DERIVED MACKEY FUNCTORS

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To Pierre Deligne, on the occasion of his 65th birthday

Abstract. For a finite group \( G \), the so-called \( G \)-Mackey functors form an abelian category \( \mathcal{M}(G) \) that has many applications in the study of \( G \)-equivariant stable homotopy. One would expect that the derived category \( \mathcal{D}(\mathcal{M}(G)) \) would be similarly important as the “homological” counterpart of the \( G \)-equivariant stable homotopy category. It turns out that this is not so — \( \mathcal{D}(\mathcal{M}(G)) \) is pathological in many respects. We propose and study a replacement for \( \mathcal{D}(\mathcal{M}(G)) \), a certain triangulated category \( \mathcal{DM}(G) \) of “derived Mackey functors” that contains \( \mathcal{M}(G) \) but is different from \( \mathcal{D}(\mathcal{M}(G)) \). We show that standard features of the \( G \)-equivariant stable homotopy category such as the fixed points functors of two types have exact analogs for the category \( \mathcal{DM}(G) \).


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Introduction

The notion of a “Mackey functor” associated to a finite group \( G \) is a standard tool both in algebraic topology and in group theory. It was originally introduced by Dress [Dr] and later clarified by several people, in particular by Lindner [Li]; the

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reader can find modern expositions in the topological context e.g. in [LMS], [M], [tD], or a more algebraic treatment in [TW].

In this paper, we will be mostly concerned with applications to algebraic topology. Of these, the main one is the following: the category of Mackey functors is the natural target for equivariant homology and cohomology.

Namely, assume given a CW complex $X$ equipped with a continuous action of a finite group $G$. Then if the action is nice enough, the cellular homology complex $C_*(X, \mathbb{Z})$ inherits a $G$-action, so that we can treat homology as a functor from $G$-equivariant CW complexes to the derived category $\mathcal{D}(G, \mathbb{Z})$ of representations of the group $G$. However, this loses some essential information. For example, for any subgroup $H \subset G$, the homotopy type of the subspace $X^H \subset X$ of $H$-fixed points is a $G$-homotopy invariant of $X$ in a suitable sense; but once we forget $X$ and remember only the object $C_*(X, \mathbb{Z}) \in \mathcal{D}(G, \mathbb{Z})$, there is no way to recover the homology $H^q(X^H, \mathbb{Z})$.

Thus, the Mackey functors: certain algebraic gadgets designed to remember not only the homology $H^q(X, \mathbb{Z})$ as a representation of $G$, but also all the groups $H^q(X^H, \mathbb{Z})$, $H \subset G$, with whatever natural group action they possess, and some natural maps between them. We recall the precise definitions in Section 2. For now, it suffices to say that in the standard approach, Mackey functors form a tensor abelian category $\mathcal{M}(G)$ such that, among other things,

(i) for any $G$-equivariant CW complex $X$, we have natural homology objects $H^G_*(X, \mathbb{Z}) \in \mathcal{M}(G)$,
(ii) there exists a forgetful exact tensor functor from $\mathcal{M}(G)$ to the category $\mathbb{Z}[G]$-mod of representations of $G$ which recovers $H_*(X, \mathbb{Z})$ with the natural $G$-action when applied to $H^G_*(X, \mathbb{Z})$,
(iii) for any subgroup $H \subset G$, the homology $H_*(X^H, \mathbb{Z})$ can also be recovered from $H^G_*(X, \mathbb{Z}) \in \mathcal{M}(G)$,
(iv) $H^G_*(X, \mathbb{Z})$ is compatible with stabilization and the tensor product, and extends to the “genuine $G$-equivariant stable homotopy category” of [LMS], here denoted by $\text{StHom}(G)$.

More precisely, for every subgroup $H \subset G$, one has an exact functor from $\mathcal{M}(G)$ to the category of abelian groups which associates an abelian group $M_H$ to every $M \in \mathcal{M}(G)$; then in (iii), there is a functorial isomorphism

$$H_*(X^H, \mathbb{Z}) \cong H^G_*(X, \mathbb{Z})^H.$$

One can use the correspondence $M \mapsto M^H$ to visualize the structure of the category $\mathcal{M}(G)$ in the following way. For any subgroup $H \subset G$, let $\mathcal{M}_H(G) \subset \mathcal{M}(G)$ be the full subcategory spanned by such $M \in \mathcal{M}(G)$ that

$$M^H = 0$$

unless $H'$ contains a conjugate of $H$. Then this is a Serre abelian subcategory, and the subcategories $\mathcal{M}_H(G)$, $H \subset G$, form an increasing “filtration by support” of the category $\mathcal{M}(G)$. The top associated graded quotient of this filtration is equivalent to the category $\mathbb{Z}[G]$-modules—that is, we have

$$\mathcal{M}(G)/\{\mathcal{M}_H(G)\}_{H \subset G} \cong \mathbb{Z}[G]$$
where \( \langle M_H(G) \rangle \} \}_{H \in G} \subseteq \mathcal{M}(G) \) is the Serre subcategory generated by \( M_H(G) \) for all subgroups \( H \subseteq G \) except for the trivial subgroup \( \{ e \} \subseteq G \). One can also compute other quotients; for example, the smallest subcategory \( \mathcal{M}_G(G) \) \( G \) corresponding to \( G \) itself is equivalent to the category \( \mathbb{Z} \text{-mod} \) of abelian groups. More generally, for any normal subgroup \( N \subseteq G \) there exists a fully faithful exact inflation functor \( \text{Infl}_N^G \) which gives an equivalence

\[
\text{Infl}_N^G : \mathcal{M}(G/N) \cong \mathcal{M}_N(G) \subseteq \mathcal{M}(G).
\]

Analogous structures also exist on the category \( \text{StHom}(G) \). Namely, for any \( G \)-spectrum \( X \in \text{StHom}(G) \), one has the so-called Lewis-May fixed points spectrum \( X^H \), so that one can define the subcategories \( \text{StHom}_H(G) \) by (\( \ast \)). Then for a normal subgroup \( N \subseteq G \), one has a fully faithful embedding \( \text{StHom}(G/N) \cong \text{StHom}_N(G) \subseteq \text{StHom}(G) \). In particular, the smallest subcategory \( \text{StHom}_e(G) \subseteq \text{StHom}(G) \) is equivalent to the non-equivariant stable homotopy category \( \text{StHom} \). Another important feature of the stable category is the geometric fixed points functor \( \Phi^H : \text{StHom}(G) \rightarrow \text{StHom} \); on the level of Mackey functors, this corresponds to the projections onto the associated graded quotients of the filtration by support.

To a person trained in homological algebra, a natural next thing to do is to consider the derived category \( D(\mathcal{M}(G)) \) of the abelian category \( \mathcal{M}(G) \), and try to extend all of the above to the “derived level”: one would like to have a natural equivariant homology functor \( C_G^*(-, \mathbb{Z}) : G-\text{StHom} \rightarrow D(\mathcal{M}(G)) \), and one would expect the category \( D(\mathcal{M}(G)) \) to imitate the natural structure of the category \( \text{StHom}(G) \).

Unfortunately, and this came as a nasty surprise to the author, this program does not work: the derived category \( D(\mathcal{M}(G)) \) is not the right thing to consider. The specific problem is that the inflation functor \( \text{Infl}_N^G \) is not fully faithful on the level of derived categories. Already in the case \( G = \mathbb{Z}/p\mathbb{Z}, p \) prime, when the only subgroups in \( G \) are the trivial subgroup \( \{ e \} \subseteq G \) and \( G \) itself, we can consider the full triangulated subcategory \( D_G(\mathcal{M}(G)) \subseteq D(\mathcal{M}(G)) \) spanned by such \( M \in D(\mathcal{M}(G)) \) that \( M^{(e)} = 0 \). Then while we do have the equivalence \( D(\mathcal{M}(G))/D_G(\mathcal{M}(G)) \cong D(G, \mathbb{Z}) \), the functor

\[
D(\mathbb{Z}\text{-mod}) \cong D(\mathcal{M}_G(G)) \rightarrow D_G(\mathcal{M}(G))
\]

is not an equivalence. The category \( D_G(\mathcal{M}(G)) \), which ought to be equivalent to the derived category of abelian groups, is in fact rather complicated and behaves badly. So, while one might be able to construct a homology functor \( \text{StHom}(G) \rightarrow D(\mathcal{M}(G)) \), it does not seem to reflect the structure of \( \text{StHom}(G) \) too closely, and in particular, one cannot expect any reasonable compatibility with the geometric fixed point functors \( \Phi^H \).

But fortunately, a moment’s reflection on the definition of a Mackey functor (to wit, the argument in the beginning paragraphs of Section 3) shows why this is so, and in fact suggests what the correct “category of derived Mackey functors” should be. This is the subject of the present paper. For any finite group \( G \), we construct a tensor triangulated category \( D\mathcal{M}(G) \) of “derived Mackey functors” which enjoys the following properties.
(i) For any subgroup $H \subset G$ and any $M \in \mathcal{D}\mathcal{M}(G)$, there exists a functorial “fixed points” object $M^H \in \mathcal{D}(\mathbb{Z}\text{-mod}).$

(ii) For any subgroup $H \subset G$, let $\mathcal{D}\mathcal{M}_H(G) \subset \mathcal{D}\mathcal{M}(G)$ be the full triangulated subcategory spanned by $M \in \mathcal{D}\mathcal{M}(G)$ satisfying $(\ast)$. Then $\mathcal{D}\mathcal{M}_H(G) \subset \mathcal{D}\mathcal{M}(G)$ is admissible in the sense of [BK] — that is, the embedding functor $\mathcal{D}\mathcal{M}_H(G) \rightarrow \mathcal{D}\mathcal{M}(G)$ has a left and a right-adjoint — and we have an equivalence

$$\mathcal{D}\mathcal{M}_H(G)/\langle \mathcal{D}\mathcal{M}_H(G) \rangle_{H \subset H', H \neq H'} \cong \mathcal{D}(W_H, \mathbb{Z}),$$

where $W_H = N_H/H$ is the quotient of the normalizer $N_H \subset G$ of $H \subset G$ by $H$ itself.

(iii) For any normal subgroup $N \subset G$, we have an equivalence

$$\text{Infl}^N_G : \mathcal{D}\mathcal{M}(G/N) \cong \mathcal{D}\mathcal{M}_N(G) \subset \mathcal{D}\mathcal{M}(G).$$

Informally speaking, the subcategories $\mathcal{D}\mathcal{M}_H(G)$ form a filtration of the category $\mathcal{D}\mathcal{M}(G)$ indexed by the lattice of conjugacy classes of subgroups in $G$; this filtration gives rise to a “semiorthogonal decomposition” in the sense of [BK], and the graded pieces of the filtration are naturally identified with $\mathcal{D}(W_H, \mathbb{Z}), H \subset G$. We explicitly construct functors

$$\tilde{\Phi}^{[G/H]} : \mathcal{D}\mathcal{M}(G) \rightarrow \mathcal{D}(W_H, \mathbb{Z})$$

that give the projections onto these graded pieces. We also compute the gluing functors between $\mathcal{D}(W_H, \mathbb{Z}) \subset \mathcal{D}\mathcal{M}(G)$; these are naturally expressed in terms of a certain generalization of Tate cohomology of finite groups.

The reader will notice that the properties of the category $\mathcal{D}\mathcal{M}(G)$ are slightly stronger than what we have mentioned for the abelian category $\mathcal{M}(G)$, in that we identify the associated graded pieces of the filtration by support, and describe how these pieces are glued. If one localizes the category $\mathcal{M}(G)$ by inverting the order of the group $G$, then the corresponding statement for the localized category $\mathcal{M}(G)$ is a theorem of Thevenaz [Th] (in this case, there is no gluing: the category $\mathcal{M}(G)$ is semisimple, and it splits into a direct sum of the categories of representations of the groups $W_H, H \subset G$). I do not know whether anything is known for $\mathcal{M}(G)$ in the general non-semisimple case. I also do not know whether analogous statements are known for the category $\text{StHom}(G)$ — namely, whether the graded pieces of the filtration by the subcategories $\text{StHom}_H(G) \subset \text{StHom}(G)$ have been computed and/or whether the gluing functors have been identified.

Moreover, in this paper we construct a natural equivariant homology functor $h^G$ from $G$-spectra to our derived Mackey functors such that

(i) the functor $h^G$ is tensor,

(ii) for any subgroup $H \subset G$, the functor $h^G$ sends $\text{StHom}_H(G) \subset \text{StHom}(G)$ into $\mathcal{D}\mathcal{M}_H(G) \subset \mathcal{D}\mathcal{M}(G)$, and

(iii) for any subgroup $H \subset G$ and a $G$-spectrum $X$, the underlying complex of $\tilde{\Phi}^{[G/H]}(h^G(X)) \in \mathcal{D}(W_H, \mathbb{Z})$ computes the homology of the geometric fixed point spectrum $\Phi^H(X)$. 
Unfortunately, we can only do all of the above for finite $G$-CW spectra $X$. I do not know whether the corresponding statements are true for the whole category $\text{StHom}(G)$. I also expect that for any $G$-spectrum $X$, $h^G(X)^H$ is the homology of the Lewis–May fixed points spectrum $X^H$, but I do not presently know how to prove it.

The paper is organized as follows. In Section 1, we give some necessary standard facts from homological algebra and category theory (this includes the notions related to semiorthogonal decompositions of triangulated categories). In Section 2, we recall the usual definition of Mackey functors and some of their basic properties. Then we give our derived version. We actually give not one but two definitions. First, we give a rather explicit definition using $A_\infty$-categories and bar resolutions — this is Section 3 (in fact, we work in a slightly larger generality of a small category $\mathcal{C}$ which has fibered products — for Mackey functors, this is the category of finite $G$-sets). Then in Section 4, we give a more invariant definition somewhat in the spirit of Waldhausen’s $S$-construction, and we show that the two definitions are equivalent. In Section 5, we show how to extend the basic properties of Mackey functors given in Section 2 to the derived setting (in particular, we construct the tensor product, the inflation functors $\hat{\text{Infl}}_N^G$ and the geometric fixed points functors $\Phi_{[G/H]}$).

Formally, this is where the paper might have ended; however, neither of our two equivalent definitions is suitable for computations. For this reason, we give a third rather explicit description of the same category, and this is the subject of the lengthy Section 6. Essentially, we do a sort of Koszul duality — we try to describe the same category $\mathcal{DM}(G)$ using the geometric fixed points functors $\Phi_{[G/H]}$ as a “fiber functor”. This is surprisingly delicate; in particular, we have to work with $A_\infty$-comodules over an $A_\infty$-coalgebra instead of the more usual $A_\infty$-modules over an $A_\infty$-algebra.

One immediate advantage is that the functors $\Phi_{[G/H]}$ are tensor, so that this new description is better compatible with the tensor product in $\mathcal{DM}(G)$. However, the real reward comes in Section 7: we are able to prove the equivalences (0.1), thus describing the category of derived Mackey functors as a successive extension of representation categories of the subquotient groups $W_H$ of the group $G$, and we express the gluing data between these representation categories in terms of a generalization of Tate (co)homology. This “generalized Tate cohomology” has the great advantage of being trivial in many cases; even when it is not trivial, it is usually possible to compute it.

In the final Section 8, we describe the relation between the category of derived Mackey functors and the $G$-equivariant stable homotopy category. Not being an expert on stable homotopy, I have kept the exposition to an absolute minimum; however, I hope that Section 8 does show that the category of derived Mackey functors imitates the equivariant stable homotopy category in a satisfactory way.

It should be stressed that a topologist would learn almost nothing from this paper: pretty much everything that we prove about derived Mackey functors is well-known for equivariant spectra (possibly in a different language). In a sense, the whole point of the paper is that so much survives in the purely homological theory,
which is usually pretty trivial compared to the richness of the sphere spectrum. On
the other hand, things standard in algebraic topology are not always well-known
outside of it; a reader with a more geometric and/or homological background can
treat the paper as an extended exercise in the theory of cohomological descent. As
such, it might even be useful, e.g. in the theory of Artin motives.

In the interest of full disclosure, I should mention that my personal main reason
for doing this research was its application to the so-called “cyclotomic spectra”, and
a comparison theorem between Topological Cyclic Homology, on one hand, and a
syntomic version of the Periodic Cyclic Homology, on the other hand. Needless to
say, in the end all of this had to be taken out and relegated to a separate paper,
which is “in preparation”.

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1. Homological Preliminaries

1.1. Generalities. Throughout the paper we will fix once and for all an abelian
category Ab; this is the category where our Mackey functors will take values. We
will assume that Ab is sufficiently nice, in that it is a Grothendieck abelian category
with enough projectives and injectives. We will also assume that Ab is equipped
with a symmetric tensor product. In addition, we will assume fixed some functorial
DG enhancement for Ab, so that for any $M, M' \in \text{Ab}$ we have a complex

\[ \text{RHom}^*(M, M') \]

of abelian groups whose homology computes $\text{Ext}^*(M, M')$; moreover, $\text{RHom}^*(-, -)$
is functorial in both arguments and equipped with functorial and associative com-
position maps

\[ \text{RHom}^*(M, M') \otimes \text{RHom}^*(M', M'') \to \text{RHom}^*(M, M'') \]

for any $M, M', M'' \in \text{Ab}$. For example, Ab may be the category of vector spaces
over a field $k$, or the category of modules over a commutative ring $R$, or indeed
simply the category of abelian groups. We will denote by $\mathcal{D}(\text{Ab})$ the (unbounded)
derived category of Ab. We will tacitly assume that the DG enhancement is ex-
tended to unbounded complexes, so that for any two complexes $M_-, M'_-$ we have a
whose 0-th homology classes are in one-to-one correspondence with maps from \( M \) to \( M' \) in \( D(\text{Ab}) \) (this is slightly delicate — to do it properly, one has to use either \( h \)-projective or \( h \)-injective replacements in the sense of [Ke1], [Sp]).

Throughout the paper, we will work with many small categories. Usually we will denote objects in a small category \( C \) by small roman letters; for any \( c, c' \in C \), we will denote by \( C(c, c') \) the set of morphisms from \( c \) to \( c' \). For any small category \( C \), we will denote by \( \text{Fun}(C, \text{Ab}) \) the category of all functors from \( C \) to \( \text{Ab} \). This is also a Grothendieck abelian category with enough projectives and injectives. We will denote its derived category by \( D(C, \text{Ab}) \).

A functor \( \gamma : C \to C' \) induces a restriction functor \( \gamma^* : \text{Fun}(C', \text{Ab}) \to \text{Fun}(C, \text{Ab}) \), and this has left and right adjoint functors \( \gamma_! : \text{Fun}(C, \text{Ab}) \to \text{Fun}(C', \text{Ab}) \) and \( \gamma_* : \text{Fun}(C, \text{Ab}) \to \text{Fun}(C', \text{Ab}) \), known as the left and right Kan extensions. By adjunction, \( \gamma^* \) is exact, \( \gamma_! \) is right-exact, and \( \gamma_* \) is left-exact, so that we have derived functors \( L^* \gamma_! , R^* \gamma_* : D(C, \text{Ab}) \to D(C', \text{Ab}) \).

In particular, if \( C' = \text{pt} \) is the point category, then \( \text{Fun}(C', \text{Ab}) = \text{Fun}(\text{pt}, \text{Ab}) \cong \text{Ab} \), and if \( \tau : C \to \text{pt} \) is the projection to the point, the Kan extensions \( \tau_! \) and \( \tau_* \) are given by the direct and inverse limit over the category \( C \). Their derived functors are known as homology \( H_\cdot(\cdot, -) \) and cohomology \( H^\cdot(\cdot, -) \) of the category \( C \).

On the other hand, if we choose an object \( c \in C \) and let \( i^c : \text{pt} \to C \) be the functor which sends \( \text{pt} \) to \( c \in C \), then the Kan extension functors \( i^c_! , i^c_* : \text{Ab} \cong \text{Fun}(\text{pt}, \text{Ab}) \to \text{Fun}(C, \text{Ab}) \) are exact. To simplify notation, we will denote \( M_c = i^c_!(M) \), \( M^c = i^c_*(M) \) for any object \( M \in \text{Ab} \). Explicitly, the functors \( M_c , M^c \in \text{Fun}(C, \text{Ab}) \) are given by

\[
M_c(c') = \bigoplus_{c(c, c')} M, \quad M^c(c') = \prod_{c(c', c)} M,
\]

for any \( c' \in C \) (the sum and the product of copies of \( M \) indexed by elements in the corresponding Hom-sets). If \( M \) is projective, then \( M_c \) is projective in \( \text{Fun}(C, \text{Ab}) \); if \( M \) is injective, then \( M^c \) is injective in \( \text{Fun}(C, \text{Ab}) \). For any \( M \in C \), a map \( f : c \to c' \) induces natural maps \( M_c \to M_e , M^c \to M^e \).

1.2. Fibrations and base change. In general, it is rather cumbersome to compute the Kan extensions explicitly; however, there is one situation introduced in [SGA] where the computations are simplified. Namely, assume given a functor \( \gamma : C' \to C \) between small categories \( C, C' \). A morphism \( f' : c_0 \to c_1 \) in \( C' \) is called Cartesian with respect to \( \gamma \) if it has the following universal property:

- any morphism \( f'' : c_0' \to c_1' \) such that \( \gamma(f'') = \gamma(f') \) factorizes uniquely as \( f'' = f' \circ i \) through a morphism \( i : c_0' \to c_0 \) such that \( \gamma(i) = \text{id} \).

The functor \( \gamma \) is called a fibration if

(i) for any morphism \( f : c_0 \to c_1 \) in the category \( C \) and any object \( c'_1 \in C' \) with \( \gamma(c'_1) = c_1 \) there exists a Cartesian map \( f' : c_0' \to c'_1 \) such that \( \gamma(f') = f \),

(ii) and moreover, the composition of two Cartesian maps is Cartesian.
Example 1.1. Let $\mathcal{C}$ be a category with fibered products, let $\mathcal{C}'$ be the category of diagrams $c_0 \to c_1$ in $\mathcal{C}$, and let $\gamma: \mathcal{C}' \to \mathcal{C}$ be the functor which sends a diagram $c_0 \to c_1$ to $c_1$. Then $\gamma$ is a fibration.

For any object $c \in \mathcal{C}$, denote by $\mathcal{C}'_c$ the fiber of the functor $\gamma$ over the object $c \in \mathcal{C}$ — that is, the category of objects $c' \in \mathcal{C}'$ such that $\gamma(c') = c$ and those morphisms $i$ between them for which $\gamma(i) = \text{id}$. Note that by the universal property of Cartesian maps, the map $f: c'_0 \to c'_1$ in (i) is unique up to a unique isomorphism, so that, if $\gamma$ is a fibration, setting $f^*(c'_1) = c'_0$ defines a functor $f^*: \mathcal{C}'_0 \to \mathcal{C}'_1$, which we will call the transition functor corresponding to $f$. The same universal property provides a canonical map $(f \circ g)^* \cong f^* \circ g^*$ for any two composable morphisms $f, g$ in $\mathcal{C}$, and the condition (ii) ensures that this is an isomorphism. These isomorphisms in turn satisfy a compatibility condition for composable triples, and the whole thing has been axiomatized by Grothendieck under the name of a “pseudo-functor.”

Assume given a fibration $\gamma: \mathcal{C}' \to \mathcal{C}$, another small category $\mathcal{C}_1$, and a functor $\eta: \mathcal{C}_1 \to \mathcal{C}$. Define a small category $\mathcal{C}'_1$ as a fibered product

$$
\begin{array}{ccc}
\mathcal{C}'_1 & \xrightarrow{\eta'} & \mathcal{C}' \\
\downarrow & & \downarrow \gamma \\
\mathcal{C}_1 & \xrightarrow{\eta} & \mathcal{C}.
\end{array}
$$

Then we have a pair of adjoint base change isomorphisms

$$
\eta^* \circ R^* \gamma_* \cong R^* \gamma'_* \circ \eta'^*, \quad L^* \eta'_! \circ \gamma'^* \cong \gamma^* \circ L^* \eta_!.
$$

For the proof, see e.g. [Ka, Lemma 1.7].

Dually, $\gamma: \mathcal{C}' \to \mathcal{C}$ is a cofibration if the corresponding functor $\gamma^{\text{opp}}: \mathcal{C}^{\text{opp}} \to \mathcal{C}^{\text{opp}}$ between the opposite categories is a fibration. Under the Grothendieck construction, cofibrations correspond to covariant pseudofunctors. If $\gamma$ is a cofibration, we have a base change isomorphism

$$
\eta^* \circ L^* \gamma_* \cong L^* \gamma'_! \circ \eta'^*.
$$

In particular, for any functor $E \in \text{Fun}(\mathcal{C}', \text{Ab})$, the value $L^* \gamma_!(E)(c)$ at an object $c \in \mathcal{C}$ is canonically given by

$$
L^* \gamma_!(E)(c) \cong H_* (\mathcal{C}'_c, E). \quad (1.2)
$$

1.3. Bar-resolutions. Another computational tool that we will need is the so-called bar-resolution. Assume given a small category $\mathcal{C}$. Then every functor $E \in \text{Fun}(\mathcal{C}, \text{Ab})$ has a canonical resolution $P_*(\mathcal{C}, E)$ with terms

$$
P_{t-1}(\mathcal{C}, E) = \bigoplus_{c_1 \to \cdots \to c_t} E(c_1)_{c_1},
$$

where the sum is taken over all the diagrams $c_1 \to \cdots \to c_t$ in the category $\mathcal{C}$, and the usual differential $\delta = d_1 - d_2 + \cdots \pm d_t$, where $d_t$ drops the object $c_t$ from the
diagram, and acts as the identity map if \(1 < l < i\), as the natural map \(E(c_1)_{c_1} \rightarrow E(c_1)_{c_{i_1}}\) induced by \(c_{i_1} \rightarrow c_i\) if \(l = i\), and as the map \(E(c_1)_{c_i} \rightarrow E(c_2)_{c_i}\) induced by the map \(E(c_1) \rightarrow E(c_2)\) if \(l = 1\). To see that this is indeed a resolution, one evaluates \(P_\bullet(C, E)\) at some object \(c \in C\). By definition, the resulting complex is given by

\[
P_{i-1}(C, E)(c) = \bigoplus_{c_1 \rightarrow \cdots \rightarrow c_i \rightarrow c} E(c_1),
\]

where the sum is now over all the diagrams ending at \(c \in C\). If one adds the term \(E(c)\) in degree \(-1\) corresponding to the diagram consisting of \(c\) itself, then the resulting complex is obviously chain-homotopic to \(0\) — the contracting homotopy \(h\) sends the term corresponding to a diagram \(c_1 \rightarrow \cdots \rightarrow c_i \rightarrow c\) to the term corresponding to \(c_1 \rightarrow \cdots \rightarrow c_i \rightarrow c \rightarrow c\), where the last map \(c \rightarrow c\) is the identity map.

Since all the objects \(M_{c}, c \in C, M \in Ab\) are obviously acyclic for the homology functor \(H_\bullet(C, \_\_\_)\), the bar resolution can be used to compute the homology \(H_\bullet(C, E)\). This results in the bar-complex \(C_\bullet(C, E)\) with terms

\[
C_{i-1}(C, E) = \bigoplus_{c_1 \rightarrow \cdots \rightarrow c_i} E(c_1).
\]  

(1.3)

This has the following standard properties.

(i) The bar-complex \(C_\bullet(C, E)\) is functorial with respect to \(E\).

(ii) For any functor \(\gamma: C \rightarrow C'\) and any \(E: C' \rightarrow Ab\), there is a natural map \(\gamma^E: C_\bullet(C, \gamma^E) \rightarrow C_\bullet(C', E)\) which induces the natural adjunction map \(H_\bullet(C, \gamma^E) \rightarrow H_\bullet(C', E)\) on homology, and for any composable pair of functors \(\gamma: C \rightarrow C', \gamma': C' \rightarrow C''\), we have \(\gamma'^E \circ \gamma^E = (\gamma' \circ \gamma)^E\).

(iii) For any two categories \(C, C'\) and functors \(E: C \rightarrow Ab, E': C' \rightarrow Ab\), we have a Künneth-type quasiisomorphism

\[
C_\bullet(C, E) \otimes C_\bullet(C', E') \rightarrow C_\bullet(C \times C', E \boxtimes E'),
\]

(1.4)

and this is associative with respect to triple products.

Of these, only (iii) is slightly non-obvious; the required quasiisomorphism is given by the shuffle product.

In addition, the bar-resolution can be used to give a canonical DG enhancement to the category \(\text{Fun}(C, Ab)\). Indeed, for any \(E, E' \in \text{Fun}(C, Ab)\) we can compute \(\text{Ext}^\bullet(E, E')\) by the bar-resolution; this results in a bicomplex \(R\text{Hom}^{\bullet \bullet}(E, E') = R\text{Hom}^\bullet_\bullet(E, E')\) with terms

\[
R\text{Hom}^{i-1,\bullet}(E, E') = \prod_{c_1 \rightarrow \cdots \rightarrow c_i} R\text{Hom}^\bullet(E(c_1), E'(c_i)).
\]  

(1.5)

We denote by \(R\text{Hom}^\bullet(E, E')\) the total complex of this bicomplex. Given three objects \(E, E', E'' \in \text{Fun}(C, Ab)\), we have a natural composition map \(R\text{Hom}^\bullet(E, E') \otimes R\text{Hom}^\bullet(E', E'') \rightarrow R\text{Hom}^\bullet(E, E'')\), and this is associative in the obvious sense.

We will also need a slightly more refined version of the bar resolution. For any \(i\), the diagrams \(c_1 \rightarrow \cdots \rightarrow c_i\) and isomorphisms between them form a groupoid \(C_i\).
Denote by \(\sigma_i, \tau_i: \mathcal{C}_i \to \mathcal{C}\) the functors which send a diagram \(c_1 \to \cdots \to c_i\) to \(c_1 \in \mathcal{C}\) resp. \(c_i \in \mathcal{C}\). For any \(E \in \text{Fun}(\mathcal{C}, \text{Ab})\), consider the complex
\[
\tilde{P}_{i-1,*}(\mathcal{C}, E) = \tau_*\sigma_i^!(\mathcal{C}_i, \sigma_i^*E).
\]
Forgetting one vertex in a diagram gives a functor \(\mathcal{C}_i \to \mathcal{C}_{i-1}\), and this construction is strictly associative. Therefore by the properties (i), (ii) of the bar complex, we can turn the collection \(\tilde{P}_{*,*}(\mathcal{C}, E)\) into a bicomplex, with the second differential given by the same formula \(\delta = d_1 - d_2 + \cdots + d_i\) as in the case of \(P_{*,*}(\mathcal{C}, E)\). The total complex \(\tilde{P}_*(\mathcal{C}, E)\) is then also a resolution on the functor \(E\). To see this, one again evaluates at an object \(c \in \mathcal{C}\), and uses the same contracting homotopy \(h\) as in the case of \(P_*(\mathcal{C}, E)\).

In the case of two functors \(E, E' \in \text{Fun}(\mathcal{C}, \text{Ab})\), we can use the resolution \(\tilde{P}_*(\mathcal{C}, E)\) to compute \(\text{RHom}^*(E, E')\); this results in the triple complex with terms
\[
\text{RHom}^i_{\mathcal{C}}(E, E') = C^i(\mathcal{C}, \text{RHom}^*(\sigma_i^*E, \tau_i^*E')),
\]
a refinement of the double complex (1.5). Its total complex is functorially quasi-isomorphic to \(\text{RHom}^*(E, E')\).

1.4. Semiorthogonal decompositions. We will also need some technology for working with triangulated categories; the standard reference here is [BK].

In light of recent advances in axiomatic homotopy theory, it is perhaps better to state explicitly that in this paper, our notion of a triangulated category is the original notion of Verdier. A full triangulated subcategory \(\mathcal{D} \subset \mathcal{D}'\) in a triangulated category \(\mathcal{D}'\) is called localizing if the quotient \(\mathcal{D}'/\mathcal{D}\) exists (in spite of the set-theoretic difficulties of the Verdier construction). A full triangulated subcategory \(\mathcal{D} \subset \mathcal{D}'\) is called left resp. right admissible if the embedding functor \(\mathcal{D} \hookrightarrow \mathcal{D}'\) admits a left resp. right adjoint; it is admissible if it is admissible both on the left and on the right. The left orthogonal \(\perp \mathcal{D}\) consists of objects \(M \in \mathcal{D}'\) such that \(\text{Hom}(M, N) = 0\) for any \(N \in \mathcal{D}\). This is also a full triangulated subcategory in \(\mathcal{D}'\), and it is known that \(\mathcal{D} \subset \mathcal{D}'\) is left-admissible if and only if \(\mathcal{D}\) and \(\perp \mathcal{D}\) generate the whole \(\mathcal{D}'\). In this case, one says that \((\perp \mathcal{D}, \mathcal{D})\) is a semi-orthogonal decomposition of the triangulated category \(\mathcal{D}'\). One shows that \(\mathcal{D}'\) is then generated by \(\mathcal{D}\) and \(\perp \mathcal{D}\) in the following strong sense: for any \(M' \in \mathcal{D}'\), there exists a unique and functorial distinguished triangle
\[
\perp M \longrightarrow M' \longrightarrow M \longrightarrow
\]
with \(M \in \mathcal{D}\) and \(\perp M \in \perp \mathcal{D}\). Moreover, the category \(\mathcal{D} \subset \mathcal{D}'\) is localizing, and we have a natural identification \(\perp \mathcal{D} \cong \mathcal{D}'/\mathcal{D}\).

Analogously, the right orthogonal \(\mathcal{D}^\perp \subset \mathcal{D}'\) consists of objects \(M \in \mathcal{D}'\) such that \(\text{Hom}(N, M) = 0\) for any \(N \in \mathcal{D}\), and \(\mathcal{D} \subset \mathcal{D}'\) is right-admissible if and only \(\mathcal{D}^\perp\) is generated by \(\mathcal{D}\) and \(\mathcal{D}^\perp\); in this case, \((\mathcal{D}, \mathcal{D}^\perp)\) is a semiorthogonal decomposition of the category \(\mathcal{D}'\). We have the following standard fact.

**Lemma 1.2.** Assume given a left-admissible triangulated subcategory \(\mathcal{D} \subset \mathcal{D}'\). Then the natural projection \(\mathcal{D}^\perp \rightarrow \mathcal{D}'/\mathcal{D} \cong \perp \mathcal{D}\) is fully faithful, and it is an equivalence if and only if \(\mathcal{D} \subset \mathcal{D}'\) is right-admissible. \(\square\)
Given an admissible subcategory \( D \subset D' \), the gluing functor \( R: D'/D \to D \) is defined as the composition
\[
D'/D \xrightarrow{\sim} D^\perp \subset D' \to D,
\]
where the second functor \( D' \to D \) is left-adjoint to the embedding \( D \subset D' \). This is a triangulated functor defined up to a canonical isomorphism. Objects \( M' \in D' \) are in natural one-to-one correspondence with triples \((M^\perp, M, r)\) of an object \( M^\perp \in D^\perp \), an object \( M \in D \), and a gluing map \( r: R(M^\perp) \to M[1] \).

We note that it is not possible to recover the category \( D' \) from \( D \), \( D'/D \) and the gluing functor \( R: D'/D \to D \) (we can recover objects, but not morphisms). However, there is the following useful fact.

**Lemma 1.3.** Assume given triangulated categories \( D'_1, D'_2 \) equipped with left-admissible subcategories \( D_1 \subset D'_1, D_2 \subset D'_2 \), and a triangulated functor \( F: D'_1 \to D'_2 \). Moreover, assume that

(i) \( F \) sends \( D_1 \) into \( D_2 \), \( \perp D_1 \) into \( \perp D_2 \), and the induced functors \( F: D_1 \to D_2 \), \( F: D'_1/D_1 \to D'_2/D_2 \) are equivalences, and

(ii) \( D_1 \subset D'_1 \) is also right-admissible, and \( F \) sends \( D_1^\perp \) into \( D_2^\perp \subset D'_2 \).

Then \( F \) is also an equivalence.

**Proof.** Since \( D'_2 \) is generated by \( D_2 = F(D_1) \) and \( \perp D_2 \cong F(\perp D_1) \), the functor \( F \) is essentially surjective, and it suffices to prove that it is fully faithful — that is, for any \( M, N \in D'_1 \), the map
\[
F: \text{Hom}(M, N) \to \text{Hom}(F(M), F(N))
\]
is an isomorphism. By (1.7), we may assume that \( M \) lies either in \( \perp D_1 \) or in \( D_1 \). If \( M \in \perp D_1 \) and \( N \in D_1 \), then both sides are 0. If \( M, N \in \perp D_1 \), then the map is bijective by (i). Thus we may assume \( M \in D_1 \). Decomposing \( N \) by (1.7) with respect to the semiorthogonal decomposition \( (D_1, D_1^\perp) \), we see that we may assume that either \( N \in D_1 \) or \( N \in D_1^\perp \). Then in the first case, the claim follows from (i), and in the second case, from (ii).

