( f \in m \)

\[
\mu(f) = 0 \iff \tau(f) = 0 \iff \text{ord}(f) = 1 \iff f \sim_c x_1.
\]

If \( \text{ord}(f) \geq 3 \) we have \( \mu(f) \geq \tau(f) \geq n + 1 \geq 2 \). If \( \mu(f) = 1 \), then \( \text{ord}(f) = 2 \) and we have the following lemma.

**Lemma 2.2.** For \( f \in m \) the following are equivalent:

(a) \( \mu(f) = 1 \).

(b) \( \tau(f) = 1 \).

(c) (1) If \( \text{char}(K) \neq 2 \), then \( f \sim_r x_1^2 + \cdots + x_n^2 \).

(2) If \( \text{char}(K) = 2 \), then \( n = 2k \) is even and \( f \sim_r x_1x_2 + \cdots + x_{2k-1}x_{2k} \).

(d) The same statement as in (c) for contact equivalence \( \sim_c \).

**Proof.** This follows from [GK90, 3.5, Prop. 3]. \( \square \)

The class of **quasihomogeneous singularities**, i.e. of singularities with a quasihomogeneous polynomial representative under right (or contact) equivalence, is an important class of singularities in characteristic zero.

In positive characteristic we have to be more careful, since the Euler relation is not helpful when the characteristic divides the weighted degree. E.g., \( f = x^p + y^{p-1} \) is quasihomogeneous of degree \( p \cdot (p-1) \) with respect to \( w = (p-1, p) \) with \( \tau(f) = p \cdot (p-2) \) and \( \mu(f) = \infty \) if \( \text{char}(K) = p \). However, when the characteristic does not divide the weighted degree some of the good properties still hold true.

**Proposition 2.3.** Let \( f \in K[\{x\}] \setminus K \) be QH of type \( (w; d) \) with \( \text{gcd}(w_1, \ldots, w_n) = 1 \).

(a) If \( f \in m \) then

\[
\mu(f) < \infty \iff \tau(f) < \infty \text{ and } \text{char}(K) \nmid d.
\]

In this case obviously \( \mu(f) = \tau(f) \).

(b) If \( \text{char}(K) \nmid d \) and \( g \in K[[x]] \), then

\[
f \sim_r g \iff f \sim_c g.
\]

**Proof.** (a) If the characteristic does not divide \( d \) and \( \tau(f) < \infty \) then we are done by the Euler formula. Conversely, if \( \mu(f) < \infty \) then \( \tau(f) < \infty \) and we have to show that the characteristic does not divide \( d \). Assume the contrary. The Euler formula then gives the identity

\[
w_1 \cdot x_1 \cdot f_{x_1} + \cdots + w_n \cdot x_n \cdot f_{x_n} = 0.
\]

Since \( \text{gcd}(w_1, \ldots, w_n) = 1 \) we may assume that \( w_n \) is not divisible by the characteristic, and we thus deduce

\[
x_n \cdot f_{x_n} = \frac{w_1}{w_n} \cdot x_1 \cdot f_{x_1} - \cdots - \frac{w_{n-1}}{w_n} \cdot x_{n-1} \cdot f_{x_{n-1}}.
\]

\( f \) being in \( m^3 \) the variable \( x_n \) is not zero in \( M = K[[x]] / j(f) \), so that \( f_{x_n} \) is a zero divisor in the \( K[[x]] / \langle f_{x_1}, \ldots, f_{x_{n-1}} \rangle \). Thus \( f_{x_1}, \ldots, f_{x_n} \) is not a regular
sequence in the Cohen–Macaulay ring $K[[x]]$, and therefore the $K$-algebra $M_f$ is not zero-dimensional, i.e. we get the contradiction $\mu(f) = \infty$.

(b) The proof works as in characteristic zero since $d$-th roots exist in $K[[x]]^*$ if $d$ is not divisible by char($K$), see e.g. [GLS07, Lemma 2.13]).

Note that the condition $f \in \mathfrak{m}^3$ cannot be avoided. To see this consider $f = xy \in K[[x,y]]$ with char($K$) = 2. It is QH of type $(1, 1; 2)$ and the Milnor number is one, yet the characteristic divides the weighted degree.

When classifying singularities with respect to right or contact equivalence the first classes one comes across have quasihomogeneous representatives. This is maybe the most important reason why they deserve attention. The next more complicated class of singularities are those which have a quasihomogeneous principal part that somehow governs its discrete part of the classification.

For a power series $f = \sum_{\alpha} a_{\alpha} x^\alpha \in K[[x]]$ and a weight vector $w \in \mathbb{Z}_{\geq 0}^n$ we denote by

$$in_w(f) = \sum_{\alpha \text{ minimal}} a_{\alpha} x^\alpha$$

the initial form or principal part of $f$ with respect to $w$. We call the power series $f$ right semi-quasihomogeneous rSQH (respectively contact semi-quasihomogeneous) cSQH with respect to $w$ if $\mu(in_w(f)) < \infty$ (respectively $\tau(in_w(f)) < \infty$). A right (resp. contact) equivalence class of singularities is called semi-quasihomogeneous if it has a semi-quasihomogeneous representative. Note that in characteristic zero the notions rSQH and cSQH coincide.

Moreover, in characteristic zero it is known that semi-quasihomogeneous singularities are always isolated and that their Milnor number coincides with the Milnor number of the principal part. In positive characteristic we get an analogous statement.

**Proposition 2.4.** Let $f \in K[[x]]$ and $w \in \mathbb{Z}_{\geq 0}^n$.

(a) If $\mu(in_w(f)) < \infty$ and $d = \deg_{in}(in_w(f))$, then

$$\mu(f) = \mu(in_w(f)) = \left(\frac{d}{w_1} - 1\right) \cdots \left(\frac{d}{w_n} - 1\right) < \infty.$$  

(b) $\tau(f) \leq \tau(in_w(f))$.

In particular, if $f$ is rSQH (resp. cSQH) then $f$ (resp. $R_f$) is an isolated singularity.

**Proof.** (a) Let $d$ be the degree of $in_w(f)$. Then

$$f' := \frac{f(t^{w_1}x_1, \ldots, t^{w_n}x_n)}{t^d} = in_w(f) + t \cdot g' \in K[[x, t]]$$

for some power series $g' \in K[[x, t]]$. We can use $f'$ to define the following local $K$-algebra homomorphism

$$K[[z, t]] \to K[[x, t]]: t \mapsto t, \ z_i \mapsto f'_z.$$  

This gives $K[[x, t]]$ the structure of a $K[[z, t]]$-algebra, and if we tensorise $K[[x, t]]$ with $K = K[[z, t]]/(z, t)$ we get

$$K[[x, t]] \otimes_{K[[z,t]]} K = K[[x, t]]/(f'_{z_1}, \ldots, f'_{z_n}, t) \cong K[[x]]/j(in_w(f)),$$
which by assumption is a finite dimensional $K$-vector space of dimension $\mu(\text{in}_w(f))$. Since $(f'_{x_1}, \ldots, f'_{x_n}, t)$ is a regular sequence, $K[[x, t]]$ is flat as a $K[[z, t]]$-module and thus it is free of rank $\mu(\text{in}_w(f))$ by Nakayama’s Lemma. Tensoring with $K[[t]] = K[[z, t]]/(z)$ we get that

$$K[[x, t]] \otimes_{K[[z, t]]} K[[x, t]]/(z) \cong K[[x, t]]/(f'_{x_1}, \ldots, f'_{x_n})$$

is a free $K[[t]]$-module of rank $\mu(\text{in}_w(f))$.

Passing to the field of fractions $L = K((t))$ of $K[[t]]$ we have the isomorphism of local $L$-algebras $\varphi: L[[x]] \to L[[x]]: x_i \mapsto t^{w_i}x_i$. Moreover $f' = \frac{\varphi(f)}{t}$, so that in $L[[x]]$ we have the equality of ideals $\langle f' \rangle = \langle \varphi(f) \rangle$ and $j(f') = j(\varphi(f)) = \varphi(j(f))$. Extending scalars in (1) to the field of fractions $L$ we get an isomorphism of $L$-vector spaces

$$K[[x, t]]/(f'_{x_1}, \ldots, f'_{x_n}) \otimes_{K[[t]]} L \cong L[[x]]/j(f') \cong L[[x]]/\varphi(j(f)).$$

By freeness, the left-hand side is of dimension $\mu(\text{in}_w(f))$ while the right-hand side has dimension $\mu(f)$. For the formula for $\mu(\text{in}_w(f))$ see [BGM, Prop. 3.8].

(b) It suffices to consider the case $\tau(\text{in}_w(f)) < \infty$, and the proof then is similar to (a), using the map

$$K[[z, t]] \to K[[x, t]]: t \mapsto t, \ z_i \mapsto f'_{x_i}$$

for $i = 1, \ldots, n$, where $z = (z_0, \ldots, z_n)$. Since $(f', f'_{x_1}, \ldots, f'_{x_n}, t)$ is not a regular sequence, $K[[x, t]]/(f', f'_{x_1}, \ldots, f'_{x_n})$ is finitely generated but may have torsion as a $K[[t]]$-module. Tensoring this module with $K$ over $K[[t]]$ gives a $K$-vector space of dimension $\tau(\text{in}_w(f))$. Tensoring with $L = K((t))$ over $K[[t]]$ kills the torsion and gives an $L$-vector space of dimension $\tau(f) \leq \tau(\text{in}_w(f))$. \hfill \square

Note that the condition on the finiteness of $\mu(\text{in}_w(f))$ in Proposition 2.4 (a) cannot be avoided, and $\tau(f)$ will in general not coincide with $\tau(\text{in}_w(f))$. Moreover, if $f \in m^d$ we get from Proposition 2.3 that $\mu(\text{in}_w(f)) < \infty$ is equivalent to $\tau(\text{in}_w(f)) < \infty$ and char($K$) $\mid d = \deg_w(\text{in}_w(f))$.