1.5. \( A_\infty \)-**structures.** To construct triangulated categories, we will use the machinery of \( A_\infty \)-algebras and \( A_\infty \)-categories (this is very well covered in the literature; a standard reference is, for example, B. Keller’s overview [Ke3]). We briefly recall the relevant notions.

1.5.1. \emph{Algebras and modules.} An \( A_\infty \)-algebra structure on a graded free \( \mathbb{Z} \)-module \( A \), is given by a coderivation \( \delta \) of the free non-unital associative coalgebra \( T_*\{A, [1]\} \) generated by \( A \), shifted by 1 such that \( \delta^2 = 0 \). Explicitly, the structure is given by a collection of operations \( b_n : A^{\otimes n} \to A_* \), \( n \geq 1 \), and \( \delta^2 = 0 \) is equivalent to
\[
\sum_{i+j+l=n} b_{i+1+l} \circ (\text{id}^{\otimes i} \otimes b_j \otimes \text{id}^{\otimes l}) = 0 \tag{1.8}
\]
for any \( n \geq 1 \). For \( n = 1 \), this reads as \( b_1^2 = 0 \), so that \( b_1 \) is a differential which turns \( A \) into a complex of \( \mathbb{Z} \)-modules. After adding some signs depending on degrees of the operands, the higher operations \( b_n, n \geq 2 \), can be arranged together into an operad \( \text{Ass}_\infty \) of complexes of \( \mathbb{Z} \)-modules. Moreover, the action of symmetric groups
on the component complexes $\widetilde{\text{Ass}}_\infty$ is irrelevant for the definition of an $A_\infty$-algebra—the operad $\text{Ass}_\infty$ is induced from an asymmetric operad $\text{Ass}_\infty$ in the sense of [Hin]. The asymmetric operad $\text{Ass}_\infty$ is equipped with a canonical surjective augmentation quasiisomorphism $\text{Ass}_\infty \to \text{Ass}$ onto the associative asymmetric operad $\text{Ass}$. If one forgets the differentials, $\text{Ass}_\infty$ is the free asymmetric operad generated by a single operation $b_n$ for each $n \geq 2$. Thus $\text{Ass}_\infty$ is cofibrant in the natural closed model structure on the category of asymmetric operads (see [Hin]), and the augmentation map $\text{Ass}_\infty \to \text{Ass}$ is a cofibrant replacement for $\text{Ass}$.

For any $A_\infty$-algebra $A_\ast$, the operation $m_2$ given by

$$m_2(x, y) = (-1)^{\text{deg}(x)}b_2(x, y)$$

(1.9)
induces an associative multiplication on the homology groups $H_\ast(A_\ast)$. A homological unit in $A_\ast$ is an element $1 \in A_0$ such that $b_1(1) = 0$, and the cohomology class of 1 is the unit for the associative algebra $H_1(A_\ast)$. A homological unit 1 is strict if it is a left unit for $m_2$, and $b_n(1, \ldots) = 0$ for $n \geq 3$. A strict unit induces a contracting homotopy $h$ for the differential $\delta$ on $T_\ast(A_\ast[1])$ by setting $h(a_1 \otimes \cdots \otimes a_n) = 1 \otimes a_1 \otimes \cdots \otimes a_n$. We will assume that all $A_\infty$-algebras are equipped with a strict unit.

An $A_\infty$-morphism $f$ between $A_\infty$-algebras $A_\ast, A'_\ast$ is a DG coalgebra morphism $T_\ast(A_\ast[1]) \to T_\ast(A'_\ast[1])$. Explicitly, $f$ is given by a collection of maps $f_n: A_{\ast \leq n} \to A'_{\ast \leq n}$, $n \geq 1$ such that

$$\sum_{i+j+l=n} f_{i+1+l} \circ (\text{id}^\otimes_i \otimes b_j \otimes \text{id}^\otimes_l) = \sum_{i_1+\cdots+i_s=s} b_s \circ (f_{i_1} \otimes \cdots \otimes f_{i_s})$$

(1.10)
for any $n \geq 1$. In particular, $f_1$ is a map of complexes, and it induces an algebra map $H_\ast(f_1): H_\ast(A_\ast) \to H_\ast(A'_\ast)$. The map $f$ is unital if so is $H_\ast(f_1)$.

Every DG algebra is automatically an $A_\infty$-algebra (with trivial $b_n$, $n \geq 3$). In particular, for a complex $M_\ast$ of objects in an abelian category $\text{Ab}$ as in Section 1.1, $\text{End}(M_\ast)$ is a DG algebra. The structure of an $A_\infty$ module over $A_\ast$ on the complex $M_\ast$ is given by a unital $A_\infty$-morphism $A_\ast \to \text{End}(M_\ast)$. Explicitly, this given by a collection of maps

$$b_n: A_{\ast \leq n-1} \otimes M_\ast \to M_\ast$$

for all $n \geq 2$ satisfying (1.8) and the unitality conditions (where $b_1$ is the differential on $M_\ast$). Equivalently, an $A_\infty$-module structure on $M_\ast$ is given by a differential $\delta$ on the cofree $T_\ast(A_\ast[1])$-comodule $T_\ast(A_\ast[1]) \otimes M_\ast$, which turns it into a DG comodule (again, with the unitality conditions).

1.5.2. **Homotopy.** The homotopy category $\text{Ho}(A_\ast, \text{Ab})$ of $A_\infty$ modules over $A_\ast$ is the full subcategory in the chain-homotopy category of DG comodules over $T_\ast(A_\ast[1])$ spanned by $A_\infty$-modules (objects are complexes $M_\ast$ equipped with an $A_\infty$-module structure, maps are chain-homotopy classes of maps between the corresponding DG comodules $T_\ast(A_\ast[1])$). Explicitly, if we are given two complexes $M_\ast, M'_{\ast \leq 1}$ equipped with $A_\infty$-module structures over $A_\ast$, then the graded group $\text{Hom}^\ast_A(M_\ast, M'_{\ast \leq 1})$ of maps between the corresponding $T_\ast(A_\ast[1])$-comodules can be canonically written
down as
\[ \text{Hom}_A^n(M_\bullet, M'_\bullet) = \prod_{n \geq 0} \text{Hom}_A^{n-1}(A^{\otimes n}_\bullet \otimes M_\bullet, M'_\bullet), \]
and it has a natural differential given by \( d(a) = \delta \circ a - a \circ \delta \). Maps in \( \text{Ho}(A_\bullet, \text{Ab}) \) are the degree-0 homology classes of this complex. The category \( \text{Ho}(A_\bullet, \text{Ab}) \) is obviously triangulated. Inside it, we have the full triangulated subcategory spanned by those \( M_\bullet \) which are acyclic as complexes of objects in \( \text{Ab} \).

**Lemma 1.4.** The subcategory of acyclic \( A_\infty \)-modules in \( \text{Ho}(A_\bullet, \text{Ab}) \) is localizing.

*Proof.* As in the case of unbounded complexes of objects studied in [Ke1], say that an \( A_\infty \)-module \( M_\bullet \) is \( h \)-injective if it is right-orthogonal in \( \text{Ho}(A_\bullet, \text{Ab}) \) to all acyclic \( A_\infty \)-modules. Then it suffices to prove that for every \( M_\bullet \in \text{Ho}(A_\infty, \text{Ab}) \), there exists an \( h \)-injective \( \tilde{M}_\bullet \in \text{Ho}(A_\bullet, \text{Ab}) \) equipped with a quasiisomorphism \( \tilde{M}_\bullet \to M_\bullet \). Choose a complex \( \tilde{M}_\bullet \) of objects in \( \text{Ab} \) which is \( h \)-injective and equipped with an injective quasiisomorphism \( M_\bullet \to \tilde{M}_\bullet \). Then since we assume that \( A_\bullet \) is a complexes of free \( \mathbb{Z} \)-modules, the map \( A^{\otimes n}_\bullet \otimes M_\bullet \to A^{\otimes n}_\bullet \otimes \tilde{M}_\bullet \) is an injective quasiisomorphism for any \( n \). Then we can solve the equations (1.8) by induction on \( n \), to obtain an \( A_\infty \)-module structure on \( \tilde{M}_\bullet \) and an \( A_\infty \)-quasiisomorphism \( M_\bullet \to \tilde{M}_\bullet \). Moreover, for any acyclic \( A_\infty \)-module \( N_\bullet \), the terms in the complex \( \text{Hom}_{A_\bullet}(N_\bullet, \tilde{M}_\bullet) \) can be rewritten as
\[ \text{Hom}_{A_\bullet}(A^{\otimes n}_\bullet \otimes N_\bullet, M_\bullet) \cong \text{Hom}_A^n(A^{\otimes n}_\bullet, \text{Hom}_{A_\bullet}(N_\bullet, \tilde{M}_\bullet)), \]
and since \( A_\bullet \) is a complex of free \( \mathbb{Z} \)-modules and \( \tilde{M}_\bullet \) is right-orthogonal to \( N_\bullet \), these are acyclic complexes. Thus \( \text{Hom}_{A_\bullet}(N_\bullet, \tilde{M}_\bullet) \) is the limit of an inverse system of acyclic complexes of abelian groups, and the transition maps in this system are surjective. Therefore the inverse limit is also acyclic, and \( \tilde{M}_\bullet \) is \( h \)-injective in \( \text{Ho}(A_\bullet, \text{Ab}) \). \( \square \)

**Definition 1.5.** The derived category \( \mathcal{D}(A_\bullet, \text{Ab}) \) is obtained by localizing the category \( \text{Ho}(A_\bullet, \text{Ab}) \) with respect to the subcategory of acyclic \( A_\infty \)-modules.

We have the obvious forgetful functor \( \mathcal{D}(A_\bullet, \text{Ab}) \to \mathcal{D}(\text{Ab}) \). It has both a left and a right-adjoint, the free and the cofree module functors; they send a complex \( M_\bullet \) into \( A_\infty \)-comodules given by \( A_\bullet \otimes M_\bullet \), resp. \( \text{Hom}_{\mathbb{Z}}(A_\bullet, M_\bullet) \), with an \( A_\infty \)-module structure induced by the structure maps \( b_n \) of the \( A_\infty \)-algebra \( A \). To see the adjunction, one notes that we have a tautological \( A_\infty \)-map \( A_\bullet \otimes M_\bullet \to M_\bullet \), whose terms are the structure maps of the \( A_\infty \)-module \( M_\bullet \), and the contracting homotopy \( h \) given by the strict unit in \( A_\bullet \) induces a homotopy which contracts
\[ \text{Hom}_{A_\bullet}(A_\bullet \otimes M_\bullet, N_\bullet) \]
to \( \text{Hom}(M_\bullet, N_\bullet) \) for any \( A_\infty \)-module \( N_\bullet \), and similarly for the cofree module \( \text{Hom}_{\mathbb{Z}}(A_\bullet, N_\bullet) \). In fact, we have
\[ \text{Hom}_{A_\bullet}^n(A^{\otimes n}_\bullet \otimes A_\bullet \otimes M_\bullet, N_\bullet) \cong \text{Hom}_{A_\bullet}^n(A^{\otimes n}_\bullet, \text{Hom}_{\mathbb{Z}}(A_\bullet, N_\bullet)) \]
Iterating the adjunction map \( A_\ast \otimes M_\ast \to M_\ast \), we obtain a version of the bar resolution for \( A_\infty \)-modules; in effect, any \( A_\infty \)-module \( M_\ast \) is quasiisomorphic to the direct limit

\[
\lim_{\to n} M_{(n)},
\]

so that the transition maps \( M_{(n)} \to M_{(n+1)} \) are injective, and their cokernels are free \( A_\infty \)-modules. Dually, we have the cobar resolution, and every \( M_\ast \) can represented as an inverse limit of a system with surjective transition maps with cofree kernels.

More generally, for any \( A_\infty \)-map \( f: A_\ast \to A'_\ast \) between \( A_\infty \)-algebras, we have an obvious restriction functor \( f^*: \mathcal{D}(A'_\ast, \text{Ab}) \to \mathcal{D}(A_\ast, \text{Ab}) \). Replacing an \( A_\infty \)-module with its free (resp., cofree) resolution, one easily shows that \( f^* \) has both a left-adjoint \( f_l \) and a right-adjoint \( f_r \).

**Lemma 1.6.** If the \( A_\infty \)-map \( f: A_\ast \to A'_\ast \) is a quasiisomorphism, then \( f^* \) and \( f_l \) are mutually inverse equivalences of categories.

**Proof.** Let \( M_\ast, N_\ast \) be two \( A_\infty \)-modules over \( A'_\ast \). Rewrite the terms of the complex (1.11) as in (1.12). Then for any \( n \geq 0 \), the natural map

\[
f^*: \text{Hom}^*(A_\ast^\otimes n, \text{Hom}^*(N_\ast, \tilde{M}_\ast)) \to \text{Hom}^*(A'_\ast^\otimes n, \text{Hom}^*(N_\ast, \tilde{M}_\ast))
\]

is a quasiisomorphism. This implies that \( f^* \) is fully faithful. To see the it is essentially surjective, note that it commutes with filtered direct limits, and for any complex \( M_\ast \), \( f^*(A'_\ast \otimes M_\ast) \) is quasiisomorphic to the free \( A_\infty \)-module \( A_\ast \otimes M_\ast \). □

1.5.3. Coalgebras. An \( A_\infty \)-coalgebra is an \( A_\infty \)-algebra in the category opposite to that of abelian groups. Explicitly, an \( A_\infty \)-coalgebra structure on a graded \( \mathbb{Z} \)-module \( A_\ast \), is given by a collection of operations

\[
b_n: A_\ast \to A_\ast^\otimes n, \quad n \geq 1,
\]

subject to relations (1.8) (where the composition \( \circ \) should be understood in the reverse order). Equivalently, this structure is encoded by a collection of maps

\[
A_\ast \otimes \text{Ass}_\infty \to A_\ast^\otimes n,
\]

where \( \text{Ass}_\infty \) is the asymmetric \( A_\infty \)-operad. As in the algebra case, we will only consider \( A_\infty \)-coalgebras which are free as \( \mathbb{Z} \)-modules. For any \( A_\infty \)-coalgebra \( A_\ast \), \( b_1 \) is a differential, that is, \( b_1^2 = 0 \), so that \( A_\ast \) becomes a complex of free abelian groups. The dual complex \((A_\ast)^* = \text{Hom}_\mathbb{Z}(A_\ast, \mathbb{Z})\) is naturally an \( A_\infty \)-algebra. A strict counit for an \( A_\infty \)-coalgebra \( A_\ast \) is a map \( 1: A_0 \to \mathbb{Z} \) such that \( 1 \circ b_1 = 0 \), and \( 1 \in (A_0)^* \) is a strict unit for \((A_\ast)^*\). We will only consider counital coalgebras.

An \( A_\infty \)-morphism \( f: A'_\ast \to A_\ast \) of \( A_\infty \)-coalgebras is given by a collection of maps

\[
f_n: A'_\ast \to A_\ast^\otimes n, \quad n \geq 1,
\]

satisfying (1.10). Such a map is a quasiisomorphism if so is its component \( f_1 \). The dual maps \( f_n^* \) give an \( A_\infty \)-map \( f^*: (A_\ast)^* \to (A'_\ast)^* \); \( f \) is counital if \( f^* \) is unital. As in the algebra case, we will only consider counital \( A_\infty \)-maps.
An $A_\infty$-comodule over an $A_\infty$-coalgebra $A$, in an abelian category $\text{Ab}$ is given by a graded object $M_\bullet$ in $\text{Ab}$ together with maps

$$b_n: M_\bullet \to M_\bullet \otimes A_{\otimes n-1} \quad n \geq 1,$$

again subject to (1.8). Again, $b_1$ is a differential on $M_\bullet$. Moreover, $M_\bullet$ is automatically an $A_\infty$-module over $(A_\bullet)^*$, and $H_*(M_\bullet)$ is a module over the cohomology algebra $H^*((A_\bullet)^*)$. If this module is unital, the $A_\infty$-comodule $M_\bullet$ is called counital. We will only consider counital comodules.

Equivalently, an $A_\infty$-structure on $A$, can be described as a square-zero derivation of the completed tensor algebra $\hat{T}^*(A, [-1])$, and an $A_\infty$-comodule structure on a complex $M_\bullet$ is the same as a differential on the completed tensor product

$$\hat{T}^*(A, [-1]) \otimes M_\bullet = \lim_{\leftarrow n} T^*(A, [-1]) / T^n(A, [-1]) \otimes M_\bullet,$$

which turns it into a DG $\hat{T}^*(A, [-1])$-module. The homotopy category $\text{Ho}(A_\bullet, \text{Ab})$ of $A_\infty$-comodules over $A$ in $\text{Ab}$ is the full subcategory in the chain-homotopy category of topological DG $\hat{T}^*(A, [-1])$-modules spanned by DG modules of the form (1.15). Given two $A_\infty$-comodules $M_\bullet$, $M'_\bullet$, we define a complex $\text{Hom}_{A_\bullet}(M_\bullet, M'_\bullet)$ as

$$\prod_{n \geq 0} \text{Hom}^*(M_\bullet, A_{\otimes n} \otimes M'_\bullet),$$

with the differential dual to that of (1.11). Then the space of maps from $M_\bullet$ to $M'_\bullet$ in $\text{Ho}(A_\bullet, \text{Ab})$ is the 0-th homology group of this complex. An object $M_\bullet \in \text{Ho}(A_\bullet, \text{Ab})$ is called acyclic if it is acyclic as a complex in $\text{Ab}$.

**Lemma 1.7.** Assume that $\text{Ab}$ is the category of modules over a ring $R$. Then the subcategory of acyclic complexes in $\text{Ho}(A_\bullet, \text{Ab})$ is localizing.

**Proof.** We adopt the method of [W, Proposition 10.4.4]; I am grateful to the referee for suggesting this reference.

As in [W, Proposition 10.4.4], it suffices to prove that for any $A_\infty$-comodule $M_\bullet \in \text{Ho}(A_\bullet, \text{Ab})$, there exists a set $A$ of quasismorphisms $r_\alpha: M_{\otimes n}^\alpha \to M_\bullet$, $\alpha \in A$, such that for any quasismorphism $r: M'_\bullet \to M_\bullet$, one of the maps $r_\alpha$ factors through $r$. Let $\kappa$ be an infinite cardinal larger than the cardinalities of $R$ and $\bigoplus M_\bullet$. There is at most a set of quasismorphisms $r_\alpha: M_{\otimes n}^\alpha \to M_\bullet$ such that the cardinality of $\bigoplus M_{\otimes n}^\alpha$ is at most $\kappa$; take them all. Assume given a quasismorphism $r: M'_\bullet \to M_\bullet$.

Every element $m \in M'_n$, $i \in \mathbb{Z}$, lies in an $A_\infty$-subcomodule $M_{n}^m \subset M'_n$ which is at most countably generated as an $R$-module—indeed, we can take the abelian subgroup generated by $m$, add to it all the left-hand sides of the structure maps $b_n$ of (1.14), and repeat the procedure by induction. Therefor there exists an $A_\infty$-subcomodule $M_{n}^{(1)} \subset M'_n$ of cardinality at most $\kappa$ such that the map

$$r_{n}^{(1)}: H_i(M_{n}^{(1)}) \to H_i(M_n)$$

induced by $r$ is surjective for every $i \in \mathbb{Z}$. Repeating the procedure, we obtain a system of subcomodules $M_{n}^{(n)} \subset M'_n$, $n \geq 1$, such that $M_{n}^{(n+1)}$ contains $M_{n}^{(n)}$, the
map
\[ r_i^{(n)} : H_i(M_i^{(n)}) \to H_i(M_i) \]
is still surjective for any \( n \geq 2, i \in \mathbb{Z} \), and the natural map
\[ H_i(M_i^{(n)}) \to H_i(M_i^{(n+1)}) \]
induced by the embedding \( M_i^{(n)} \to M_i^{(n+1)} \) annihilates \( \text{Ker} r_i^{(n)} \) for any \( i \in \mathbb{Z} \). Let \( M_i'' = \bigcup M_i^{(n)} \subset M_i' \). Then the natural map
\[ r'' : M_i'' \to M_i \]
is a quasiisomorphism by construction, and it is of the form \( r^\alpha \) for some \( \alpha \) in the indexing set \( A \).

\[ \square \]

**Remark 1.8.** The proof can probably be modified so that it only requires our original assumptions on \( \text{Ab} \), but I haven’t pursued it for lack of interesting examples.

**Definition 1.9.** The derived category \( \mathcal{D}(A_\infty, \text{Ab}) \) of \( A_\infty \)-comodules over \( A_\infty \) in \( \text{Ab} \) is the quotient of the homotopy category \( \text{Ho}(A_\infty, \text{Ab}) \) by the subcategory of acyclic objects.

As in the algebra case, an \( A_\infty \)-comodule \( M_\infty \in \text{Ho}(A_\infty, \text{Ab}) \) is called \( h \)-injective if it is right-orthogonal to all acyclic objects. We have an obvious forgetful functor \( \text{Ho}(A_\infty, \text{Ab}) \to \text{Ho}(\text{Ab}) \) onto the chain-homotopy category of complexes in \( \text{Ab} \), and it has a right-adjoint which sends \( M_\infty \in \text{Ho}(\text{Ab}) \) into the cofree comodule \( A_\infty \otimes M_\infty \), with the comodule structure maps \( b_n \) given by the structure maps of \( A_\infty \). As in the algebra case, the counit on \( A_\infty \) induces a homotopy which contracts the complex
\[ \text{Hom}_A^\infty (N_\infty, A_\infty \otimes M_\infty) \]
onoonto \( \text{Hom}(N_\infty, M_\infty) \) for any \( N_\infty, M_\infty \in \text{Ho}(A_\infty, \text{Ab}) \), and this gives the adjunction. In particular, if \( M_\infty \in \mathcal{D}(\text{Ab}) \) is \( h \)-injective, the cofree \( A_\infty \)-comodule \( A_\infty \otimes M_\infty \) is \( h \)-injective in \( \text{Ho}(A_\infty, \text{Ab}) \). One is tempted now to dualize the bar-construction and obtain an \( h \)-injective replacement for any \( M_\infty \in \text{Ho}(A_\infty, \text{Ab}) \). However, \textit{this does not work}. The reason is the following: to be \( h \)-injective, the cobar resolution of an \( A_\infty \)-comodule \( M_\infty \) has to be a projective limit of \( h \)-injective comodules. Thus as a graded object in \( \text{Ab} \), it is of the form
\[ A^* \otimes \prod_{n \geq 0} (A_\infty^* \otimes M_\infty) \]
However, this is different from
\[ \prod_{n \geq 0} (A_\infty^* \otimes A_\infty^* \otimes M_\infty) \]
and it is the latter, not the former, which can be contracted onto \( M_\infty \) by the counit of \( A_\infty \). This is why the existence of \( \mathcal{D}(A_\infty, \text{Ab}) \) has to be proved by an indirect method. And even when \( \text{Ab} \) is as in Lemma 1.7, it is not clear at present whether for an arbitrary \( A_\infty \), any \( M_\infty \in \text{Ho}(A_\infty, \text{Ab}) \) is quasiisomorphic to an \( h \)-injective \( M_\infty' \).
Remark 1.10. If $A$ is a DG coalgebra, then more is known, since a very comprehensive study of the homological properties of unbounded complexes of comodules has been done recently by L. Positselski [P]. In particular, it has been proved in [P] that $h$-injective replacements do exist. However, the proof is very indirect, and it is not clear at all whether it can be generalized to the $A_{\infty}$-case.

As in the algebra case, an $A_{\infty}$-map $f: A \to A'$ induces a natural corestriction functor $f^* : D(A, Ab) \to D(A', Ab)$, assuming that Ab is as in Lemma 1.7 so that both categories are well-defined. If $A' = \mathbb{Z}$ and $f$ is the counit, this is the forgetful functor $D(A, Ab) \to D(Ab)$, and it has a right-adjoint given by the cofree comodule construction. In general, it is not clear whether $f^*$ admits a right-adjoint (and the situation with the left-adjoint is even worse).

Let us list some other things that do not work for coalgebras.

(i) There are no free comodules, only the cofree ones ((1.13) does not work, since it would involve double dualization).

(ii) The proof of Lemma 1.4 breaks down because an analog of (1.12) is not an isomorphism.

(iii) For those interested in such things, the category of $A_{\infty}$-comodules for a general $A$, does not admit a closed model structure (or at least, none such is known).

(iv) Finally, Lemma 1.6 fails. Not only does its proof break down, the statement itself is false. In fact, one case where this happens will be the main subject of Section 6.

1.5.4. Categories. Informally, a (small) $A_{\infty}$-category is an $A_{\infty}$-algebra “with many objects”. To keep track of the combinatorics, it is convenient to use the approach of [Le]. For any set $S$, denote by $\mathbb{Z}_S$-mod the category of $S$-graded $\mathbb{Z}$-modules. Fix a set $S$, and consider the category $\mathbb{Z}_{S \times S}$-mod. Equip it with a tensor product by setting

$$(A' \otimes A'')[s, s'] = \bigoplus_{s \in S} A'_{s, s} \otimes A''_{s, s'}$$

for any $A', A'' \in \mathbb{Z}_{S \times S}$-mod. This is an associative and unital tensor product, with the unit object $I_S$ given by

$$I_{S s, s'} = \begin{cases} \mathbb{Z}, & s = s', \\ 0, & \text{otherwise.} \end{cases}$$

The tensor product not symmetric, but neither are the operads $\text{Ass}$ and $\text{Ass}_{\infty}$, so that speaking about associative and $A_{\infty}$-algebras in $\mathbb{Z}_{S \times S}$-mod makes perfect sense. For any small additive category $\mathcal{B}$ with the set of objects $S$, the sum $\bigoplus_{s, s'} \mathcal{B}(s, s')$ is a unital associative algebra in $\mathbb{Z}_{S \times S}$-mod, and conversely, any such unital associative algebra in $\mathbb{Z}_{S \times S}$-mod defines an additive category with the set of objects $S$. Then an $A_{\infty}$-category $\mathcal{B}$ consists of

(i) a set of objects $S$,
(ii) an $A_\infty$-algebra $\mathcal{B}$, in $\mathbb{Z}[S\times S]\text{-mod}$, and

(iii) a map $I_\mathcal{S} \to \mathcal{B}$, which is a strict unit for the $A_\infty$-structure on $\mathcal{B}$.

As in the algebra case, we will assume that $\mathcal{B}(b, b')$ is a complex of free $\mathbb{Z}$-modules for any two objects $b, b' \in \mathcal{B}$. For every $A_\infty$-category $\mathcal{B}_\ast$, its homology $H_\ast(\mathcal{B}_\ast)$ is an additive category with the set of objects $S$.

For any map of sets $f: S \to S'$, we have an obvious pseudotensor restriction functor $f^*: \mathbb{Z}[S\times S]\text{-mod} \to \mathbb{Z}[S'\times S']\text{-mod}$; an $A_\infty$-functor between $A_\infty$-categories $\mathcal{B}_\ast$, $\mathcal{B}'_\ast$ consists of a map $f: S \to S'$ and a unital $A_\infty$-morphism $\mathcal{B}_\ast \to f^*\mathcal{B}'_\ast$.

For any abelian category $\text{Ab}$ as in Section 1.1, the category $\text{Ab}$ is generated by representable and corepresentable functors: for any object $b \in \mathcal{B}$, and any $M_\ast \in \text{Ho}(\text{Ab})$, these are given by

$$M^\prime_\ast(b') = M_\ast \otimes \mathcal{B}_\ast(b, b'), \quad M_\ast(b') = \text{Hom}_\mathbb{Z}(\mathcal{B}(b', b), M_\ast)$$

for any $b' \in \mathcal{B}$. We also have the bar and cobar resolution, so that the derived category $\mathcal{D}(\mathcal{B}_\ast, \text{Ab})$ is generated by representable resp. corepresentable functors in the same sense as the category of $A_\infty$-modules is generated by free resp. cofree modules. For any $A_\infty$-functor $f: \mathcal{B}_\ast \to \mathcal{B}'_\ast$, between two $A_\infty$-categories, we have the restriction functor $f^*: \mathcal{D}(\mathcal{B}'_\ast, \text{Ab}) \to \mathcal{D}(\mathcal{B}_\ast, \text{Ab})$ and its two adjoints

$$f_!: \mathcal{D}(\mathcal{B}_\ast, \text{Ab}) \to \mathcal{D}(\mathcal{B}'_\ast, \text{Ab}), \quad f_*: \mathcal{D}(\mathcal{B}_\ast, \text{Ab}) \to \mathcal{D}(\mathcal{B}'_\ast, \text{Ab}).$$

An $A_\infty$-functor $f: \mathcal{B}_\ast \to \mathcal{B}'_\ast$ is a \textit{quasiisomorphism} if the corresponding functor $H_\ast(\mathcal{B}_\ast) \to H_\ast(\mathcal{B}'_\ast)$ is an equivalence, and the natural map

$$f_!: \mathcal{B}_\ast(b, b') \to \mathcal{B}'_\ast(f(b), f(b'))$$

is a quasiisomorphism for any $b, b' \in \mathcal{B}$. It is not difficult to generalize Lemma 1.6 and show that for a quasiisomorphism $f$, the functors $f^*$ and $f_!$ are mutually inverse equivalences of the derived categories.

We note that any small category $\mathcal{C}$ defines an additive category $\mathbb{Z}[\mathcal{C}]$ with the same objects, and morphisms given by $\mathbb{Z}[\mathcal{C}][c, c'] = \mathbb{Z}[\mathcal{C}(c, c')]$, where $\mathbb{Z}[S]$ for a set $S$ means the free abelian group generated by $S$. Then $\mathbb{Z}[\mathcal{C}]$ can be treated as an $A_\infty$-category (placed in homological degree $0$), and we of course have $\mathcal{D}(\mathcal{C}, \text{Ab}) \cong \mathcal{D}(\mathbb{Z}[\mathcal{C}], \text{Ab})$.

We will also need a version of this for coalgebras. In fact, it will more convenient to use a slightly more refined notion. Assume given a small category $\mathcal{C}$, with the set of objects $\mathcal{C}^{\text{ob}}$ and the set of morphisms $\mathcal{C}^{\text{mor}}$. Introduce a tensor product on the category $\mathbb{Z}[\mathcal{C}^{\text{mor}}]\text{-mod}$ by setting

$$(A' \otimes A'')_f = \bigoplus_{f', f'' \in \mathcal{C}^{\text{mor}}, f' \circ f'' = f} A'_f \otimes A''_{f''}$$

for any $A', A'' \in \mathbb{Z}[\mathcal{C}^{\text{mor}}]\text{-mod}$ and $f \in \mathcal{C}^{\text{mor}}$. Then $\mathbb{Z}[\mathcal{C}^{\text{mor}}]\text{-mod}$ is a monoidal category, and for any abelian category $\text{Ab}$ as in Section 1.1, $\text{Ab}_{\mathbb{Z}[\mathcal{C}^{\text{ob}}]}$ is a module category over $\mathbb{Z}[\mathcal{C}^{\text{ob}}]\text{-mod}$. 

**Definition 1.11.** A $C$-graded $A_{\infty}$-coalgebra $A_*$ is an $A_{\infty}$-coalgebra in the monoidal category $Z_{\text{Comod}}$-mod. An $A_{\infty}$-comodule in $\text{Ab}$ over $A_*$ is an $A_{\infty}$-module over $A_*$ in the module category $\text{Ab}_{\text{Com}}$.

As in the $A_{\infty}$-category case, we will always assume that our graded coalgebras consist of free $Z$-modules; we will also assume that coalgebras and comodules are counital in the same sense as in the non-graded case. Explicitly, a $C$-graded $A_{\infty}$-coalgebra $A_*$ is given by a collection $A_*(f)$ of complexes of free $Z$-modules numbered by morphisms $f$ of the category $C$, together with a homological counit map $A_*(\text{id}_c) \to Z$ for any object $c \in C$ and a comultiplication map

$$b_n: A_*(f_1 \circ \cdots \circ f_n) \to A_*(f_1) \otimes \cdots \otimes A_*(f_n)$$

for any $n \geq 2$ and any $n$-tuple of composable maps $f_1, \ldots, f_n$ in $C$, subject to (1.8).

An $\text{Ab}$-valued $A_*$-comodule $E_*$ in the category $\text{Ab}$ is a collection of complexes $E_*(c)$ in $\text{Ab}$, one for each object $c \in C$, together with maps

$$b_n: E_*(c) \to A_*(f_1) \otimes \cdots \otimes A_*(f_n) \otimes E_*(c')$$

for any $n \geq 1$ and any $n$-tuple of composable maps $f_0, \ldots, f_n, f_0 \circ \cdots \circ f_n: c \to c'$ in $C$, again subject to (1.8).

As in the non-graded case, any $C$-graded $A_{\infty}$-coalgebra $A_*$ produces the homotopy and the derived categories of $A_*$-comodules, denoted $\text{Ho}(A_*, \text{Ab})$ resp. $\mathcal{D}(A_*, \text{Ab})$. For any object $c \in C$ and any complex $M_*$ in $\text{Ab}$, we have the corepresentable $A_*$-comodule $M'_*(c)$ given by

$$M'_*(c') = \prod_{f: c' \to c} A_*(f) \otimes M_*.$$  \hspace{1cm} (1.16)

For any $C$-graded $A_{\infty}$-map $f: A_* \to A'_*$, the restriction functor $f^*: \mathcal{D}(A_*, \text{Ab}) \to \mathcal{D}(A'_*, \text{Ab})$ is defined. As in the coalgebra case, it does not have to be an equivalence even if $f$ is a quasiisomorphism in a suitable sense. On the other hand, assume given another small category $C'$ and a functor $\rho: C' \to C$. Define a $C'$-graded $A_{\infty}$-coalgebra $\rho^*A_*$ by setting

$$\rho^*A_*(f) = A_*(\rho(f))$$

for any morphism $f$ in $C'$, with the same structure maps $b_n$. We then have an obvious pullback functor

$$\rho^*: \mathcal{D}(A_*, \text{Ab}) \to \mathcal{D}(\rho^*A_*, \text{Ab}).$$

I do not know under what assumptions, if any, either of the functors $\rho^*$, $f^*$ has a right adjoint.

**1.6. 2-categories.** To produce $A_{\infty}$-categories, we will use 2-categories; we end the preliminaries with a brief sketch of the corresponding construction.

Assume given a small monoidal category $C$; consider the bar complex $C_*(C, Z)$. If $C$ is strictly associative, then the tensor product functor $m: C \times C \to C$ induces an associative DG algebra structure on $C_*(C, Z)$ (apply the properties (ii) and (iii) of Section 1.3). More generally, if we also have an object $T \in \text{Fun}(C, Z)$ and a map

$$T \otimes T \to m^*T$$  \hspace{1cm} (1.17)
which is associative on triple products, then \( C_\ast(T) \) becomes an associative DG algebra (plug in the property (i)).

However, monoidal categories in nature are usually associative only up to an isomorphism — there is an associativity isomorphism \( m \circ m_{12} \cong m \circ m_{23} \) satisfying the pentagon equation. We observe that in this case \( C_\ast(C, \mathbb{Z}) \) is no longer a DG algebra, but it has an \( A_\infty \)-algebra structure.

Indeed, for any \( n \geq 2 \), let \( I_n \) be the groupoid whose objects are all possible \( n \)-ary operations obtained from a single binary operation, and which has exactly one morphism between every two objects. Then \( I_n, n \geq 2 \), form an asymmetric operad of categories, and any weakly associative monoidal category \( C \) is an algebra over this operad: we have natural functors

\[
I_n \times C^n \to C
\]

for every \( n \). The bar complexes \( C_\ast(I_n, \mathbb{Z}) \) form an asymmetric operad of complexes of \( \mathbb{Z} \)-modules, \( C_\ast(C, \mathbb{Z}) \) is an algebra over this operad, and the operad itself is a resolution of the trivial asymmetric operad \( \text{Ass} \). The asymmetric operad \( \text{Ass}_\infty \) is another such resolution, and it is cofibrant. Therefore the augmentation map \( \text{Ass}_\infty \to \text{Ass} \) factors through a map \( \text{Ass}_\infty \to C_\ast(I_n, \mathbb{Z}) \). Fixing this map once and for all, we turn the bar complex \( C_\ast(C, \mathbb{Z}) \) for any monoidal category \( C \) into an \( A_\infty \)-algebra.