3. PIECEWISE FILTRATIONS AND GRADED ALGEBRAS

Fixing a weight vector $w \in \mathbb{Z}^n_{\geq 0}$ we get in a natural way a filtration on $K[[x]]$. If a singularity is semi-quasihomogeneous with respect to $w$ then this filtration is perfectly suited to study the singularity and in general $w$ singles out a unique facet of the Newton diagram of the defining power series. However, in general we will have to consider more complicated filtrations since there is no single facet of the Newton diagram which captures enough information on the singularity. This was noted by Arnold and he introduced in [Arn75] piecewise filtrations which are used to study non-degeneracy conditions by Kouchnirenko in [Kou76].

Given weight vectors $w_i \in \mathbb{Q}^n_{\geq 0}$ with positive entries, $i = 1, \ldots, k$, they define linear functions

$$\lambda_i: \mathbb{R}^n \to \mathbb{R}: r \mapsto w_1 \cdot r + \sum_{j=1}^n w_{i,j} \cdot r_j,$$
and their minimum defines a convex piecewise linear function

\[ \lambda : \mathbb{R}^n \to \mathbb{R} : r \mapsto \min\{\lambda_1(r), \ldots, \lambda_k(r)\}. \]

We will always assume that the set of weights is irredundant, i.e., that none of the \( \lambda_i \) is superfluous in the definition of \( \lambda \). The set

\[ P_\lambda = \{r \in \mathbb{R}^n_{\geq 0} | \lambda(r) = 1\} \]

is a compact rational polytope of dimension \( n - 1 \) in the positive orthant \( \mathbb{R}^n_{\geq 0} \), and its facets are given by

\[ \Delta_i = \{r \in P_\lambda | \lambda_i(r) = 1\}. \]

\( P_\lambda \) has the property that each ray in \( \mathbb{R}^n_{\geq 0} \) emanating from the origin meets \( P_\lambda \) in precisely one point and that the region in \( \mathbb{R}^n_{\geq 0} \) lying above \( P_\lambda \) is convex. Following the convention of Wall (see [Wal99]) we call such polytopes \( C \)-polytopes. Thus, irredundant sets of weight vectors define \( C \)-polytopes.

Conversely, given a \( C \)-polytope \( P \) the suitably scaled inner normal vectors of its facets define an irredundant set of weight vectors such that \( P = P_\lambda \) for the corresponding piecewise linear function \( \lambda \). We denote by \( \lambda_P \) the piecewise linear function defined by \( P \), and by \( \lambda_\Delta \) the linear function corresponding to a facet \( \Delta \) of \( P \).

To each power series \( f = \sum a_\alpha x^\alpha \in K[[x]] \) we can associate its Newton polyhedron \( \Gamma_+(f) \) as the convex hull of the set \( \bigcup_{\alpha \in \text{supp}(f)} (\alpha + \mathbb{R}^n_{\geq 0}) \), where \( \text{supp}(f) = \{\alpha | a_\alpha \neq 0\} \) denotes the support of \( f \). Following Arnol’d we call the union \( \Gamma(f) \) of its compact faces the Newton diagram of \( f \), some authors call it the Newton polytope resp. the Newton polygon if \( n = 2 \). Note that the Newton diagram of a QH or SQH polynomial has exactly one facet, where a facet is a face of dimension \( n - 1 \). For later use we denote by \( \Gamma_-(f) \) the union of line segments joining points on \( \Gamma(f) \) with the origin. (See Figure 1 for an example.)

![Figure 1](image-url)  

\( \Gamma_+(f) \)  \( \Gamma(f) \)  \( \Gamma_-(f) \)

**Figure 1.** The Newton diagram of \( x \cdot (y^4 + xy^3 + x^2y^2 - x^3y^2 + x^6) \).

Note that \( \Gamma(f) \) is a \( C \)-polytope if and only if \( f \in K[[x]] \) is a convenient power series, i.e. if the support of \( f \) contains a point on each coordinate axis. \( C \)-Polytopes should thus be thought of as generalising the Newton diagram, and in our applications they will basically arise by extending Newton diagrams of non-convenient power series in a suitable way.

For a \( C \)-polytope \( P \) we denote by \( N_P \) the lowest common multiple of the denominators of all entries in the weight vectors corresponding to \( P \), so that \( N_P \cdot \lambda_P \)
takes non-negative integer values on \( \mathbb{N}^n \). We then define a piecewise valuation on \( K[[x]] \) by
\[
v_P(f) := \min\{N_P \cdot \lambda_P(\alpha) \mid a_\alpha \neq 0\} \in \mathbb{N}
\]
for \( f = \sum_\alpha a_\alpha x^\alpha \in K[[x]] \) and \( v_P(0) := \infty \). \( v_P \) satisfies
\[
v_P(f \cdot g) \geq v_P(f) + v_P(g) \quad \text{and} \quad v_P(f + g) \geq \min\{v_P(f), v_P(g)\}.
\]
Indeed we have
\[
v_P(f \cdot g) = v_P(f) + v_P(g) \iff v_P(f) = v_\Delta(f) \text{ and } v_P(g) = v_\Delta(g)
\]
for some facet \( \Delta \) of \( P \). The sets
\[
F_d := F_{P,d} := \{ f \in K[[x]] \mid v_P(f) \geq d \}
\]
with \( d \in \mathbb{N} \) are thus ideals in \( K[[x]] \) and satisfy \( F_d \cdot F_e \subseteq F_{d+e} \), i.e., they form a filtration on \( K[[x]] \). Note also that \( F_0 = K[[x]] \) and \( F_1 = m \). Moreover, since all weight vectors corresponding to \( P \) have only positive entries for each \( d \) there is a positive integer \( m \) such that
\[
m^m \subseteq F_d,
\]
and also for any \( k \) there is a \( d \) such that each monomial of valuation degree \( d \) has a degree bigger than or equal to \( k \), i.e., such that
\[
F_d \subseteq m^k.
\]
Given any \( C \)-polytope \( P \) and a power series \( f \in K[[x]] \), we call the polynomial
\[
\text{in}_P(f) = \sum_{\lambda_P(\alpha) \text{ minimal}} a_\alpha x^\alpha
\]
the initial form or the principal part of \( f \) with respect to \( P \). \( f \) is said to be piecewise homogeneous PH of degree \( d \in \mathbb{Q}_{\geq 0} \) with respect to \( P \) if \( \lambda_P(\alpha) = d \) for all \( \alpha \in \text{supp}(f) \). Note then that \( f = \text{in}_P(f) \) is a polynomial. The power series \( f \) is called right semi-piecewise homogeneous rSPH (respectively contact semi-piecewise homogeneous) cSPH with respect to \( P \) if \( \mu(\text{in}_P(f)) < \infty \) (respectively \( \tau(\text{in}_P(f)) < \infty \)). Even though PH, rSPH and cSPH are straightforward generalisations of QH, rSQH and cSQH things get more complicated. One of the reasons is that the product of two PH polynomials need no longer be so, as Example 3.1 shows.

**Example 3.1.** Consider the weights \( w_1 = (1, 2) \) and \( w_2 = (3, 1) \) together with the polynomials \( f = x^7 + y^7 \) and \( g = x \). The corresponding \( C \)-polytope \( P \) is the black polygon shown in Figure 2. Both \( f \) and \( g \) are PH with respect to \( P \), of degrees 7 and 1. However,
\[
\text{in}_P(f \cdot g) = \text{in}_P(f^8) = x^8 \neq x^8 + xy^7 = f \cdot g \in F_{P,8},
\]
so that \( f \cdot g \) is no longer piecewise homogeneous.

This example shows also that there cannot be any monomial ordering \( > \) (see [GP08]) which refines the piecewise degree with respect to \( P \) if \( P \) has more than one side. In fact, suppose there is, then either \( x^7 \) or \( y^7 \) is the leading term of \( f \). However, since \( > \) refines the piecewise degree, \( xf \) definitely will have leading term \( x^8 \) and \( yf \)
will have leading term $y^8$, in contradiction to the fact that the leading term must be compatible with the multiplication by monomials. This makes computations with piecewise filtrations difficult, in particular, we cannot use Gröbner basis methods directly.

We also should like to point out that a polynomial can be PH with respect to many different $C$-polytopes. E.g. consider for $f = x^5 + x^2y^2 + y^5$ the two $C$-polytopes shown in Figure 3.

If $I \subseteq K[[x]]$ is an ideal in $K[[x]]$ and $P$ is a $C$-polytope then the filtration induced by $P$ on $K[[x]]$ leads to the filtration

$$F_0 + I/I \supseteq F_1 + I/I \supseteq F_2 + I/I \supseteq \cdots$$

on $K[[x]]/I$, and induces thus the associated graded $K$-algebra

$$\text{gr}_P(K[[x]]/I) = \bigoplus_{d \in \mathbb{N}} (F_d + I)/(F_{d+1} + I)) \cong \bigoplus_{d \in \mathbb{N}} F_d/((I \cap F_d) + F_{d+1}).$$

The product of the classes of two monomials $x^\alpha$ and $x^\beta$ in $\text{gr}_P(K[[x]]/I)$ satisfies

$$x^\alpha \cdot x^\beta = \begin{cases} x^{\alpha+\beta}, & \text{if } v_P(x^{\alpha+\beta}) = v_P(x^\alpha) + v_P(x^\beta), \\ 0, & \text{else.} \end{cases} \quad (5)$$

We will show next that there are isomorphisms of vector spaces,

$$M_f \cong \text{gr}_P(M_f), \quad T_f \cong \text{gr}_P(T_f),$$

if the graded algebras are finite dimensional. Therefore, these graded algebras are natural means to study the singularity defined by $f$. Arnol’d [Arn75] has shown how to use a monomial basis of $\text{gr}_P(M_f)$ under suitable conditions on $f$ to compute a normal form for $f$. We will generalise this in Section 4.
**Proposition 3.2.** Let \( I \subseteq K[[x]] \) be an ideal and let \( P \) be a \( C \)-polytope.

(a) Then \( \dim_K(\text{gr}_P(K[[x]])/I) = \dim_K(K[[x]]/I) \).
(b) If \( \dim_K(\text{gr}_P(K[[x]])/I) < \infty \), then any monomial basis of \( \text{gr}_P(K[[x]])/I \) is a basis \( K[[x]]/I \) as \( K \)-vector space.

**Proof.** (a) The sequence of ideals

\[
K[[x]] = F_0 + I \supseteq F_1 + I \supseteq \cdots \supseteq F_d + I \supseteq F_{d+1} + I \supseteq \cdots \supseteq I
\]

shows that \( \dim_K(K[[x]]/I) < \infty \) if and only if there are only finitely many \( d \) such that \( 0 < \dim_K(F_d + I/F_{d+1} + I) < \infty \); this is equivalent to \( \dim_K(\text{gr}_P(K[[x]])/I) < \infty \). In this case the dimensions obviously coincide.

(b) Let \( B \) be any set of monomials whose residue classes in \( \text{gr}_P(K[[x]])/I \) form a \( K \)-vector space basis. We have to show that the residue classes of the elements of \( B \) in \( K[[x]]/I \) generate \( K[[x]]/I \) as a \( K \)-vector space.

Let \( f \in K[[x]] \) be given and let \( d = v_P(f) \) be its piecewise valuation. We then can write \( f \) as

\[
f = \sum_{\nu_P(x^\alpha) = d} c_\alpha x^\alpha + g_d + h_d
\]

with \( c_\alpha \in K, \ g_d \in I \cap F_d \) and \( h_d \in F_{d+1} \).

We continue with \( h_d \) in the same way, and thus for any \( k \geq d \) there are \( c_\alpha \in K, \ g_k \in I \cap F_k \) and \( h_k \in F_{k+1} \) such that

\[
f = \sum_{d \leq v_P(x^\alpha) \leq k} c_\alpha x^\alpha + (g_d + g_{d+1} + \cdots + g_k) + h_k,
\]

where \( g = g_d + g_{d+1} + \cdots + g_k \in I \). Since \( \dim_K(\text{gr}_P(K[[x]])/I) < \infty \), there is a \( d_0 \) such that \( (I \cap F_d) + F_{d+1} = F_d \) for all \( d \geq d_0 \), i.e., \( B \cap F_{d_0} = \emptyset \). For \( k \geq d_0 \) we thus have

\[
f = \sum_{d \leq v_P(x^\alpha) < d_0} c_\alpha x^\alpha = (g_d + g_{d+1} + \cdots + g_k) + h_k \in I + F_{k+1},
\]

where the left-hand side does not depend on \( k \). Using Krull’s Intersection Theorem this shows that

\[
f = \sum_{d \leq v_P(x^\alpha) < d_0} c_\alpha x^\alpha \in \bigcap_{k \geq 0} (I + F_k) \overset{(3),(4)}{=} \bigcap_{k \geq 0} (I + \mathfrak{m}^k) = I,
\]

and hence the claim. \( \square \)

If we apply Proposition 3.2 to \( M_f \) and \( T_f \) we get the following corollary.

**Corollary 3.3.** Let \( f \in K[[x]] \) be a power series and \( P \) a \( C \)-polytope.

(a) \( \mu(f) = \dim_K(\text{gr}_P(M_f)) \).
(b) \( \tau(f) = \dim_K(\text{gr}_P(T_f)) \).
For any $C$-polytope $P$ the piecewise valuation $v_P$ on $K[[x]]$ can easily be extended to the $K[[x]]$-module $\text{Der}_K(K[[x]])$ of derivations on $K[[x]]$. For this we define

$$v_P(\xi) = \min\{\lambda_P(\alpha - e_i) \mid a_{i,\alpha} \neq 0\},$$

where

$$\xi = \sum_{i=1}^{n} \sum_{\alpha \in \mathbb{N}^n} a_{i,\alpha} \cdot x^\alpha \cdot \partial_{x_i} \neq 0$$

and where $e_i$ is the $i$-th standard basis vector of $\mathbb{Z}^n$, i.e., we naturally extend $v_P(x^\alpha \partial_{x_i}) = \lambda_P(\alpha - e_i)$ to all derivations where the derivation $\partial_{x_i}$ lowers the exponent of $x_i$ in $x^\alpha$ by one. Note that $v_P(\partial_{x_i})$ is negative.

Straightforward computations show that $v_P$ then satisfies

$$v_P(\xi f) \geq v_P(\xi) + v_P(f) \quad (6)$$

for any $0 \neq f \in K[[x]]$ and any $0 \neq \xi \in \text{Der}_K(K[[x]])$. Moreover (see [Arn75, Lemma 6.6]), if $f \in m^2$ and $g_1, \ldots, g_n \in m$ with $v_P(g_i) > v_P(x_i)$ then $\varphi: K[[x]] \to K[[x]]$: $x_i \mapsto x_i + g_i$ is an isomorphism and

$$\varphi(f) = f + \xi f + h, \quad (7)$$

where

$$\xi = \sum_{i=1}^{n} g_i \partial_{x_i} \quad \text{and} \quad v_P(h) > v_P(\xi) + v_P(f).$$

The fact that we do not always have $v_P(\xi f) = v_P(\xi) + v_P(f)$ is somewhat annoying and forces us to adapt the filtrations induced by $v_P$ on the ideals

$$j(f) = \{\xi f \mid \xi \in \text{Der}_K(K[[x]])\}$$

and

$$tj(f) = \{g \cdot f + \xi f \mid g \in K[[x]], \xi \in \text{Der}_K(K[[x]])\}.$$

In the following definitions we will restrict our attention to $j(f) \cap F_d$ and $tj(f) \cap F_d$ to those elements whose valuation is expected to be at least $d$, avoiding those who do so simply by bad luck.

**Definition 3.4.** (a) For $d \geq 0$ we define the ideals

$$j^d_P(f)_d := \{\xi f \mid v_P(\xi) + v_P(f) \geq d\} \subset K[[x]]$$

and

$$tj^AC_P(f)_d := \{g \cdot f + \xi f \mid \min\{v_P(g) + v_P(f), v_P(\xi) + v_P(f)\} \geq d\} \subset K[[x]].$$

(b) Replacing $j(f) \cap F_d$ resp. $tj(f) \cap F_d$ in the definition of $\text{gr}_P(M_f)$ resp. $\text{gr}_P(T_f)$ by $j^d_P(f)_d$ resp. $tj^AC_P(f)_d$ we define the graded $K$-algebras

$$\text{gr}_P^d(M_f) := \bigoplus_{d \geq 0} F_d/(j^d_P(f)_d + F_{d+1})$$

and

$$\text{gr}_P^{AC}(T_f) := \bigoplus_{d \geq 0} F_d/(tj^AC_P(f)_d + F_{d+1}).$$
(c) A monomial basis of \( \text{gr}_A^A(M_f) \) (respectively \( \text{gr}_A^{AC}(T_f) \)) is called a regular basis for \( M_f \) (respectively \( T_f \)).

**Remark 3.5.** (a) These notions were introduced for \( M_f \) and \( j(f) \) implicitly by Arnol’d [Arn75] and explicitly by Wall [Wal99]. We introduce in this paper the modification to \( T_f \) and \( tj(f) \) in order to treat contact equivalence. The upper index \( A \) refers to Arnol’d’s condition \( A \) and \( AC \) to our condition \( AC \) (see Definition 4.1).

(b) We obviously have the inclusions

\[
\begin{align*}
\text{tj}(f) \cap F_d & \supseteq \text{tj}_P^{AC}(f)_d \supseteq \text{j}_P(f)_d \subseteq \text{j}(f) \cap F_d, \\
\end{align*}
\]

and hence canonical surjections

\[
\begin{align*}
\text{gr}_A^A(M_f) & \twoheadrightarrow \text{gr}_P(M_f), \\
\text{gr}_A^A(T_f) & \twoheadrightarrow \text{gr}_P(T_f).
\end{align*}
\]

Due to Proposition 3.2 this yields together with Corollary 3.3 the following result.