Analogously, a (weakly) monoidal functor between monoidal categories \( C, C' \) induces an \( A_\infty \)-map between the \( A_\infty \)-algebras \( C_\ast(C, \mathbb{Z}), C_\ast(C', \mathbb{Z}) \).

The same construction obviously works for bar complex \( C_\ast(C, T) \) with coefficients, where \( T \in \text{Fun}(T, \mathbb{Z}) \) is equipped with an associative map (1.17).

If \( C \) has a strict unit object, then the corresponding \( A_\infty \)-algebras are strictly unital (a unit object in the usual sense only gives a homological unit).

Moreover, if we have a 2-category \( Q \) with a certain set of objects \( \{c\} \) and categories of 1-morphisms \( Q(c, c') \), \( c, c' \in C \), then the same construction produces an \( A_\infty \)-category with the same objects, and with the bar complexes \( C_\ast(Q(c, c'), \mathbb{Z}) \) as complexes of morphisms. This is also functorial with respect to 2-functors, and has an obvious version with coefficients. In order to ensure the unitality properties, we need to assume that the identity 1-morphisms in \( Q \) are strict.

2. Recall on Mackey Functors

This ends the preliminaries. For the convenience of the reader, we start the paper itself by briefly recalling the definitions and known facts about Mackey functors (we more-or-less follow the expositions in [M] and [TW]).

2.1. Definitions. Assume given a group \( G \), and let \( \Gamma_G \) be the category of finite sets equipped with a \( G \)-action. This category obviously has pullbacks. Define a bigger category \( Q\Gamma_G \) as follows: objects are the same as in \( \Gamma_G \), maps from \( S_1 \) to \( S_2 \) are isomorphism classes of diagrams \( S_1 \leftarrow S'_1 \to S_2 \), composition of \( S_1 \leftarrow S'_1 \to S_2 \) and \( S_2 \leftarrow S'_2 \to S_3 \) is given by the diagram \( S_1 \leftarrow S'_1 \times_{S_2} S'_2 \to S_3 \).

The category \( Q\Gamma_G \) is self-dual, \( Q\Gamma_G \cong Q\Gamma_G^{\text{opp}} \). Moreover, disjoint unions of sets give finite coproducts both in the category \( \Gamma_G \) and in the category \( Q\Gamma_G \). Every \( G \)-
finite set \( S \in \Gamma_G \) can be canonically decomposed into such a disjoint union
\[
S = \coprod_{p \in S/G} S_p
\]  
(2.1)
of subsets \( S_p \) on which \( G \) acts transitively; we call them \( G \)-orbits. This decomposition is valid both in \( \Gamma_G \) and in \( Q\Gamma_G \).

**Definition 2.1.** A \( G \)-Mackey functor \( M \) is a functor \( M : Q\Gamma_G \to \text{Ab} \) to the category \( \text{Ab} \) of abelian groups which is additive in the following sense: for any \( S \in Q\Gamma \), the natural map
\[
\bigoplus_{p \in S/G} M(S_p) \to M(S)
\]induced by the decomposition (2.1) is an isomorphism.

Mackey functors clearly form an abelian category which we denote by \( \mathcal{M}(G, \text{Ab}) \), or simply by \( \mathcal{M}(G) \) and which is a full subcategory in \( \text{Fun}(Q\Gamma_G, \text{Ab}) \). One checks easily that the embedding \( \mathcal{M}(G, \text{Ab}) \to \text{Fun}(Q\Gamma_G, \text{Ab}) \) admits a left adjoint, which we call **additivization** and denote by \( \text{Add} : \text{Fun}(Q\Gamma_G, \text{Ab}) \to \mathcal{M}(G, \text{Ab}) \) (in fact, \( \text{Add} \) is also right-adjoint to the embedding).

For any cofinite subgroup \( H \in G \), the value \( M([G/H]) \) of a \( G \)-Mackey functor \( M \) on the \( G \)-orbit \([G/H]\) is usually denoted by \( M^H \). By the additivity property, \( M^H \) for all \( H \subset G \) completely define \( M \). Explicitly, a \( G \)-Mackey functor \( M \) is given by
\[
\begin{align*}
(\text{i}) & \quad \text{an abelian group } M^H \text{ for any cofinite subgroup } H \subset G, \text{ and } \\
(\text{ii}) & \quad \text{two maps } f_* : M^{H_1} \to M^{H_2}, \ f^* : M^{H_2} \to M^{H_1} \text{ for any two cofinite subgroups } H_1, H_2 \subset G \text{ and a } G\text{-equivariant map } f : [G/H_1] \to [G/H_2], \end{align*}
\]
such that \( f^* \circ g^* = (g \circ f)^* \) and \( g_* \circ f_* = (g \circ f)_* \) for any two composable maps \( f : [G/H_1] \to [G/H_2], \ g : [G/H_2] \to [G/H_3] \), and for any two maps \( f : [G/H_1] \to [G/H], \ g : [G/H_2] \to [G/H] \), we have
\[
g_* \circ f_* = \sum_{p \in S/G} f_{p*} \circ g_{p*}, \tag{2.2}
\]where we let \( S = [G/H_1] \times [G/H_2] \), take its decomposition (2.1), and let \( g_p : S_p \to [G/H_1], \ f_p : S_p \to [G/H_2] \) be the natural projections. We note that since \( S/G = G/(G \times G)/(H_1 \times H_2) \cong H_1 \backslash G/H_2 \), the components \( S_p \) correspond to double cosets \( H_1 g H_2 \subset G \); for this reason, (2.2) is known as the **double coset formula**. This is the original definition of Mackey functors introduced by Dress [Dr]; the version with the category \( Q\Gamma_G \) is due to Lindner [Li]. The collection \( (f_*, f^*) \) without the condition (2.2) is sometimes called a **bifunctor** (from the category of finite \( G \)-orbits to \( \text{Ab} \)).

**Example 2.2.** Representation ring: setting \([G/H] \mapsto R_H\), the representation ring of the group \( H \), defines a Mackey functor, with \( f^* \) given by restriction and \( f_* \) given by induction. This is the origin of the notion and the name: the double coset formula for \( R_H \) was found by Mackey.

**Example 2.3.** Cohomology: setting \([G/H] \mapsto H^*(H, \mathbb{Z})\), \( f^* \) given by restriction, \( f_* \) given by corestriction, defines a (graded) Mackey functor.
To obtain a third useful definition of Mackey functors, one considers an additive category $\mathcal{B}^G$ defined as follows: objects of $\mathcal{B}^G$ are finite $G$-orbits $[G/H]$, and the set of maps from $S_1$ to $S_2$ is the free abelian group generated by isomorphism classes of diagrams $S_1 \leftarrow S \rightarrow S_2$, where $S$ is another finite $G$-orbit. Composition $g \circ f$ of two maps $f: [G/H_1] \rightarrow [G/H_2]$, $g: [G/H_2] \rightarrow [G/H_3]$ represented by diagrams $[G/H_1] \leftarrow S_f \rightarrow [G/H_2]$, $[G/H_2] \leftarrow S_g \rightarrow [G/H_3]$ is given by

$$g \circ f = \sum_p (g \circ f)_p,$$

where we consider the decomposition (2.1) of the fibered product $S = S_f \times_{[G/H_2]} S_g$, and let $(g \circ f)_p$ be the map represented by the diagram $[G/H_1] \leftarrow S_p \rightarrow [G/H_2]$. Then a $G$-Mackey functor $M$ is obviously the same thing as an additive functor $\mathcal{B}^G \rightarrow \text{Ab}$. The category $\mathcal{B}^G$ can be described more explicitly in terms of the so-called “Burnside rings”.

**Definition 2.4.** The Burnside ring $\mathcal{A}^G$ of a group $G$ is the abelian group generated by isomorphism classes $[S]$ of objects $S \in \Gamma_S$ in the category $\Gamma_G$, modulo the relations $[S_1] + [S_2] = [S_1 \cup S_2]$, and with the product given by $[S_1] \cdot [S_2] = [S_1 \times S_2]$.

Then the endomorphism ring of the trivial $G$-orbit $[G/G] \in \mathcal{B}^G$ obviously coincides with the Burnside ring $\mathcal{A}^G$. And more generally, given two finite orbits $S_1, S_2 \in \mathcal{B}^G$, we have

$$\mathcal{B}^G(S_1, S_2) = \bigoplus_{p \in (S_1 \times S_2)/G} \mathcal{A}^H_p,$$  

(2.3)

where $S_p = [G/H_p]$ are the components in the decomposition (2.1) of the product $S = S_1 \times S_2$.

**Remark 2.5.** Normally the definitions in this subsection are given for a finite group $G$; however, everything works in a slightly wider generality, and this is sometimes useful. Of course, the category $\mathcal{M}(G)$ as defined here only depends on the profinite completion of the group $G$. There is also a version of Mackey functors for topological groups such as finite-dimensional Lie groups, see e.g. [tD]; however, this is beyond the scope of the present paper.

### 2.2. Functoriality

For any cofinite subgroup $H \subset G$ of a group $G$ and a finite $H$-set $S$, the product $S \times_H G = (S \times G)/H$ is naturally a finite $G$-set, with the $G$-action through the second factor. This defines a functor $\gamma^G_H: \Gamma_G \rightarrow \Gamma_G$. In fact, if we denote $S = [G/H]$, then $\gamma^G_H$ is an equivalence between $\Gamma_G$ and the category $\Gamma_G/S$ of finite $G$-sets equipped with a map to $S$. The functor $\gamma^G_H$ obviously commutes with fibered products, thus extends to a functor $\gamma^G_H: Q\Gamma_G \rightarrow Q\Gamma_G$. It also commutes with disjoint unions, so that we can define an exact functor

$$\text{Restr}^G_H: \mathcal{M}(G) \rightarrow \mathcal{M}(H)$$

which sends $M: Q\Gamma_G \rightarrow \text{Ab}$ to $\gamma^G_H \circ M: Q\Gamma_G \rightarrow \text{Ab}$. For any $G$-Mackey functor $M \in \mathcal{M}(G)$, the $H$-Mackey functor $\text{Restr}^G_H(M)$ is called the **restriction** of $M$ to $H \subset G$.

Assume that a subgroup $H \subset G$ is normal, and let $N = G/H$ be the quotient. Then any $N$-set is also a $G$-set, so that we have an obvious full embedding $\Gamma_N \rightarrow \Gamma_G$.
which induces a full embedding $Q\Gamma_N \to Q\Gamma_G$ compatible with disjoint unions. This induces a functor $\Psi^H: \mathcal{M}(G) \to \mathcal{M}(N)$ (usually $\Psi^H(M)$ is denoted simply by $M^H$, but this might cause confusion). However, we also have a functor $\text{Infl}_G^N: \mathcal{M}(N) \to \mathcal{M}(G)$ called inflation and given by “extension by 0”: we set

$$\text{Infl}_G^N(M)^K = \begin{cases} M^{K/H}, & H \subset K, \\ 0, & \text{otherwise}. \end{cases}$$

This is an exact full embedding. It has a left-adjoint denoted by $\Phi^H: \mathcal{M}(G) \to \mathcal{M}(N)$.

If the subgroup $H \subset G$ is not normal, we can consider its normalizer $N_H \subset G$. Assume that the normalizer $N_H \subset G$ is cofinite in $G$. Then we can define the functor $\Phi^H: \mathcal{M}(G) \to \mathcal{M}(N_H/H)$ by first restricting to $N_G(H)$:

$$\Phi^H(M) = \Phi^H(\text{Restr}^G_{N_H}(M)).$$

Analogously, $M^H$ has a natural structure of a $(N_H/H)$-Mackey functor.

**2.3. Products.** Both in Example 2.2 and Example 2.3, the Mackey functors have an additional structure— an associative product. This is axiomatized as follows (this definition is taken from [D, Subsection 6.2]).

**Definition 2.6.** A **Green functor** is a Mackey functor $M \in \mathcal{M}(G)$ equipped with an associative product in each $M^H$ such that

(i) for any $f$, the map $f^*$ preserves the product,

(ii) for any $f: [G/H_1] \to [G/H_2]$, $x \in M^{H_2}$, $y \in M^{H_1}$, we have

$$x \cdot f_*(y) = f_*(x \cdot y), \quad f_*(y) \cdot x = f_*(y \cdot f^*(x)).$$

We note that the Cartesian product of finite sets defines a functor $m: Q\Gamma_G \times Q\Gamma_G \to Q\Gamma_G$; this induces a symmetric tensor product on the category $\mathcal{M}(G)$ by

$$M \otimes N = \text{Add}(m(M \boxtimes N)).$$

A Green functor is then the same as a Mackey functor equipped with an algebra structure in the symmetric tensor category $\mathcal{M}(G)$. For example, to see the condition (i), one can argue as follows. Consider the natural embedding $i: \Gamma^*_G \to Q\Gamma_G$. Then for any $M, N, K \in \mathcal{M}(G)$ we have

$$\text{Hom}(M \otimes N, K) = \text{Hom}(\text{Add}(m(M \boxtimes N)), K) = \text{Hom}(m_!(M \boxtimes N), K) = \text{Hom}(M \boxtimes N, m^*K),$$

and since $m^*$ commutes with $i^*$, every map $M \otimes N \to K$ induces a map $i^*M \otimes i^*N \to i^*K$, where the tensor product on $\text{Fun}(\Gamma^*_G, \text{Ab})$ is again given by the direct image $m_!$ with respect to the product functor $m: \Gamma^*_G \times \Gamma^*_G \to \Gamma^*_G$. However, on $\Gamma^*_G$, this product functor is left-adjoint to the diagonal embedding $\delta: \Gamma^*_G \to \Gamma^*_G \times \Gamma^*_G$; therefore $m_! \cong \delta^*$, and the tensor structure on $\text{Fun}(\Gamma^*_G, \text{Ab})$ is in fact given by the pointwise product, so that for any Green functor $M \in \mathcal{M}(G)$, the restriction $i^*M$ is simply a functor from $\Gamma^*_G$ to the category of rings.

Since the point orbit $[G/G] \in Q\Gamma_G$ is the unit object for the product functor $m$, the Mackey functor $A, [G/H] \to \mathcal{B}^G([G/G], [G/H])$ it represents is the unit object for the tensor product of Mackey functors and in particular, a Green functor. This
Green functor $A$ is called the Burnside ring Green functor — indeed, by (2.3), the component $A^H$ is exactly the Burnside ring of the finite group $H$. Every Mackey functor $M \in \mathcal{M}(G)$ is then a module over the Burnside ring Green functor $A$.

3. The Derived Version

Since the category $\mathcal{M}(G)$ of $G$-Mackey functors is abelian, one can consider its derived category $D(\mathcal{M}(G))$. However, as we have explained in the Introduction, its formal properties are somewhat deficient. The goal of this paper is to suggest a cure for this by defining a certain triangulated category which contains $\mathcal{M}(G)$ but differs from $D(\mathcal{M}(G))$, and has all the properties one would like to have.

The idea behind the construction is very simple. In the definition of the Burnside ring $A^G$, and more generally, in the definition of the additive category $B^G$, we take abelian groups spanned by isomorphism classes of certain objects — in other words, we take the 0-th homology group $H_0$ of a certain groupoid. The correct thing to take at the level of triangulated categories is the full homology, not just its degree-0 part.

There are several ways to make this precise. In this section, we give a construction which uses $A_\infty$ methods and bar-resolutions.

3.1. The quotient construction. Assume given a small category $\mathcal{C}$ which has fibered products. Then we can obviously define a category $Q\mathcal{C}$ as follows: objects are objects of $\mathcal{C}$, maps from $c_1 \in Q\mathcal{C}$ to $c_2 \in Q\mathcal{C}$ are given by isomorphism classes of diagrams $c_1 \to c \to c_2$, and compositions are given by the fibered products, as in Section 2.1.

However, we can refine the construction. Let $Q\mathcal{C}$ be the 2-category whose objects are again the objects of $\mathcal{C}$, and such that for any $c_1, c_2 \in \mathcal{C}$, the category $Q\mathcal{C}(c_1, c_2)$ of maps from $c_1$ to $c_2$ in $Q\mathcal{C}$ is the category of diagrams $c_1 \to c \to c_2$ and their isomorphisms. If $\mathcal{C}$ also has a terminal object, thus all products, we can equivalently set $Q(c_1, c_2) = Q(c_1 \times c_2)$, where $Q(c)$ for an object $c \in \mathcal{C}$ is the category of objects in $\mathcal{C}$ equipped with a map to $c$ and their isomorphisms. The composition is again given by fibered products.

Since fibered products are associative up to a canonical isomorphism, $Q\mathcal{C}$ is a well-defined 2-category. The identity 1-morphisms are given by the diagrams $c \to c \to c$ with both maps being the identity maps; these identity 1-morphisms are obviously strict. We denote by $B^\mathcal{C}$ the $A_\infty$-category in the sense of Section 1.5 associated to $Q\mathcal{C}$ by the procedure described in Section 1.6. Thus the objects in $B^\mathcal{C}$ are again the same as in $\mathcal{C}$, maps from $c_1$ to $c_2$ are given by

$$B^\mathcal{C}(c_1, c_2) = C_* (Q\mathcal{C}(c_1, c_2), \mathbb{Z}),$$

where $\mathbb{Z}$ in the right-hand side is the constant functor with value $\mathbb{Z}$, and compositions are induced by the 2-category structure on $Q\mathcal{C}$. We note that by definition, this $A_\infty$-category is concentrated in non-positive cohomological degrees.

**Definition 3.1.** The derived category $D(B^\mathcal{C}, \text{Ab})$ of $A_\infty$-functors from $B^\mathcal{C}$ to $\text{Ab}$ is denoted by $DQ(\mathcal{C}, \text{Ab})$. 
We note that every map $f : c_1 \to c_2$ in the category $C$ canonically defines a 1-map from $c_1$ to $c_2$ in the 2-category $QC$, and we have a 2-functor $C \to QC$, where $C$ is understood as a discrete 2-category. The bar complex of a discrete category with the set of objects $S$ is canonically quasiisomorphic to the free abelian group $\mathbb{Z}[S]$ generated by $S$; therefore by restriction, we obtain a canonical functor $DQ(C, \text{Ab}) \to D(C, \text{Ab})$. Analogously, we have a canonical functor $DQ(C, \text{Ab}) \to D(C^{\text{opp}}, \text{Ab})$.

Assume that the category $C$, in addition to fibered products, has finite coproducts.

**Definition 3.2.** An $A_\infty$-functor $M \in DQ(C, \text{Ab})$ is additive if its restriction $\overline{M} \in D(C^{\text{opp}}, \text{Ab})$ is additive in the sense of Section 2.1: for any $c, c' \in C$, the natural map

$$\overline{M}_*(c \sqcup c') \to \overline{M}_*(c) \oplus \overline{M}_*(c')$$

induced by the embeddings $c \to c \sqcup c'$, $c' \to c \sqcup c'$ is a quasiisomorphism. The full subcategory in $DQ(C, \text{Ab})$ spanned by additive $A_\infty$-functors is denoted by $DQ_{\text{add}}(C, \text{Ab}) \subset DQ(C, \text{Ab})$.

In particular, let $G$ be a group, and let $\Gamma_G$ be the category of finite $G$-sets.

**Definition 3.3.** A derived $G$-Mackey functor is an additive object in the category $DQ(\Gamma_G, \text{Ab})$. The category $DQ_{\text{add}}(\Gamma_G, \text{Ab})$ of derived $G$-Mackey functors is denoted by $DM(G, \text{Ab})$. If $\text{Ab} = \mathbb{Z}$-$\text{mod}$ is the category of abelian groups $DM(G, \mathbb{Z}$-$\text{mod}$) is denoted simply by $DM(G)$.

3.2. **Example: the trivial group.** To illustrate the general notion of a derived Mackey functor, consider the case of the trivial group $G = \{e\}$, so that $\Gamma_G = \Gamma$. Of course, $M(\{e\}, \text{Ab}) = \text{Ab}$; we would expect the same to hold of the derived level. Let us check that this is indeed so.

To do this, consider the subcategory $\Gamma_+ \subset Q\Gamma$ with the same objects, and those 1-morphisms $S_1 \leftarrow S \to S_2$ for which the map $S \to S_1$ is injective. We note that such diagrams have no non-trivial automorphisms; therefore the 2-category structure on $\Gamma_+$ is trivial and we can treat it as a usual category. Here are two equivalent descriptions of $\Gamma_+$.

(i) The category whose objects are finite sets $S$, and whose morphisms from $S_1 \to S_2$ are “partial maps” $f : S_1 \to S_2$ — that is, maps to $S_2$ defined on a subset $S \subseteq S_1$.

(ii) The category of finite sets $S$ with a fixed point $1 \in S$.

Here (i) is just a restatement of the definition, and the passage from (i) to (ii) is by formally adding the fixed point (on morphisms, all elements in the set $S_1$ where a partial map $f : S_1 \to S_2$ is undefined go into the added fixed point). Denote by

$$\lambda^* : DQ(\Gamma, \text{Ab}) \to D(\Gamma_+, \text{Ab})$$

the functor given by restriction with respect to the embedding $\lambda : \Gamma_+ \to Q\Gamma$.

For any finite set $S$, denote by $T(S) = \mathbb{Z}[S]$ the free abelian group it generates. Then the correspondence $S \mapsto T(S)$ obviously defines an object $T \in$
for any element $s \in S_2$. The object $T \in \mathcal{DQ}(\Gamma, \mathbb{Z}\text{-mod})$ is additive in the sense of Definition 3.2. Restricting it to $\Gamma_+$ gives an object $\lambda^*(T) \in \text{Fun}(\Gamma_+, \mathbb{Z}\text{-mod})$, which we will denote by the same letter $T$ by abuse of notation.

**Lemma 3.4.** (i) For any $M_\bullet \in \mathcal{D}(\text{Ab})$ and any $M'_\bullet \in \mathcal{DQ}(\Gamma, \text{Ab})$, the natural map

\[ \text{RHom}_{\mathcal{DQ}(\Gamma, \text{Ab})}(M'_\bullet, T \otimes M_\bullet) \to \text{RHom}_{\mathcal{DQ}(\Gamma_+, \text{Ab})}(\lambda^*(M'_\bullet), T \otimes M_\bullet) \]  

(3.1) is an isomorphism.

(ii) The functor $\mathcal{D}(\text{Ab}) \to \mathcal{DQ}(\Gamma, \text{Ab})$ given by $M_\bullet \mapsto T \otimes M_\bullet$ is the full embedding onto $\mathcal{DQ}_{\text{add}}(\Gamma, \text{Ab})$.

**Proof.** We will need some semi-obvious facts on the structure of the category $\text{Fun}(\Gamma_+, \mathbb{Z}\text{-mod})$ (see e.g. [Ka, Section 3.2]). A standard set of projective generators of this category is given by representable functors $T_n$, $n \geq 0$, explicitly described by

\[ T_n(S) = \mathbb{Z}[(S \sqcup \{1\})^n]. \]

We have $T_0 = T_1^\otimes$. In particular, $T_0 = \mathbb{Z}$, the constant functor with value $\mathbb{Z}$. Moreover, we have a direct sum decomposition $T_1 = T \oplus T_0$. Therefore the tensor powers $T^\otimes n$ are also projective, and give another set of generators for the category $\text{Fun}(\Gamma_+, \mathbb{Z}\text{-mod})$. These generators are semi-orthogonal: we have $\text{Hom}(T^\otimes n, T^\otimes m) = 0$ when $n > m$. In addition, $\text{Hom}(T_0, T^\otimes n) = 0$ for any $n \geq 1$.

Explicitly,

\[ T^\otimes n(S) = \mathbb{Z}[S^n] \]  

(3.2)

for all $S \in \Gamma_+$. We also note that we have $\text{Hom}(T, T) = \mathbb{Z}$, which immediately implies that $M_\bullet \mapsto T \otimes M$ gives fully faithful embeddings $\text{Ab} \subset \text{Fun}(\Gamma_+, \text{Ab})$, $\mathcal{D}(\text{Ab}) \subset \mathcal{D}(\Gamma_+, \text{Ab})$.

The category $\mathcal{DQ}(\Gamma, \text{Ab})$ is generated by representable $A_\infty$-functors $M^S_\bullet$ of the form

\[ M^S_\bullet(S') = C_\bullet(Q(S, S'), \mathbb{Z}) \otimes M \]

for all $M \in \text{Ab}$, $S \in \Gamma$. Therefore it is sufficient to prove (i) for objects of this form. Fix a finite set $S' \in \Gamma$ and an object $M' \in \text{Ab}$, and let $M'_\bullet = M^S_\bullet$. Explicitly, we have

\[ M'_\bullet(S) = \bigoplus_{S \in \Gamma} C_\bullet(\Sigma_{\tilde{S}}, M' \otimes \mathbb{Z}[\Gamma(\tilde{S}, S \times S')]), \]

where $\Sigma_{\tilde{S}}$ is the group of automorphisms of the finite set $\tilde{S}$. This direct sum decomposition is not functorial with respect to $S$. However, if we restrict to $\Gamma_+$, then the increasing filtration $F^\star_n$ on $\lambda^*(M'_\bullet)$ given by

\[ F^*_n \lambda^*(M'_\bullet)(S) = \bigoplus_{|S| \leq n} C_\bullet(\Sigma_{\tilde{S}}, M' \otimes \mathbb{Z}[\Gamma(\tilde{S}, S \times S')]) \]
is functorial (\(|\tilde{S}|\) denotes the cardinality of the set \(\tilde{S}\)). The associated graded quotient is given by
\[
\text{gr}_n^F \lambda^\ast(M'_s) = C_s(\Sigma \tilde{S}_s, M' \otimes \mathbb{Z}[\Gamma(\tilde{S}, S')] \otimes T^{\otimes n}),
\] (3.3)
where we have used (3.2), and \(\tilde{S}\) is the set of cardinality \(n\). By semi-orthogonality of the generators \(T^{\otimes n}\), this implies that
\[
\text{RHom}^\ast(\text{gr}_n^F \lambda^\ast(M'_s), T \otimes M_s) = 0
\]
for \(n \neq 1\), so that
\[
\text{RHom}^\ast(\lambda^\ast(M'_s), T \otimes M_s) = \text{RHom}^\ast(\text{gr}_1^F \lambda^\ast(M'_s), T \otimes M_s)
= \text{RHom}_{\mathcal{D}\mathcal{Q}(\Gamma, \text{Ab})}(T \otimes M', T \otimes M_s) \otimes \mathbb{Z}[S']
= \text{RHom}_{\text{add}}(M', M_s) \otimes \mathbb{Z}[S'].
\]
Since \(M'_s = M'_s^{\text{Ad}}\) is representable, this is exactly the left-hand side of (3.1), so that we have proved (i).

As for (ii), (i) immediately implies that the functor \(\mathcal{D}(\text{Ab}) \rightarrow \mathcal{D}(\Gamma, \text{Ab})\) is a full embedding, and since \(T \in \mathcal{D}(\Gamma, \text{Z-mod})\) is additive, we in fact have a full embedding \(\mathcal{D}(\text{Ab}) \subset \mathcal{D}_{\text{add}}(\Gamma, \text{Ab})\). To prove that it is essentially surjective, it suffices to prove that its contains all objects \(M_s \in \mathcal{D}_{\text{add}}(\Gamma, \text{Ab})\) which are concentrated in a single cohomological degree. But such objects are Mackey functors in the usual non-derived sense, so they are all of the form \(T \otimes M, M \in \text{Ab}\). \(\Box\)

3.3. Wreath products. Definition 3.3 is a DG version of the first definition of a Mackey functor given in Section 2.1. To get a more explicit description of the category \(\mathcal{D}(\mathcal{M}(G))\), we need to somehow use the additivity condition and replace the \(A_{\infty}\)-category \(\mathcal{B}^\Gamma_{\text{co}}\) with an \(A_{\infty}\)-category whose objects are \(G\)-orbits, not all finite \(G\)-sets. We do it by using the structure of a “wreath product” of the category \(\Gamma_G\) of finite \(G\)-sets.

For any small category \(\mathcal{C}\), by the wreath product \(\mathcal{C} \wr \Gamma\) of \(\mathcal{C}\) with the category \(\Gamma\) of finite sets we will understand the category of pairs \(\langle S, \{c_s\} \rangle\) of a finite set \(S\) and a collection of objects \(c_s \in \mathcal{C}\), one for each element \(s \in S\), with maps from \(\langle S, \{c_s\} \rangle\) to \(\langle S', \{c'_s\} \rangle\) being a pair \((f, f_S)\) of a map \(f : S \rightarrow S'\) and a collection of maps \(f_s : c_s \rightarrow c'_s(f(s))\), one for each \(s \in S\).

By definition, we have a forgetful functor \(\rho : \mathcal{C} \wr \Gamma \rightarrow \Gamma, \langle S, \{c_s\} \rangle \mapsto S\). The functor \(\rho\) is a fibration; its fiber \(\rho_S\) over a finite set \(S \in \Gamma\) is canonically identified with \(\mathcal{C}^S\), the product of copies of \(\mathcal{C}\) indexed by elements of the set \(S\). In particular, \(\mathcal{C}\) itself is naturally embedded into \(\mathcal{C} \wr \Gamma\) as the fiber over the one-element set \(\text{pt} \in \Gamma\). We will denote this embedding by \(j^\mathcal{C} : \mathcal{C} \rightarrow \mathcal{C} \wr \Gamma\).

Irrespective of the properties of the category \(\mathcal{C}\), the category \(\mathcal{C} \wr \Gamma\) has finite coproducts. In a sense, it is obtained by formally adjoining finite coproducts to \(\mathcal{C}\)—this can be formulated precisely as a certain universal property of wreath products, but we will not need this. Another way to characterize \(\mathcal{C} \wr \Gamma\) by a universal property is the following: for any category \(\mathcal{C}'\) fibered over \(\Gamma\), any functor \(f : \mathcal{C}'_{\text{pt}} \rightarrow \mathcal{C}\) from the fiber \(\mathcal{C}'_{\text{pt}}\) over the one-element set \(\text{pt} \in \Gamma\) extends uniquely to a Cartesian functor \(\mathcal{C}' \rightarrow \mathcal{C} \wr \Gamma\). We will need the following easy corollary of this fact.
Lemma 3.5. Assume given a small category $C$ and an object $S \in C \wr \Gamma$. Then the category $(C \wr \Gamma)/S$ of objects $S' \in C \wr \Gamma$ equipped with a map $S' \to S$ is naturally equivalent to the wreath product $(C/S)\wr \Gamma$, where $C/S$ is the category of objects $c \in C$ equipped with a map $j^C(c) \to S$.

Proof. The projection $(C \wr \Gamma)/S \to \Gamma$ which sends $S' \to S$ to $\rho(S')$ is obviously a fibration; the universal property of wreath products then gives a Cartesian comparison functor

$$(C \wr \Gamma)/S \to (C/S)\wr \Gamma,$$

which is obviously an equivalence on all the fibers, thus an equivalence. □

In the assumptions of Lemma 3.5, denote by $Q'(S) \subset (C \wr \Gamma)/S$ the subcategory with the same objects as $(C \wr \Gamma)/S$ and those maps which are Cartesian with respect to the fibration $(C \wr \Gamma)/S \to \Gamma$. Equivalently, we have $Q'(S) \cong C/S \wr \Gamma$, where $C/S \subset C/S$ is the subcategory whose objects are all objects in $C/S$, and whose maps are isomorphisms in $C/S$.

More generally, given two objects $S_1, S_2 \in C \wr \Gamma$, denote by $(C \wr \Gamma)/(S_1, S_2)$ the category of objects $S \in C \wr \Gamma$ equipped with maps $S \to S_1, S \to S_2$, let $C/(S_1, S_2) \subset (C \wr \Gamma)/(S_1, S_2)$ be the fiber of the projection $(C \wr \Gamma)/(S_1, S_2) \to \Gamma$ given by $\langle S_1 \leftarrow S \to S_2 \rangle \mapsto \rho(S)$, so that $(C \wr \Gamma)/(S_1, S_2) \cong (C/(S_1, S_2)) \wr \Gamma$, and let $C/(S_1, S_2) \subset C/(S_1, S_2)$ be the groupoid of isomorphisms of the category $C/(S_1, S_2)$. Denote

$$Q'(S_1, S_2) = C/(S_1, S_2) \wr \Gamma \subset (C \wr \Gamma)/(S_1, S_2),$$

and assume that the category $C \wr \Gamma$ has fibered products. Then these fibered products define associative composition functors

$$m: (C \wr \Gamma)/(S_1, S_2) \times (C \wr \Gamma)/(S_2, S_3) \to (C \wr \Gamma)/(S_1, S_3)$$

(3.4)

which induce composition functors on the categories $Q'(-, -)$. This allows to define a 2-category which we denote by $Q'C$: its objects are the objects of $C \wr \Gamma$, and its categories of morphisms are $Q'(-, -)$.

Note that for any $S_1, S_2 \in C \wr \Gamma$, we have a natural embedding

$$F_C(S_1, S_2): Q(S_1, S_2) \to Q'(S_1, S_2),$$

and these embeddings are compatible with fibered products, thus glue together to a 2-functor

$$F_C: Q(C \wr \Gamma) \to Q'(C).$$

Both 2-categories here have the same objects and the same 1-morphisms; the difference is that the right-hand side has more 2-morphisms — the embeddings $F_C(-, -)$ are identical on objects and faithful, but not full.

We prove right away one technical result on the categories $Q'(-, -)$, which we will need later on. Any object $c \in C \subset C \wr \Gamma$ gives a functor $j^c: \Gamma \to C \wr \Gamma$ which sends a finite set $S$ to the union of $S$ copies of $c$. This functor preserves fibered products, thus gives a 2-functor $j^c: Q\Gamma \to Q(C \wr \Gamma)$, a restriction functor $j^c*: DQ(C \wr \Gamma, Ab) \to DQ(\Gamma, Ab)$, and a right-adjoint functor $j^c^*: DQ(\Gamma, Ab) \to DQ(C \wr \Gamma, Ab)$.
Lemma 3.6. For any $E_i \in \mathcal{D}\mathcal{Q}(\Gamma, \text{Ab})$ and any $S \in \mathcal{C} \uparrow \Gamma$, we have a natural quasiisomorphism

$$j^*_\ast E_i(S) \cong C_\ast(Q'(c, S)^{\text{opp}}, \rho^{\text{opp}} E_i),$$

where $E_i$ in the right-hand side is restricted to $\Gamma^{\text{opp}} \subset \Gamma$ and then pulled back to $Q'(c, S)^{\text{opp}}$ by the opposite $\rho^{\text{opp}}$ to the projection $p: Q'(c, S) = \mathcal{C}/(c, S) i \Gamma \to \Gamma$.

Proof. Denote the embedding $\Gamma^{\text{opp}} \to \Gamma$ by $\lambda$; let $\lambda_j: \mathcal{D}(\Gamma^{\text{opp}}, \text{Ab}) \to \mathcal{D}\mathcal{Q}(\Gamma, \text{Ab})$ be the left-adjoint functor to the restriction functor $\lambda^\ast$. Then for any $M_i \in \mathcal{D}(\text{Ab})$ we have

$$\text{RHom}^\ast(M_i, C_\ast(Q'(c, S)^{\text{opp}}, \rho^{\text{opp}} E_i)) \cong \text{RHom}^\ast(\lambda_j \rho^{\text{opp}} \tau^* M_i, E_i),$$

where $\tau: Q'(c, S)^{\text{opp}} \to \text{pt}$ is the tautological projection. Thus by adjunction, it suffices to prove that

$$\lambda_j \rho^{\text{opp}} \tau^* M_i \cong j^\ast M_i^S.$$

To construct a map $f: \lambda_j \rho^{\text{opp}} \tau^* M_i \to j^\ast M_i^S$, it suffices by adjunction to construct a map $\tau^* M_i \to \rho^{\text{opp}} \lambda^\ast j^\ast M_i^S$; that is, a compatible system of maps

$$M_i \to \rho^{\text{opp}} \lambda^\ast j^\ast M_i^S(S') \cong H_\ast(Q(S, j^\ast(S'))),$$

for any $S' \in Q'(c, S)$; these maps are induced by obvious tautological maps $Z \to H_\ast(Q(S, j^\ast(S'))), Z$.