**Corollary 3.6.** Let \( f \in K[[x]] \) be a power series and let \( P \) be a \( C \)-polytope.

(a) Any monomial basis \( B \) of \( \text{gr}_A^A(M_f) \) generates \( \text{gr}_P(M_f) \), and if \( \mu(f) < \infty \) then \( B \) also generates \( M_f \). In particular,

\[
\mu(f) = \dim_K(\text{gr}_P(M_f)) \leq \dim_K(\text{gr}_A^A(M_f)).
\]

(b) Any monomial basis \( B \) of \( \text{gr}_A^{AC}(T_f) \) generates \( \text{gr}_P(T_f) \), and if \( \tau(f) < \infty \) then \( B \) also generates \( T_f \). In particular,

\[
\tau(f) = \dim_K(\text{gr}_P(T_f)) \leq \dim_K(\text{gr}_A^{AC}(T_f)).
\]

We should point out that the finiteness of \( \mu(f) \) or \( \tau(f) \) does not suffice in general to guarantee the finite dimensionality of \( \text{gr}_P(M_f) \) or \( \text{gr}_P^{AC}(T_f) \). The reason for this is that elements of valuation \( d \) in \( j(f) \) or \( \text{tj}(f) \) may not be contained in \( \text{tj}_P(f)_d \) or \( \text{tj}_P^{AC}(f)_d \), as in the following example.

**Example 3.7.** \((T_2)-\text{Singularity in characteristic} 2\). Let \( \text{char}(K) = 2 \) and let the \( C \)-polytope \( P \) be defined by the weights \( w_1 = (4, 6) \) and \( w_2 = (5, 5) \). The polynomial \( f = x^2 + x^2 y^2 + y^4 \) is \( \text{PH} \) of degree 20 with respect to \( P \) with Tjurina number \( \tau(f) = 16 \). For \( n \geq 4 \) we have

\[
y^{4n} = g \cdot f + \xi f \in \text{tj}(f),
\]

where \( g = y^{4n-4} + x^2 y^{4n-6} + x^4 y^{4n-8} \) is \( \text{QH} \) of degree \( 20n - 20 = v_p(y^{4n}) - v_p(f) \) and \( \xi = (x \cdot g + x^2 y^{4n-6}) \cdot \partial_x \). Thus \( g \) guarantees that \( y^{4n} \) is indeed in \( \text{tj}_P^{AC}(f)_{20n} \), however,

\[
v_p(\xi) = v_p(x^2 y^{4n-6} \partial_x) = 20n - 25 < v_p(y^{4n}) - v_p(f)
\]

has a valuation which is too small. Moreover, we cannot do any better, i.e., \( y^{4n} \notin \text{tj}_P^{AC}(f) + F_{20n+1} \) and thus \( \dim_K(\text{gr}_P^{AC}(T_f)) = \infty \). See also Example 4.12.

The following lemma shows that \( \text{gr}_P^A(M_f) \) and \( \text{gr}_P^{AC}(T_f) \) depend only on the initial part of \( f \) w.r.t. the \( C \)-polytope \( P \).

**Lemma 3.8.** If \( P \) is a \( C \)-polytope and \( f \in K[[x]] \) then

\[
\text{gr}_P^A(M_f) = \text{gr}_P^A(M_{\text{in}_P(f)}) \quad \text{and} \quad \text{gr}_P^{AC}(T_f) = \text{gr}_P^{AC}(T_{\text{in}_P(f)}).
\]
Proof. For this we write $f = \text{in}_P(f) + h$ for some $h \in K[[x]]$ with $v_P(h) \geq v_P(f) + 1$, and we note that for any $\xi \in \text{Der}_K(K[[x]])$

$$v_P(\xi h) \geq v_P(\xi) + v_P(h) \geq v_P(\xi) + v_P(f) + 1. \quad (8)$$

In order to show $\text{gr}^A_P(M_f) = \text{gr}^A_P(M_{\text{in}_P(f)})$ we have to show

$$j^A_P(f)_d + F_{d+1} = j^A_P(\text{in}_P(f))_d + F_{d+1} \quad (9)$$

for all $d \geq 0$.

If $g \in j^A_P(f)_d$ then there is a derivation $\xi$ such that $g = \xi f$ with

$$d \leq v_P(\xi) + v_P(f) = v_P(\xi) + v_P(\text{in}_P(f)).$$

In view of (8) we thus have

$$g = \xi \text{in}_P(f) + \xi h \in j^A_P(\text{in}_P(f))_d + F_{d+1},$$

which shows that the left-hand side in (9) is contained in the right hand side.

On the other hand, if $g \in j^A_P(\text{in}_P(f))_d$ then there is a derivative $\xi$ such that $g = \xi \text{in}_P(f)$ with

$$d \leq v_P(\xi) + v_P(\text{in}_P(f)) = v_P(\xi) + v_P(f).$$

Again, in view of (8) we thus have

$$g = \xi f - \xi h \in j^A_P(\text{in}_P(f))_d + F_{d+1},$$

which shows that the left-hand side in (9) is contained in the right hand side.

The proof for $\text{gr}^{AC}_P(T_f) = \text{gr}^{AC}_P(T_{\text{in}_P(f)})$ works analogously. \hfill \blacksquare

**Corollary 3.9.** Let $f \in K[x]$ be QH of type $(w; d)$ and let $P$ be the C-polytope defined by the single weight vector $w$.

(a) Then $\Gamma(f) \subseteq P$, $\text{gr}^A_P(M_f) = \text{gr}_P(M_f)$ and $\text{gr}^{AC}_P(T_f) = \text{gr}_P(T_f)$

(b) If moreover $\mu(f) < \infty$ (respectively $\tau(f) < \infty$), then a set of monomials is a $K$-vector space basis for $\text{gr}^A_P(M_f)$ (respectively for $\text{gr}^{AC}_P(T_f)$) if and only if it is one for $M_f$ (respectively for $T_f$).

**Proof.** Since $P$ has only one side it induces a grading on $K[x]$ and a filtration on $K[[x]]$.

(a) We note that the partial derivative $f_{x_i}$ is QH of type $(w; d - w_i)$ if it does not vanish. Thus the ideals $i(f)$ and $tj(f)$ are generated by weighted homogeneous elements. This implies that $(j(f) \cap F_k) + F_{k+1} = j^A_P(f)_k + F_{k+1}$ and $(tj(f) \cap F_k) + F_{k+1} = tj^{AC}_P(f)_k + F_{k+1}$ for all $k \geq 0$ as required.

(b) By Corollary 3.6 any monomial basis $B$ of $\text{gr}^A_P(M_f)$ is a generating set of $M_f$. However, by (a) and Corollary 3.3 these vector spaces have the same dimension. Hence, $B$ is a basis of $M_f$.

For the converse we note that $j(f)$ is generated by weighted homogeneous polynomials. If $B$ is a monomial basis $M_f$ and $x^\beta$ is any monomial, then there are $c_\alpha \in K$ and a $g \in j(f)$ such that

$$x^\beta = \sum_{x^\alpha \in B} c_\alpha \cdot x^\alpha + g,$$
and all \( x^a \) as well as \( g \) are weighted homogeneous polynomials of the same weighted degree as \( x^b \). In particular \( g \in j(f) \cap F_d \) with \( d = v_P(x^b) \), and thus \( x^b \) is a linear combination of the elements of \( B \) in \( \text{gr}_P(M_f) \). This shows that \( B \) generates \( \text{gr}_P(M_f) = \text{gr}_A^P(M_f) \), and since \( M_f \) and \( \text{gr}_P(M_f) \) have the same dimension by Corollary 3.3 \( B \) must be a basis of \( \text{gr}_A^P(M_f) \).

The proof for \( \text{gr}_A^C(T_f) \) and \( T_f \) works in the same way. \( \Box \)

4. Normal Forms

To obtain normal forms of power series which are not right semi-quasihomogeneous, the only known method was introduced by Arnol’d in \[\text{Arn75}\] over the complex numbers and slightly generalised by Wall in \[\text{Wal99}\]. It requires the principal part in \( P(f) \) of the power series (with respect to some \( C \)-polytope \( P \)) to be an isolated singularity and its Milnor algebra to have a finite regular basis. Arnol’d actually gives a more restrictive condition but his proof shows that this suffices as was pointed out by Wall. We generalise Arnol’d’s condition both in the strict and in the weak form to the situation of contact equivalence and derive normal forms for right as well as for contact equivalence in arbitrary characteristic.

**Definition 4.1.** Let \( P \) be a \( C \)-polytope and let \( f \in K[[x]] \) be a power series.

(a) We say that \( f \) satisfies condition A w.r.t. \( P \), if for any \( g \in j(f) \) there exists a derivation \( \xi \in \text{Der}_K(K[[x]]) \) such that
\[
v_P(g) = v_P(\xi) + v_P(f) < v_P(g - \xi f).
\]
(b) \( f \) satisfies condition AA w.r.t. \( P \), if \( \dim_K(\text{gr}_A^P(M_f)) < \infty \).
(c) We say \( f \) satisfies condition AC w.r.t. \( P \), if for all \( h \in \text{tj}(f) \) there exists a \( g \in K[[x]] \) and a derivation \( \xi \in \text{Der}_K(K[[x]]) \) such that
\[
v_P(h) = \min\{v_P(g) + v_P(f), v_P(\xi) + v_P(f)\} < v_P(h - g \cdot f - \xi f).
\]
(d) Finally, we say \( f \) satisfies condition AAC w.r.t. \( P \), if \( \dim_K(\text{gr}_A^C(T_f)) < \infty \).

**Remark 4.2.** (a) Condition A is due to Arnol’d and is equivalent to
\[
(j(f) \cap F_d) + F_{d+1} = \text{gr}_P^A(f)_d + F_{d+1}, \quad \text{for all } d \geq 0,
\]
i.e., if
\[
\text{gr}_P(M_f) = \text{gr}_A^P(M_f).
\]
i.e., if
\[
\mu(f) = \dim_K(\text{gr}_A^P(M_f)).
\]
(b) Condition AA (AA stands for almost A), which is obviously weaker than A, is due to Wall \[\text{Wal99}\] and says that the Milnor algebra of \( f \) has a regular basis.
(c) While condition A is meant to deal with right equivalence, our condition AC deals with contact equivalence. AC is equivalent to
\[
(\text{tj}(f) \cap F_d) + F_{d+1} = \text{tj}_A^C(f)_d + F_{d+1} \quad \text{for all } d \geq 0,
\]
i.e., if
\[
\text{gr}_P(T_f) = \text{gr}_A^C(T_f).
i.e., if
\[ \tau(f) = \dim_K(\text{gr}^{AC} P(T_f)). \]

(d) AAC means *almost AC* and says that the Tjurina algebra of \( f \) has a regular basis.

**Remark 4.3.** The above equivalence of the characterisations of condition A and AC uses Corollary 3.6. Corollary 3.3 and 3.6 together with Example 5.9 show that for isolated singularities the *almost* conditions are indeed strictly weaker. Moreover,
\[ \mu(f) < \infty \text{ and } f \text{ satisfies } A \implies f \text{ satisfies } \text{AA} \implies \mu(f) < \infty \]  \( (10) \)
and
\[ \tau(f) < \infty \text{ and } f \text{ satisfies } AC \implies f \text{ satisfies } \text{AAC} \implies \tau(f) < \infty. \]  \( (11) \)

We point out that by Lemma 3.8
\[ f \text{ satisfies } \text{AA resp. } \text{AAC} \iff \text{ in}_P(f) \text{ satisfies } \text{AA resp. } \text{AAC}, \]  \( (12) \)
i.e., the conditions AA and AAC only depend on the principal part of \( f \).

We now formulate our main result on normal forms without referring to the conditions AA or AAC, and we will see later how they come in useful. The statement for right equivalence in this form without referring to condition A or AA was first stated over the complex numbers in [Wal99, Theorem 2.1], but was already proved by Arnol’d in [Arn75, Theorem 9.5]. We generalise the statement to contact equivalence and give a different proof which works for any algebraically closed field making an “Ansatz” with power series. Recall that \( \text{ord}(f) \) denotes the order of the power series \( f \).

**Theorem 4.4** (Normal forms with respect to right equivalence). Let \( f \in m, P \) be a \( C \)-polytope and \( B = \{x^\alpha \mid \alpha \in \Lambda \} \) a regular basis for \( M_{\text{in}_P(f)} \).

If \( m^{k+2} \leq m^2 \cdot j(f) \) then
\[ f \sim_c \text{ in}_P(f) + \sum_{\alpha \in \Lambda_f} c_\alpha x^\alpha. \]  \( (13) \)
for suitable \( c_\alpha \in K \), where \( \Lambda_f \) is the finite set
\[ \Lambda_f = \{ \alpha \in \Lambda \mid \text{deg}(x^\alpha) \leq 2k - \text{ord}(f) + 2, v_P(x^\alpha) \geq v_P(f - \text{in}_P(f)) \}. \]

**Theorem 4.5** (Normal forms with respect to contact equivalence). Let \( f \in m, P \) be a \( C \)-polytope and \( B = \{x^\alpha \mid \alpha \in \Lambda \} \) a regular basis for \( T_{\text{in}_P(f)} \).

If \( m^{k+2} \leq m \cdot \langle f \rangle + m^2 \cdot j(f) \) then
\[ f \sim_c \text{ in}_P(f) + \sum_{\alpha \in \Lambda_f} c_\alpha x^\alpha. \]  \( (14) \)
for suitable \( c_\alpha \in K \), where \( \Lambda_f \) is the finite set
\[ \Lambda_f = \{ \alpha \in \Lambda \mid \text{deg}(x^\alpha) \leq 2k - \text{ord}(f) + 2, v_P(x^\alpha) \geq v_P(f - \text{in}_P(f)) \}. \]

We will only prove Theorem 4.5 since the proof of Theorem 4.4 works along the same lines.
Proof of Theorem 4.5. In the proof we will write $f_P$ instead of $\text{in}_P(f)$ to shorten the notation. The basic idea is to construct a finite sequence $(f_i)_{i=0}^m$ with $f_0 = f$ such that $f_i \sim_c f$ for all $i$ and that

$$f_m \equiv f_P + \sum_{\alpha \in \Lambda_f} c_\alpha x^\alpha \pmod{m^{2k - \text{ord}(f) + 3}}.$$  

We try to do so by eliminating terms in $f$ (piecewise) degree by degree. If we succeed then by [BGM, Theorem 2.1] we have

$$f \sim_c f_m \sim_c f_P + \sum_{\alpha \in \Lambda_f} c_\alpha x^\alpha$$

as desired since $2k - \text{ord}(f) + 2$ is a bound for the determinacy of $f$.

We start our construction by denoting by $g = \text{in}_P(f - f_P)$ the principal part of $g_1 = f - f_P$ with respect to $P$ and setting $h = f - f_P - g$. If we set $d_0 = v_P(f) = v_P(f_P)$ and $d_1 = v_P(f - f_P) > d_0$ then $f_P \in F_{d_0}$ is PH of degree $d_0$, $g \in F_{d_1}$ is PH of degree $d_1$ and $h \in F_{d_1+1}$. Moreover, since $B$ is a $K$-vector space basis of $\text{gr}_P^A(T_{f_P})$ we have

$$g = \sum_{\alpha \in \Lambda \text{ for suitable } c_\alpha \in K, b_0 \in K[[x]], \xi = \sum_{i=1}^n b_i \cdot \partial_i \in \text{Der}_K(K[[x]]), \text{and } h' \in K[[x]]} c_\alpha x^\alpha + b_0 \cdot f_P + \xi f_P + h'$$

satisfying

$$d_1 = \min\{v_P(b_0) + d_0, v_P(\xi) + d_0\} \quad \text{and} \quad v_P(h') > d_1. \quad (15)$$

Form (15) we deduce that $v_P(b_0) \geq d_1 - d_0 > 0$ and thus

$$b_0 \in \mathfrak{m}, \quad (16)$$

and also

$$v_P(b_i) - v_P(x_i) = v_P(b_i \partial_{x_i}) \geq v_P(\xi) \geq d_1 - d_0 > 0. \quad (17)$$

From (7) we know that then

$$\varphi : K[[x]] \to K[[x]] : x_i \mapsto x_i - b_i$$

is a local $K$-algebra isomorphism of $K[[x]]$.

Moreover, applying (7) to $\varphi(f_P)$ and to $\varphi(g)$ we get

$$\varphi(f) = \varphi(f_P) + \varphi(g) + \varphi(h)$$

$$= f_P - \xi f_P + h_1 + g - \xi g + h_2 + \varphi(h)$$

$$= (1 + b_0) \cdot f_P + \sum_{\alpha \in \Lambda \text{ for suitable } c_\alpha \in K, h_1, h_2, \varphi(h) \in F_{d_1+1} \text{ taking (6) and (17) into account.}} c_\alpha x^\alpha = (1 + b_0) \cdot f_P + \sum_{\alpha \in \Lambda \text{ for suitable } c_\alpha \in F_{d_1+1} \text{ taking (6) and (17) into account.}} c_\alpha x^\alpha + h_2 + \varphi(h) - \xi g$$

with $h', h_1, h_2, \varphi(h), \xi g \in F_{d_1+1}$, taking (6) and (17) into account.

Since by (16) $b_0 \in \mathfrak{m}$ we can multiply by the inverse of the unit $1 + b_0$, which is of the form $1 + b$ with $v_P(b) \geq d_1 - d_0 > 0$, so that we get

$$(1 + b) \cdot \varphi(f) = f_P + \sum_{\alpha \in \Lambda \text{ for suitable } c_\alpha \in K, h_1, h_2, \varphi(h) \in F_{d_1+1} \text{ taking (6) and (17) into account.}} c_\alpha x^\alpha + g_2 \quad \text{with } v_P(g_2) > d_1.$$
Setting \( f_1 = (1 + b) \cdot \varphi(f) \) we have \( f_1 \sim_c f \), and we can go on inductively treating \( g_2 \) as we have treated \( g_1 = f - f_P \) before. That way is how we construct power series

\[
f_m = f_P + \sum_{\alpha \in \Lambda, d_1 \leq \lambda_P(\alpha) \leq d_m} c_\alpha x^\alpha + g_{m+1}
\]

with \( g_{m+1} \in F_{d_m} \) and \( d_1 < d_2 < \cdots < d_m \), \( m \geq 0 \). By (4) we eventually have that \( F_{d_m} \subseteq m^{d_{k-\ord(f)+3}} \) and we are done. \( \square \)

Theorem 4.4 has as an easy corollary a result of Arnol’d which he proved over the complex numbers in [Arn75, Theorem 9.5], even though he used condition \( \Lambda \) and \( \mu(\text{in}_P(f)) < \infty \).

**Corollary 4.6 (Normal forms for right equivalence).** Let \( P \) be a \( C \)-polytope and \( f \in \mathfrak{m} \) be a power series such that \( \text{in}_P(f) \) satisfies AA then \( f \) is finitely right determined, \( rSPH \) and

\[
f \sim_r \text{in}_P(f) + \sum_{x^\alpha \in B, v_P(x^\alpha) > d} c_\alpha x^\alpha
\]

for suitable \( c_\alpha \in \mathbb{K} \), where \( B \) is a finite regular basis for \( T_{\text{in}_P(f)} \) and \( d = v_P(\text{in}_P(f)) \).