To prove that the map $f$ is an isomorphism, we first need a way to control the functor $\lambda_j$. For any finite set $S \in \Gamma$, let $\mathcal{C}/S$ be the category of all finite sets $\mathcal{S}$ equipped with a map $\mathcal{S} \to S$ and isomorphisms between them, and let $\kappa^\mathcal{S}: (\mathcal{C}/\mathcal{S})^{\text{opp}} \to \Gamma^{\text{opp}}$ be the natural projection which sends $[\mathcal{S} \to S]$ to $\mathcal{S}$. Then we obviously have

$$\lambda^\ast \mathcal{S}^\mathcal{S} \cong \kappa^\mathcal{S} Z \in \mathcal{D}(\Gamma^{\text{opp}}, \mathcal{Z}\text{-mod}),$$

and by adjunction, this yields a canonical isomorphism

$$\lambda_j N_i(\mathcal{S}) \cong H_\ast((\mathcal{C}/\mathcal{S})^{\text{opp}}, \kappa^\mathcal{S} N_i),$$

for any $N_i \in \mathcal{D}(\Gamma^{\text{opp}}, \text{Ab})$.

Now apply this to $N_i = \rho^{\text{opp}} \tau^* M_i$. Since $\rho^{\text{opp}}$ is a cofibration, we may compute $\kappa^\mathcal{S} \rho^{\text{opp}}$ by base change; this gives a canonical isomorphism

$$H_\ast((\mathcal{C}/\mathcal{S})^{\text{opp}}, \kappa^\mathcal{S} \rho^{\text{opp}} \tau^* M_i) \cong H_\ast(Q'(c, S, \mathcal{S})^{\text{opp}}, M_i),$$

where $Q'(c, S, \mathcal{S})$ is the category obtained as the Cartesian product

$$Q'(c, S, \mathcal{S}) \longrightarrow \Gamma/\mathcal{S},$$

$$\downarrow \quad \downarrow \quad \downarrow \Gamma.$$

It remains to notice that the category $Q'(c, S, \mathcal{S})$ is canonically identified with $Q(S, j^\ast(S))$, so that

$$H_\ast(Q'(c, S, \mathcal{S})^{\text{opp}}, M_i) \cong H_\ast(Q(S, j^\ast(S))^{\text{opp}}, M_i) \cong j^\ast M_i^S(\mathcal{S}).$$
Thus the map $f$ becomes an isomorphism after evaluating at every object $S \in Q\Gamma$, as required. \hfill \Box

### 3.4. Additivization of the quotient construction.

Now let us consider again the functor $T \in DQ(\Gamma, Z\text{-mod})$ of Section 3.2, and let us restrict it to an object $T \in \text{Fun}(\Gamma, Z\text{-mod})$ by the embedding $\Gamma \to Q\Gamma$. This $T \in \text{Fun}(\Gamma, Z\text{-mod})$ is isomorphic to the functor $Z_{[1]}$ represented by the set $[1] \in \Gamma$ with a single element. For any $S \in \Gamma$, let $\tau_S : Z \to Z[S] = T(S)$ be the diagonal embedding. The maps $\tau_S$ are not functorial with respect to arbitrary maps of finite sets $S$; however, they are functorial with respect to isomorphisms. Thus if we denote by $\Gamma \subset \Gamma$ the category of finite sets and their isomorphisms, then we have a map of functors

$$\tau : Z \to \iota^* T,$$

where $Z \in \text{Fun}(\Gamma, Z\text{-mod})$ is the constant functor with value $Z$, and $\iota : \Gamma \to \Gamma$ is the embedding.

For any small category $\mathcal{C}$, we will denote the pullback $\rho^* T \in \text{Fun}(\mathcal{C} \wr \Gamma, Z)$ with respect to the forgetful functor $\rho : \mathcal{C} \wr \Gamma \to \Gamma$ by the same letter $T$. By base change, we have

$$T \cong \rho^* Z_{[1]} \cong j_!^\Gamma Z_C,$$

where $Z_C \in \text{Fun}(\mathcal{C}, Z)$ is the constant functor with value $Z$, and $L^i j_!^\Gamma Z_C = 0$ for $i \geq 1$. Therefore the natural map

$$H_*(\mathcal{C}, Z_C) \to H_*(\mathcal{C} \wr \Gamma, T) \quad (3.6)$$

is an isomorphism.

Assume that the wreath product category $\mathcal{C} \wr \Gamma$ has fibered products, and consider the 2-category $Q'(\mathcal{C})$. Then for any $S_1, S_2 \in \mathcal{C} \wr \Gamma$, the category $Q'(S_1, S_2) \cong \mathcal{C}/(S_1, S_2) \wr \Gamma$ carries a natural $Z\text{-mod}$-valued functor

$$T \cong \rho^* T \cong j_!^{S_1, S_2} Z \in \text{Fun}(Q'(S_1, S_2), Z\text{-mod}),$$

where $j_!^{S_1, S_2} : \mathcal{C}/(S_1, S_2) \to Q'(S_1, S_2)$ denotes the natural embedding. Moreover, for any $S_1, S_2, S_3 \in \mathcal{C} \wr \Gamma$, the composition functors (3.4) induce functors

$$\overline{m} \cong (j_!^{S_1, S_2} \times j_!^{S_2, S_3}) \circ m : \mathcal{C}/(S_1, S_2) \times \mathcal{C}/(S_2, S_3) \to \mathcal{C}/(S_1, S_3),$$

and by definition, all the maps in the category $\mathcal{C}/(S_1, S_2)$ are invertible, so that the composition $\rho \circ \overline{m}$ actually goes into $\Gamma \subset \Gamma$. Therefore the canonical map $\tau$ of (3.5) induces a map $\tau_{S_1, S_2, S_3} : Z \to \overline{m}^* T$. These maps are associative on triple products in the obvious sense. By adjunction, they induce maps

$$\mu_{S_1, S_2, S_3} : T \boxtimes T \cong (j_!^{S_1, S_2} \times j_!^{S_1, S_3})_! Z \to m^* T, \quad (3.7)$$

and these maps are also associative on triple products.

As in the Section 3.1, the procedure of Section 1.6 gives an $A_\infty$-category $\mathcal{B}^{\mathcal{C}}$, with the same objects as $\mathcal{C} \wr \Gamma$ by setting

$$\mathcal{B}^{\mathcal{C}}(S_1, S_2) = C_*(Q'(\mathcal{C}(S_1, S_2), T),$$

with compositions induced by the 2-category structure on $Q'(\mathcal{C}$ and the canonical maps $\mu$ of (3.7).
Definition 3.7. The derived category $\mathcal{D}(\mathcal{B}_c^\infty, \text{Ab})$ of $A_\infty$-functors from $\mathcal{B}_c^\infty$ to $\text{Ab}$ is denoted by $\mathcal{D}Q'(\mathcal{C}, \text{Ab})$.

For any $S_1, S_2 \in \mathcal{C} \times \Gamma$, the natural functor $F: Q'(S_1, S_2) \rightarrow Q'(S_1, S_2)$ again goes into $\tilde{\Gamma} \subset \Gamma$ when composed with the projection $\rho: Q'(S_1, S_2) \rightarrow \Gamma$; therefore the map $\tilde{\tau}$ of (3.5) induces maps $\mathbb{Z} \rightarrow F^*T$, and the 2-functor $F_C: Q(\mathcal{C} \times \Gamma) \rightarrow Q\mathcal{C}$ extends to an $A_\infty$-functor $F_C: \mathcal{B}_c^\infty \rightarrow \mathcal{B}_c^\infty$. The main comparison result that we want to prove is the following.

Proposition 3.8. The functor $F^*_C: \mathcal{D}Q'(\mathcal{C}, \text{Ab}) \rightarrow \mathcal{D}Q(\mathcal{C} \times \Gamma, \text{Ab})$ induced by $F_C$ gives an equivalence

$$\mathcal{D}Q'(\mathcal{C}, \text{Ab}) \rightarrow \mathcal{D}Q(\mathcal{C} \times \Gamma, \text{Ab}).$$

Remark 3.9. The appearance of the functor $T$ in the definition of the $A_\infty$-category $\mathcal{B}_c^\infty(-, -)$ looks like a trick. One motivation for this comes from topology. The categories $Q\mathcal{C}(-, -)$ are symmetric monoidal categories with respect to the disjoint union. Applying group completion to their classifying spaces $|Q\mathcal{C}(-, -)|$, one obtains infinite loop spaces. Then the complex $\mathcal{B}_c^\infty(-, -)$ simply computes the homology of the corresponding $\Omega$-spectrum (as opposed to $\mathcal{B}^\infty(-, -)$), which computes the homology of the classifying space $|Q\mathcal{C}(-, -)|).$ We do not prove this since we will not need it, but the proof is not difficult (for example, it can be done along the lines of [Ka, Section 3.2]).

3.5. The proofs of the comparison results. Before we prove Proposition 3.8, let us explain why it is useful. Assume given a small category $\mathcal{C}$ such that $\mathcal{C} \times \Gamma$ has fibered products, and denote by

$$\mathcal{B}_c^\infty \subset \mathcal{B}_c^\infty$$

the full subcategory spanned by $\mathcal{C} \subset \mathcal{C} \times \Gamma$. Let $\mathcal{D}Q(\mathcal{C}, \text{Ab})$ be the derived category of $A_\infty$-functors from $\mathcal{B}_c^\infty$ to $\text{Ab}$.

Lemma 3.10. Restriction with respect to the natural embedding $\mathcal{B}_c^\infty \rightarrow \mathcal{B}_c^\infty$ induces an equivalence

$$R: \mathcal{D}Q'(\mathcal{C}, \text{Ab}) \rightarrow \mathcal{D}Q(\mathcal{C} \times \Gamma, \text{Ab}).$$

Proof. Let $E: \mathcal{D}Q(\mathcal{C}, \text{Ab}) \rightarrow \mathcal{D}Q'(\mathcal{C}, \text{Ab})$ be the left-adjoint functor to $R$. It suffices to prove that $E$ is essentially surjective, and that the adjunction map $\text{Id} \rightarrow RE$ is an isomorphism. The second fact is obvious: by adjunction, $E$ sends representable functors into representable functors, and since $\mathcal{B}_c^\infty \subset \mathcal{B}_c^\infty$ is a full subcategory, $E$ does not change the spaces of maps between them. It remains to prove that $E$ is essentially surjective. Since $\mathcal{D}Q'(\mathcal{C}, \text{Ab})$ is generated by representable functors $M^S_\Gamma, S \in \mathcal{C} \times \Gamma, \ M \in \text{Ab}$, it suffices to prove that all these functors lie in the essential image of $E$. By induction on the cardinality $|\rho(S)|$, it suffices to prove that

$$M^{S_1 \cup S_2}_\Gamma \cong M^{S_1}_\Gamma \oplus M^{S_1}_\Gamma$$

for any $M \in \text{Ab}, S_1, S_2 \in \mathcal{C} \times \Gamma$.

Indeed, assume given such an $M$ and $S_1, S_2$. Explicitly, we have

$$M^{S_1 \cup S_2}_\Gamma(S') = C_\Gamma(Q'(S, S'), T) \otimes M$$
for any $S' \in \mathcal{C} \downarrow \Gamma$. We have

$$Q'(S_1 \sqcup S_2, S') \cong Q'(S_1, S') \times Q'(S_2, S'),$$

(3.9)

and

$$T \cong (\pi_1^T \boxtimes \pi_2^2) \oplus (\pi_1^2 \boxtimes \pi_2^1 T),$$

(3.10)

where $\pi_1$ and $\pi_2$ are the projections onto the factors of the decomposition (3.9), and $\mathbb{Z}$ means the constant functor with value $\mathbb{Z}$. The direct sum decomposition (3.10) is functorial with respect to $S'$, thus induces a certain direct sum decomposition $M_{S_1 \sqcup S_2} \cong M_1 \oplus M_2$. Here $M_1$ is given by

$$M_1(S') = C_*(Q'(S_1, S') \times Q'(S_2, S'), \pi_1^T \boxtimes \pi_2^2),$$

and by the Künneth formula, this is canonically quasiisomorphic to

$$M_{S_1}(S') \otimes C_*(Q'(S_2, S'), \mathbb{Z}).$$

Since the category $Q'(S_2, S')$ is a wreath product, it has an initial object, thus no homology with constant coefficients, $H_*(Q'(S_2, S'), \mathbb{Z}) = \mathbb{Z}$, and we conclude that $M_1 \cong M_{S_1}$. Analogously $M_2 \cong M_{S_2}$.

Lemma 3.10 shows that Proposition 3.8 allows one to get rid of the additivity assumption in the definition of the category $DQ_{ad}(\mathcal{C} \downarrow \Gamma, \text{Ab})$ and reduces everything to an $A_\infty$-category whose objects are those of $\mathcal{C}$, not of $\mathcal{C} \downarrow \Gamma$. This has the following immediate corollary.

**Corollary 3.11.** Assume that the small category $\mathcal{C}$ itself has fibered products. Then there exists a natural equivalence of triangulated categories

$$DQ(\mathcal{C}, \text{Ab}) \cong DQ'(\mathcal{C}, \text{Ab}).$$

**Proof.** By virtue of Lemma 3.10, it suffices to construct an equivalence $DQ(\mathcal{C}, \text{Ab}) \cong DQ(\mathcal{C}, \text{Ab})$, or equivalently, a quasiisomorphism of $A_\infty$-categories $\mathcal{B}_c^* \cong \mathcal{B}_c$. Both $A_\infty$-categories have the same objects, the objects of the category $\mathcal{C}$. For any $c_1, c_2 \in \mathcal{C}$, the natural embedding $\mathcal{Q}(c_1, c_2) \to \mathcal{Q}'(c_1, c_2) = \mathcal{Q}(c_1, c_2) \downarrow \Gamma$ induces a map

$$\mathcal{B}_c^*(c_1, c_2) = C_*(\mathcal{Q}(c_1, c_2), \mathbb{Z}) \to \mathcal{B}_c^*(c_1, c_2) = C_*(\mathcal{Q}'(c_1, c_2), T),$$

these maps are obviously compatible with compositions, and they are all quasiisomorphisms by (3.6).

We will now prove Proposition 3.8. To do this, recall that for any object $c \in \mathcal{C}$, we have the embedding $j^c: \mathcal{Q} \to \mathcal{Q}(\mathcal{C} \downarrow \Gamma)$ and the corresponding restriction functor $j^{\ast c}: DQ(\mathcal{C} \downarrow \Gamma, \text{Ab}) \to DQ(\mathcal{C}, \text{Ab})$. Moreover, $j^c$ obviously extends to an embedding $j^\ast: \mathcal{Q}(\text{pt}) \to \mathcal{Q}(\mathcal{C})$, and we have a restriction functor $j^{\ast c}: DQ(\mathcal{C}, \text{Ab}) \to DQ(\text{pt}, \text{Ab})$. These are compatible with the functors $F_\mathcal{C}^\ast$ of Proposition 3.8: we have a commutative diagram

$$\begin{array}{ccc}
DQ(\Gamma, \text{Ab}) & \xrightarrow{j^{\ast \mathcal{C}}} & DQ(\mathcal{C} \downarrow \Gamma, \text{Ab}) \\
F_\mathcal{C}^\ast \uparrow & & \uparrow F_\mathcal{C} \\
DQ'(\text{pt}, \text{Ab}) & \xleftarrow{j^{\ast \mathcal{C}}} & DQ'(\mathcal{C}, \text{Ab}).
\end{array}$$

(3.11)
Of course, by Corollary 3.11 we have $\mathcal{D}Q(\text{pt}, \text{Ab}) \cong \mathcal{D}Q(\text{pt}, \text{Ab}) \cong \mathcal{D}(\text{Ab})$, and $j_c^* : \mathcal{D}Q(C, \text{Ab}) \to \mathcal{D}(\text{Ab})$ is simply the evaluation at $c \in \mathcal{Q}(C)$.

**Lemma 3.12.** Denote by $F_{\text{pt}}$, $F_{\text{pt}}^*$ the functors left-adjoint to $F_C^*$ and $F_{\text{pt}}^*$. Then the base change map

$$F_{\text{pt}}^* \circ j_c^* \to j_{c*}^* \circ F_C^!$$

obtained by adjunction from (3.11) is an isomorphism.

**Proof.** Since $\mathcal{D}Q(C \uparrow \Gamma, \text{Ab})$ is generated by representable objects $M^S_c$, it suffices to prove that

$$R\text{Hom}^* (F_{\text{pt}}^* j_c^* M^S_c, M'_c) \cong R\text{Hom}^* (j_{c*}^* F_C^* M^S_c, M'_c)$$

for any $M_c, M'_c \in \mathcal{D}(\text{Ab}), S \in C$. By adjunction, $F_C^* M^S_c = M^F_c(S)$, so that the right-hand side is isomorphic to

$$R\text{Hom}^*_{\mathcal{Q}(S,c)}(T, Z) \otimes R\text{Hom}^* (M_c, M'_c).$$

The left-hand side by adjunction is isomorphic to

$$R\text{Hom}^* (j_c^* M^S_c, F_{\text{pt}}^* M'_c) \cong R\text{Hom}^* (j_{c*}^* M^S_c, M'_c \otimes T),$$

which is isomorphic to

$$R\text{Hom}^*_{\mathcal{Q}(S,c)}(Z, T) \otimes R\text{Hom}^* (M_c, M'_c)$$

by Lemma 3.6. It remains to notice that

$$R\text{Hom}^*_{\mathcal{Q}(S,c)}(Z, T) = R\text{Hom}^*_{\mathcal{Q}(S,c)}(T, Z))$$

identically, if both sides are understood in the sense of (1.5). □

**Proof of Proposition 3.8.** It suffices to check that

(i) the adjunction map $F_C^* \circ F_C^* \to \text{Id}$ is an isomorphism, so that $F_C^*$ is fully faithful, and

(ii) an additive object $M_c \in \mathcal{D}Q_{\text{add}}(C \uparrow \Gamma, \text{Ab})$ with trivial $F_C^* (M_c)$ is itself trivial, so that $F_C^*$ is essentially surjective.

By Lemma 3.10, it suffices to check (i) after evaluating on all $c \in C \subset C \uparrow \Gamma$, and by Lemma 3.12, this amount to checking (i) with $C$ replaced by pt. Analogously, the restriction functor $j^*$ obviously sends $\mathcal{D}Q_{\text{add}}(C \uparrow \Gamma, \text{Ab})$ into $\mathcal{D}Q_{\text{add}}(\Gamma, \text{Ab})$, and an object $M_c \in \mathcal{D}Q(C \uparrow \Gamma, \text{Ab})$ with trivial $j^* (M_c)$ for all $c \in C$ is itself trivial; thus by Lemma 3.12, (ii) can also be only checked for $C = \text{pt}$. Conclusion: it suffices to prove the Proposition for $C = \text{pt}$. This has been done already—combine Lemma 3.10 and Corollary 3.11, on one hand, and Lemma 3.4, on the other hand. □

**3.6. Derived Burnside rings.** Now again fix a finite group $G$ and take $C = \Gamma_G$, the category of finite $G$-sets. Define a functor $p : \Gamma_G \to \Gamma$ by setting $S \mapsto S/G$, the set of $G$-orbits on $S$. This functor is a fibration, and moreover, we actually have an identification $\Gamma_G \cong O_G \uparrow \Gamma$, where $O_G$ is the category of finite $G$-orbits, and $p$ is the tautological projection $O_G \uparrow \Gamma \to \Gamma$. Therefore we can also consider the 2-category $\mathcal{Q}(O_G)$ and the associated $\mathcal{A}_\infty$-category $\mathcal{B}^{\mathcal{Q}_G}$ of (3.8) whose objects are finite $G$-orbits. To simplify notation, denote

$$\mathcal{B}^{\mathcal{Q}_G} = \mathcal{B}^{\mathcal{Q}_G}.$$
then the following is a reformulation of Proposition 3.8 and Lemma 3.10 (with \( C = O_G \)).

**Proposition 3.13.** The triangulated category \( \mathcal{DM}(G) \) of derived \( G \)-Mackey functors is equivalent to the derived category of \( A_\infty \)-functors \( B^G \to \text{Ab} \).

This proposition allows us to do some computations in the category \( \mathcal{DM}(G) \); in particular, spelling out the definitions, we can now define a derived version of the Burnside ring \( A^G \). Let \( T = \rho^* T \in \text{Fun}(\Gamma_G, \mathbb{Z}\text{-mod}) \), so that \( T(S) = \mathbb{Z}[S/G] = \mathbb{Z}[S]^G \). For every \( S_1, S_2 \in \Gamma_G \), let

\[
\mu_{S_1, S_2} : \mathbb{Z}[S_1]^G \otimes \mathbb{Z}[S_2]^G = \mathbb{Z}[S_1 \times S_2]^G \to \mathbb{Z}[S_1 \times S_2]^G
\]

be the natural embedding. Taken together, these maps give a map

\[
\mu : T \boxtimes T \to m^* T,
\]

where, as in Section 2, \( m : \Gamma_G \times \Gamma_G \to \Gamma_G \) is the product functor (this map \( \mu \) is of course the special case of (3.7) for \( S_1 = S_2 = S_3 = (\text{pt}) \)).

**Definition 3.14.** The derived Burnside ring \( A^G \) of the group \( G \) is the complex \( C_\ast (\Gamma_G, T) \), with the \( A_\infty \)-structure induced by the Cartesian product of \( G \)-sets and the canonical map

\[
m^T \circ \mu : C_\ast (\Gamma_G \times \Gamma_G, T \boxtimes T) \to C_\ast (\Gamma_G \times \Gamma_G, m^* T) \to C_\ast (\Gamma_G, T),
\]

where \( \mu \) is as in (3.12).

By definition, \( A^G \) is a \( A_\infty \)-algebra over \( \mathbb{Z} \) (and its homology algebra \( H_\ast (A^G) \) is commutative). It is isomorphic to \( B^G ([G/G], [G/G]) \), where \([G/G]\) is the trivial \( G \)-orbit (the point set \( \text{pt} \) with the trivial \( G \)-action).

**Lemma 3.15.** Assume given a group \( G \).

\( i \) The 0-th homology \( H_0(\mathcal{C}) \) of the derived Burnside ring \( \mathcal{C} \) is isomorphic to the usual Burnside ring \( \mathcal{C} \), and the 0-th homology \( H_0(B^G) \) of the \( A_\infty \)-category \( B^G \) is isomorphic to the additive category \( B^G \) of Section 2.1.

\( ii \) For any two \( G \)-orbits \( S_1, S_2 \), we have a natural quasiisomorphism

\[
\mathcal{B}^G(S_1, S_2) = \bigoplus_{p \in (S_1 \times S_2)/G} \mathcal{A}^H_p,
\]

where \( S_p = [G/H_p] \) are the components in the decomposition (2.1) of the product \( S = S_1 \times S_2 \), and this quasiisomorphism induces the isomorphism (2.3) on 0-th homology.

**Proof.** The quasiisomorphism (3.6) in our new notation reads as

\[ \mathcal{C}^G \cong C_\ast (\mathcal{O}_G, \mathbb{Z}) \]

so that the 0-th homology of \( \mathcal{C}^G \) is the 0-th homology of the groupoid \( \mathcal{O}_G \) of \( G \)-orbits; this is precisely the Burnside ring \( \mathcal{C}^G \). The decomposition (3.13) follows from the decomposition

\[ Q'(S_1, S_2) \cong \prod_{p \in (S_1 \times S_2)/G} Q'(S_p) \]
by the same argument as in the proof of Lemma 3.10. Combining together (3.13), (2.3) and the isomorphism $H_0(\mathcal{A}^G) \cong \mathcal{A}^G$ gives the isomorphism $H_0(B^G) \cong B^G$. It remains to prove that this isomorphism is compatible with the compositions; this is a straightforward check, which we leave to the reader. □

4. WALDHAUSEN-TYPE DESCRIPTION

The construction of the triangulated category $\mathcal{DM}(G)$ of derived Mackey functors given in the last Section is very explicit but somewhat deficient, since it relies on explicit resolutions and $A_\infty$ methods. We will now give a more invariant construction. To do this, we modify the quotient construction $\Gamma_G \mapsto \mathcal{Q}\Gamma_G$ of Section 3.1 in a way which is similar to the passage from Quillen’s $Q$-construction to Waldhausen’s $S$-construction in algebraic $K$-theory.

4.1. Heuristic explanation. Let us first explain informally what we are going to do (this Subsection is purely heuristic and may be skipped, formally, nothing in the rest of the paper depends on it). Recall that a simplicial set $X$ is by definition a contravariant functor from the category $\Delta$ of finite non-empty totally ordered sets to the category of all sets. For any non-negative integer $n \geq 1$, we will denote by $[n] \in \Delta$ the set with $n$ elements, or, to be specific, the set of all integers $i$, $1 \leq i \leq n$; we will also denote $X_i = X([i])$ for any $i \geq 1$. An Ab-valued sheaf $M$ on $X$ is a collection of

(i) a functor $M_n : X_n \to \text{Ab}$ for every $n \geq 1$, and
(ii) a map $M(f) : M_{n'} \to X(f)^*M_n$ for every map $f : [n] \to [n']$, subject to standard compatibility conditions. Here in (i), we treat the set $X_n$ as a discrete category, so that $M_n$ is effectively just an $X_n$-graded object in $\text{Ab}$; in (ii), $X(f) : X_{n'} \to X_n$ is value of the functor $X : \Delta^{\text{op}} \to \text{Sets}$ on the map $f$.

There is the following convenient way to pack together these data (i), (ii), and also the compatibility conditions. Let us not only treat the sets $X_n$ as discrete categories, but also treat $X$ as a functor $\Delta^{\text{op}} \to \text{Cat}$. Then we can apply the Grothendieck construction of Section 1.2. The result is a category $S(X)$ fibered over $\Delta$; explicitly, objects in $S(X)$ are pairs $\langle n, x \in X_n \rangle$, and a map from $\langle n, x \in X_n \rangle$ to $\langle n', x' \in X_{n'} \rangle$ is given by a map $f : [n] \to [n']$ such that $X(f)(x') = x$. One immediately checks that in this notation, a sheaf $M$ on $X$ is exactly the same thing as a functor $M : S(X) \to \text{Ab}$.

Now assume given a small category $\mathcal{C}$. Recall that to $\mathcal{C}$, one canonically associates a simplicial set $N(\mathcal{C})$ called the nerve of $\mathcal{C}$, and to any functor $E \in \text{Fun}(\mathcal{C}, \text{Ab})$, one associates an Ab-valued sheaf $\hat{E}$ on the nerve—in other words, we have a natural embedding

$$\text{Fun}(\mathcal{C}, \text{Ab}) \to \text{Fun}(S(N(\mathcal{C})), \text{Ab}).$$

(4.1)

Explicitly, $N(\mathcal{C})_n$ is the set of diagrams $c_1 \to \cdots \to c_n$ in the category $\mathcal{C}$; the functor $\hat{E} : S(N(\mathcal{C})) \to \text{Ab}$ sends a diagram $c_1 \to \cdots \to c_n$ to $E(c_n)$. For any map $f : [n'] \to [n]$, the map $N(\mathcal{C})(f)$ sends a diagram $c_1 \to \cdots \to c_n$ to the diagram $c_{f(1)} \to \cdots \to c_{f(n')}$, and the corresponding map $\hat{E}(f) : E(c_{f(n')}) \to E(c_n)$ is induces by the natural map $c_{f(n')} \to c_n$. 


The embedding (4.1) is fully faithful but not essentially surjective—not all sheaves on \( N(\mathcal{C}) \) come from functors \( E \in \text{Fun}(\mathcal{C}, \text{Ab}) \). Indeed, for any sheaf of the form \( \tilde{E} \), the map \( \tilde{E}(f) \) is an isomorphism whenever the map \( f: [n'] \to [n] \) sends \( n' \) to \( n \)—that is, preserves the last elements of the totally ordered sets. However, this is the only condition: the essential image of embedding (4.1) consists of such \( M \in \text{Fun}(S(N(\mathcal{C})), \text{Ab}) \) that \( M(f) \) is a quasiisomorphism whenever \( f \) preserves the last elements. Indeed, it is easy to see that such a sheaf \( M \) is completely defined by the following part of (i), (ii):

(i) the functor \( M_1: N(\mathcal{C})_1 \to \text{Ab} \), and

(ii) the map

\[
M(s): s^*M_1 \to t^*M_1,
\]

where the maps \( s, t: [1] \to [2] \) send \( 1 \in [1] \) to \( 1 \in [2] \), resp. to \( 2 \in [2] \), subject to some conditions. This is exactly the same as a functor from \( \mathcal{C} \) to \( \text{Ab} \): \( M_0 \) gives its values on objects of the category \( \mathcal{C} \), and the map \( s^*M_1 \to t^*M_1 \) encodes the action of its morphisms.

The same comparison result is true on the level of derived categories: the functor

\[
\mathcal{D}(\mathcal{C}, \text{Ab}) \to \mathcal{D}(S(N(\mathcal{C})), \text{Ab})
\]

is a fully faithful embedding; moreover, its essential image consists of such \( M_* \in \mathcal{D}(S(N(\mathcal{C})), \text{Ab}) \) that \( M_*(f) \) is a quasiisomorphism whenever \( f \) preserves the last elements.

Now, the observation that we would like make is that small 2-categories also have nerves. Of course, the nerve \( N(\mathcal{C}) \) of a 2-category \( \mathcal{C} \) is a simplicial category rather than a simplicial set; however, the Grothendieck construction still applies, and the fibered category \( S(N(\mathcal{C}))/\Delta \) is perfectly well defined. The only new thing is that the fibers of this fibration are no longer discrete. As it happens, this changes nothing: if we define the triangulated category \( \mathcal{D}(\mathcal{C}, \text{Ab}) \) by the bar-construction, as in Section 1.6, then we still have a fully faithful embedding (4.3), with the same characterization of its essential image. However, since \( S(N(\mathcal{C})) \) is a usual category, not a 2-category, the right-hand side \( \mathcal{D}(S(N(\mathcal{C})), \text{Ab}) \) can be defined in the usual way, with no recourse to 2-categorical machinery and \( A_\infty \) methods.

Return now to the situation of Section 3.1—assume given a small category \( \mathcal{C} \) which has fibered products. In principle, we could simple apply the above discussion to the 2-category \( Q(\mathcal{C}) \). However, we will actually do something slightly different. Namely, we can define nerves in an even greater generality: instead of a 2-category, we can consider a “category in categories”, where not only morphisms form a category, not just a set, but also objects do the same thing.

It is probably not worth the effort to axiomatize the situation; instead, let us give the specific example we will use. Let \( \mathcal{C}^{[2]} \) be the category of arrows \( c_1 \to c_2 \) in \( \mathcal{C} \), with morphisms given by Cartesian squares

\[
\begin{array}{ccc}
c_1 & \longrightarrow & c_2 \\
\downarrow & & \downarrow \\
c'_1 & \longrightarrow & c'_2.
\end{array}
\]
Consider the opposite category $C^{[2]}_{\text{opp}}$. We have two projections $s$, resp. $t$ from $C^{[2]}_{\text{opp}}$ to $C^{\text{opp}}$, which send an arrow $c_1 \to c_2$ to its source $c_1$, resp. its target $c_2$. Moreover, composition of the arrows defines a functor

$$m: C^{[2]}_{\text{opp}} \times C^{[2]}_{\text{opp}} \to C^{[2]}_{\text{opp}},$$

where in the left-hand side, the projection to $C^{\text{opp}}$ is by $t$ in the left factor, and by $s$ in the right factor. This is our “category in categories”: $C^{\text{opp}}$ is its category of objects, $C^{[2]}_{\text{opp}}$ is its category of morphisms, and the functor $m$ defines the composition.

A functor from this “category in categories” to $\text{Ab}$ is, then, given by the following data:

- (i) a functor $M_1: C^{\text{opp}} \to \text{Ab}$, and
- (ii) a map $s^*M_1 \to t^*M_1$ (the analog of the action map (4.2)).

We now notice that these data define exactly a functor $M$ from the quotient category $QC$ of Section 3 to $\text{Ab}$. The functor $M_1$ defines the values of $M$ at objects of the category $C$ and the action of the maps $f^*$, $f$ a morphism of $C$, while the action map $s^*M_1 \to t^*M_1$ adds the maps $f_*$ to the picture.

This description breaks the symmetry between $f_*$ and $f^*$ manifestly present in the definition of the category $QC$, but this is not necessarily a bad thing: constructions such as the tensor product of Section 2.3 also break this symmetry, and one may hope that some constructions actually look better in the new description. As we shall see, this is indeed the case.

We now resume rigorous exposition. We start by constructing the nerve of our “category in categories”; we will denote this nerve simply by $SC$. 

### 4.2. The $S$-construction

Assume given a small category $C$ which has fibered products. For every integer $n \geq 2$, let $C^{[n]}$ be the category of diagrams of the form $c_1 \to \cdots \to c_n$ in the category $C$—in other words, $C^{[n]}$ is the category of $C$-valued functors from the totally ordered set $[n]$ with $n$ elements considered as a small category in the usual way. Sending a diagram $c_1 \to \cdots \to c_n$ to $c_n \in C$ defines a projection $C^{[n]} \to C$, and since $C$ has fibered products, this projection is a fibration in the sense of [SGA]. Denote by $\overline{C}^{[n]} \subset C^{[n]}$ the subcategory with the same objects as $C^{[n]}$ and the maps which are Cartesian with respect to the fibration $C^{[n]} \to C$. Thus for example, $\overline{C}^{[2]}$ is the category whose objects are arrows in $C$, and whose morphisms are Cartesian squares.

As usual, denote by $\Delta$ the category of non-empty finite totally ordered sets. Then for any order-preserving map $f: [n] \to [n']$, we have a natural functor $f^*: \overline{C}^{[n']} \to \overline{C}^{[n]}$. Moreover, we can also treat it as a functor $f^*: \overline{C}^{[n']}_{\text{opp}} \to \overline{C}^{[n]}_{\text{opp}}$ between the opposite categories. Then the collection $\overline{C}^{[n]}_{\text{opp}}$ with the transition functors $f^*$ forms a simplicial category, that is, a category fibered over $\Delta$. We denote this category by $SC$.

Explicitly, objects of $SC$ are pairs of a finite non-empty totally ordered set $[n]$ and a diagram $c_1 \to \cdots \to c_n$ in $C$; a map from $([n], c_1 \to \cdots \to c_n)$ to $([n'], c_1' \to \cdots \to c_n')$ is given by an order-preserving map $f: [n] \to [n']$ and a collection of
maps $f_i : c'_i f(i) \to c_i$ in $C$, one for each $i \in [n]$, such that the square

$$
\begin{array}{ccc}
  c'_i f(i) & \xrightarrow{f_i} & c_i \\
  \downarrow & & \downarrow \\
  c'_j f(j) & \xrightarrow{f_j} & c_j
\end{array}
$$

is commutative and Cartesian for any $i, j \in [n], i \leq j$. In particular, we have a natural embedding $C^{\text{opp}} \to SC$ which sends $c \in C^{\text{opp}}$ to $\langle 1, c \rangle$.

We can define a natural projection functor $\varphi : SC \to QC$ as follows: an object $\langle [n], c_1 \to \cdots \to c_n \rangle \in SC$ goes to $c_n \in QC$, and a map $\langle f, \{f_i\} : \langle [n], c_1 \to \cdots \to c_n \rangle \to \langle [n'], c'_1 \to \cdots \to c'_{n'} \rangle \rangle$ goes to a map represented by the diagram

$$
c_n \xrightarrow{f(n)} c'_n \to c'_{n'}.
$$

We will say that a map $f : [n'] \to [n]$ in the category $\Delta$ is special if it is an isomorphism between $[n']$ and a final segment of the ordinal $[n]$ (in other words, we have $f(l) = n + l - n'$ for any $l \in [n']$). We will say that a map $\langle f, \{f_i\} \rangle$ in the category $SC$ is special if it is Cartesian with respect to the fibration $SC \to \Delta^{\text{opp}}$ — that is, all the maps $f_i$ are invertible and the component $f : [n'] \to [n]$ is special.

One checks easily that if a map $f$ is special, then $\varphi(f)$ is invertible in $QC$. Moreover, we will say that a functor $F : SC \to Ab$ is special if $F(f)$ is invertible for every special $f$. Then setting $F \mapsto F \circ \varphi$ gives an equivalence

$$
\text{Fun}(QC, Ab) \cong \text{Fun}_{\text{sp}}(SC, Ab),
$$

where $\text{Fun}_{\text{sp}}(SC, Ab) \subset \text{Fun}(SC, Ab)$ is the full subcategory spanned by special functors. In this sense, the category $QC$ is obtained from the category $SC$ by inverting all special maps. Explicitly, a functor $M \in \text{Fun}(S(C), Ab)$ is given by the following data:

(i) A functor $M_n \in \text{Fun}(C^{[n]}^{\text{opp}}, Ab)$ for every $[n] \in \Delta$.

(ii) A transition map

$$
(f^*)^* M_n \to M_{n'},
$$

for any map $f : [n] \to [n']$, where $f^* : C^{[n']^{\text{opp}}} \to C^{[n]^{\text{opp}}}$ is the transition functor corresponding to the map $f$.

The functor $M$ is special if the transition map $M(f)$ is an isomorphism for every special map $f$ (it is clearly sufficient to check this for the maps $f : [1] \to [n], f(1) = n$).

Consider now the derived category $\mathcal{D}(SC, Ab)$.

**Definition 4.1.** An object $M \in \mathcal{D}(SC, Ab)$ is called special if for any special map $f$ in the category $SC$, the corresponding map $M(f)$ is a quasiisomorphism. The full subcategory in $\mathcal{D}(SC, Ab)$ spanned by special complexes is denoted by $\mathcal{D}S(C, Ab) \subset \mathcal{D}(SC, Ab)$.

The triangulated category $\mathcal{D}S(C, Ab)$ obviously contains the derived category $\mathcal{D}(\text{Fun}_{\text{sp}}(SC, Ab)) \cong \mathcal{D}(QC, Ab)$; however, there is no reason why they should be same, or indeed, even why the functor $\mathcal{D}(QC, Ab) \subset \mathcal{D}S(C, Ab)$ should be full and
faithful. It turns out that in general, it is not. This is exactly the difference between the naive derived category $\mathcal{D}(Q(C), \text{Ab})$ and the triangulated category $\mathcal{D}Q(C, \text{Ab})$ of Section 3.1.

**Theorem 4.2.** For any small category $C$ which has fibered products, we have a natural equivalence

$$\mathcal{D}S(C, \text{Ab}) \cong \mathcal{D}Q(C, \text{Ab}).$$

**4.3. Digression: complementary pairs.** To prove Theorem 4.2, we need to develop some combinatorial machinery for inverting special maps in the category $SC$. Unfortunately, the class of special maps does not admit a calculus of fractions in the usual sense. However, there is the following substitute.

**Definition 4.3.** Assume given a category $\Phi$ and two classes of maps $P, I$ in $\Phi$. Then $(P, I)$ is a **complementary pair** if the following conditions are satisfied:

(i) The classes $P$ and $I$ are closed under the composition and contain all isomorphisms.

(ii) For every object $b \in \Phi$, the category $\Phi_b^I$ of diagrams $i: b' \to b$, $i \in I$, has an initial object $i_0: \iota(b) \to b$.

(iii) Every map $f$ in $\Phi$ factorizes as $f = p(f) \circ i(f)$, $p(f) \in P$, $i(f) \in I$, and such a factorization is unique up to a unique isomorphism.

(iv) Every diagram $b_1 \xrightarrow{b} b_2$ in $\Phi$ with $p \in P$, $i \in I$, fits into a cocartesian square

$$\begin{array}{ccc}
b & \xrightarrow{p} & b_1 \\
\downarrow & & \downarrow \psi \\
\downarrow & & \\
\downarrow & & \\
\downarrow & & \\
b_2 & \xrightarrow{p'} & b_{22}
\end{array}$$

in $\Phi$ with special $p' \in P$, $i' \in I$.

**Remark 4.4.** If $\Phi$ has an initial object 0, then (ii) follows from (iii) (the map $i_0: \iota(c) \to c$ is a part of the decomposition $0 \to \iota(c) \to c$ of the map $0 \to c$).

**Remark 4.5.** A complementary pair in the sense of Definition 4.3 is an example of a **factorization system** in the sense of [Bon].

Assume given a small category $\Phi$ and a complementary pair $(P, I)$ of classes of maps in $\Phi$. Let $\mathcal{D}_I(\Phi, \text{Ab})$ be the full subcategory in the derived category $\mathcal{D}(\Phi, \text{Ab})$ spanned by those $M_i \in \mathcal{D}(\Phi, \text{Ab})$ for which $M_i(i)$ is a quasiisomorphism for any $i \in I$. Let $\mathcal{R}\Phi$ be the category of diagrams $c_1 \xrightarrow{c} c_2$ in $\Phi$ with maps $i_1: c \to c_1$, $i_2: c \to c_2$ in the class $I$. Let $\pi_1, \pi_2: \mathcal{R}\Phi \to \Phi$, be the projections which send $c_1 \xrightarrow{c} c_2$ to $c_1$, resp. to $c_2$, and denote by $\text{Sp}: \mathcal{D}(\Phi, \text{Ab}) \to \mathcal{D}(\Phi, \text{Ab})$ the functor given by $\text{Sp} = L^*\pi_1\pi_2^*.$

**Lemma 4.6.** There exists a map of functors $\psi: \text{Id} \to \text{Sp}$ such that $\psi \circ \text{Sp} = \text{Id}$, and the map $\psi: M_i \to \text{Sp}(M_i)$ for an object $M_i \in \mathcal{D}(\Phi, \text{Ab})$ is a quasiisomorphism if and only if $M_i$ lies in $\mathcal{D}_I(\Phi, \text{Ab}) \subset \mathcal{D}(\Phi, \text{Ab})$.

In other words, $\text{Sp}: \mathcal{D}(\Phi, \text{Ab}) \to \mathcal{D}(\Phi, \text{Ab})$ is the canonical projection onto the left-admissible full subcategory $\mathcal{D}_I(\Phi, \text{Ab})$ — that is, the composition of the embedding $\mathcal{D}_I(\Phi, \text{Ab}) \subset \mathcal{D}(\Phi, \text{Ab})$ and the left-adjoint functor $\mathcal{D}(\Phi, \text{Ab}) \to \mathcal{D}_I(\Phi, \text{Ab})$. 
Proof. Both projections $\pi_1, \pi_2$ have a common section $\delta : \Phi \to \mathcal{R}\Phi$ which sends $c \in \Phi$ to the diagram $c \leftarrow c \to c$ with identity maps. The isomorphism $\text{Id} \cong \delta^* \pi_2^*$ induces by adjunction a map $L^* \delta \to \pi_2^*$; the functorial map $\psi$ is obtained by applying $L^* \pi_1$ to this map.

To prove the required properties of the map $\psi$, we start by noting that due to the conditions (iii) and (iv) of Definition 4.3, the projection $\pi_1 : \mathcal{R}\Phi \to \Phi$ is a cofibration. Therefore for any $M_i \in \mathcal{D}(\Phi, \text{Ab})$ and any object $c \in \Phi$, we have a canonical base change isomorphism

$$\text{Sp}(M_i)(c) \cong H_*(\mathcal{R}\Phi, \pi_2^*M_i),$$

where $\mathcal{R}\Phi \subset \mathcal{R}\Phi$ is the fiber of the cofibration $\pi_1$ — that is, the category of diagrams $c \leftarrow c' \to c''$ in $\Phi$ with maps $i' : c' \to c, i'' : c' \to c''$ lying in $I$. This fiber $\mathcal{R}\Phi$ projects to the category $\Phi^I$ of Definition 4.3 (iii) by forgetting $i'$ and $i''$; denote by $F\Phi$ the fiber of this projection over the initial object $\iota(c) \in \Phi^I$. Explicitly, $F\Phi$ is the category of objects $c'' \in \Phi$ equipped with a map $i : \iota(c) \to c''$, $i \in I$.

By definition, for any special map $i' : c' \to c$, the map $i_2 : \iota(c) \to c$ canonically factorizes as $i_2 = i' \circ i_1 : \iota(c) \to c' \to c$. Moreover, it is easy to see that if we have two maps $i_1, i_2 \in \Phi$ such that $i_1 \in I$ and $i_1 \circ i_2 \in I$, then $i_2 \circ i_1$ may not lie in $I$ (take its factorization $i_2 = i_2 \circ p$ of (iii)), compose with $i_1$ to obtain a factorization of $i_1 \circ i_2$, use the uniqueness to deduce that $p$ is invertible. This implies that $\iota(c) \cong i(c)$ canonically, and $i_0 = i_1$. Then sending a diagram $c \leftarrow c' \to c''$ to $i'' \circ i_2 : \iota(c) \to c''$ defines a projection $\pi : \mathcal{R}\Phi \to F\Phi$, and this projection is right-adjoint to the embedding $i : F\Phi \to \mathcal{R}\Phi$.

The projection $\pi_2$ factors as $\pi_2 = \pi \circ \iota : \mathcal{R}\Phi \to F\Phi \to \Phi$, where $\pi : F\Phi \to \Phi$ sends a diagram $\iota(c) \to c''$ to $c''$. Therefore we have

$$H_*(\mathcal{R}\Phi, \pi_2^*M_i) \cong H_*(\mathcal{R}\Phi, \pi^* \pi_2^*M_i) \cong H_*(\mathcal{R}\Phi, L^* i_1^* \pi^*M_i),$$

so that $\text{Sp}(M_i)(c) \cong H_*(F\Phi, \pi^*M_i)$. The canonical map $\psi : M_i(c) \to \text{Sp}(M_i)(c)$ is then induced by the inclusion $\iota(c) \to c \in F\Phi$. The point to the diagram $\iota(c) \to c \in F\Phi$.

Now, for every map $i : c \to c', i \in I$, the categories $F\Phi$ and $F\Phi'$ are canonically equivalent; therefore the map $\text{Sp}(M_i)(c) \to \text{Sp}(M_i)(c')$ is a quasi-isomorphism, and we have $\text{Sp}(M_i) \in \mathcal{D}(\Phi, \text{Ab})$. And if we know in advance that $\mathcal{D}(\Phi, \text{Ab})$, then the pullback $\pi^*M_i \in \mathcal{D}(F\Phi, \text{Ab})$ is constant, so that $\text{Sp}(M_i)(c) \cong M_i(c) \otimes H_*(F\Phi, \mathbb{Z})$. Since $F\Phi$ has an initial object, we have $H_*(F\Phi, \mathbb{Z}) \cong \mathbb{Z}$. □

For any two objects $c_1, c_2 \in \Phi$, let $Q_I(c_1, c_2)$ be the category of diagrams $c_1 \leftarrow c_2 \in \Phi$ such that the map $c_1 \to c$ lies in the class $I$, and the map $c_2 \to c$ lies in the class $P$. Then for any three objects $c_1, c_2, c_3$, we have natural composition functors

$$Q_I(c_1, c_2) \times Q_I(c_2, c_3) \to Q_I(c_1, c_3) \quad (4.5)$$

given by the cocartesian squares, which exist by Definition 4.3 (iv). These functors are associative. Say that an object $c \in \Phi$ is simple if the canonical map $i_c : c \to c$ is an isomorphism, and let $Q_I(\Phi)$ be the 2-category whose objects are simple objects in $\Phi$, and whose morphism categories are $Q_I(c_1, c_2)$, with composition functors
Applying the procedure of Section 1.6, construct an $A_{\infty}$-category $B^I_2(\Phi)$ with the same objects as $Q_I(\Phi)$, and with morphisms given by
\[ B^I_2(c_1, c_2) = C_* (Q_I(c_1, c_2), \mathbb{Z}). \]

Let $DQ_I(\Phi, \text{Ab})$ be the triangulated category of $A_{\infty}$-functors from the $A_{\infty}$-category $B^I_2(\Phi)$ to $\text{Ab}$.

**Proposition 4.7.** There exists a natural equivalence of triangulated categories
\[ DQ_I(\Phi, \text{Ab}) \cong D_I(\Phi, \text{Ab}). \]

**Proof.** To define a comparison functor $\varphi: D_I(\Phi, \text{Ab}) \to DQ_I(\Phi, \text{Ab})$, let $\tilde{\Phi}$ be $\Phi$ with a formally added initial object $\emptyset$, and declare that the map $\emptyset \to c$ is in the class $P$ for every $c \in \Phi$. Then all the conditions of Definition 4.3 are satisfied for $\tilde{\Phi}$, so that we can form the 2-category $Q_I(\tilde{\Phi})$. For any simple $c \in \Phi$, the category $Q_I(\emptyset, c)$ is the category $F\Phi_c$ of the proof of Lemma 4.6, and for any simple $c_1, c_2 \in \Phi$, we have the composition functors
\[ m_{c_1, c_2}: F\Phi_{c_1} \times Q_I(c_1, c_2) = Q_I(\emptyset, c_1) \times Q_I(c_1, c_2) \to F\Phi_{c_2} = Q_I(\emptyset, c_2). \]

For every simple $c \in \Phi$, a functor $M \in \text{Fun}(\Phi, \text{Ab})$ gives by restriction a functor $M_c = \pi_2^* M \in \text{Fun}(F\Phi_c, \text{Ab})$. For every simple $c_1, c_2 \in \Phi$, we have a natural map
\[ \mu_{c_1, c_2}: M_{c_2} \boxtimes \mathbb{Z} \to m_{c_1, c_2}^* M_{c_1}, \]

and these maps are associative in the obvious sense. We define $\varphi(M)_* \in DQ_I(\Phi, \text{Ab})$ by
\[ \varphi(M)_*(c) = C_* (F\Phi_c, M_c), \]

with the $A_{\infty}$-functor structure induced by the functors $m$ and the maps $\mu$. This extends to a functor $\varphi: D(\Phi, \text{Ab}) \to DQ_I(\Phi, \text{Ab})$. Comparing the definitions of the functors $\text{Sp}$ and $\varphi$, we see that the canonical map $\psi: M_c \to \text{Sp}(M_c)$ induces a quasiisomorphism
\[ \varphi(M_c) \cong \varphi(\text{Sp}(M_c)) \]

for any $M_c \in D(\Phi, \text{Ab})$, so that the functor $\varphi$ factors through the projection $\text{Sp}: D(\Phi, \text{Ab}) \to D_I(\Phi, \text{Ab})$.

To show that the functor $\varphi: D_I(\Phi, \text{Ab}) \to DQ_I(\Phi, \text{Ab})$ is an equivalence, we have to prove that it is full, faithful, and essentially surjective. The triangulated category $DQ_I(\Phi, \text{Ab})$ is generated by the representable $A_{\infty}$-functors $M_c^Q$, for all simple $c \in \Phi$ and complexes $M \in \text{Ho}(\text{Ab})$, given by $M_c^Q(c', c) = B^I_2(c, c') \boxtimes M$. The triangulated category $DQ_I(\Phi, \text{Ab})$ is generated by the objects $\text{Sp}(M_c)$, for all $c \in \Phi$ and $M \in \text{Ho}(\text{Ab})$, where $M_c \in \text{Fun}(\Phi, \text{Ab})$ is the representable functor given by $M_c(c') = \mathbb{Z} [\Phi(c, c')] \boxtimes M$. Therefore it suffices to prove that
(i) for any $c \in \Phi$ and $M \in \text{Ho}(\text{Ab})$, we have $\text{Sp}(M_c) \cong \text{Sp}(M_{c(c)})$,
(ii) for any simple $c \in \Phi$ and $M \in \text{Ho}(\text{Ab})$, we have $\varphi(M_c) \cong M_c^Q$, and
(iii) for any simple $c, c' \in \Phi$ and $M \in \text{Ho}(\text{Ab})$, the natural map
\[ \text{RHom}^*(\text{Sp}(M_c), \text{Sp}(M_{c'})) \to \text{RHom}^*(M_c^Q, M_{c'}^Q) \]

induced by the functor $\varphi$ is a quasiisomorphism.
To prove (i), note that by adjunction, we have
\[ \text{RHom}^*(\text{Sp}(M_c), M') \cong \text{RHom}^*(M_c, M') \cong \text{RHom}^*(M, M'(c)) \]
for any \( c \in \Phi, M \in \text{H}(\text{Ab}) \), and any \( M' \in \mathcal{D}_I(\Phi, \text{Ab}) \subset \mathcal{D}(\Phi, \text{Ab}) \). In particular, \( \text{RHom}^*(\text{Sp}(M_c), M') \cong \text{RHom}^*(\text{Sp}(M\langle c \rangle), M') \). Since this is true for any \( M' \in \mathcal{D}_I(\Phi, \text{Ab}) \), this implies (i). To prove (ii), note that for any simple \( c' \in \Phi \), we have an embedding \( j: \mathcal{Q}_I(c, c') \to F\Phi_c \), and by Definition 4.3 (iii) there is a natural isomorphism
\[ M_c|_{F\Phi_c} \cong j!M_{\text{const}}, \]
where \( M_{\text{const}} \in \text{Fun}(\mathcal{Q}_I(c, c'), \text{Ab}) \) is the constant functor with value \( M \). Therefore
\[ \text{Sp}(M_c)(c') \cong C_*(F\Phi_c, j!M_{\text{const}}) \cong C_*(\mathcal{Q}_I(c, c'), M_{\text{const}}) = \mathcal{B}^I(c, c') \otimes M, \]
as required. Finally, for (iii), note that by adjunction
\[ \text{RHom}^*(\text{Sp}(M_c), \text{Sp}(M_c)) \cong \text{RHom}^*(M_c, \text{Sp}(M_c)) \cong \text{Sp}(M_c)(c), \]
so that (iii) follows from (ii). \( \square \)

4.4. The comparison theorem. Now again, assume given a small category \( \mathcal{C} \) which has fibered products, and take \( \Phi = SC \). Say that a map \( \langle f, \{ f_i \} \rangle \) in \( SC \) is co-special if the underlying map \( f: [n] \to [n'] \) sends the first element in \( [n] \) to the first element in \( [n'] \), \( f(1) = 1 \). Let \( I, P \) be the classes of special, resp. co-special maps in the category \( SC \).

Lemma 4.8. The pair \((P, I)\) is a complementary pair in the sense of Definition 4.3.

Proof. Definition 4.3 (i) is obvious. For (ii), let \( c = \langle [n], c_1 \to \cdots \to c_n \rangle \); then \( i(c) = \langle [1], c_n \rangle \), with the obvious map \( i_c: i(c) \to c \). For (iii), take a map \( f = \langle f, \{ f_i \} \rangle: \langle [n], \{ c_i \} \rangle \to \langle [n'], \{ c_i' \} \rangle \), and let \( n_0 = f(1) \in [n'] \); then the decomposition is given by
\[ \langle [n], \{ c_i \} \rangle \longrightarrow \langle [n' - n_0], \{ c_i'' \} \rangle \longrightarrow \langle [n'], \{ c_i' \} \rangle, \]
where \( c_i'' = c_{i+n_0} \), \( 1 \leq l \leq n' - n_0 \), with the obvious maps. Finally, for (iv), let \( b = \langle [n], \{ b_i \} \rangle, b_1 = \langle [n_1], \{ b_{i_1} \} \rangle, b_2 = \langle [n_2], \{ b_{i_2} \} \rangle \). Then \( b_{12} = \langle [n_2], \{ b_{i_2} \} \rangle \) is given by \( n_2 = n_1 + n_2 - n, b_{i_2} = b_{i_2+1-n_2} \) for \( l = n - n_1 + 1, \ldots, n_1, \) and for \( l = 1, \ldots, n - n_2, b_{i_2} \) is obtained as the fibered product
\[
\begin{array}{ccc}
b_1^{i_2} & \longrightarrow & b_2^{i_2} \\
\downarrow & & \downarrow \\
b_1^{i_1} & \longrightarrow & b_1 \cong b_{n_2+1-n}^{i_1} 
\end{array}
\]
in the category \( \mathcal{C} \). \( \square \)

Proof of Theorem 4.2. By Lemma 4.8, Proposition 4.7 can be applied to the category \( SC \). An object \( c = \langle [n], c_i \rangle \) is simple if and only if \( n = 1 \), thus simple objects in \( SC \) are the same as objects in \( \mathcal{C} \subset SC \). Then comparing the definitions of \( A_\infty \)-categories \( B^S_c \) of Section 3.1 and \( B^I_*(SC) \) of Section 4.3, we see that it remains to prove the following: for any \( c_1, c_2 \in \mathcal{C} \subset SC \), there exists a natural quasisomorphism
\[ \eta^{c_1,c_2}: C_*(\mathcal{Q}_I(c_1, c_2), \mathcal{Z}) \cong C_*(\mathcal{Q}(c_1, c_2), \mathcal{Z}), \]
and these quasiisomorphisms extend to an $A_\infty$-functor. The functor $\varphi: SC \to QC$ of Section 3.1 obviously extends to a 2-functor $QI(SC) \to QC$; this gives maps $\eta_{c_1,c_2}$ which form an $A_\infty$-functor. To prove that $\eta_{c_1,c_2}$ is a quasiisomorphism, it remains to notice that the functor $\varphi: QI(c_1, c_2) \to QC(c_1, c_2)$ has a left-adjoint which sends a diagram $c_1 \leftarrow c \rightarrow c_2$ in $C$ to the diagram $c_1 \rightarrow \langle[2], c_2 \rightarrow c_1 \rangle \leftarrow c_2$ in $SC$.

\section{Additivity.} Assume now given a small category $C$ such that the wreath product $C\wr \Gamma$ has fibered products. Then we can form the category $S(C\wr \Gamma)$ and the derived category $DS(C\wr \Gamma, Ab)$. By definition, $(C\wr \Gamma)^{\text{opp}}$ is embedded into $S(C\wr \Gamma)$ as the fiber over $[1] \in \Delta$, so that we have the restriction functor $DS(C\wr \Gamma, Ab) \to D((C\wr \Gamma)^{\text{opp}}, Ab)$.

\begin{definition}
An object $M_\ast \in DS(C\wr \Gamma)$ is called additive if its restriction to $D((C\wr \Gamma)^{\text{opp}}, Ab)$ is additive in the sense of Definition 3.2.
\end{definition}

\begin{proposition}
Assume that the small category itself $C$ has fibered products. Then the full subcategory $DS_{\text{add}}(C\wr \Gamma, Ab) \subset DS(C\wr \Gamma, Ab)$ spanned by additive objects is canonically equivalent to the category $DS(C, Ab)$.
\end{proposition}

\begin{proof}
The statement immediately follows from Theorem 4.2, Proposition 3.8 and Corollary 3.11.
\end{proof}

\begin{definition}
Assume given a small category $C$ such that $C\wr \Gamma$ has fibered products. Then the full subcategory $DS_{\text{add}}(C\wr \Gamma, Ab) \subset DS(C\wr \Gamma, Ab)$ spanned by additive objects is denoted by $DS(C, Ab)$.
\end{definition}

We note that by Proposition 4.10, this is consistent with our earlier Definition 4.1.

\begin{remark}
For the sake of methodological purity, it would be nice to have a direct proof of Proposition 4.10 which does not use the material of Section 3. Unfortunately, I was not able to find such a proof. All I could come up with essentially repeats the proof of Proposition 3.8, with additional complications (which arise because we do not have a Waldhausen-type interpretation of the category $DQI(C, Ab)$ of Definition 3.7).

In the particular case $C = O_G$, the category of finite $G$-orbits for a group $G$, we have $C\wr \Gamma \cong \Gamma_G$. Combining Theorem 4.2 and Proposition 4.10, we get the following.

\begin{corollary}
For any finite group $G$, let $O_G$ be the category of $G$-orbits. Then the category $DM(G)$ of derived $G$-Mackey functors is naturally equivalent to the category $DS(O_G, Ab)$ of Definition 4.11.
\end{corollary}

\section{Functoriality and Products}
We will now describe some basic properties of derived Mackey functors, mostly analogous to the material in Section 2.2 and Section 2.3.
5.1. Functoriality. Assume given two small categories $C, C'$ such that $C \wr \Gamma$ and $C' \wr \Gamma$ have fibered products. Then any functor $\gamma: C' \to C \wr \Gamma$ uniquely extends to a coproduct-preserving functor $\gamma: C' \wr \Gamma \to C \wr \Gamma$. In either of the two constructions of the category $\mathcal{D}\mathcal{S}(\mathcal{-}, \text{Ab}) \cong \mathcal{D}\mathcal{Q}_{\text{add}}(\mathcal{-} \wr \Gamma, \text{Ab})$ we have the following obvious functoriality property.

- If the extended functor $\gamma: C' \wr \Gamma \to C \wr \Gamma$ preserves fibered products, then we have a natural restriction functor

$$\gamma^*: \mathcal{D}\mathcal{S}(C, \text{Ab}) \cong \mathcal{D}\mathcal{Q}_{\text{add}}(C \wr \Gamma, \text{Ab}) \to \mathcal{D}\mathcal{S}(C', \text{Ab}) \cong \mathcal{D}\mathcal{Q}_{\text{add}}(C' \wr \Gamma, \text{Ab}),$$

and the left-adjoint induction functor $\gamma_!: \mathcal{D}\mathcal{S}(C', \text{Ab}) \to \mathcal{D}\mathcal{S}(C, \text{Ab}).$

It turns out that this yields the derived versions of both the functor $\Psi$ and the functor $\Phi$ of Section 2.2.

In fact, the functor $\Psi$ has already appeared in Section 3 under a different name. For any object $c \in C$, we have its embedding functor $j_c: \text{pt} \to C \wr \Gamma$, and the extended functor $j^c: \Gamma \to C \wr \Gamma$ preserves fibered products for semi-trivial reasons.

**Definition 5.1.** The naive fixed point functor $\Psi^c$ is the functor

$$\Psi^c = j^c* : \mathcal{D}\mathcal{S}(C, \text{Ab}) \to \mathcal{D}\mathcal{S}(\text{pt}, \text{Ab}) \cong \mathcal{D}(\text{Ab}).$$

In the Mackey functor case $C = \mathcal{O}_G$, $c = [G/H]$, the functor $\Psi^c$ is the derived version of the functor $\Psi^H$ of Section 2.2.

We can also slightly refine the construction. Let $(c) \subset C$ be the groupoid of objects in $C$ isomorphic to $c$, and invertible maps between them. Then $j^c$ extends to an embedding $\tilde{j}^c: (c) \to C \wr \Gamma$ whose natural extension $\tilde{j}^c: (c) \wr \Gamma \to C \wr \Gamma$ still preserves fibered products. We denote the corresponding restriction functor by

$$\tilde{\Psi}^c = \tilde{j}^c*: \mathcal{D}\mathcal{S}(C, \text{Ab}) \to \mathcal{D}\mathcal{S}((c), \text{Ab}) \cong \mathcal{D}(\text{Aut}(c), \text{Ab}).$$

(5.1)

It takes values in the category $\mathcal{D}\mathcal{S}((c), \text{Ab})$, which is obviously equivalent to the derived category $\mathcal{D}(\text{Aut}(c), \text{Ab})$ of Ab-valued representations of the group $\text{Aut}(c)$ of automorphisms of the object $c$.

To proceed further, we need to impose a restriction on the category $C$.

**Definition 5.2.** A category $C$ is called Hom-finite if the set of maps $C(c, c')$ is finite for any two objects $c, c' \in C$.

Assume that the category $C$ is Hom-finite. Then so is the wreath product $C \wr \Gamma$, and an object $c$ also represents a functor $\tau_c: C \to \Gamma$,

$$\tau_c(c') = C(c, c').$$

Its natural extension $\tau_c: C \wr \Gamma \to \Gamma$ is represented by the same object $c$, thus preserves fibered products.

**Definition 5.3.** The geometric fixed points functor $\Phi^c$ is given by

$$\Phi^c = \tau_{cd} : \mathcal{D}\mathcal{S}(C, \text{Ab}) \to \mathcal{D}\mathcal{S}(\text{pt}, \text{Ab}) \cong \mathcal{D}(\text{Ab}).$$
In the case $C = O_G$, $c = [G/H]$, this is the derived version of the functor $\Phi^H$ of Section 2.2. The adjoint functor $\tau^*_c : D(Ab) \to D\Sigma(C, Ab)$ can be explicitly described as follows. Denote

$$T^c = \tau^*_c(T) \in \text{Fun}_{sp}(S(C \wr \Gamma), \mathbb{Z}\text{-mod}) \subset D\Sigma(C, \mathbb{Z}\text{-mod}),$$

where the functor $T \in \text{Fun}_{sp}(S(\Gamma), \mathbb{Z}\text{-mod})$ is the natural generator of the category $D\Sigma_{\text{add}}(\Gamma, \mathbb{Z}\text{-mod})$ introduced in Lemma 3.4. Then for any $M_\ast \in D(Ab)$, we have

$$\tau^*_c(M_\ast) \cong M_\ast \otimes T^c.$$

This can also be refined to incorporate $\text{Aut}(c)$. Indeed, the group $\text{Aut}(c)$ obviously acts on the object $T^c$. We define the inflation functor

$$\text{Infl}^c : \text{Fun}(\text{Aut}(c), Ab) \to \text{Fun}(S(C \wr \Gamma), Ab)$$

by $\text{Infl}^c(M) = (M \otimes T^c)^{\text{Aut}(c)}$, and we note that $R^*\text{Infl}^c$ sends $D(Aut(c), Ab)$ into the category $D\Sigma(C, Ab) \subset D(S(C \wr \Gamma), Ab)$. By abuse of notation, we will drop $R^*$ and denote the derived functor simply by

$$\text{Infl}^c : D(Aut(c), A) \cong D(\langle c \rangle, Ab) \to D\Sigma(C, Ab),$$

and we denote by

$$\tilde{\Phi}^c : D\Sigma(C, Ab) \to D(Aut(c), Ab) \cong D(\langle c \rangle, Ab)$$

its left-adjoint.

**Remark 5.4.** The names *naïve* and *geometric* attached to fixed points functors come from equivariant stable homotopy theory, whose part in the story we will explain in Section 8. It seems that the geometric fixed point functor is the more important of the two; thus from now, “fixed point functor” without an adjective will mean the functor $\Phi^c$.

We can further refine this construction by the following observation: the group $\text{Aut}(c)$ acts not only on the functor $T^c$, but on the functor $\tau^*_c$, too — for any $c' \in C \Gamma$, $\text{Aut}(c)$ naturally act on the finite set $\tau^*_c(c) = C(c, c')$. Thus we actually have a functor $\tilde{\tau}^*_c : C \to \Gamma_{\text{Aut}(c)} = O_{\text{Aut}(c)} \wr \Gamma$ which induces functors

$$\tilde{\Phi}^c = \tilde{\tau}^*_{cd} : D\Sigma(C, Ab) \to D\Sigma(O_{\text{Aut}(c)}, Ab) = D\Sigma(\text{Aut}(c), Ab),$$

$$\text{Infl}^c = \tilde{\tau}^*_c : D\Sigma(\text{Aut}(c), Ab) \to D\Sigma(C, Ab).$$

In the case $C = O_G$, $c = [G/H]$, we have $\text{Aut}(c) = N_H/H$, where $N_H \subset G$ is the normalizer of the subgroup $H$. Then the extended inflation functor takes the form

$$\text{Infl}^N_G : D\Sigma(N, Ab) \to D\Sigma(G, Ab),$$

where $N = N_H/H$. If $H \subset G$ is normal, so that $N_H = G$ and $N = G/H$, this is the derived version of the fully faithful functor $\text{Infl}^N_G$ of Section 2.2. As we will prove in Section 7, Lemma 7.13, in this case the functor $\text{Infl}^N_G$ is fully faithful.
5.2. Products. To introduce a tensor product on the category \( DS(C, \text{Ab}) \), we impose the same additional assumption as in Section 2.3—we require that the small category \( C \) has a terminal object. Then so does \( C \wr \Gamma \); of course, we still assume that \( C \wr \Gamma \) has fibered products, so that altogether, it has all finite limits (in particular, products).

It is convenient to identify \( DS(C, \text{Ab}) \cong DQ_{\text{add}}(C \wr \Gamma, \text{Ab}) \) and use the \( A_{\infty} \) methods of Section 3. By the definition of the 2-category \( Q(C \wr \Gamma) \), we have an isomorphism
\[
Q((C \wr \Gamma) \times (C \wr \Gamma)) \cong Q(C \wr \Gamma) \times Q(C \wr \Gamma).
\]
Since the bar construction is compatible with products, this means that for any two objects \( M_q, M'_q \in DQ(C \wr \Gamma, \text{Ab}) \), we have a well-defined box product
\[
M_q \boxtimes M'_q \in DQ((C \wr \Gamma) \times (C \wr \Gamma), \text{Ab}).
\]
We also have the product functor
\[
m: (C \wr \Gamma) \times (C \wr \Gamma) \to C \wr \Gamma,
\]
and since it preserves fibered products, it induces the restriction functor
\[
m^*: DS(C, \text{Ab}) \cong DQ_{\text{add}}(C \wr \Gamma, \text{Ab}) \to DQ((C \wr \Gamma) \times (C \wr \Gamma), \text{Ab}).
\]
Consider the left-adjoint functor
\[
m^{\text{add}}: DQ((C \wr \Gamma) \times (C \wr \Gamma), \text{Ab}) \to DS(C, \text{Ab}).
\]

**Definition 5.5.** The tensor product \( M \otimes M' \) of two objects \( M_q, M'_q \in DS(C, \text{Ab}) \) is given by
\[
M_q \otimes M'_q = m^{\text{add}}(M_q \boxtimes M'_q).
\]

Under our assumptions on \( \text{Ab} \), this obviously gives a well-defined symmetric tensor product structure on the triangulated category \( DS(C, \text{Ab}) \).

We have the following result on compatibility between tensor products and fixed points. Assume that the small category \( C \) is Hom-finite, so that for any object \( c \in C \), we have a well-defined geometric fixed points functor \( \Phi^c: DS(C, \text{Ab}) \to D(\text{Ab}) \).

**Proposition 5.6.** For any \( c \in C \), the geometric fixed points functor \( \Phi^c \) is a tensor functor—that is, for any two objects \( M_q, M'_q \in DS(C, \text{Ab}) \) we have an isomorphism
\[
\Phi^c(M_q \otimes M'_q) \cong \Phi^c(M_q) \otimes \Phi^c(M'_q),
\]
and this isomorphism is functorial in \( M_q \) and \( M'_q \).

**Proof.** The functor \( \tau_c: C \wr \Gamma \to \Gamma \), being representable, commutes with products, so that we have a commutative diagram
\[
\begin{array}{ccc}
Q(C \wr \Gamma) \times Q(C \wr \Gamma) & \xrightarrow{m} & Q(C \wr \Gamma) \\
\tau_c \times \tau_c \downarrow & & \downarrow \tau_c \\
Q(\Gamma) \times Q(\Gamma) & \xrightarrow{m} & Q(\Gamma),
\end{array}
\]
and \( m^* \circ \tau^*_c \cong (\tau^*_c \times \tau^*_c) \circ m^* \). By adjunction, this gives a functorial isomorphism
\[
\Phi^c(M_q \otimes M'_q) \cong \Phi^c(M_q) \otimes \Phi^c(M'_q),
\]
where the product in the right-hand side is taken in the category $\mathcal{DS}(\text{pt}, \text{Ab})$ in the sense of Definition 5.5. It remains to notice that the canonical equivalence of Lemma 3.4 identifies this product with the tensor product in the derived category $\mathcal{D}(\text{Ab}).$ □

**Remark 5.7.** The situation with the tensor product in $\mathcal{D}Q(\mathcal{C}, \text{Ab})$ is somewhat reminiscent of the tensor product of $\mathcal{D}$-modules on a smooth algebraic variety $X$: while a $\mathcal{D}$-module is simply a sheaf of modules over the ring $\mathcal{D}_X$ of differential operators on $X$, this ring is not commutative, and one has to take special care to define a symmetric tensor product on the category $\mathcal{D}_X$-mod. The product becomes much more natural if one passes to a Koszul-dual interpretation and replaces $\mathcal{D}$-modules with sheaves of DG modules over the de Rham complex $\Omega^*_X$. We will obtain an analogous Koszul-dual interpretation of Mackey functors in Section 6.

**5.3. Induction.** We finish this section with one more functoriality result. It seems that it does not appear in the standard theory of Mackey functors; however, it will be very useful in Section 8.