**Proof.** If \( \text{in}_P(f) \) satisfies AA then \( f \) does so as well by Lemma 3.8. By (10) thus \( \mu(f) \) is finite and therefore \( j(f) \) contains some power of the maximal ideal. Hence we are done by Theorem 4.4 since a regular basis for \( M_{\text{in}_P(f)} \) is finite due condition AA. \( \square \)

Using Lemma 3.8, (11) and Theorem 4.5 we get the analogous statement for contact equivalence.

**Corollary 4.7 (Normal forms for contact equivalence).** Let \( P \) be a \( C \)-polytope and \( f \in \mathfrak{m} \) be a power series such that \( \text{in}_P(f) \) satisfies AAC then \( f \) is finitely contact determined, \( cSPH \) and

\[
f \sim_c \text{in}_P(f) + \sum_{x^\alpha \in B, v_P(x^\alpha) > d} c_\alpha x^\alpha
\]

for suitable \( c_\alpha \in \mathbb{K} \), where \( B \) is a finite regular basis for \( T_{\text{in}_P(f)} \) and \( d = v_P(\text{in}_P(f)) \).

The proof of Theorem 4.4 and 4.5 actually gives a more precise bound on the determinacy if AA or AAC is fulfilled.

**Corollary 4.8 (Finite determinacy bound for right equivalence).** Let \( P \) be a \( C \)-polytope, \( f \in \mathfrak{m} \) be a power series such that \( \text{in}_P(f) \) satisfies AA and let \( B \) be a regular basis for \( M_{\text{in}_P(f)} \). Then

\[
d := \max\{v_P(\text{in}_P(f)), v_P(x^\alpha) \mid x^\alpha \in B\}
\]

is finite and \( f \sim_r g \) for any \( g \in \mathbb{K}[x] \) with \( v_P(f - g) > d \). In particular, if \( \mathfrak{m}^{k+1} \subseteq F_{d+1} \), then \( f \) is right \( k \)-determined.
Corollary 4.9 (Finite determinacy bound for contact equivalence). Let $P$ be a $C$-polytope, $f \in \mathfrak{m}$ be a power series such that $\text{in}_P(f)$ satisfies AAC and let $B$ be a regular basis for $T_{\text{in}_P(f)}$. Then

$$d := \max \{v_P(\text{in}_P(f)), v_P(x^\alpha) \mid x^\alpha \in B\}$$

is finite and $f \sim_c g$ for any $g \in K[[x]]$ with $v_P(f - g) > d$. In particular, if $\mathfrak{m}^{k+1} \subseteq F_{d+1}$, then $f$ is contact $k$-determined.

We present the proof for these corollaries only in the case of contact equivalence.

Proof. By Corollary 4.7 $f$ is finitely determined, and thus by [BGM, Theorem 2.5] some power of $\mathfrak{m}$ lies in $\langle f \rangle + m \cdot j(f)$, so that we are in the situation of Theorem 4.5. The proof of Theorem 4.5 shows that

$$f \sim_c \text{in}_P(f) + \sum_{x^\alpha \in B, v_P(x^\alpha) > v_P(\text{in}_P(f))} c_\alpha x^\alpha + g_l$$

for suitable $c_\alpha \in K$ and with $g_l \in F_l$ for $l$ arbitrarily large. Moreover, in the process of constructing the transformations we see that terms of piecewise valuation larger than $d$ do not have any influence on the coefficients $c_\alpha$ of the above normal form. Thus any power series $g$ which coincides with $f$ up to valuation $d$ will give the same normal form and is thus contact equivalent to $f$. □

The determinacy bounds from Corollaries 4.8 resp. 4.9 for power series satisfying conditions AA resp. AC are in general much better than those for arbitrary isolated singularities given in [BGM] (see Example 5.1), hence the conditions AA and AAC are desirable. In the following we give many examples of power series satisfying these conditions. We will first consider quasihomogeneous polynomials.

Proposition 4.10. If $f \in K[x]$ is QH of type $(w; d)$ and $P$ is the $C$-polytope defined by the single weight vector $w$, then $f$ satisfies the conditions $A$ and $AC$ with respect to $P$.

Proof. This was proved in from Corollary 3.9 (a). □

From (10) and (11) together with (12) it follows that any power series with an isolated quasihomogeneous principal part satisfies AA and AAC.

Corollary 4.11. If $f \in K[[x]]$ is rSQH (respectively cSQH) w.r.t. $w$ then $f$ is AA (respectively AAC) w.r.t. the $C$-polytope defined by $w$.

Example 4.12 (T_{15}-Singularity in characteristic 2). The condition in Corollary 4.11 that the $C$-polytope has only one facet, i.e., that the principal part is quasihomogeneous, is essential. Let $\text{char}(K) = 2$ and $f = x^5 + x^2y^2 + y^4 + x^3y^2 \in K[[x, y]]$. Then $f$ is cSPH with respect to $P = \Gamma(f)$ with principal part $\text{in}_P(f) = x^5 + x^2y^2 + y^4$ and $\tau(\text{in}_P(f)) = 16$. However, $\tau(f) = \infty$, which is an alternative proof of the fact $f$ is not AAC, as we have already seen in Example 3.7.

Remark 4.13. In [Wal99] Wall introduces the notion of inner Newton non-degeneracy, which turns out to be a sufficient condition for AA and AAC. Let us recall the definition here. A face $\Delta$ of $P$ is called an inner face if it is not contained in
any coordinate hyperplane. Each point \( q \in \mathbb{K}^n \) determines a coordinate hyperspace \( H_q = \bigcap_{i=0}^{n} \{ x_i = 0 \} \subseteq \mathbb{R}^n \). We call \( f \) inner non-degenerate IND along \( \Delta \) if for no common zero \( q \) of \( |(\text{in}_\Delta(f))| \) the polytope \( \Delta \) contains a point on \( H_q \), and we call \( f \) inner Newton non-degenerate INND w.r.t. \( P \) if \( f \) is non-degenerate of type IND along each inner face of \( P \).

Inner Newton non-degeneracy can be formulated differently so that the connection to \( \text{gr}_P(M_f) \) is more evident. Each face \( \Delta \) of the Newton diagram of \( f \) determines a finitely generated semigroup \( C_\Delta \) in \( \mathbb{Z}^n \) by considering those lattice points which lie in the cone over \( \Delta \) with the origin as base. This semigroup then determines a finitely generated \( \mathbb{K} \)-algebra \( \mathbb{K}[C_\Delta] = \mathbb{K}[x_\alpha | \alpha \in C_\Delta] \) and a \( \mathbb{K}[C_\Delta] \)-module \( D_\Delta = \langle x_\alpha \cdot \partial_{x_i} | x_\alpha \cdot \partial_{x_i} x_\gamma \in \mathbb{K}[C_\Delta] \forall \gamma \in C_\Delta \rangle \mathbb{K}[C_\Delta] \) generated by monomial derivations which leave \( \mathbb{K}[C_\Delta] \) invariant. Applying all elements in \( D_\Delta \) to \( \text{in}_\Delta(f) \) leads to an ideal \( J_\Delta \) in \( \mathbb{K}[C_\Delta] \), and Wall then shows that (see [Wal99, Prop. 2.2])

\[
\dim_{\mathbb{K}}(\mathbb{K}[C_\Delta]/J_\Delta) < \infty \iff f \text{ is IND along all inner faces of } \Delta.
\]

The rings \( \mathbb{K}[C_\Delta]/J_\Delta \) can be stacked neatly in an exact sequence of complexes whose homology Wall uses to show that (see [Wal99, Prop. 2.3])

\[
f \text{ is INND } \implies \dim_{\mathbb{K}}(\text{gr}_P(M_f)) < \infty.
\]

Wall’s arguments use only standard facts from toric geometry and homological algebra and do not depend on the characteristic of the base field. This proves Theorem 4.14, which shows that inner Newton non-degenerate singularities possess good normal forms w.r.t. right equivalence and also w.r.t. contact equivalence (see Corollaries 4.6 and 4.7).

We refer to [Wal99] and [BGM, Sec. 3] for more information on inner Newton non-degeneracy. See also [GN11].

**Theorem 4.14** (Wall, [Wal99]). If \( f \in \mathbb{K}[[x]] \) is INND w.r.t. \( P \), then \( f \) is AA and AAC w.r.t. \( P \), and hence

\[
\tau(f) \leq \mu(f) < \infty.
\]

5. Examples

In this section we apply the results of the previous sections to the classification of singularities of low modality in positive characteristic. A full classification of hypersurface singularities of right modality at most 2 and of contact modality at most 1 is still missing in positive characteristic, although a big part of this classification was achieved in [GK90] and [Bou02].