We again fix a small category $\mathcal{C}$ such that $\mathcal{C} \ltimes \Gamma$ has fibered products. By the definition of the 2-category $\mathcal{Q}(\mathcal{C} \ltimes \Gamma)$, we have a natural functor $\mathcal{C} \ltimes \Gamma \to \mathcal{Q}(\mathcal{C} \ltimes \Gamma)$ which is identical on objects, and sends a map $f: S' \to S$ to the diagram $S' \xleftarrow{id} S' \xrightarrow{f} S$. Composing this with the natural embedding $j: \mathcal{C} \to \mathcal{C} \ltimes \Gamma$, we obtain a functor

$$q: \mathcal{C} \to \mathcal{Q}(\mathcal{C} \ltimes \Gamma).$$

Since the 2-category $\mathcal{Q}(\mathcal{C} \ltimes \Gamma)$ is manifestly self-dual, we also have the opposite functor $q^{\text{opp}}: \mathcal{C}^{\text{opp}} \to \mathcal{Q}(\mathcal{C} \ltimes \Gamma)$, and the corresponding restriction functor

$$q^{\text{opp}}: \mathcal{DS}(\mathcal{C}, \text{Ab}) \cong \mathcal{D}Q_{\text{add}}(\mathcal{C} \ltimes \Gamma, \text{Ab}) \subset \mathcal{D}Q(\mathcal{C} \ltimes \Gamma, \text{Ab}) \to \mathcal{D}(\mathcal{C}^{\text{opp}}, \text{Ab}).$$

We will call the adjoint functor

$$q_!^{\text{opp}}: \mathcal{D}(\mathcal{C}^{\text{opp}}, \text{Ab}) \to \mathcal{DS}(\mathcal{C}, \text{Ab})$$

the induction functor, and we will say that an object $M_\ast \in \mathcal{DS}(\mathcal{C}, \text{Ab})$ is induced if $M_\ast \cong q_!^{\text{opp}}(E_\ast)$ for some $E_\ast \in \mathcal{D}(\mathcal{C}^{\text{opp}}, \text{Ab}).$

If the target category $\text{Ab}$ is a symmetric tensor category, then the functor category $\text{Fun}(\mathcal{C}^{\text{opp}}, \text{Ab})$ has a natural “pointwise” tensor product that induces a tensor product on the derived category $\mathcal{D}(\mathcal{C}^{\text{opp}}, \text{Ab}).$

**Proposition 5.8.** (i) Assume that the category $\mathcal{C}$ is Hom-finite. Then for any object $c \in \mathcal{C}$ and any $E_\ast \in \mathcal{D}(\mathcal{C}^{\text{opp}}, \text{Ab})$, we have an isomorphism

$$\Phi^c(q_!^{\text{opp}}(E_\ast)) \cong E_\ast(c),$$

and this isomorphism is functorial in $E_\ast$. (ii) Assume that the category $\mathcal{C}$ has a terminal object and that the target category $\text{Ab}$ is a symmetric tensor category. Then the induction functor $q_!^{\text{opp}}$ is a tensor functor — we have an isomorphism

$$q_!^{\text{opp}}(E_\ast \otimes E'_\ast) \cong q_!^{\text{opp}} E_\ast \otimes q_!^{\text{opp}} E'_\ast$$

for any $E_\ast, E'_\ast \in \mathcal{D}(\mathcal{C}^{\text{opp}}, \text{Ab})$, and this isomorphism is functorial in $E_\ast$ and $E'_\ast$. 


Proof. For (i), note that by adjunction, it suffices to prove that 

\[ q^{opp} \ast Tc \in \text{Fun}(C^{opp}, Z\text{-mod}) \]

is the object co-represented by \( c \in C^{opp} \), that is, 

\[ Tc(c') \cong Z[C^{opp}(c', c)] = Z[C(c, c')]. \]

This is the definition of the object \( Tc \).

For (ii), we repeat the argument of Section 2.3. Firstly, as noted several times already, we have a natural equivalence between \( D(C^{opp}, Ab) \) and the full subcategory \( D_{add}((C \wr \Gamma)^{opp}, Ab) \subset D((C \wr \Gamma)^{opp}, Ab) \) spanned by additive objects. Secondly, we have a commutative diagram

\[
\begin{array}{ccc}
(C \wr \Gamma)^{opp} \times (C \wr \Gamma)^{opp} & \xrightarrow{m} & (C \wr \Gamma)^{opp} \\
q^{opp} \downarrow & & \downarrow q^{opp} \\
Q(C \wr \Gamma) \times Q(C \wr \Gamma) & \xrightarrow{m} & Q(C \wr \Gamma).
\end{array}
\]

Thus, it suffices to prove that for any \( E, E' \in D(C^{opp}, Ab) \cong D_{add}((C \wr \Gamma)^{opp}, Ab) \) we have a functorial isomorphism

\[ m_!(E, \boxtimes E') \cong E, \otimes E'. \]

This is obvious: the product functor \( m: (C \wr \Gamma)^{opp} \times (C \wr \Gamma)^{opp} \rightarrow (C \wr \Gamma)^{opp} \times (C \wr \Gamma)^{opp} \) is left-adjoint to the diagonal embedding \( \delta: (C \wr \Gamma)^{opp} \rightarrow (C \wr \Gamma)^{opp} \times (C \wr \Gamma)^{opp} \), so that \( m_! \cong \delta^* \), and by definition, \( E, \otimes E' \cong \delta^*(E, \boxtimes E') \). \(\square\)

6. Categories of Galois Type

We will now give yet another description of the category \( DS(C, Ab) \) of Definition 4.11, to complement those given in Section 3 and Section 4. It is based on an explicit DG model, as in Section 3; however, this new DG model is more economical and more convenient for applications. It will require some additional assumptions on \( C \) (which are satisfied in the Mackey functor case \( C = O_G \)). In addition, from now we will assume that the target category \( Ab \) is the category of modules over a ring, so that Lemma 1.7 applies.

6.1. Galois-type categories and fixed points. We begin by imposing our conditions on the small category \( C \).

Definition 6.1. A category \( C \) is lattice-like if it is Hom-finite, and all its morphisms are surjective.

Lemma 6.2. In a lattice-like category \( C \), every right-inverse \( f': c \rightarrow c' \) to a morphism \( f: c \rightarrow c' \) is also a left-inverse. Moreover, every endomorphism \( f: c \rightarrow c \) of an object \( c \in C \) is invertible.

Proof. For the first claim, note that \( (f' \circ f) \circ f' = f' \circ (f \circ f') = f' \circ \text{id} = \text{id} \circ f' \); since \( f' \) is surjective, this implies \( f' \circ f = \text{id} \). For the second claim, note that since \( f \) is surjective, the natural map

\[ C(c, c) \xrightarrow{-\circ f} C(c, c) \]
is injective. Since $\mathcal{C}(c, c)$ is a finite set, this map must also be surjective, so that there exists $f' \in \mathcal{C}(c, c)$ such that $f' \circ f = \text{id}$. 

\textbf{Definition 6.3.} A category $\mathcal{C}$ is of \textit{Galois type} if it is lattice-like, and the wreath product $\mathcal{C} \wr \Gamma$ has fibered products.

\textbf{Example 6.4.} Here are some examples of categories of Galois type.

(i) $\mathcal{C} = O_G$, the category of finite $G$-orbits for a group $G$.

(ii) $\mathcal{C}$ is the category opposite to that of finite separable extensions of a field $k$.

(iii) $\mathcal{C}$ is the category $\Gamma$ of finite sets and surjective maps between them.

Of course, Galois theory shows that (ii) is a particular case of (i), thus the name “Galois-type”; however, the very interesting example (iii) is not of this form.

Assume given a small category $\mathcal{C}$ of Galois type, and consider the category $\mathcal{DS}(\mathcal{C}, \text{Ab})$ of Definition 4.11. Since $\mathcal{C}$ is Hom-finite, we can apply the constructions of Section 5.1; thus for any $c \in \mathcal{C}$, we have the object $T^c \in \mathcal{DS}(\mathcal{C}, \mathbb{Z}\text{-mod})$, the inflation functor

$$\text{Infl}^c: \mathcal{D}(\langle c \rangle, \text{Ab}) \to \mathcal{DS}(\mathcal{C}, \text{Ab})$$

and the left-adjoint fixed point functor

$$\tilde{\Phi}^c: \mathcal{DS}(\mathcal{C}, \text{Ab}) \to \mathcal{D}(\langle c \rangle, \text{Ab}).$$

In general, computing the functors $\tilde{\Phi}^c$ explicitly seems to be rather difficult; however, in the case of Galois-type categories, there is a drastic simplification. Namely, suppose that $q: \mathcal{C} \to \mathcal{C} \wr \Gamma \to Q(\mathcal{C})$ is the natural embedding of Section 5.3, and let $q^*: \mathcal{DS}(\mathcal{C}, \text{Ab}) \to \mathcal{D}(\mathcal{C}, \text{Ab})$ be the corresponding restriction functor. Moreover, define an embedding

$$\nu_c: \mathcal{D}(\langle c \rangle, \text{Ab}) \to \mathcal{D}(\mathcal{C}, \text{Ab})$$

by

$$\nu_c(M_q(c')) = \begin{cases} M_q(c'), & c' \in \langle c \rangle, \\ 0, & \text{otherwise,} \end{cases}$$

and let $\tilde{\varphi}^c: \mathcal{D}(\mathcal{C}, \text{Ab}) \to \mathcal{D}(\langle c \rangle, \text{Ab})$ be the left-adjoint functor. Then we have the following.

\textbf{Proposition 6.5.} We have a functorial isomorphism

$$\tilde{\Phi}^c \cong \tilde{\varphi}^c \circ q^*.$$

In order to prove this proposition, we need some preliminary results and constructions.

Consider the simplicial category $S(\mathcal{C} \wr \Gamma)/\Delta$. Note that by the definition of the wreath product, we have a natural fibration $S(\mathcal{C} \wr \Gamma) \to \Delta$, so that altogether, $S(\mathcal{C} \wr \Gamma)$ is fibered over $\Delta \times \Gamma$. Let $S(\mathcal{C} \wr \Gamma) \subset S(\mathcal{C} \wr \Gamma)$ be the subcategory with the same objects, and those morphisms which are Cartesian with respect to the fibration $S(\mathcal{C} \wr \Gamma) \to \Delta \times \Gamma$. By definition, $S(\mathcal{C} \wr \Gamma)$ is also fibered over $\Delta$, and we have a natural Cartesian embedding

$$\eta: S(\mathcal{C} \wr \Gamma) \to S(\mathcal{C} \wr \Gamma).$$
While the category $\overline{S(C \Gamma)}$ is not of the form $S(C')$ for some small category $C'$, most of the material of Section 4 still applies to it, with appropriate changes. In particular, say that a map $f$ in $\overline{S(C \Gamma)}$ is special, resp. co-special if so is $\eta(f)$. Then it is easy to check that the classes of special and co-special maps form a complementary pair in the sense of Definition 4.3. We can thus consider the full subcategory

$$\overline{DS}(C, Ab) \subset D(S(C \Gamma), Ab)$$

spanned by special functors. Since $\eta$ is a Cartesian functor which respects special maps, we have a restriction functor $\eta^* : DS(C \Gamma, Ab) \to \overline{DS}(C, Ab)$, and the adjoint functor

$$R^* \eta_* : D(S(C \Gamma), Ab) \to D(S(C \Gamma), Ab)$$

sends special functors into special functors.

Moreover, we also have a 2-category description of $\overline{DS}(C, Ab)$. Namely, the fiber $\overline{S(C \Gamma)}_1$ over $[1] \in \Delta$ is obviously equivalent to $(\overline{C \Gamma})^{opp}$ (where $\overline{C} \subset C$ is the subcategory with the same objects and invertible maps between them). Consider the 2-category $\mathcal{Q}(C \Gamma)$ of Section 3, and let $\overline{\mathcal{Q}(C \Gamma)} \subset \mathcal{Q}(C \Gamma)$ be the 2-subcategory with the same objects, those 1-morphisms $c_1 \to c_2$ for which the map $c_1 \to c_2$ actually lies in $\overline{C \Gamma} \subset C \Gamma$, and all 2-morphisms between them. One checks easily that this condition is compatible with the pullbacks, so that we indeed have a well-defined 2-category. Applying the machinery of Section 1.6, we produce the $A_\infty$-category $\overline{Valid}$, and the derived category $\overline{DQ}(C \Gamma, Ab)$ of $A_\infty$-functors from $\overline{Valid}$ to $Ab$. Note that we have a natural functor $(\overline{C \Gamma})^{opp} \to \overline{\mathcal{Q}(C \Gamma)}$, thus a restriction functor $\overline{DQ}(C \Gamma, Ab) \to D((\overline{C \Gamma})^{opp}, Ab)$. By definition, we also have a natural embedding $\eta : \overline{Q}(C \Gamma) \to Q(C \Gamma)$.

**Lemma 6.6.** There exists a natural equivalence

$$\overline{DS}(C, Ab) \cong \overline{DQ}(C \Gamma, Ab)$$

which is compatible with $\eta^*$ and with the natural restrictions to $(\overline{C \Gamma})^{opp}$.

**Proof.** Same as Theorem 4.2. \qed

Now fix an object $c \in C$, and define a 2-functor $\varepsilon_c : \overline{Q}(C \Gamma) \to Q(\langle c \rangle; \Gamma)$ as follows. On objects, $\varepsilon_c$ sends $\langle S, \{c_a\} \rangle \in C \Gamma$ to the formal union of those components $c_a$ which are isomorphic to $c$. On morphisms, $\varepsilon_c$ sends a diagram $c_1 \to c_2$ to $\varepsilon_c(c_1) \to \varepsilon_c(c_2)$ (if $c_1, c_2 \in \langle c \rangle \Gamma$, this is well-defined). Moreover, let

$$\mathcal{T}^c \equiv \varepsilon_c^*(T_{\langle c \rangle}) \in \overline{DQ}(C \Gamma, Z\text{-mod})$$

be the pullback of the standard object $T_{\langle c \rangle} \in \overline{DQ}(\langle c \rangle; \Gamma, Z\text{-mod})$, $T_{\langle c \rangle}(c') = Z[(\langle c \rangle; \Gamma)/(c, c')]$, and define an embedding $\nu_c^* : D(\langle c \rangle, Ab) \to \overline{DQ}(C \Gamma, Ab)$ by

$$\nu_c^*(M^c) = (\mathcal{T}^c \otimes M^c)^{Aut(c)}$$

(since $\mathcal{T}^c(c')$ is a free $Aut(c)$-module for any $c' \in C$, the group $Aut(c)$ has no higher cohomology with coefficients in $\mathcal{T}^c \otimes M^c$, so that taking $Aut(c)$-invariants is well-defined on the level of derived categories).
Lemma 6.7. We have a natural isomorphism

\[ T^c \cong R^* \eta_* T^c \in DS(C \wr \Gamma, \text{Ab}) \cong DQ(C \wr \Gamma, \text{Ab}). \]

Proof. To construct a map \( T^c \to R^* \eta_* T^c \), it suffices by adjunction to construct a map \( \eta_* T^c \to T^c \). Such a map is obvious — it is identical on \( c' \in \langle c \rangle \wr \Gamma \subset C \wr \Gamma \), and 0 otherwise. To prove that the induced map \( T^c \to R^* \eta_* T^c \) is an isomorphism, use Theorem 4.2 and Lemma 6.6 to pass to the Waldhausen-type interpretation. Then since both \( T^c \) and \( R^* \eta_* T^c \) are special, it suffices to check that \( T^c \to R^* \eta_* T^c \) is an isomorphism on the fiber \((C \wr \Gamma)_{\text{opp}} \) over \( [1] \in \Delta \). This is again obvious: on this fiber, \( T^c \) is the functor represented by \( c \in (C \wr \Gamma)_{\text{opp}} \), and \( T^c \) is the functor represented by the same \( c \) considered as an object in \( S(C \wr \Gamma) \). \( \square \)

It remains to analyse the 2-category \( \overline{Q}(C \wr \Gamma) \). As in Section 3.2, consider the full subcategory in \( \overline{Q}(C \wr \Gamma) \) whose 1-morphisms are diagrams \( S \leftarrow S' \to S'' \) with injective map \( S_1 \to S \). This is actually a 1-category which we denote by \( C \wr \Gamma^+ \) by abuse of notation. The embedding \( q : C \to \overline{Q}(C \wr \Gamma) \) factors through the embedding \( \eta : Q(C \wr \Gamma) \) by means of the embeddings \( C \hookrightarrow C \wr \Gamma \) and \( \lambda : \overline{Q}(C \wr \Gamma) \). Say that an object \( M_q \in DS(C \wr \Gamma, \text{Ab}) \) is additive if its restriction to \( C \wr \Gamma^+ \) is additive in the sense of Definition 3.2.

Lemma 6.8. Assume given two objects \( M_*, M'_* \in \overline{DQ}(C \wr \Gamma, \text{Ab}) \). If \( M'_* \) is additive, then the natural map

\[ \text{Hom}(M_*, M'_*) \to \text{Hom}(\lambda^* M_*, \lambda^* M'_*) \]  

is an isomorphism. If \( M_* \) is also additive, then the natural map

\[ \text{Hom}(\lambda^* M_*, \lambda^* M'_*) \to \text{Hom}(j^* \lambda^* M_*, j^* \lambda^* M'_*) \]

is also an isomorphism.

Proof. As in the proof of Lemma 3.4, we note that the category \( \text{Fun}(C \wr \Gamma^+, \mathbb{Z}-\text{mod}) \) is generated by representable functors \( \mathbb{Z}^S, S \in C \wr \Gamma^+ \). For any such \( S = \coprod c_s, c_s \in C \), we have

\[ \mathbb{Z}^S \cong \bigotimes c^c, \]

and for any \( c \in C \), we have a canonical direct sum decomposition \( \mathbb{Z}^c = T^c \oplus \mathbb{Z} \). This induces direct sum decompositions

\[ \mathbb{Z}^S = \bigoplus_{S' \subset S} T^{S'}, \]

where the objects

\[ T^{S'} = \bigotimes c^{c_s}, \quad S = \coprod c_s \in C \wr \Gamma^+ \]

give a smaller set of projective generators of the category \( \text{Fun}(C \wr \Gamma^+, \mathbb{Z}-\text{mod}) \). For any \( S_1, S_2 \in C \wr \Gamma^+ \), we have \( \mathbb{Z}^{S_1 \cup S_2} \cong \mathbb{Z}^{S_1} \oplus \mathbb{Z}^{S_2} \); therefore the additivity condition
on an object \(E_c \in \mathcal{D}(C \amalg \Gamma, Ab)\) means that the natural map \(Z^{S_1} \times Z^{S_2} \to Z^{S_1 \cup S_2} = Z^{S_1} \otimes Z^{S_2}\) induces an isomorphism

\[\text{RHom}^*(Z^{S_1} \otimes Z^{S_2} \otimes M_c, E_c) \to \text{RHom}^*(Z^{S_1} \otimes M_c, E_c) \oplus \text{RHom}^*(Z^{S_2} \otimes M_c, E_c)\]

for any \(M_c \in \mathcal{D}(Ab)\). In terms of the generators \(T^S\), this is equivalent to saying that \(\text{RHom}^*(T^S \otimes M_c, E_c) = 0\) unless the decomposition \(S = \coprod c_c\) has exactly one term, \(S = c\) for some \(c \in C \subseteq C \amalg \Gamma_+\):

\[\text{RHom}^*(T^S \otimes M_c, E_c) = \begin{cases} \text{RHom}^*(M_c, E_c), & S = c \in C \subseteq C \amalg \Gamma_+, \\ 0, & \text{otherwise}. \end{cases} \tag{6.4}\]

(If \(C = \{\text{pt}\}\), as in Lemma 3.4, these are exactly the orthogonality conditions on the generators \(T^{S_{\text{pt}}}\).) Then as in the proof of Lemma 3.4, it suffices to check the first claim for a representable \(M_c = M_c^S \in \mathcal{D}(C, Ab)\); we have a filtration on \(\text{Hom}(M^S_c, M'_c)\) with associated graded quotient of the form (3.3), and checking that the map (6.3) is an isomorphism amounts to applying (6.4) to \(E_c = \lambda^* M'_c\).

To prove the second claim, it remains to notice that for any additive object \(E_c \in \mathcal{D}(C \amalg \Gamma_+, Ab)\), the adjunction map \(L^* j^* E_c \to E_c\) is an isomorphism. \(\Box\)

**Proof of Proposition 6.5.** By definition, for any \(M_c \in \mathcal{D}(C, Ab)\), \(M'_c \in \mathcal{D}(\{c\}, Ab)\) we have

\[\text{Hom}(\bar{\Phi}^c M_c, M'_c) \cong \text{Hom}(M_c, \text{Infl}^c(M'_c)),\]

and by Lemma 6.7, we have

\[\text{Infl}^c(M'_c) \cong R^* \eta_c (\bar{T}^c \otimes M'_c)_{\text{Aut}(c)} = R^* \eta_c \nu^c(M'_c),\]

so that

\[\text{Hom}(\bar{\Phi}^c M_c, M'_c) \cong \text{Hom}(\eta^* M_c, \nu^c(M'_c)).\]

It remains to apply Lemma 6.6 and Lemma 6.8, and notice that by definition, \(q^* \nu^c(M'_c) \cong \nu_c(M'_c)\). \(\Box\)

**6.2. Filtration by support.** We will now give some corollaries of Proposition 6.5.

First, the following simple observation. Suppose we are not interested in the functor \(\bar{\Phi}^c\) of (6.1), but only in the fixed point functor \(\Phi^c\) of Definition 5.3— that is, we are prepared to forget the \(\text{Aut}(c)\)-action on \(\Phi^c\). Then there is a more convenient way to compute \(\Phi^c : \mathcal{D}(C, Ab) \to \mathcal{D}(Ab)\). Namely, denote by \(\mathcal{C}_c\) the category of objects \(c' \in C\) equipped with a map \(c' \to c\). We have a projection \(s_c : \mathcal{C}_c \to C\) which forgets the map. Composing \(s_c\) with the embedding \(q\) gives a functor

\[q_c : \mathcal{C}_c \to \mathcal{Q}(C \amalg \Gamma)\]

and a corresponding restriction functor \(q^c : \mathcal{D}(C, Ab) \to \mathcal{D}(C_c, Ab)\). Let \(T^c \in \text{Fun}(C_c, \mathbb{Z} \text{-mod})\) be given by

\[T^c(c') = \begin{cases} \mathbb{Z}, & c' \cong c, \\ 0, & \text{otherwise}, \end{cases} \tag{6.5}\]

and let \(\varphi^c : \mathcal{D}(C, Ab) \to \mathcal{D}(Ab)\) be the functor left-adjoint to the embedding \(\mathcal{D}(Ab) \to \mathcal{D}(C_c, Ab)\), \(M_c \mapsto M_c \otimes T^c)\).
Corollary 6.9. We have
\[ \Phi^c \cong \varphi^c \circ q^*_c. \]
In particular, \( \Phi^c \cdot M_c = 0 \) if \( q^*_c \cdot M_c = 0 \).

Proof. The right-adjoint to the forgetful functor \( \mathcal{D}(\langle c \rangle, \text{Ab}) \to \mathcal{D}(\text{Ab}) \) sends \( M_c \in \mathcal{D}(\text{Ab}) \) to \( M_c \otimes \mathbb{Z}[\text{Aut}(c)] \), where \( \mathbb{Z}[\text{Aut}(c)] \) is the regular representation of the group \( \text{Aut}(c) \). Then by Proposition 6.5, it suffices to prove that \( R^* s_c(\overline{T}) \cong \nu_c(\mathbb{Z}[\text{Aut}(c)]) \).

This is clear: \( s^c \) is obviously a discrete fibration, so that \( R^* s^c_c \) can be computed fiberwise, \( \overline{T} \) is non-trivial on the fiber over \( c' \in \mathcal{C} \) if and only if \( c' \) lies in \( \langle c \rangle \), and for every such \( c' \), the fiber is a torsor over \( \text{Aut}(c) \).

Now, by Lemma 6.2, the set \( [\mathcal{C}] \) of isomorphism classes of objects of the lattice-like category \( \mathcal{C} \) has a natural partial order, given by \( [c] \geq [c'] \) if and only if there exists a map \( c \to c' \).

Definition 6.10. A subset \( U \subset [\mathcal{C}] \) is closed if it is closed with respect to the standard \( T_0 \)-topology associated to the partial order — that is, for any \( [c], [c'] \in [\mathcal{C}] \) with \( [c] \geq [c'] \), \( [c] \in U \) implies \( [c'] \in U \).

Definition 6.11. For any subset \( U \subset [\mathcal{C}] \), an object \( E_* \in \mathcal{DS}(\mathcal{C}, \text{Ab}) \) is supported in \( U \) if \( E_∗(c) = 0 \) for any \( c \in \mathcal{C} \) whose isomorphism class is outside \( U \). An object \( E_* \in \mathcal{DS}(\text{Ab}) \) has finite support if it is supported in a finite closed subset \( U \subset [\mathcal{C}] \). The full subcategory in \( \mathcal{DS}(\mathcal{C}, \text{Ab}) \) spanned by objects supported in a subset \( U \subset [\mathcal{C}] \) is denoted by
\[ \mathcal{DS}_U(\mathcal{C}, \text{Ab}) \subset \mathcal{DS}(\mathcal{C}, \text{Ab}), \]
and the full subcategory in \( \mathcal{DS}(\mathcal{C}, \text{Ab}) \) spanned by objects with finite support is denoted by \( \mathcal{DS}_{\text{fs}}(\mathcal{C}, \text{Ab}) \subset \mathcal{DS}(\mathcal{C}, \text{Ab}) \).

For any two subsets \( U' \subset U \subset [\mathcal{C}] \), there exists an obvious full embedding \( \mathcal{DS}_{U'}(\mathcal{C}, \text{Ab}) \subset \mathcal{DS}_U(\mathcal{C}, \text{Ab}) \).

Corollary 6.12. For any closed subset \( U \subset [\mathcal{C}] \), the fixed points functor
\[ \Phi^c : \mathcal{DS}_U(\mathcal{C}, \text{Ab}) \to \mathcal{D}(\langle c \rangle, \text{Ab}) \]
is trivial unless \( [c] \in U \). If \( [c] \in U \) is maximal, so that \( U' = U \setminus \{ [c] \} \) is also closed, then \( \Phi^c \cong \Psi^c \) on \( \mathcal{DS}_U(\mathcal{C}, \text{Ab}) \), where \( \Psi^c \) is as in (5.1). The subcategory \( \mathcal{DS}_{U'}(\mathcal{C}, \text{Ab}) \subset \mathcal{DS}_U(\mathcal{C}, \text{Ab}) \) is right-admissible, and \( \Phi^c \cong \Psi^c \) factors through an equivalence
\[ \mathcal{DS}_U(\mathcal{C}, \text{Ab})/\mathcal{DS}_{U'}(\mathcal{C}, \text{Ab}) \cong \mathcal{D}(\langle c \rangle, \text{Ab}). \quad (6.6) \]

Proof. Consider an object \( E_* \in \mathcal{DS}_U(\mathcal{C}, \text{Ab}) \). By definition, for any \( c \in \mathcal{C} \) whose isomorphism class is not in \( U \), we have \( E_*^c(c) = 0 \). Since \( U \) is closed, we also have \( E_*^c(c') = 0 \) for any \( c' \) with \( [c'] \geq [c] \), so that \( q^*_c E = 0 \). By Corollary 6.9, this proves that \( \Phi^c(E_*) = 0 \); since the forgetful functor \( \mathcal{D}(\langle c \rangle, \text{Ab}) \to \mathcal{D}(\text{Ab}) \) is obviously conservative, we also have \( \Phi^c(E_*) \). This proves the first claim.

If the isomorphism class of an object \( c \in \mathcal{C} \) is maximal in \( U \), we have \( q^*_c(E_*)^c(c') = 0 \) unless \( c' \in \mathcal{C}_c \) is isomorphic to \( c \), so that \( q^*_c(E_*)^c \cong \Psi^c(E_*) \otimes \overline{T}^c \). Since the correspondence \( E_* \mapsto E_* \otimes \overline{T}^c \) is obviously a full embedding from \( \mathcal{D}(\text{Ab}) \) to \( \mathcal{D}(\mathcal{C}_c, \text{Ab}) \),
implies that
\[ \varphi^c(q_*^c E_*) \cong \Psi^c(E_*) , \]
so that by Corollary 6.9, the natural map \( \Psi^c(E_*) \to \Phi^c(E_*) \) is an isomorphism. Since the forgetful functor \( \mathcal{D}((c), \text{Ab}) \to \mathcal{D}(\text{Ab}) \) is conservative, this proves the second claim.

To prove the rest of the claims, note that the inflation functor \( \text{Infl}^c : \mathcal{D}((c), \text{Ab}) \to \mathcal{D}_U(\mathcal{C}, \text{Ab}) \) obviously sends \( \mathcal{D}((c), \text{Ab}) \) into the full subcategory \( \mathcal{D}_U(\mathcal{C}, \text{Ab}) \). By definition, \( \text{Infl}^c \) is right-adjoint to \( \Phi^c \cong \Psi^c \). Moreover, we obviously have
\[ \Psi^c \circ \text{Infl}^c \cong \text{Id}. \]
Therefore \( \text{Infl}^c : \mathcal{D}((c), \text{Ab}) \to \mathcal{D}_U(\mathcal{C}, \text{Ab}) \) is a fully faithful embedding with left-admissible image. But by definition, for any \( E \in \mathcal{D}_U(\mathcal{C}, \text{Ab}) \), \( \Psi^c(E) = 0 \) if and only if \( E \) lies in \( \mathcal{D}_U(\mathcal{C}, \text{Ab}) \). Thus the orthogonal \( \perp \text{Infl}^c(\mathcal{D}((c), \text{Ab})) \) is exactly \( \mathcal{D}_U(\mathcal{C}, \text{Ab}) \), and the category \( \mathcal{D}_U(\mathcal{C}, \text{Ab}) \) has a semiorthogonal decomposition (\( \text{Infl}^c(\mathcal{D}((c), \text{Ab})), \mathcal{D}_U(\mathcal{C}, \text{Ab}) \)), which finishes the proof.

\[ \square \]

**Remark 6.13.** In the particular case \( \mathcal{C} = \mathcal{O}_G, c = [G/H] \) for a cofinite subgroup \( H \subset G \), and \( U = \{ [c'] \in [\mathcal{C}] | [c'] \subseteq [c] \} \), (6.6) is exactly (0.1) of the Introduction.

**Corollary 6.14.** For any two finite closed subsets \( U' \subset U \subset [\mathcal{C}] \), the category \( \mathcal{D}_U(\mathcal{C}, \text{Ab}) \) is an admissible full subcategory in \( \mathcal{D}_U(\mathcal{C}, \text{Ab}) \), and the left orthogonal \( \perp \mathcal{D}_U(\mathcal{C}, \text{Ab}) \) consists of those objects \( E \in \mathcal{D}_U(\mathcal{C}, \text{Ab}) \) which are supported in \( U \setminus U' \).

**Proof.** The fact that \( \mathcal{D}_U(\mathcal{C}, \text{Ab}) \subset \mathcal{D}_U(\mathcal{C}, \text{Ab}) \) is right-admissible immediately follows by induction from Corollary 6.12. Moreover, the semiorthogonality statement of Corollary 6.12 implies that \( \mathcal{D}_U(\mathcal{C}, \text{Ab}) \) is generated by objects of the form \( \text{Infl}^c(M), c' \in U', M \in \mathcal{D}((c'), \text{Ab}) \). Thus \( E \in \mathcal{D}_U(\mathcal{C}, \text{Ab}) \) lies in the orthogonal \( \perp \mathcal{D}_U(\mathcal{C}, \text{Ab}) \) if and only if it is orthogonal to all such \( \text{Infl}^c(M) \). By adjunction this is equivalent to \( \Phi^c(E) = 0, c' \in U' \), which by definition means exactly that \( E \) is supported in \( U \setminus U' \).

Finally, to prove that \( \mathcal{D}_U(\mathcal{C}, \text{Ab}) \subset \mathcal{D}_U(\mathcal{C}, \text{Ab}) \) is left-admissible, it suffices by induction to consider the case \( U' = U \setminus \{ [c] \} \) for a maximal \( c \in U \). By Lemma 1.2, it suffices to prove that the projection
\[ \perp \mathcal{D}_U(\mathcal{C}, \text{Ab}) \to \mathcal{D}_U(\mathcal{C}, \text{Ab}) \]
is essentially surjective. Define a functor \( I : \text{Fun}(\text{Aut}(c), \text{Ab}) \to \text{Fun}(\mathcal{C} \setminus \{ \Gamma \}, \text{Ab}) \) by
\[ I(M) = (I^c \otimes M)_{\text{Aut}(c)}, \]
and let
\[ L^*I : \mathcal{D}((c), \text{Ab}) \cong \mathcal{D}(\langle c \rangle, \text{Ab}) \to \mathcal{D}_U(\mathcal{C}, \text{Ab}) \subset \mathcal{D}(\text{Fun}(\mathcal{C} \setminus \{ \Gamma \}, \text{Ab})) \]
be its left-derived functor (this is dual to the definition of the inflation functor \( \text{Infl}^c \) in Section 5.1). We obviously have \( \tilde{\Phi}^c \circ L^*I \cong \tilde{\Psi}^c \circ L^*I \cong \text{Id} \), so that it suffices to prove that \( L^*I \) sends \( \mathcal{D}(\text{Aut}(c), \text{Ab}) \) into the orthogonal \( \perp \mathcal{D}_U(\mathcal{C}, \text{Ab}) \). As we have already proved, this is equivalent to proving that
\[ \Phi^c \circ L^*I = 0 \]
for any $c' \in U'$. Thus by Corollary 6.9, we have to prove that
\[
\varphi^{c'} \circ q^*_{c'} \circ L^* I = 0 \tag{6.7}
\]
for any $c' \in U'$. By definition, we have
\[
q^*_{c'}(T^c \otimes M) \cong s^*_{c'} j^f_M
\]
for any $M \in \text{Fun}(\langle c \rangle, \text{Ab})$, where $j^c : \langle c \rangle \to C$ is the natural embedding. Passing to the derived functors, we obtain
\[
q^*_{c'} \circ L^* I \cong s^*_{c'} \circ L^* j^f,
\]
and by the definition of the functor $\varphi^{c'}$, (6.7) is equivalent to
\[
j^f R^* s^*_{c'} T^c = 0,
\]
where $T^c' \in \text{Fun}(C_{c'}, \text{Z-mod})$ is as in (6.5). Since the projection $s_{c'} : C_{c'} \to C$ is a fibration, this immediately follows by base change. \hfill \Box

6.3. DG models. Corollaries 6.12 and 6.14 show that the category $\mathcal{DS}_{fs}(C, \text{Ab})$ admits a filtration by admissible full triangulated subcategories $\mathcal{DS}_{U}(C, \text{Ab})$ indexed by finite closed subsets $U \subset [C]$, and the associated graded quotients $\mathcal{DS}_c(C, \text{Ab}) = \mathcal{DS}_{U}(C, \text{Ab})/\mathcal{DS}_{U \setminus \{c\}}(C, \text{Ab})$, $c \in U$ maximal, of this filtration are identified with the derived categories $\mathcal{D}(\text{Aut}(c), \text{Ab})$. In the remainder of this section, we will obtain a full description of the category $\mathcal{DS}_{fs}(C, \text{Ab})$ in terms of the functors $\tilde{\Phi}^c$. Among other things, this will later allow us to describe the gluing data between these associated graded quotients.

The general strategy is rather straightforward — we apply a version of the standard Tannakian formalism. Namely, we treat the collection $\tilde{\Phi}^c$ as a “fiber functor” for the category $\mathcal{DS}_{fs}(C, \text{Ab})$, and

(i) lift the functors $\tilde{\Phi}^c$ to DG functors $\tilde{\Phi}^c$, and

(ii) present a natural $C$-graded $A_\infty$-coalgebra $T^c_\cdot(-)$ which acts on $\tilde{\Phi}^c_\cdot(-)$ (in particular, this will include the Aut$(c)$-action).

The functor $\Phi^c : \mathcal{DS}_{fs}(C, \text{Ab}) \to \mathcal{D}(\text{Ab})$ is by definition left-adjoint to the functor $\mathcal{D}(\text{Ab}) \to \mathcal{DS}_{fs}(C, \text{Ab})$ given by $M \mapsto M \otimes T^c$; the “natural $A_\infty$-coalgebra” in (ii) is then an $A_\infty$ model for $\Phi^c(T^c)$, $c, c' \in C$,

with the comultiplication given by adjunction. The same adjunction gives a comparison functor from $\mathcal{DS}_{fs}(C, \text{Ab})$ to the derived category of $A_\infty$-comodules over $T^c_\cdot$, and we check that this comparison functor is an equivalence (this is Theorem 6.17).

We note that here it is crucial to work with $A_\infty$-coalgebras rather than $A_\infty$-algebras. In fact, the complex $T^c_\cdot(f)$ is acyclic for any non-invertible map $f : c \to c'$. However, as we have noted in Section 1.5, quasiisomorphic $A_\infty$-coalgebras may have different categories of $A_\infty$-comodules, and in particular, acyclic complexes $T^c_\cdot(f)$ cannot be replaced with 0.
6.3.1. The complexes. Fix an object \( c \in \mathcal{C} \). For any \( n \geq 1 \), let \( \mathcal{C}_n(c) \) be the groupoid of diagrams \( c_1 \to \cdots \to c_n \to c \) in \( \mathcal{C} \) and invertible maps between them, and let \( \mathcal{C}_0(c) \subset \mathcal{C}_n(c) \) be the full subcategory spanned by such diagrams that

- the map \( c_n \to c \) is not an isomorphism.

As in Section 1.3, let \( \sigma_n : \mathcal{C}_n(c) \to \mathcal{C} \) be the functor which sends a diagram to \( c_1 \in \mathcal{C} \).

Fix an object \( \mathcal{C}_n(c) \to \mathcal{C} \). For any \( E \in \text{Fun}(S(\mathcal{C} \times \Gamma), \text{Ab}) \), let

\[
\Phi^c_\bullet(E) = C_\bullet(C_n(c), \sigma_n^* q^* E).
\] (6.8)

For \( n = 0 \), set \( \Phi^c_0(E) = E(c) \) and \( \Phi^c_0(E) = 0 \) for \( i \geq 1 \). As in Section 1.3, define a second differential \( d : \Phi^c_\bullet(E) \to \Phi^c_\bullet(E) \) as the alternating sum of maps \( d^i \), \( 1 \leq i \leq n \), where \( d^i \) removes the object \( c_i \) from the diagram, and let \( \Phi^c(E) \) be the total complex of the resulting bicomplex.

Any map \( f : c \to c \) acts on the diagrams \( c_1 \to \cdots \to c_n \to c \) by composition on the right-hand side; this turns \( \Phi^c(E) \) into a complex \( \tilde{\Phi}^c(E) \) of representations of the group \( \text{Aut}(c) \).

Lemma 6.15. The complex \( \tilde{\Phi}^c(E) \) computes \( \tilde{\Phi}^c(E) \).

Proof. By Proposition 6.5, it suffices to construct a functorial quasiisomorphism

\[ \text{RHom}^\bullet_\langle (c) \rangle (\tilde{\Phi}^c(E), M) \cong \text{RHom}^\bullet (q^* E, \nu_c(M)) \] (6.9)

for any \( M \in \text{Fun}((c), \text{Ab}) \). Computing the left-hand side of (6.9) by (1.6), we get

\[
\bigoplus_{0 \leq i} \text{RHom}^\bullet_\langle (c) \rangle (\tilde{\Phi}^c(E), M) \cong \\
\cong \bigoplus_{0 \leq i} \bigoplus_{0 \leq n \leq i < n} C^\bullet((c)_{n-i}, \text{RHom}^{i+1-n} (C_\bullet(C_i(c), \sigma_i^* q^* E), M)) \\
\cong \bigoplus_{0 \leq i} \bigoplus_{0 \leq n \leq i < n} C^\bullet(C_i(c) \times (c)_{n-i}, \text{RHom}^{i+1-n} (\sigma_n^* q^* E, M)),
\] (6.10)

where \( (c)_i \) is as in Section 1.3. For every diagram \( c_1 \to \cdots \to c_n \to c \) in \( \mathcal{C}_n(c) \), there exists a unique \( i \), \( 0 \leq i < n \) such that \( c_i \to c_{i+1} \) is not an isomorphism, but \( c_j \cong c \) for \( j \geq i+1 \). Therefore we have a functorial decomposition

\[ \mathcal{C}_n(c) = \coprod_{0 \leq i < n} C_i(c) \times (c)_{n-i}, \]

and (6.10) can be rewritten as

\[
\bigoplus_{n \geq 0} C^\bullet(C_n'(c), \text{RHom}^{i+n-1-n} (\sigma_n^* q^* E, M)).
\]

To compute the right-hand side of (6.9), we can also use (1.6); this gives the same formula, except that \( \mathcal{C}_n(c) \) is replaced with the groupoid \( \mathcal{C}_n \) of Section 1.3, and \( M \) is replaced by \( \tau_n^* \nu_c(M) \). It remains to notice that by the definition of the functor \( \nu_c \), the terms corresponding to \( \mathcal{C}_n \setminus \mathcal{C}_n(c) \) are all equal to 0. \( \square \)
Now consider two objects \( c, c' \in \mathcal{C} \), and apply Lemma 6.15 to the object \( T^{c'} \in \text{Fun}(S(\mathcal{C} \wr \Gamma), \mathbb{Z}\text{-mod}) \). Recall that by definition, for any diagram \( \alpha = [c_1 \to \cdots \to c_n] \in \mathcal{C}_n(c) \) the group \( T^{c'}(\alpha) = T^{c'}(c_1) \) is the free abelian group generated by the space of maps from \( c' \) to \( c_1 \). These maps can be incorporated into the diagram. For any map \( f : c' \to c \) and any \( n \geq 1 \), denote by \( \mathcal{C}_n(f) \) the groupoid of all diagrams \( c' \to c_1 \to \cdots \to c_n \to c \) such that the composition of all the maps is equal to \( f \), and the last map \( c_n \to c \) is not an isomorphism. Let

\[
T^c_n,\ast(f) = \mathcal{C}_n(\mathcal{C}_n(f), \mathbb{Z}),
\]

and extend it to \( n = 0 \) by setting \( T^c_0(f) = \mathbb{Z} \) for \( i = 0 \) and 0 otherwise. For any \( 1 \leq i \leq n \), forgetting the object \( c_i \) in the diagram gives functors \( \mathcal{C}_n(f) \to \mathcal{C}_{n-1}(f) \) and the corresponding maps \( d_i : T^c_n(f) \to T^c_{n-1}(f) \); define a second differential \( d : \mathcal{C}_n(f) \to \mathcal{C}_{n-1}(f) \) as their alternating sum, and let \( T^c(\alpha) \) be the total complex of the corresponding bicomplex. By definition, we have an isomorphism of complexes

\[
\Phi^c(T^{c'}) \cong \bigoplus_{f \in \mathcal{C}(c',c)} T^c(\alpha).
\]

6.3.2. Coaction. Consider a diagram \( \alpha = [c_1 \to \cdots \to c_n] \in \mathcal{C}_n(c) \). Since \( \mathcal{C} \wr \Gamma \) has fibered products, we can form the fibered product diagram \( c_1 \times_c c' \to \cdots \to c_n \times_c c' \to c' \). By definition, for any \( 1 \leq i \leq n \), \( c_i \times_c c' \) is the formal union

\[
c_i \times_c c' = \bigsqcup_{s \in S_i} c_{i,s}
\]

of some objects \( c_{i,s} \in \mathcal{C} \), numbered by elements in a finite set \( S_i \). Denote \( S(\alpha) = S_1 \). Picking up an element \( s \in S(\alpha) \) uniquely determines an element \( s \in S_i \) for any \( i \), and we obtain a diagram \( c_{i,s} \to \cdots \to c_n,s \to c' \). Moreover, we have a commutative diagram

\[
\begin{array}{ccccc}
\vdots & \cdots & \cdots & \cdots & \vdots \\
| & & & | \\
\alpha_{i,s} & \cdots & \cdots & \cdots & \alpha_{n,s} \\
\downarrow \alpha_{i,s} & \cdots & \cdots & \cdots & \downarrow \alpha_{n,s} \\
\end{array}
\]

By assumption, the map \( c_i \to c \) is not an isomorphism for any \( i \), but the map \( c_{i,s} \to c' \) might well be. So, let \( i \) be the smallest integer such that all the maps \( c_{j,s} \to c \), \( j > i \) are isomorphisms. Then \( c_{i,s} \to \cdots \to c_{n,s} \to c' \) is a well-defined object of the groupoid \( \mathcal{C}(c) \), which we denote by \( \alpha^{(1)}_s \). Moreover, composing the map \( \alpha_{i,s} : c_{i,s} \to c_i \) with the isomorphism \( c' \to c_{i,s} \) inverse to the map \( c_{i,s} \to c' \), we obtain a diagram

\[
c' \cong c_{i,s} \xrightarrow{\alpha_{i,s}} c_i \to \cdots \to c_n \to c
\]

which gives an object in the groupoid \( \mathcal{C}_{n-1}(f) \); we denote it by \( \alpha^{(2)}_s \).

These constructions are functorial, so that sending a diagram \( \alpha \) to the formal union of the products \( \alpha_s^{(1)} \times \alpha_s^{(2)} \), \( s \in S_1 \) gives a functor

\[
\beta_n : \mathcal{C}_n(c) \to \left( \prod_{0 \leq i \leq n} \mathcal{C}_i(c') \times \mathcal{C}_{n-1}(f) \right) \wr \Gamma.
\]
For any $E \in \text{Fun}(S(C \otimes \Gamma), \text{Ab})$, we have a natural map
\[
\sigma_n^* q^n E \to \beta_n^* \alpha_n^{* n} E,
\]
where $\sigma_n^*$ is obtained by extending the natural functor
\[
\sigma_n^* : C_n(c') \times C_{n-1}(f) \to C_n(c') \xrightarrow{\sigma_n} C
\]
to wreath products. If $E$ is additive in the sense of Definition 4.9, then $\sigma_n^* E$ is addi-
tive, so that we have a canonical decomposition
\[
\sigma_n^* E \left( \prod_s \alpha_s^{(1)} \times \alpha_s^{(2)} \right) \cong \bigoplus E(\alpha_s^{(1)}).
\]
We can then compose the map $C_n(\beta_i)$ induced by the functors $\beta_i$, with the projec-
tions onto the terms of this decomposition and obtain a canonical ma-
p of complexes
\[
b_f : \Phi^c(E) \to \Phi^c(E) \otimes T^c(f).
\]
6.3.3. Comultiplication and higher operations. Now assume given a third object $c'' \in C$ and a map $g: c'' \to c'$, and apply (6.12) to $E = T^c''$. This gives a comulti-
plication map
\[
b_{f,g} : T^c(f \circ g) \to T(f) \otimes T^c(g)
\]
by the following procedure. Consider again a diagram $\alpha = [c'' \to c_1 \to \cdots \to c_n \to c] \in C_n(f \circ g)$, and form the pullback diagram $c_1 \times_c c' \to \cdots \to c_n \times_c c' \to c'$. In this case, we also have a natural map $c'' \to c_1 \times_c c'$; this distinguishes a component $c'_i$ in the decom-
position $c_1 \times_c c' = \prod c_i$, hence also a component $c'_i$ in the decom-
position $c_i \times_c c' = \prod c_i', s$ for any $1 \leq i \leq n$, and we have a commutative diagram
\[
\begin{array}{cccccc}
c'' & \longrightarrow & c_1 & \longrightarrow & \cdots & \longrightarrow & c_n & \longrightarrow & c \\
\| & & \| & & \| & & \| & & \| \\
c'' & \longrightarrow & c'_1 & \longrightarrow & \cdots & \longrightarrow & c'_n & \longrightarrow & c'.
\end{array}
\]
We again take the smallest integer $i$ such that $c'_{i+1} \to c'$ is an isomorphism, and
obtain a functor
\[
\beta_n^I : \text{C}_n(f \circ g) \to \prod_{0 \leq i \leq n} \text{C}_i(g) \times \text{C}_{n-i}(f).
\]
This functor induces our map $b_{f,g}$.

Moreover, since our functors $\beta_n, \beta_n'$ are defined by pullbacks, they are associativ-
e up to a canonical isomorphism. This means that for any composable $l$-tuple of maps
$f_1, \ldots, f_l$ in the category $\text{C}_l$, we have canonical functors
\[
\beta_n^I : \text{C}_n(f_1 \circ \cdots \circ f_l) \to \prod_{n_1 + \cdots + n_l = n} \text{C}_{n_1}(f_1) \times \cdots \times \text{C}_{n_l}(f_l),
\]
where $I_l$ are the terms of the asymmetric operad of categories from Section 1.6, and these functors are compatible with the operad structure on $I_l$. Taking the bar complexes and using the operad map $\text{Ass}_\infty \to \text{C}(I_*, \mathbb{Z})$ of Section 1.6, we turn $T^c(f)$ into a $C$-graded $A_\infty$-coalgebra in the sense of Definition 1.11. The analogous functors $\beta_n,f$ turn the collection $\Phi^c(E), c \in C$, into an Ab-valued $A_\infty$-comodule
over $T^c(-)$. 
6.4. The comparison theorem. We can now prove the comparison theorem expressing $\mathcal{D}S_{fs}(C, Ab)$ in terms of the coalgebra $T^C_{\Gamma}(-)$.

**Definition 6.16.** The category $\mathcal{D}T(C, Ab) = \mathcal{D}(T^C_{\Gamma}, Ab)$ is the derived category of Ab-valued $A_{\infty}$-comodules over the $C$-graded $A_{\infty}$-coalgebra $T^C_{\Gamma}(-)$. An object $E, \in \mathcal{D}T(C, Ab)$ is finitely supported if $E_c(c) = 0$ except for $c$ with isomorphism class in a finite closed subset $U \subset [C]$; the full subcategory spanned by finitely supported objects is denoted by $\mathcal{D}T_{fs}(C, Ab) \subset \mathcal{D}T(C, Ab)$.

In Section 6.3, we constructed a functor $\Phi_\ast : \text{Fun}(\mathcal{S}(C \otimes \Gamma), Ab) \to \mathcal{D}T(C, Ab)$. Taking the total complex of a double complex, we extend it to a functor

$$\Phi_\ast : \mathcal{D}S(C, Ab) \to \mathcal{D}T(C, Ab).$$

(There is the usual ambiguity in taking the total complex of a possibly infinite bicomplex — we can either take the sum of the terms on a diagonal, or the product of these terms. Here we take the sum.)

**Theorem 6.17.** The functor $\Phi_\ast$ induces an equivalence

$$\Phi_\ast : \mathcal{D}S_{fs}(C, Ab) \to \mathcal{T}_{fs}(C, Ab).$$

**Proof.** For any subset $U \subset [C]$, let $C_U \subset C$ be the full subcategory spanned by objects with isomorphism classes in $U$, with the embedding functor $\iota_U : C_U \to C$, and let $T_{\Gamma U}^C(-) = \iota_U^* T^C_{\Gamma}(-)$ be the $C_U$-graded $A_{\infty}$-coalgebra made up of the complexes $T^C_{\Gamma f} (f)$ with $f$ in $C_U$. Let $\mathcal{D}T_{\Gamma U}(C, Ab)$ be the triangulated category of $A_{\infty}$-comodules over $T_{\Gamma U}^C$. We have the restriction functor $\iota_U^* : \mathcal{D}T(C, Ab) \to \mathcal{D}T_{\Gamma U}(C, Ab)$, and for any two subsets $U' \subset U$, we have a restriction functor $\iota_{U' U}^* : \mathcal{D}T_{U}(C, Ab) \to \mathcal{D}T_{U'}(C, Ab)$. If $U$ is closed, and either $U' \subset U$ or the complement $U \setminus U'$ is also closed, then we have an obvious inverse functor $\mathcal{D}T_{U'}(C, Ab) \to \mathcal{D}T_{U}(C, Ab)$ given by extension by 0 — explicitly, $M \in \mathcal{D}T_{U'}(C, Ab)$ is sent to $M' \in \mathcal{D}T_{U}(C, Ab)$ such that

$$M'(c) = \begin{cases} M(c), & [c] \in U', \\ 0, & \text{otherwise}, \end{cases}$$

with the obvious structure maps. If it was $U'$ which was closed, then this extension functor is right-adjoint to the restriction functor $\iota_{U' U}^*$ (this is obvious for the homotopy categories $\text{Ho}(T_{\Gamma U}^C, Ab)$, $\text{Ho}(T_{\Gamma U'}^C, Ab)$, and since both restriction and extension preserve acyclic comodules, the adjunction descends to the derived categories).

**Lemma 6.18.** For any two finite closed subsets $U' \subset U \subset [C]$, the category $\mathcal{D}T_{U'}(C, Ab)$ admits a semiorthogonal decomposition

$$\langle \mathcal{D}T_{U \setminus U'}(C, Ab), \mathcal{D}T_{U'}(C, Ab) \rangle,$$

where the embeddings

$$\mathcal{D}T_{U \setminus U'}(C, Ab) \subset \mathcal{D}T_{U}(C, Ab) \text{ and } \mathcal{D}T_{U'}(C, Ab) \subset \mathcal{D}T_{U}(C, Ab)$$

are given by extension by 0.
Lemma 6.18 implies by induction that $\mathcal{DT}_U(\mathcal{C}, \text{Ab}) = \cup_U \mathcal{DT}_U(\mathcal{C}, \text{Ab})$, the union over all finite closed $U \subset [\mathcal{C}]$, where $\mathcal{DT}_U(\mathcal{C}, \text{Ab})$ is identified with the full subcategory in $\mathcal{DT}(\mathcal{C}, \text{Ab})$ spanned by objects $E_*$ such that $E_*(c) = 0$ unless $[c] \in U$. The comparison functor $\Phi_*$, obviously sends $\mathcal{DS}_U(\mathcal{C}, \text{Ab})$ into $\mathcal{DT}_U(\mathcal{C}, \text{Ab})$, and it suffices to prove that it is an equivalence for any finite closed $U \subset [\mathcal{C}]$. Fix such a subset $U$, take an object $c \in \mathcal{C}$ whose isomorphism class $[c]$ is a maximal element in $U$, and let $U' = U \setminus \{[c]\}$. By induction on the cardinality of $U$, we may assume that $\Phi_* : \mathcal{DS}_U(\mathcal{C}, \text{Ab}) \to \mathcal{DT}_U(\mathcal{C}, \text{Ab})$ is an equivalence. Moreover, for any invertible map $f$, the groupoids $C_n(f)$, $n \geq 1$ are obviously empty, so that $\mathcal{T}_c^f(f)$ is $\mathbb{Z}$ placed in degree 0. Therefore the category $\mathcal{DT}_{[c]}(\mathcal{C}, \text{Ab})$ is equivalent to $\mathcal{D}(\langle c \rangle, \text{Ab})$, and the functor

$$\Phi_* : \mathcal{DS}_U(\mathcal{C}, \text{Ab}) \cong \mathcal{D}(\langle c \rangle, \text{Ab}) \to \mathcal{DT}_{[c]}(\mathcal{C}, \text{Ab})$$

is also an equivalence. Then by Corollary 6.14, $\mathcal{DS}_U(\mathcal{C}, \text{Ab}) \subset \mathcal{DS}_U(\mathcal{C}, \text{Ab})$ is admissible, so that by Lemma 1.3, it suffices to prove that $\Phi_*$ sends $\mathcal{DS}_U(\mathcal{C}, \text{Ab})^\perp = \text{Inff}(\mathcal{D}(\langle c \rangle, \text{Ab})) \subset \mathcal{DS}_U(\mathcal{C}, \text{Ab})$ into $\mathcal{DT}_U(\mathcal{C}, \text{Ab})^\perp \subset \mathcal{DT}_U(\mathcal{C}, \text{Ab})$.

Indeed, denote $\text{Ho}(\langle c \rangle) \cong \text{Ho}(\mathcal{T}_c^f(\langle c \rangle), \text{Ab})$, $\text{Ho}_U = \text{Ho}(\mathcal{T}_c^U, \text{Ab})$. Then the restriction functor $\text{Ho}_U \to \text{Ho}(\langle c \rangle)$ has an obvious right-adjoint $\text{Ho}(\langle c \rangle) \to \text{Ho}_U$ which sends a complex $M_*$ of $\text{Aut}(c)$-comodules into an $A_{\infty}$-comodule $\text{I}(M_*)$ such that

$$\text{I}(M_*(c')) = \left( \bigoplus_{f \in \mathcal{C}(c,c')} \mathcal{T}_c^f(f) \otimes M_*(c) \right)^{\text{Aut}(c)}.$$  \hfill (6.13)

If the complex $M_*$ is $h$-injective, then $\text{I}(M_*)$ is also $h$-injective by adjunction, so that its image in the quotient category $\mathcal{DT}_U(\mathcal{C}, \text{Ab})$ lies in $\mathcal{DT}_U(\mathcal{C}, \text{Ab})^\perp \subset \mathcal{DT}_U(\mathcal{C}, \text{Ab})$. However, since $\text{Ho}(\langle c \rangle)$ is equivalent to the category of unbounded complexes of functors in $\text{Fun}(\langle c \rangle, \text{Ab})$, every object has an $h$-injective replacement. Thus we can take any object $M' = \text{Inff}(M) \in \mathcal{DS}_U(\mathcal{C}, \text{Ab})^\perp$, and choose an $h$-injective complex $M_*$ of $\text{Aut}(c)$-comodules representing $M = \Phi_*(M') \in \mathcal{D}(\langle c \rangle, \text{Ab}) \cong \mathcal{D}(\text{Aut}(c), \text{Ab})$. Then by definition, $M' = \text{Inff}(M)$ is represented by the complex

$$\text{Inff}(M_*) = (M_* \otimes T^c)^{\text{Aut}(c)},$$

and we see that $\Phi_*(M_*) \in \mathcal{DT}_U(\mathcal{C}, \text{Ab})$ is precisely isomorphic to $\text{I}(M_*)$. \hfill $\square$

Remark 6.19. As the proof of Theorem 6.17 shows, every finitely supported object in the category $\text{Ho}(\mathcal{T}_*, \text{Ab})$ does have an $h$-injective replacement. As we have mentioned in Section 1.5.3, this property seems rather special.

6.5. Induction and products. Assume now that the category $\mathcal{C}$ has a terminal object, as in Section 5.2. Then $\mathcal{DS}(\mathcal{C}, \text{Ab})$ is a tensor category, and by Proposition 5.6 all the fixed points functors $\Phi^\sigma$ are tensor functors, so that the subcategory $\mathcal{DS}_0(\mathcal{C}, \text{Ab}) \subset \mathcal{DS}(\mathcal{C}, \text{Ab})$ is closed under tensor products. To describe the tensor product structure on $\mathcal{DS}_0(\mathcal{C}, \text{Ab})$ in terms of the equivalent category $\mathcal{DT}_0(\mathcal{C}, \text{Ab})$,
one would have to introduce a product on the $C$-graded $A_\infty$-coalgebra $T^C$ — more precisely, an associative product on the complex $T^C(f)$ for any map $f$ in $C$. In keeping with the Tannakian formalism, $T^C$ then becomes a “$C$-graded Hopf algebra”, with an induced tensor product on the category $DT(C, \text{Ab})$.

Unfortunately, it seems that the particular $C$-graded $A_\infty$-coalgebra, or using a Segal-like notion of a “special $A_\infty$ $\Gamma$-coalgebra” instead of a commutative $A_\infty$ Hopf algebra, but I haven’t pursued this. Nevertheless, we have a simpler result which we will need for applications in Section 8.

Consider the trivial $C$-graded coalgebra $Z^C$ given by $Z^C(f) = Z$ for any map $f$ in $C$. Then since $C$ is assumed to be Hom-finite, the derived category $D(Z^C, \text{Ab})$ of $\text{Ab}$-valued $A_\infty$-comodules over $Z^C$ is equivalent to the derived category $D(C^{\text{op}}, \text{Ab})$. Moreover, since $T^C_0(f) = Z = Z^C(f)$ for any $f$, we have a natural map

$$\lambda: Z^C = F^0T^C \rightarrow T^C,$$

where $F^0$ stands for the 0-th term of the stupid filtration. This map induces a corestriction functor $\lambda^*: D(C^{\text{op}}, \text{Ab}) \cong D(Z^C, \text{Ab}) \rightarrow DT(C, \text{Ab})$. Explicitly, for any $Z^C$-comodule $E$, we have $\lambda^*E_*(c) \cong E_*c$, $c \in C$, and for any map $f: c' \rightarrow c$ in $C$, the comodule structure map of Definition 1.11 is the natural map

$$E_*(c) \xrightarrow{E_*(f)} E_*(c') \otimes Z = E_*(c') \otimes Z^C(f) \xrightarrow{\lambda(f)} E_*(c') \otimes T^C(f),$$

with all the higher maps being equal to 0. Moreover, for any complex $E_*$ of pointwise-flat functors in $D(C^{\text{op}}, \text{Ab})$ and any $A_\infty$-comodule $M_*$ over $T^C_*$, we have an obvious $A_\infty$-comodule $E_* \otimes M_*$ over $T^C_*$, with $(E_* \otimes M_*)(c) = E_*(c) \otimes M_*(c)$, and with the comodule structure maps obtained as the products of the comodule structure maps for $M_*$ and the canonical maps

$$E_*(c) \rightarrow E_*(c')$$

for any composable n-tuple $f_1, \ldots, f_n$ of maps in $C$ with the composition $f_1 \circ \cdots \circ f_n$ being a map from $c$ to $c'$. This preserves quasiisomorphisms, thus descents to a tensor product functor

$$D(C^{\text{op}}, \text{Ab}) \times DT(C, \text{Ab}) \rightarrow DT(C, \text{Ab}). \quad (6.14)$$

**Lemma 6.20.** Under the comparison functor $\Phi_*: DS(C, \text{Ab}) \rightarrow DT(C, \text{Ab})$ of Theorem 6.17, the induction functor $q_*^{\text{op}}: D(C^{\text{op}}, \text{Ab}) \rightarrow DS(C, \text{Ab})$ of Section 5.3 goes to the corestriction functor $\lambda^*$ — that is, we have a natural functorial quasiisomorphism

$$\Phi_* (q_*^{\text{op}}E_*) \cong \lambda^*E_* \quad (6.15)$$

for any $E_* \in D(C^{\text{op}}, \text{Ab})$. Moreover, for any objects $E_* \in D(C^{\text{op}}, \text{Ab})$, $M_* \in DS(C, \text{Ab})$, we have a natural isomorphism

$$\Phi_* ((q_*^{\text{op}}E_*) \otimes M_*) \cong E_* \otimes \Phi_* (M_*), \quad (6.16)$$

where the product in the right-hand side is the product (6.14).

**Proof.** Both claims immediately follow from Proposition 5.8. \qed
7. Tate Homology Description

We will now use Theorem 6.17 to obtain an explicit and useful description of the gluing data in the semiorthogonal decomposition of Corollary 6.12.

7.1. Generalized Tate cohomology. Assume given a finite group \( G \), and a family \( \{ H_i \} \) of subgroups \( H_i \subset G \). For every \( H_i \), we have the induction functor

\[
\text{Ind}_i^G : \mathcal{D}^b( H_i, \mathbb{Z} ) \rightarrow \mathcal{D}^b( G, \mathbb{Z} ),
\]

where \( \mathcal{D}^b( G, \mathbb{Z} ) \) and \( \mathcal{D}^b( H_i, \mathbb{Z} ) \) are the bounded derived categories of \( \mathbb{Z}[G] \)-modules, resp. \( \mathbb{Z}[H_i] \)-modules; this functor is adjoint both on the left and on the right to the natural restriction functor \( \mathcal{D}^b( G, \mathbb{Z} ) \rightarrow \mathcal{D}^b( H_i, \mathbb{Z} ) \). Let

\[
\mathcal{D}^b_{\{ H_i \}}( G, \mathbb{Z} ) \subset \mathcal{D}^b( G, \mathbb{Z} )
\]

be the smallest thick triangulated subcategory containing all objects of the form \( \text{Ind}_i^G( V_i ) \), \( V_i \in \mathcal{D}^b( H_i, \mathbb{Z} ) \), \( H_i \in \{ H_i \} \), and denote by

\[
\mathcal{D}( G, \{ H_i \}, \mathbb{Z} ) = \mathcal{D}^b( G, \mathbb{Z} ) / \mathcal{D}^b_{\{ H_i \}}( G, \mathbb{Z} )
\]

the quotient category.

**Definition 7.1.** The generalized Tate cohomology \( \check{H}^\ast( G, \{ H_i \}, M_\ast ) \) of the group \( G \) with coefficients in an object \( M_\ast \in \mathcal{D}^b( G, \mathbb{Z} ) \) with respect to the family \( \{ H_i \} \) is given by

\[
\check{H}^\ast( G, \{ H_i \}, M_\ast ) = \text{Ext}^\ast_{\mathcal{D}( G, \{ H_i \}, \mathbb{Z} )}( \mathbb{Z}, M_\ast ),
\]

where \( \mathbb{Z} \) is the trivial representation of the group \( G \), and \( \text{Ext}^\ast( -, - ) \) is computed in the quotient category \( \mathcal{D}( G, \{ H_i \}, \mathbb{Z} ) \).

By the definition of the quotient category, \( \check{H}^\ast( G, \{ H_i \}, M_\ast ) \) is expressed as follows. Consider the category \( I( G, \{ H_i \} ) \) of objects \( V_i \in \mathcal{D}^b( G, \mathbb{Z} ) \) equipped with a map \( V_i \rightarrow \mathbb{Z} \) whose cone lies in \( \mathcal{D}^b_{\{ H_i \}}( G, \mathbb{Z} ) \subset \mathcal{D}^b( G, \mathbb{Z} ) \). Let \( I = I( G, \{ H_i \} )^{\text{opp}} \) be the opposite category, and let

\[
I^{\text{fg}} \subset I = I( G, \{ H_i \} )^{\text{opp}} \quad (7.1)
\]

be the subcategory spanned by those \( V_i \) which can be represented by complexes of finitely generated \( \mathbb{Z}[G] \)-modules. Then we have

\[
\check{H}^\ast( G, \{ H_i \}, M_\ast ) = \lim_{\leftarrow i} \text{Ext}^\ast( V_i, M_\ast ) \quad (7.2)
\]

where the limit is taken over the category \( I^{\text{fg}} \), and \( \text{Ext}^\ast( -, -, - ) \) is computed in the category \( \mathcal{D}( G, \mathbb{Z} ) \). Sending \( V_i \) to the dual complex \( V_i^\ast \) identifies \( I^{\text{fg}} \) with the category of objects \( V_i^\ast \) in \( \mathcal{D}^b( G, \mathbb{Z} ) \) which are represented by finitely generated \( \mathbb{Z}[G] \)-modules and equipped with a map \( \mathbb{Z} \rightarrow V_i^\ast \) whose cone lies in \( \mathcal{D}^b_{\{ H_i \}}( G, \mathbb{Z} ) \). Then (7.2) can be rewritten as

\[
\check{H}^\ast( G, \{ H_i \}, M_\ast ) = \lim_{\leftarrow i} H^\ast( G, M_\ast \otimes V_i ),
\]

where again the limit is over \( I^{\text{fg}} \). Since \( I^{\text{fg}} \) is small and filtered, the limit is well-defined, and gives a cohomological functor.
We can use the same expression to define Tate cohomology with coefficients. Namely, let \( D^b(G, \text{Ab}) \) be the derived category of representations of the group \( G \) in the abelian category \( \text{Ab} \).

**Definition 7.2.** The generalized Tate cohomology \( \check{H}^\ast(G, \{ H_i \}, M_\ast) \) of the group \( G \) with coefficients in an object \( M_\ast \in D^b(G, \text{Ab}) \) with respect to the family \( \{ H_i \} \) is given by

\[
\check{H}^\ast(G, \{ H_i \}, M_\ast) = \lim_{\rightarrow} H^\ast(G, M_\ast \otimes V_{\ast}),
\]

where the limit is taken over the category \( I_{\text{fg}} \subset I = I(G, \{ H_i \}) \) of \([7.1]\).

Again, since \( \text{Ab} \) is by assumption a Grothendieck category, and \( I_{\text{fg}} \) is small and filtered, generalized Tate cohomology is a well-defined cohomological functor from \( D(G, \text{Ab}) \) to \( \text{Ab} \).

**Lemma 7.3.** For any \( M \in D(G, \text{Ab}) \) and any \( W_{\ast} \in D_{\{ H_i \}}(G, \text{Z-mod}) \) which can be represented by a bounded complex of finitely generated \( \text{Z}[G] \)-modules, we have

\[
\check{H}^\ast(G, \{ H_i \}, M \otimes W_{\ast}) = 0.
\]

**Proof.** Let \( W'_{\ast} = \text{Z} \oplus W_{\ast} \), and let \( w: I_{\text{fg}} \to I_{\text{fg}} \) be the functor which sends \( V_{\ast} \in I_{\text{fg}} \) to \( W'_{\ast} \otimes V_{\ast} = V_{\ast} \oplus (V_{\ast} \otimes W_{\ast}) \). Then \( w \) has a left-adjoint functor \( I_{\text{fg}} \to I_{\text{fg}} \) given by \( V_{\ast} \mapsto \text{Z} \oplus (V_{\ast} \otimes W_{\ast}) \), and since \( I_{\text{fg}} \) is filtered, this implies that \( w \) is cofinal in the sense of \([KS]\). Therefore the natural map \( \text{Id} \to w \) induced by \( \text{Z} \to W'_{\ast} \) gives an isomorphism

\[
\lim_{\rightarrow} H^\ast(G, M \otimes V_{\ast}) \cong \lim_{\rightarrow} H^\ast(G, M \otimes W'_{\ast} \otimes V_{\ast}),
\]

where both limits are over \( V_{\ast} \in I_{\text{fg}} \). Since \( \check{H}^\ast \) is a cohomological functor, this proves the claim. \( \square \)

### 7.2. Adapted complexes.

In practice, the index category \( I_{\text{fg}} \) is still too big. To be able to compute generalized Tate cohomology more efficiently, we use the following gadget.

**Lemma 7.4.** Assume given a bounded from above complex \( P_{\ast} \) of finitely generated \( \text{Z}[G] \)-modules. For any integer \( l \), denote by \( F^l P_{\ast} \) the \((-l)\)-th term of the stupid filtration on \( P_{\ast} \). Then the following conditions are equivalent.

(i) For any of the subgroups \( H_i \in \{ H_i \} \) and any integer \( l \), there exists an integer \( l' \geq l \) such that the map \( F^l P_{\ast} \to F^{l'} P_{\ast} \) becomes 0 after restricting to \( D(H_i, \text{Z}) \).

(ii) For any \( V_{\ast} \in D_{\{ H_i \}}(G, \text{Z}) \), we have

\[
\lim_{\rightarrow} H^\ast(G, V_{\ast} \otimes F^l P_{\ast}) = 0.
\]

(iii) For any \( V_{\ast} \in D_{\{ H_i \}}(G, \text{Z}) \), any Grothendieck abelian category \( \text{Ab} \), and any \( M \in D(G, \text{Ab}) \), we have

\[
\lim_{\rightarrow} H^\ast(G, M \otimes V_{\ast} \otimes F^l P_{\ast}) = 0.
\]
Proof. The condition (iii) contains (ii) as a particular case. By adjunction, (i) implies (iii) for \( V_* \) of the form \( \text{Ind}_G^H(V'_*) \), \( V'_* \in \mathcal{D}(H, \mathbb{Z}) \). Since \( \mathcal{D}_{(H)}(G, \mathbb{Z}) \) consists of direct summands of sums of such \( V_* \), this condition on \( V_* \) can be dropped. Finally, (ii) applied to a bounded complex \( V_* \) of finitely generated \( \mathbb{Z}[G] \)-modules can be rewritten as follows: for any \( l \), any map \( \kappa : V_* \to F^lP_* \), becomes 0 after composing with the natural map \( F^lP_* \to F^lP_* \) for sufficiently large \( l' \geq l \). Applying this to \( V_* = \text{Ind}_G^H(F^lP) \) with \( \kappa \) being the adjunction map yields (i).

Definition 7.5. A complex \( P_* \), of \( \mathbb{Z}[G] \)-modules is said to be adapted to the family \( \{H_i\} \) if

(i) \( P_i = 0 \) for \( i < 0 \), \( P_0 \cong \mathbb{Z} \), and for any \( i \geq 0 \), \( P_i \) is a flat \( \mathbb{Z} \)-modules of finite rank, and is a direct summand of a sum of \( \mathbb{Z}[G] \)-modules induced from one of the subgroups \( H_i \), and

(ii) the complex \( P_* \) satisfies the equivalent conditions of Lemma 7.4.

Proposition 7.6. Assume given a finite group \( G \), a family \( \{H_i\} \) of subgroups \( H_i \subset G \), and complex \( P_* \), of \( \mathbb{Z}[G] \)-modules adapted to the family \( \{H_i\} \). Then for any \( M_* \in \mathcal{D}^b(G, \text{Ab}) \), we have an isomorphism

\[
\hat{H}^*(G, \{H_i\}, M_*) \cong \lim_{\rightarrow l} \hat{H}^*(G, M_*, F^lP_*),
\]

(7.3)

where \( F^lP_* \) is as in Definition 7.5 (ii), and this isomorphism is functorial in \( M_* \).