**Example 5.1** (Q10-Singularity in characteristic 2). Let \( \text{char}(\mathbb{K}) = 2 \) and assume that \( f \in \mathbb{K}[[x, y, z]] \) is cSQH with respect to the \( C \)-polytope \( P \) containing \( \Gamma(f) \) and with principal part \( \text{in}_P(f) = x^2z + y^3 + z^4 \). Using SINGULAR [DGPS10] we see that

\[
B = \{ 1, x, y, z, xy, xz, yz, z^2, xyz, x^2z, yz^2, z^3, xyz^2, xz^3, yz^3, xyz^3 \}
\]
is a $K$-vector space basis of $T_{\text{in}_P(f)}$. By Proposition 4.10 we see that $B$ is indeed a regular basis for $T_f$ and that $f$ is AC with
\[ \dim_K(\text{gr}^{AC}(T_f)) = \tau(\text{in}_P(f)) = 16. \]
Corollary 4.7 then shows that
\[ f \sim c x^2 z + y^3 + z^4 + c_1 \cdot xyz^2 + c_2 \cdot xz^3 + c_3 \cdot yz^3 + c_4 \cdot xyz^3 \]
for some $c_1, \ldots, c_4 \in K$. Moreover, using the weight vector $w = (9, 8, 6)$ to determine the filtration induced by $P$ then
\[ \max \{ v_P(f), v_P(b) \mid b \in B \} = 35 \]
and an easy computation shows that $m^6 \in F_{36}$. Thus $f$ is contact $5$-determined, and this bound of determinacy is better than that obtained from [BGM, Theorem 2.1], which would be 11.

\begin{figure}[h]
\centering
\includegraphics[width=0.2\textwidth]{newton_diagram.png}
\caption{The Newton diagram of $xz^2 + y^3 + z^4$.}
\end{figure}

When checking if certain monomials are zero in $\text{gr}^{AC}_P(M_f)$ or $\text{gr}^{AC}_P(T_f)$ the following lemma is very helpful.

**Lemma 5.2.** Let $P$ be a $C$-polytope, $\Delta$ a facet of $P$ and $f \in K[[x]]$. Moreover, denote by $C_{\Delta}$ the cone over $\Delta$, and assume that $\alpha, \beta \in C_{\Delta} \cap \mathbb{Z}^n$. If $x^\alpha$ is zero in $\text{gr}^{AC}_P(M_f)$ (respectively in $\text{gr}^{AC}_P(T_f)$) then $x^{\alpha + \beta}$ is so.

**Proof.** Since $\alpha$ and $\beta$ belong to the same cone $C_{\Delta}$ Equation (2) shows that
\[ v_P(x^{\alpha + \beta}) = v_P(x^\alpha) + v_P(x^\beta). \]
Thus in the graded algebra $\text{gr}^{AC}_P(M_f)$ or $\text{gr}^{AC}_P(T_f)$ the class of $x^{\alpha + \beta}$ is the product of the classes of $x^\alpha$ and $x^\beta$ (see (5)). Since the former is zero by assumption so is the product. \qed

**Remark 5.3.** With the notation and assumptions of Lemma 5.2 it follows that all monomials corresponding to the cone $\alpha + C_{\Delta}$ vanish in $\text{gr}^{AC}_P(M_f)$ (respectively in $\text{gr}^{AC}_P(T_f)$).

**Example 5.4** ($T_{pq}$- Singularities). Arnol’d considered in [Arn75, Example 9.6] the power series $f = x^p + \lambda x^q y + y^9 \in \mathbb{C}[[x]]$ with $\lambda \neq 0$ and $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ or equivalently
\[ pq - 2p - 2q > 0. \]
$f$ is PH with respect to its Newton diagram $P = \Gamma(f)$ depicted in Figure 6. If we scale the corresponding weight vectors to length $2pq$ instead of one they are
\[ w_1 = (2q, pq - 2q) \quad \text{and} \quad w_2 = (pq - 2p, 2p), \]
and the piecewise degree of $f$ is then $\deg_P(f) = 2pq$. Arnol’d describes in his paper a geometric procedure to compute a regular basis for $M_f$ if the partial derivatives have only two terms as in the example, and he deduces that $f$ satisfies condition A and AA and that
\[ B = \{1, x, \ldots, x^p, y, y^2, \ldots, y^{q-1}, xy\} \]
is a regular basis for $M_f$. In particular,
\[ \mu(f) = \dim_K(\operatorname{gr}_P^A(M_f)) = p + q + 1. \]
Since no monomial in $B$ lies above $\Gamma(f)$, it follows, as seen in Corollary 4.6 and 4.8, that any power series whose principal part with respect to the above $P$ coincides with $T_{pq}$ actually is right equivalent to $T_{pq}$.
Arnol’d’s arguments actually work for any field $K$ where the characteristic is neither two, nor divides $p$ or $q$. If the characteristic divides $p$ or $q$ then $\mu(f) = \infty$, and in the characteristic two case the Jacobian ideal is generated by the monomials $x^{p-1}$ and $y^{q-1}$, so that $\mu(f) = pq$. 

**Figure 5.** The lattice points in $\alpha + C_\Delta$ correspond to monomials $x^\gamma$ vanishing in $\operatorname{gr}_P^A(M_f)$ (respectively in $\operatorname{gr}_P^{AC}(T_f)$).

**Figure 6.** The Newton diagram of $x^p + \lambda x^2 y^2 + y^q$. 
We now want to investigate $f = T_{pq}$ with respect to contact equivalence and the condition AC, and we first want to show that

\[ \text{char}(K) \neq 2 \implies f \text{ satisfies AC and hence AAC}. \]

Assume first that in addition \( \text{char}(K) \) does neither divide \( p \) nor \( q \) nor \( pq - 2 \cdot (p + q) \). Then \( \mu(f) < \infty \) and thus also \( \tau(f) < \infty \). Moreover, by Corollary 3.6 the above \( B \) generates \( \text{gr}^{\text{AC}}(T_f) \) and

\[ T_f = K[[x, y]]/(f, f_x, f_y). \]

It is clear that other than \( x^p \) the monomials in \( B \) will stay linearly independent modulo \( \text{tj}(f) \), and all monomials \( x^i y^j \) which are in \( j(f) + F_{d+1} \) with \( d = v_P(x^i y^j) \) are also in \( \text{tj}(f) + F_{d+1} \) (since they are a regular basis for \( M_f \), see also [Bou09, Proposition 3.2.14]). To see that \( f \) satisfies AC it thus suffices to show that there are \( a, b, c \in K \) such that

\[ x^p = a \cdot f + b \cdot x \cdot f_x + c \cdot y \cdot f_y \]

since then

\[ x^p \in \text{tj}(f). \]

Considering the coefficients for \( x^p, x^2 y^2 \) and \( y^p \) this leads to a linear system of equations with extended coefficient matrix

\[ M = \begin{pmatrix} 1 & p & 0 & 1 \\ \lambda & 2\lambda & 2\lambda & 0 \\ 1 & 0 & q & 0 \end{pmatrix}. \]

This system is solvable if and only if the equation

\[ \lambda \cdot (pq - 2 \cdot (p + q)) = 2\lambda \]

has a solution, i.e., if the first \((3 \times 3)\)-minor \(-\lambda \cdot (pq - 2 \cdot (p + q)) \neq 0 \). This shows that

\[ p + q = \tau(f) = \dim_K(\text{gr}^{\text{AC}}(T_f)), \]

where for the latter equality we take into account that \( \tau(f) \) is a lower bound for the dimension. Moreover,

\[ B' = \{ 1, x, \ldots, x^{p-1}, y, y^2, \ldots, y^{q-1}, xy \} \]

is a regular basis for \( T_f \) and \( f \) satisfies AC.

Assume next that \( \text{char}(K) \) does neither divide \( p \) nor \( q \), but it divides \( pq - 2 \cdot (p + q) \). We have already seen in the first case that the system of linear equations with extended coefficient matrix \( M \) is not solvable under the given hypotheses. It follows that \( x^p \) does not lie in \( \text{tj}(f) \). Therefore, \( B \) is a regular basis for \( T_f \) and

\[ p + q + 1 = \tau(f) = \dim_K(\text{gr}^{\text{AC}}(T_f)). \]

Assume now that \( \text{char}(K) \) divides \( p \) but not \( q \). Then it is straightforward to see that

\[ \text{tj}(f) = \langle x^p, y^q, xy^2, qy^{q-1} - 2\lambda x^2 y \rangle, \]

and thus \( B' \) is a \( K \)-vector space basis of \( T_f \). We claim that \( B' \) also generates \( \text{gr}^{\text{AC}}(T_f) \), so that \( f \) satisfies AC with Tjurina number \( \tau(f) = p + q \). By Lemma 5.2
Figure 7. On the left-hand side the elements of $B'$ are depicted by large white dots and the elements in $B_c$ are depicted by large black dots. The right-hand side shows the union of $(2, 2) + C_\Delta$, $(2, 1) + C_\Delta$ and $(p, 0) + C_\Delta$, which covers all lattice points in $C_\Delta$ which are not in $B'$.

it suffices to check that the monomials in

$$B^c = \{x^2y^2, xy^2, x^2y, xp, yq\}$$

are zero in $\text{gr}_P^{AC}(T_f)$ (see Figure 7). However, we have that

$$x^2y^2 = \frac{1}{2\lambda} \cdot x \cdot \partial_x f \in \text{tj}(f)_{2pq}$$

and

$$xp = f - \left(\frac{1}{2} + \frac{1}{q}\right) \cdot x \cdot \partial_x f - \frac{1}{q} \cdot y \cdot \partial_y f \in \text{tj}(f)_{2pq}.$$ 

Moreover, $v_P(xy^2) = pq + 2q$ and $v_P(\partial_x) = 2p - pq$, so that

$$xy^2 = \frac{1}{2\lambda} \cdot \partial_x f \in \text{tj}(f)_{pq + 2p}.$$ 

Similar arguments hold for $x^2y$ and $y^q$. This finishes this case.

Assume now that $\text{char}(K)$ divides $q$ but not $p$. This case follows by symmetry from the previous case, i.e., $f$ is AC with Tjurina number $\tau(f) = p + q$.

Assume finally that $\text{char}(K)$ divides both $p$ and $q$. Then $\text{tj}(f) = \langle xy^2, x^2y, x^p + y^q, x^{p+1}, y^{q+1}\rangle$ and $B$ is a $K$-vector space basis of $T_f$. Moreover, we claim that it is a regular basis for $T_f$ as well. By Lemma 5.2 it suffices to show that the monomials $x^{p+1}, y^{q+1}, x^2y^2, xy^2$ and $x^2y$ as well as the binomial $x^p + y^q$ are zero in $\text{gr}_P^{AC}(T_f)$. This can be achieved in the same way as above. In particular we have

$$p + q + 1 = \tau(f) = \dim_K(\text{gr}_P^{AC}(T_f))$$

and $f$ satisfies AC.

Conclusion: In each of the above cases the regular basis $B$ (respectively $B'$) consists of monomials on or below the Newton diagram $P = \Gamma(f)$. Therefore, the normal form algorithm shows that any power series with principal part $f$ with respect to $P$ has indeed $f$ as normal form.
Corollary 5.5 (Normal form of $T_{pq}$-singularities). Suppose that $\text{char}(K) \neq 2$ and let $f \in K[[x, y]]$ be a power series with $\text{in}_\Gamma(f) = x^p + \lambda \cdot x^2y^2 + y^q$, $\frac{1}{p} + \frac{1}{q} < \frac{1}{2}$ and $\lambda \neq 0$. Then $f$ is AC and

$$f \sim_c x^p + \lambda \cdot x^2y^2 + y^q.$$ 

The contact determinacy of $f$ is $\max\{p, q\}$.

Proof. That $f$ satisfies AC and is contact equivalent to its principal part was shown in Example 5.4. It is obvious that any monomial above the Newton diagram has strictly larger piecewise valuation than $f$. Using the notation from Example 5.4 it follows that $m^{k+1} \subset F_{pq+1}$ for $k = \max\{p, q\}$, so that by Corollary 4.9 the degree of contact determinacy is at most $k$. To see that it cannot be less we may assume the contrary and we may assume moreover that $p \geq q$. Then $f \sim_c \text{in}_\Gamma(f) = x^p = \lambda \cdot x^2y^2 + y^q$, but the latter is non-reduced and has thus infinite Tjurina number. This is clearly a contradiction. \qed

Remark 5.6. If $\text{char}(K) = 2$ neither the conclusion in Corollary 5.5 nor the investigation in Example 5.4 hold in general as we can see from Example 3.7.

For the $T_{pq}$-singularities we considered the conditions A and AC and deduced a normal form. However, for normal forms we only need a good way to choose a small regular basis for $M_{\text{in}_R(f)}$ or $T_{\text{in}_R(f)}$. There the following observations are useful.

There is only finitely many zero-dimensional faces and each facet is the convex hull of some of these. The cones over these zero-dimensional faces are rays, and for each facet $\Delta$ of $P$ the cone $C_\Delta$ is spanned by a finite number of these rays none of which is superfluous, i.e., they are the extremal rays of the cone.

Then $f \in K[[x]]$ satisfies condition AA (respectively AAC) w.r.t. $P$ if and only if on each ray spanned by a zero-dimensional face of $P$ there is a lattice point $\alpha$ such that $x^\alpha$ is zero in $\text{gr}_P^\lambda(M_f)$ (respectively in $\text{gr}_P^{AC}(T_f)$).

Proof. Consider first the two-dimensional situation such that each cone $C_\Delta$ is spanned by two rays. Suppose that a cone $C_\Delta$ is given and it is spanned by the rays $r$ and $s$, and suppose that $\alpha$ is a lattice point on $r$ and $\beta$ is a lattice point on $s$. The shifted rays $\alpha + s$ and $\beta + r$ will intersect, since $r$ and $s$ are not parallel, and thus the rays $r$, $s$, $\alpha + s$ and $\beta + r$ bound a finite region in the cone $C_\Delta$ (see Figure 8). By Lemma 5.2 the lattice points which are not inside the bounded region will be zero in $\text{gr}_P^\lambda(M_f)$ (respectively in $\text{gr}_P^{AC}(T_f)$). We can play this game for each facet of $P$, and thus there are only finitely many monomials whose class is not zero.

The argument generalises right away to higher dimensions. \qed

Corollary 5.8. Let $f = x^a + y^b + \lambda \cdot x^c y^d \in K[[x, y]]$ with $\lambda \in K$, $a > c > 1$, $b > d \geq 1$ and $ad + bc < ab$, and let $P = \Gamma(f)$. Then $f$ satisfies AA (respectively AAC) w.r.t. $P$ if and only if there are natural numbers $k, m, n$ such that $x^m$, $y^n$ and $x^k y^l$ are zero in $\text{gr}_P^\lambda(M_f)$ (respectively in $\text{gr}_P^{AC}(T_f)$).

Proof. The Newton diagram of $f$ is schematically shown in Figure 9, and the result follows from Remark 5.7. \qed
Example 5.9 (E₃,₃-Singularity in characteristic 3). Let char(K) = 3 and consider the equation

\[ f = x^{12} + x^3y^2 + y^3 \in K[[x, y]]. \]

\( f \) is piecewise homogeneous with respect to its Newton diagram and using the procedure \texttt{isAC} from the SINGULAR library \texttt{gradalg.lib} we can check that \( x^{15}, y^{15} \) and \( x^3y^6 \) are zero in \( \text{gr}^{\text{AC}}(T_f) \). Thus \( f \) is AAC with respect to \( \Gamma(f) \) by Corollary 5.8.

Moreover, using the procedure \texttt{ACgrbase} from the same library we can compute the regular basis

\[ B = \{ 1, x, \ldots, x^{12}, y, xy, x^2y, y^2, xy^2, x^2y^2, xy^3, x^2y^3, x^2y^4 \} \]

for \( T_f \). Hence, \( \dim_K(\text{gr}^{\text{AC}}(T_f)) = |B| = 22 \) while \( \tau(f) = 21 \). This shows that \( f \) is not AC.

Theorem 4.5 shows that any power series \( g \) whose principal part with respect to \( \Gamma(f) \) is \( f \) satisfies

\[ g \sim e f + c_1 \cdot xy^3 + c_2 \cdot x^2y^3 + c_3 \cdot x^2y^4 \]

for suitable \( c_1, c_2, c_3 \in K \).

Scaling the weight vectors corresponding to the facets of \( \Gamma(f) \) suitably they are \( w_1 = (6, 27) \) and \( w_2 = (8, 24) \), and \( f \) is PH of piecewise degree 72. Moreover, the
maximum of the piecewise degree of the monomials in $B$ is $d = 112$, and an easy computation shows that $m^{19} \subseteq F_{113}$. By Corollary 4.9 we therefore know that the contact determinacy bound of $f$ is at most 18. That is much better than the bound $2 \cdot \tau(f) - \ord(f) + 2 = 41$,

which [BGM, Theorem 2.1] gives.

**Example 5.10** $(E_7$-Singularity). Let $f \in K[[x, y, z]]$ and let $P$ be a $C$-polytope containing $\Gamma(f)$ and suppose that the principal part of $f$ is $\text{in}_P(f) = x^3 + xy^3 + z^2$. Then $f$ is cSQH and our methods show the following (for the details we refer to [Bou09, Example 3.3.22]):

1ST CASE: $\text{char}(K) \notin \{2, 3\}$. Then $f \sim c \cdot \text{in}_P(f)$, $\tau(f) = \tau(\text{in}_P(f)) = 7$ and the contact determinacy is 4.

2ND CASE: $\text{char}(K) = 3$. Then $f \sim c \cdot \text{in}_P(f) + c \cdot x^2 y^2$ for some $c \in K$, $\tau(\text{in}_P(f)) = 9$ and the contact determinacy is again 4. If $c \neq 0$ then $\tau(f) = 7$.

3RD CASE: $\text{char}(K) = 2$. Then $f \sim c \cdot \text{in}_P(f) + c_1 \cdot y^3 z + c_4 \cdot y^4 z$, $\tau(\text{in}_P(f)) = 14$ and the determinacy is 5.

**Example 5.11** $(W_{1,1}$-Singularities). Let $f \in K[[x, y]]$ be such that the principal part with respect to $P = \Gamma(f)$ is $\text{in}_P(f) = x^7 + x^3 y^2 + y^4$. Then $f$ is cSPH and our methods give the following normal forms (for the details we refer to [Bou09, Example 3.3.9]):

1ST CASE: $\text{char}(K) \notin \{2, 3\}$. Then $f \sim c \cdot \text{in}_P(f)$.

2ND CASE: $\text{char}(K) = 3$. Then $f \sim c \cdot \text{in}_P(f) + c_1 \cdot x y^4 + c_2 \cdot x^2 y^3 + c_3 \cdot x^2 y^4 + c_4 \cdot x^2 y^5$ for some $c_1, \ldots, c_4 \in K$. However, considering parametrisations it can be shown that actually $f \sim c \cdot \text{in}_P(f) + c \cdot x^2 y^3$ for some $c \in K$ (see [Bou02]).

3RD CASE: $\text{char}(K) = 2$. Then $f \sim c \cdot \text{in}_P(f) + c \cdot x^6 y$ for some $c \in K$.

**Figure 10.** The Newton diagram of $x^7 + x^3 y^2 + y^4$ for $\text{char}(K) \neq 2, 3, 7$.

**References**


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