Proof. Denote for the moment

\[
\hat{H}^*_0(G, \{H_i\}, M_*) \cong \lim_{\rightarrow l} \hat{H}^*_0(G, M_*, F^lP_*).
\]

Consider the product \( I^{fs} \times \mathbb{N} \) of the index category \( I^{fs} \) of (7.1) and the partially ordered set of non-negative integers. Computing the double limit first in one order, then in another, we obtain an isomorphism

\[
\lim_{\rightarrow l} \hat{H}^*_0(G, \{H_i\}, M_*, F^lP_*) \cong \hat{H}^*_0(G, \{H_i\}, M_*, F^lP_*, V_*)
\]

\[
\cong \lim_{\rightarrow l} \hat{H}^*_0(G, \{H_i\}, M_*, V_*),
\]

where the last limit is over \( V_* \in I^{fs} \), and the intermediate limit is over \( V_* \times l \in I^{fs} \times \mathbb{N} \). By Lemma 7.3 and Definition 7.5 (i), \( \hat{H}^*_0(G, \{H_i\}, M_*, F^lP_*) \) does not depend on \( l \), so that

\[
\lim_{\rightarrow l} \hat{H}^*_0(G, \{H_i\}, M_*, F^lP_*) \cong \hat{H}^*_0(G, \{H_i\}, M_*).
\]

By Definition 7.5 (ii) in the form of Lemma 7.4 (iii), \( \hat{H}^*_0(G, \{H_i\}, M_*, V_*) \) does not depend on \( V_* \in I^{fs} \), so that

\[
\lim_{\rightarrow l} \hat{H}^*_0(G, \{H_i\}, M_*, V_*) \cong \hat{H}^*_0(G, \{H_i\}, M_*).
\]

This finishes the proof.

Example 7.7. Take as \( \{H_i\} \) the family consisting only of the trivial subgroup \( \{e\} \subset G \). A \( \mathbb{Z}[G] \)-module which is free over \( \mathbb{Z} \) is induced from \( \{e\} \) if and only if it is free over \( \mathbb{Z}[G] \), so that the category \( \mathcal{D}^b_{\{H_i\}}(G, \mathbb{Z}) \) consists of direct summands of...
finite complexes of free \(\mathbb{Z}[G]\)-modules — equivalently, these are the perfect objects in \(D^b(G, \mathbb{Z})\). Thus in this case, \(\tilde{H}^\ast(G, \{c\}, -)\) is the standard Tate (co)homology functors. Further, Definition 7.5 (ii) simply means that the complex \(P_i\) is acyclic (thus contractible as a complex of abelian groups), and Definition 7.5 (i) means that \(P_i\) is a finitely generated projective \(\mathbb{Z}[G]\)-module for any \(i \geq 1\). This gives the standard procedure for computing Tate homology: take a projective resolution \(P_i\) of the trivial representation \(\mathbb{Z}\) and let \(\tilde{P}_i\) be the cone of the augmentation map \(P_i \to \mathbb{Z}\); then

\[
\tilde{H}^\ast(G, V_\ast) \cong \lim_{\leftarrow i} H^\ast(G, V_\ast \otimes F^i \tilde{P}_i)
\]

for any \(V_\ast \in D^b(G, \mathbb{Z}\text{-mod})\).

In general, there are different ways to construct an adapted complex for a finite group \(G\) and a family \(\{H_i\}\); we will present one construction later in Section 7.5.

7.3. Tate cohomology and fixed points functors. Now assume given a small category \(C\) of Galois type, and consider the category \(\mathcal{D}S_{\mathit{Gr}}(C, \mathbb{Ab}) \cong \mathcal{D}T_{\mathit{Gr}}(C, \mathbb{Ab})\), as in Theorem 6.17. We want to express the gluing data between the pieces of the semiorthogonal decomposition of Corollary 6.12 in terms of generalized Tate cohomology.

The main result is as follows. Take a morphism \(f: c' \to c\) in \(C\), and consider the corresponding complex \(T^c_C(f)\) of (6.11). Denote by \(\text{Aut}(f) \subset \text{Aut}(c')\) the subgroup of those automorphisms \(\sigma: c' \to c'\) that commute with \(f, f \circ \sigma = f\). This group acts on every set \(\mathcal{C}_n(f)\) by left composition, that is, a diagram \(c' \xrightarrow{f_1} c_1 \to \cdots \to c_n \to c\) goes to \(c' \xrightarrow{f \circ \sigma} c_1 \to \cdots \to c_n \to c\). Thus \(\text{Aut}(f)\) also acts on \(T^c_C(f)\). Moreover, for any diagram \(\alpha = [c' \to c_1 \to c] \in C_1(f)\), let \(\text{Aut}(\alpha) \subset \text{Aut}(f)\) be its stabilizer in the group \(\text{Aut}(f)\).

Proposition 7.8. The complex \(T^c_C(f)\) equipped with the natural action of the group \(\text{Aut}(f)\) is adapted to the family \(\{\text{Aut}(\alpha) \mid \alpha \in C_1(f)\}\) in the sense of Definition 7.5.

Proof. The condition (i) of Definition 7.5 is obvious: the stabilizer of any diagram \(c' \to c_1 \to \cdots \to c_n \to c\) is contained in the stabilizer of the diagram \(c' \to c_1 \to c\) obtained by forgetting \(c_i, i \geq 2\), so that it suffices to use the following trivial observation.

Lemma 7.9. Assume given a finite group \(G\) with a family of subgroups \(\{H_i\}\), \(H_i \subset G\), and let \(X\) be a finite \(G\)-set such that the stabilizer \(G_x\) of any point \(x \in X\) is contained in one of the subgroups \(H_i\). Then the set \(\mathbb{Z}[X]\) with the natural \(G\)-action lies in the subcategory \(D^b_{\{H_i\}}(G, \mathbb{Z}\text{-mod}) \subset D^b(G, \mathbb{Z}\text{-mod})\).

To check the condition (ii) in the form of Lemma 7.4 (ii), consider the category \(\mathcal{C}_C\) of Corollary 6.9. Let \(\mathcal{C}_f\) be the category of diagrams \(c' \to c_1 \to c\) such that the composition map \(c' \to c\) is equal to \(f\), and let \(\pi^f: \mathcal{C}_f \to \mathcal{C}_C\) be the projection functor which sends an object \([c' \to c_1 \to c]\) of \(\mathcal{C}_f\) to the object \([c_1 \to c]\) of \(\mathcal{C}_C\). This is a discrete cofibration. Let \(T^f = \pi^f_! \mathbb{Z} \in \text{Fun}(\mathcal{C}_C, \mathbb{Z}\text{-mod})\). The group \(\text{Aut}(f)\) acts on \(T^f\), and we have

\[
T^c_C(f) \cong \Phi^C_*(T^f),
\]
where \( \Phi^c(\tau) \) is the complex (6.8). For any subgroup \( H \subset \text{Aut}(f) \) and any \( h \)-injective complex \( V_c \) of \( \mathbb{Z}[H] \)-modules, we have

\[
\lim_{l} \left( V_c \otimes F^lT^c(f) \right)^H \cong \lim_{l} F^l\Phi_c(T^f \otimes V_c)^H \cong \Phi_c(T^f \otimes V_c)^H,
\]

and by Lemma 6.15, this computes

\[
\Phi^c(V_c \otimes T^f)^H. \tag{7.4}
\]

It suffices to prove that this is trivial whenever \( H = \text{Aut}(\alpha) \) for some \( \alpha = [e' \rightarrow c_1 \rightarrow c] \in C_1(f) \). Indeed, fix such an \( \alpha \), and denote its component maps by \( f_1: c' \rightarrow c_1 \), \( f_2: c_1 \rightarrow c \). Then we have the category \( C_{c_1} \) and the functor \( T^{f_1} \in \text{Fun}(C_{c_1}, \mathbb{Z}\text{-mod}) \), and the group \( \text{Aut}(\alpha) = \text{Aut}(f_1) \) acts on \( T^{f_1} \). Moreover, composition with \( f_2 \) defines functors \( \rho: C_{c_1} \rightarrow C_c, \rho': C_{c_1} \rightarrow C_f \), and we have a commutative diagram

\[
\begin{array}{ccc}
C_{f_1} & \xrightarrow{\rho'} & C_f \\
\downarrow{\pi^{f_1}} & & \downarrow{\pi^f} \\
C_{c_1} & \xrightarrow{\rho} & C_c
\end{array}
\]

Since both \( C_{f_1} \) and \( C_f \) have an initial object, and \( \rho' \) preserves them, we have \( L^*\rho'\mathbb{Z} \cong \rho'\mathbb{Z} \cong \mathbb{Z} \), so that we have an isomorphism

\[
L^*\rho T^{f_1} \cong \rho T^{f_1} \cong \rho \pi^{f_1} \mathbb{Z} \cong \pi^{f_1} \rho' \mathbb{Z} \cong \pi^f \mathbb{Z} \cong T^f,
\]

and this isomorphism is \( \text{Aut}(\alpha) \)-equivariant. This yields a quasiisomorphism

\[
(V_c \otimes T^f)^{\text{Aut}(\alpha)} \cong L^*\rho \pi(T^{f_1} \otimes V_c)^{\text{Aut}(\alpha)},
\]

so that by Corollary 6.9, (7.4) is of the form \( \varphi^cL^*\rho \pi E_c \) for some \( E_c \in D(C_{f_1}, \mathbb{Z}\text{-mod}) \). This is equal to 0: by definition of the functor \( \varphi^c \), we have

\[
\text{Hom}(\varphi^cL^*\rho \pi E_c, M_c) \cong \text{Hom}(L^*\rho \pi E_c, M_c \otimes \mathcal{T}) \cong \text{Hom}(E_c, \rho^c \mathcal{T} \otimes M_c)
\]

for any \( M_c \in D(\text{Ab}) \), and \( \rho^c \mathcal{T} = 0 \). \( \square \)

**Corollary 7.10.** For any Galois-type category \( C \) and two objects \( c, c' \in C \), the homology \( H^*(\Phi^c\text{Infl}^{c'}(M_c)) \) of the object \( \Phi^c\text{Infl}^{c'}(M_c) \in D(\text{Ab}) \) is given by

\[
H^*(\Phi^c\text{Infl}^{c'}(M_c)) \cong \bigoplus_{f \in \mathcal{C}((c', c))} H^*(\text{Aut}(f), \{\text{Aut}(\alpha) \mid \alpha \in C_1(f)\}, M_c) \tag{7.5}
\]

for any \( M_c \in D((c'), \text{Ab}) \). \( \square \)

To write down a formula for \( \tilde{\Phi}^c\text{Infl}^{c'} \), the gluing functor in the semi-orthogonal decomposition of Corollary 6.12, one has to incorporate the natural \( \text{Aut}(c) \)-action on \( \Phi^c \) into (7.5); we leave it to the reader.
7.4. Invertible objects. Let us say that an object \( M \in \mathcal{D} \) is a triangulated tensor category \( \mathcal{D} \) is invertible if the functor \( M \otimes - : \mathcal{D} \to \mathcal{D} \) given by multiplication by \( M \) is an equivalence (for example, a unit object \( I \in \mathcal{D} \) is invertible, and so are all its shifts \( I[l], l \in \mathbb{Z} \)). For simplicity, for the moment we restrict our attention to the case \( \text{Ab} = \mathbb{Z}\text{-mod} \). Keep the assumptions of the previous subsection, and assume also that \( \mathcal{C} \) has a terminal object, so that \( \mathcal{DS}(\mathcal{C}, \mathbb{Z}) \) is a tensor triangulated category. In addition, assume that \( [\mathcal{C}] \) is finite, so that \( \mathcal{DS}_{\text{fs}}(\mathcal{C}, \mathbb{Z}) = \mathcal{DS}(\mathcal{C}, \mathbb{Z}) \). As an application of Proposition 7.8, let us prove the following criterion of invertibility for objects in \( \mathcal{DT}(\mathcal{C}, \mathbb{Z}) \cong \mathcal{DS}(\mathcal{C}, \mathbb{Z}) \).

**Proposition 7.11.** Assume given an object \( M \in \mathcal{D}(\mathcal{C}^{\text{opp}}, \mathbb{Z}) \) such that

(i) the restriction \( M(c) = j^*M \in \mathcal{D}(\langle c \rangle, \mathbb{Z}) \) with respect to the inclusion \( j^*: \langle c \rangle \to \mathcal{C} \) is invertible for any \( c \in \mathcal{C} \), and

(ii) for any map \( f: c' \to c \) in \( \mathcal{C} \), equip \( M(c) \) with the trivial \( \text{Aut}(f) \)-action, so that the map \( M(f): M(c) \to M(c') \) is \( \text{Aut}(f) \)-equivariant. Then the cone of this map lies in

\[
\mathcal{D}_{\{\text{Aut}(a)\mid a \in \mathcal{C}(f)\}}(\text{Aut}(f), \mathbb{Z}\text{-mod}) \subset \mathcal{D}^p(\text{Aut}(f), \mathbb{Z}\text{-mod}).
\]

Then the induced object \( q_1^{\text{opp}}M \in \mathcal{DS}(\mathcal{C}, \mathbb{Z}) \) is invertible.

**Proof.** Let \( \mathcal{DT}(\mathcal{C}, \mathbb{Z}) \to \mathcal{DT}(\mathcal{C}, \mathbb{Z}) \) be the functor given by multiplication by \( M \), \( M(E) = M \otimes E \). By (6.15) of Lemma 6.20, \( M \) preserves supports: if \( E \in \mathcal{DS}(\mathcal{C}, \mathbb{Z}) \) is supported in some subset \( U \subset [\mathcal{C}] \), then \( M(E) \) is also supported in \( U \). Thus as in the proof of Theorem 6.17, it suffices to prove that

\[ M: \mathcal{DT}_U(\mathcal{C}, \mathbb{Z}) \to \mathcal{DT}_U(\mathcal{C}, \mathbb{Z}) \]

is an equivalence for any finite closed \( U \subset [\mathcal{C}] \). Take a maximal element \([c] \in U \) and let \( U' = U \setminus \{[c]\} \). Then \( M \) sends the pieces of the semiorthogonal decomposition of Lemma 6.18 into themselves. By induction, \( M \) is an autoequivalence of \( \mathcal{DT}_{U'}(\mathcal{C}, \mathbb{Z}) \), and by the condition (i), \( M \) is also an autoequivalence of the orthogonal \( \mathcal{DT}_U(\mathcal{C}, \mathbb{Z}) \cong \mathcal{D}(\langle c \rangle, \mathbb{Z}) \). Again as in the proof of Theorem 6.17, by Lemma 1.3 it suffices to prove that \( M \) sends the orthogonal \( \mathcal{DT}_U(\mathcal{C}, \mathbb{Z}) \) into itself. But every object in this orthogonal is of the form \( l(E_c) \) for some \( h \)-injective complex \( E_c \), of \( \text{Aut}(c) \)-modules, where \( l \) is as in (6.13), and by Corollary 7.10 and Theorem 7.3, the condition (ii) then implies that \( M(l(E_c)) \) is quasiisomorphic to \( l(E_c \otimes M(c)) \). \( \square \)

7.5. The case of Mackey functors. We now turn to the case of Mackey functors — we assume given a group \( G \), and take \( \mathcal{C} = O_G \), the category of finite \( G \)-orbits. First, we fulfill two earlier promises — give a construction of an adapted complex, and prove that inflation is fully faithful.

**Lemma 7.12.** For any finite group \( G \) with a family of subgroups \( \{H_i\} \), there exists a complex \( P \) of \( \mathbb{Z}[G] \)-modules adapted to the family \( \{H_i\} \) in the sense of Definition 7.5.

**Proof.** Note that we may assume that together with any subgroup \( H_i \subset G \), the family \( \{H_i\} \) contains all the subgroups \( H \subset H_i \) — adding such subgroups to the family does not change the conditions of Definition 7.5 (nor of Definition 7.2).
Having assumed this, consider the full subcategory \( \mathcal{C}' \subset O_G \) spanned by the one-point orbit \([G/G]\) and all the orbits \([G/H_i]\), \(H_i \in \{H_1\}\). For any subgroup \(H_1, H_2 \subset G\), every \(G\)-orbit in the product \(G/H_1 \times G/H_2\) of \(O_G\) is of the form \(G/H'\) for some \(H' \subset H_1\). Therefore the category \( \mathcal{C}' \) is of Galois type, so that Proposition 7.8 applies.

Take as \( f \) the unique map \([G/\{e\}] \to [G/G]\); then \(G = \text{Aut}(f)\), \(\text{Aut}(\alpha) \in \{H_1\}\) for any \(\alpha \in \mathcal{C}'(f)\), and \(T^\vee_\alpha\) is the required adapted complex.

**Lemma 7.13.** Assume given a normal subgroup \(H \subset G\) with the quotient \(N = G/H\). Then the corestriction functor

\[
\text{Infl}_G^N: \text{DM}(N, \text{Ab}) \to \text{DM}(G, \text{Ab})
\]

is fully faithful.

**Proof.** Let \(U \subset \{O_G\}\) be the set of orbits \([G/H']\) such \([G/H] \geq [G/H']\), that is, there exists a \(G\)-equivariant map \(G/H \to G/H'\), that is, \(H' \subset G\) contains a conjugate of \(H \subset G\). Since \(H \subset G\) is assumed to be normal, \(H\) itself must be contained in \(H'\). Therefore the orbits \([G/H'], [G/H'] \in U\) are in one-to-one correspondence with orbits \(N/(H'/H)\) of the quotient group \(N = G/H\), and the full subcategory in \(O_G\) spanned by objects with classes in \(U\) is equivalent to the category \(O_N\). Moreover, it is easy to see that the embedding \(O_N \subset O_G\) is compatible with the graded \(A_\infty\)-coalgebras \(T\) — for any map \(f \in O_N\), we have

\[
T^\vee_{\text{Infl}^N_G}(f) \cong T^\vee_{O_N}(f).
\]

Thus \(\text{DM}(N, \text{Ab}) \cong \text{DM}(G, \text{Ab})\) and \(\text{Infl}_G^N\) is the full embedding \(\text{DM}(O_G, \text{Ab}) \subset \text{DM}(G, \text{Ab}) = \text{DM}(G, \text{Ab})\).

Now we make the following easy observation.

**Lemma 7.14.** Assume given a Galois-type category \(\mathcal{C}\) with the corresponding \(C\)-graded \(A_\infty\)-coalgebra \(T^\vee_\mathcal{C}\) of Definition 6.16, another \(C\)-graded \(A_\infty\)-coalgebra \(T^\vee_\mathcal{C}'\), and an \(A_\infty\)-map \(\nu: T^\vee_\mathcal{C}' \to T^\vee_\mathcal{C}\) such that

(i) if \(f\) is invertible, the map \(\nu: T^\vee_\mathcal{C}'(f) \to T^\vee_\mathcal{C}(f)\) is an isomorphism,

(ii) for a non-invertible \(f: c' \to c\) and any \(\mathbb{Z}[\text{Aut}(f)]\)-module \(V\), the map \(\nu\) induces an isomorphism

\[
\lim_{\mathcal{C}} H^*(\text{Aut}(f), V \otimes F^i T^\vee_\mathcal{C}'(f)) \cong \lim_{\mathcal{C}} H^*(\text{Aut}(f), V \otimes F^i T^\vee_\mathcal{C}(f)),
\]

where \(F^i T^\vee_\mathcal{C}, F^i T^\vee_\mathcal{C}'\) are the stupid filtrations.

Then the corestriction functor \(\nu^*\) induces an equivalence between the derived category \(\text{DT}^\vee_\mathcal{C}(C, \text{Ab})\) of finitely supported \(A_\infty\)-comodules over \(T^\vee_\mathcal{C}\) and the category \(\text{DT}^\vee_\mathcal{C}'(C, \text{Ab})\) of finitely supported \(A_\infty\)-comodules over \(T^\vee_\mathcal{C}'\).

**Proof.** The same as Theorem 6.17 and Proposition 7.11: since \(\mathcal{C}\) is lattice-like, both categories have filtrations by support and the corresponding semi-orthogonal decompositions, and the corestriction functor \(\nu^*\) is compatible with these decompositions. Then (i) ensures that \(\nu^*\) is an equivalence of the associated graded pieces, and (ii) ensures that \(\nu^*\) is compatible with the gluing.
The condition (ii) of this Lemma is satisfied, for example, when $\mathcal{T}_i(f)$ is adapted to $\{\text{Aut}(g)\}$ for any $f$, as in Proposition 7.5. However, it can also be satisfied for other reasons: for instance, if the right-hand side of (7.7) is equal to 0, the left-hand side may just be trivial, $\mathcal{T}_i(f) = 0$. This is especially useful in the Mackey functors case, for the following reason.

**Lemma 7.15.** Let $G$ be a finite group, and let $\{H_i\}$ be the family of all proper subgroups $H \subset G$.

(i) The generalized Tate cohomology $\hat{H}^\ast(G, \{H_i\}, V)$ with coefficients in any $\mathbb{Z}[G]$-module $V$ is trivial unless $G$ is a $p$-group for some prime $p$, and in this case, it is annihilated by $p$.

(ii) If $G$ is a cyclic prime group, $G = \mathbb{Z}/p^n$ for some prime $p$ and integer $n \geq 2$, then $\hat{H}^\ast(G, \{H_i\}, V) = 0$ for any $\mathbb{Z}[G]$-module $V$.

**Proof.** For a proper subgroup $H \subset G$, we have the natural induction and coinduction maps $\mathbb{Z} \to \text{Ind}_G^H(\mathbb{Z})$, $\text{Ind}_G^H(\mathbb{Z}) \to \mathbb{Z}$, and their composition is multiplication by the index $|G/H|$ of the subgroup $H$. Therefore multiplication by $|G/H|$ is trivial in the quotient category $\mathcal{D}(G, \{H_i\}, \mathbb{Z}\text{-mod})$, hence annihilates generalized Tate cohomology. If $G$ is not a $p$-group, then the greatest common denominator of the indices of its Sylow subgroups is 1, so that 1 also annihilates Tate cohomology. If $G$ is a $p$-group, then it contains a subgroup of index $p$. This proves (i).

For (ii), note that for any $n$, the second cohomology group $H^2(\mathbb{Z}/p^n\mathbb{Z}, \mathbb{Z})$ contains a canonical periodicity element $u_n$ represented by Yoneda by the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}] \xrightarrow{1-\sigma} \mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}] \to \mathbb{Z} \to 0,$$

where $\sigma$ is the generator of the cyclic group $\mathbb{Z}/p^n\mathbb{Z}$. For any positive integer $n' < n$, we have a natural quotient map $q_{n,n'}: \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n'}\mathbb{Z}$, and one easily checks that $q_{n,n'}(u_n) = p^{n-n'}u_n$. In the quotient category $\mathcal{D}(G, \{H_i\}, \mathbb{Z}\text{-mod})$, all the objects $q_{n,n'}^*\mathbb{Z}[\mathbb{Z}/p^n\mathbb{Z}]$ become trivial, so that all the maps $q_{n,n'}^*u_n$ are invertible. If $n \geq 2$, this implies that $p$ is invertible. Since $p$ annihilates Tate cohomology by (i), this is only possible if $\hat{H}^\ast(G, \{H_i\}, -) = 0$. $\square$

**Corollary 7.16.** Assume that $G$ is a finite group whose order is invertible in the category $\text{Ab}$. Then

$$\mathcal{D}\mathcal{M}(G, \text{Ab}) \cong \bigoplus_{H \subset G} \mathcal{D}(N_H/H, \text{Ab}),$$

where the sum is over all conjugacy classes of subgroups $H \subset G$, and $\mathcal{D}(N_H/H, \text{Ab})$ is the derived category of the representations of the quotient $N_H/H$ of the normalizer $N_H$ of the subgroup $H \subset G$ by $H$ itself.

**Proof.** Since $G$ is finite, everything has finite support,

$$\mathcal{D}\mathcal{S}(O_G, \text{Ab}) \cong \mathcal{D}\mathcal{S}_{\text{fin}}(O_G, \text{Ab}) \cong \mathcal{D}\mathcal{F}(O_G, \text{Ab}).$$

Moreover, every map $f$ between two $G$-orbits is isomorphic to the quotient map $[G/H] \to [G/H']$ for some subgroups $H \subset H' \subset G$, the automorphisms group of an object $[G/H]$ is precisely $N_H/H$, and for any map $f: [G/H] \to [G/H']$, any
proper subgroup in the automorphisms group $\text{Aut}(f)$ is of the form $\text{Aut}(\alpha)$ for some $\alpha \in C_1(f)$. Now take $T'_c(f) = T^c_\varepsilon(f)$ for invertible $f$, $T'_c(f) = 0$ otherwise, and apply Lemma 7.14. The condition (ii) is satisfied, since the left-hand side of (7.7) is isomorphic to $H^*(\text{Aut}(f), \{\text{Aut}(\alpha)\}, -)$, and this is trivial by Lemma 7.15. □

In the ordinary, non-derived Mackey functor case, this is a theorem of Thevenaz [Th].

Now take $A$ arbitrary, $G$ possibly infinite, $C = O_G$. In this case, we can still use Lemma 7.14 and Lemma 7.15 to simplify the $A_\infty$-coalgebra $T^c$. Namely, any map between two finite $G$-orbits is isomorphic to the quotient map $f : [G/H] \to [G/H']$, where $H < H' \subset G$ are subgroups of finite index. Denote by $\text{Ind}(f)$ the index of $H$ in $H'$. Then one can replace $T^c_\varepsilon$ with the $C$-graded $A_\infty$-coalgebra $T^c$ defined as

$$T^c_\varepsilon(f) = \begin{cases} T^c_\varepsilon(f), & \text{Ind}(f) = p^n \text{ for some prime } p, \\ 0, & \text{otherwise}. \end{cases}$$

Again, by Lemma 7.15 (i) this will not change the category $\mathcal{DT}(C; A)$.

7.6. Cyclic group. We can go even further in the important special case $G = \mathbb{Z}$, the infinite cyclic group. For every integer $n$, choose a projective resolution $P^\bullet_n$ of the trivial $\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}]$-module $\mathbb{Z}$. For example, we can take the standard periodic resolution

$$\frac{\mathbb{Z}_{\mathbb{Z}/n\mathbb{Z}}}{{\mathbb{Z}/n\mathbb{Z}}} \xrightarrow{id-\sigma} \frac{\mathbb{Z}_{\mathbb{Z}/n\mathbb{Z}}}{{\mathbb{Z}/n\mathbb{Z}}} \xrightarrow{id-\sigma} \frac{\mathbb{Z}_{\mathbb{Z}/n\mathbb{Z}}}{{\mathbb{Z}/n\mathbb{Z}}} \xrightarrow{id-\sigma} \frac{\mathbb{Z}_{\mathbb{Z}/n\mathbb{Z}}}{{\mathbb{Z}/n\mathbb{Z}}},$$

(7.8)

where $\sigma \in \mathbb{Z}/n\mathbb{Z}$ is the generator. Let $\tilde{P}^\bullet_n$ be the cone of the augmentation map $P^\bullet_n \rightarrow \mathbb{Z}$ (that is, $\tilde{P}_0 = \mathbb{Z}$, and $\tilde{P}^l_n = P^l_{n-1}$ for $l \geq 1$). For any morphism $f : c' \rightarrow c$ in $O_G$, choose $c' = [\mathbb{Z}/n'\mathbb{Z}]$ to $c = [\mathbb{Z}/n\mathbb{Z}]$, define the complex $\tilde{T}_\varepsilon(f)$ by

$$\tilde{T}_\varepsilon(f) = \begin{cases} \mathbb{Z}_{\mathbb{Z}/n\mathbb{Z}}, & \text{Ind}(f) = 1, \\ \tilde{P}^\bullet_{n'}, & \text{Ind}(f) = \frac{n'}{n} = p \text{ is prime}, \\ 0, & \text{otherwise}. \end{cases}$$

To extend this to an $A_\infty$-coalgebra, we need to define co-multiplication maps, and the only possible non-trivial comultiplication maps are those that encode $\text{Aut}(c') \times \text{Aut}(c)$-action on $\bigoplus_{f \in O_G(c'; c)} \tilde{T}_\varepsilon(f)$, in the case Ind$(f) = p$, $n' = np$. For any $c = [\mathbb{Z}/n\mathbb{Z}] \in O_G$, we have $\text{Aut}(c) = \mathbb{Z}/n\mathbb{Z}$. More generally, for any $c' = [\mathbb{Z}/n'\mathbb{Z}]$, $c = [\mathbb{Z}/n\mathbb{Z}]$, $n'$ divisible by $n$, we have $O_G(c', c) = \mathbb{Z}/n\mathbb{Z}$, and the group $\text{Aut}(c') \times \text{Aut}(c)$ acts transitively on this set; we have

$$O_G(c', c) = (\text{Aut}(c') \times \text{Aut}(c))/\text{Aut}(c'),$$

where the embedding $\text{Aut}(c') \subset \text{Aut}(c') \times \text{Aut}(c)$ is the product of the identity map $\text{Aut}(c') \rightarrow \text{Aut}(c')$ and the natural projection $\text{Aut}(c') = \mathbb{Z}/n'\mathbb{Z} \rightarrow \text{Aut}(c) = \mathbb{Z}/n\mathbb{Z}$.

In the particular case $n' = np$, we take the given $\mathbb{Z}/n'\mathbb{Z}$-action on $\tilde{P}^\bullet_{n'}$, and induce the $\text{Aut}(c') \times \text{Aut}(c)$-action on $\bigoplus_{f \in O_G(c'; c)} \tilde{T}_\varepsilon(f)$ via the identification

$$\bigoplus_{f \in O_G(c'; c)} \tilde{T}_\varepsilon(f) = \bigoplus_{f \in O_G(c'; c)} \tilde{P}^\bullet_{n'} \cong \text{Ind}_{\text{Aut}(c') \times \text{Aut}(c)}^{\text{Aut}(c')} \tilde{P}^\bullet_{n'}.$$

All the higher maps $b_n$, $n \geq 3$ are set to be trivial, $b_n = 0$. 

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Proposition 7.17. With the notation above, the category $\mathcal{DS}_{\text{fs}}(O_G, \text{Ab})$ is equivalent to the derived category of finitely supported $\text{Ab}$-valued $A_{\infty}$-comodules over the $O_G$-graded $A_{\infty}$-coalgebra $\tilde{T}$. 

Proof. First, we construct a map of $O_G$-graded $A_{\infty}$-coalgebras $\nu: \tilde{T} \to \mathcal{T}^{O_G}$. If $\text{Ind}(f) = 1$, in other words $f$ is invertible, we have $\tilde{T}_*(f) = \mathcal{T}^{O_G}_*(f) = \mathbb{Z}$ by definition. If $\text{Ind}(f)$ is not prime, $\tilde{T}_*(f) = 0$, so that there is nothing to define. It remains to define an $\text{Aut}(c') \times \text{Aut}(c)$-equivariant map

$$\nu: \tilde{T}_*(f) \to \mathcal{T}^{O_G}_*(f)$$

for any map $f: c' \to c$ of prime index $\text{Ind}(f) = p$, $c = [\mathbb{Z}/n\mathbb{Z}], c' = \mathbb{Z}/m\mathbb{Z}$. We have $\text{Aut}(c') = \mathbb{Z}/np\mathbb{Z}$, and by adjunction, defining $\nu$ is equivalent to defining an $\mathbb{Z}/m\mathbb{Z}$-equivariant map

$$\nu: \tilde{T}_*(f) \to \mathcal{T}^{O_G}_*(f).$$

Both complexes are acyclic. In degree 0, $\tilde{T}^{np}_0 = \mathbb{Z}, \mathcal{T}^{O_G}_0(c', c) = \mathbb{Z}[O_G(c', c)]$ is a free $\mathbb{Z}/np\mathbb{Z}$-module, and we take as $\varpi$ the natural map which identifies $\mathbb{Z}$ with the subgroup of $\mathbb{Z}/np\mathbb{Z}$-invariants in $\mathcal{T}^{O_G}_0(c', c)$. In higher degrees, $\tilde{P}^{np}_{*+1}$ is a resolution of $\mathbb{Z}$, and $\mathcal{T}^{O_G}_{*+1}(c', c)$ is a resolution of $\mathcal{T}^{O_G}_0(c', c)$. But the resolution $\tilde{P}^{np}_{*+1}$ is by definition projective; therefore the given map $\nu: \tilde{P}^{np}_0 \to \mathcal{T}^{O_G}_0(c', c)$ extends to a map of resolutions $\nu: \tilde{P}^{np}_{*+1} \to \mathcal{T}^{O_G}_{*+1}(c', c)$.

To finish the proof, it remains to check the conditions of Lemma 7.14. The condition (i) is satisfied by definition. As for (ii), it is satisfied unless $\text{Ind}(f)$ is prime by virtue of Lemma 7.15 (ii). And if $\text{Ind}(f) = p$ is prime, both $\mathcal{T}^{O_G}_*(f)$ and $\tilde{T}_*(f)$ are complexes of $\mathbb{Z}[\text{Aut}(f)]$-modules adapted to the family of Lemma 7.15 (which in this case consist of the unique proper subgroup in $\text{Aut}(f) = \mathbb{Z}/p\mathbb{Z}$, namely, the unit subgroup $\{c\} \subset \text{Aut}(f)$). This also implies (ii) of Lemma 7.14. □

By Lemma 7.13, this description of the category $\mathcal{DS}_{\text{fs}}(O_G, \text{Ab})$ also yields a description of the category $\mathcal{DM}(\mathbb{Z}/n\mathbb{Z}, \text{Ab})$ for every finite cyclic group $\mathbb{Z}/n\mathbb{Z}$ — one just restricts to the orbits $[\mathbb{Z}/l\mathbb{Z}]$ with $l$ a divisor of $n$ (for a finite group, the “finite support” condition becomes vacuous). In the simplest possible case $G = \mathbb{Z}/p\mathbb{Z}$, $p$ prime, this boils down to the following.

Corollary 7.18. Let $G = \mathbb{Z}/p\mathbb{Z}$. Then the category $\mathcal{DM}(G, \text{Ab})$ is obtained by inverting quasiisomorphisms from the category of triples $(V_*, W_*, \rho)$ of a complex $W_*$ of objects in $\text{Ab}$, a complex $V_*$ of $G$-representations in $\text{Ab}$, and a map

$$\rho: W_* \to \widehat{C}^*(G, V_*),$$

where $\widehat{C}^*(G, V_*)$ is the standard 2-periodic complex which computes Tate cohomology $\check{H}^*(G, V_*)$ by means of the resolution (7.8).

Proof. Clear. □
8. Relation to Stable Homotopy

To finish the paper, we fix a finite group $G$ and explain the relation between the category of derived Mackey functors $\mathcal{DM}(G)$ and $G$-equivariant stable homotopy theory. We will only give a skeleton exposition so as to present the general principles. In particular, we will restrict our attention to finite CW spectra when giving proofs. To keep the text accessible to a person without knowledge of equivariant stable homotopy, we do recall some of the basics; for further information, the reader should consult the original references, which are [Br] for Section 8.1 and [LMS] for Section 8.2.

8.1. Spaces. The homotopy category $\text{Hom}(G)$ of $G$-spaces is defined in a straightforward way: objects are topological spaces equipped with a continuous action of the group $G$, maps between them are continuous $G$-equivariant maps, a homotopy between two maps $f, f': X \rightarrow Y$ is a continuous $G$-equivariant map $F: X \times I \rightarrow Y$ with $F = f$ on $X \times \{0\}$ and $F = f'$ on $X \times \{1\}$, where $I$ is the unit interval $[0, 1]$ with the standard topology and trivial $G$-action, morphisms in the homotopy category are homotopy classes of maps. Note that in this category, for any subgroup $H \subset G$ the functor $X \mapsto X^H$ which sends a space $X$ to the space $X^H$ of $H$-invariant points is well-defined. As usual, to get a useful theory one passes to the category of pointed spaces, and restricts one's attention to the full subcategory spanned by CW complexes. Base points are assumed to be $G$-invariant. One also needs some compatibility condition between the $G$-action and the CW structure; a convenient notion is that of a $G$-CW complex — this is a CW complex $X$ equipped with a continuous $G$-action such that for any $g \in G$, the fixed points subset $X^g \subset X$ is a subcomplex (that is, a union of cells). From now on, we will by abuse of notation denote the homotopy category of pointed $G$-CW complexes by $\text{Hom}(G)$.

For any subgroup $H \subset G$ and any $G$-CW complex $X$, the fixed points subset $X^H \subset X$ is obviously a CW complex. The geometric realization of a simplicial $G$-set is a $G$-CW complex; therefore any $G$-set which is homotopy equivalent to a $G$-equivariant CW complex is also equivalent to a $G$-CW complex. For any two pointed $G$-spaces $X, Y$, their smash product $X \wedge Y$ is also a pointed $G$-space, and if $X$ and $Y$ are $G$-CW complexes, then so is their product $X \wedge Y$.

Equivalently, the fixed points subset $X^H \subset X$ of a $G$-space $X$ can be described as follows: take the orbit $[G/H]$, treat it as a $G$-space with the discrete topology, and consider the space of continuous maps $\text{Maps}([G/H], X)$ with a natural topology on it. Then we have

$$X^H \cong \text{Maps}([G/H], X). \quad (8.1)$$

This shows that for any $G$-space $X$, the correspondence $H \mapsto X^H$ is a actually a functor from the category $O^\text{opp}_G$ opposite to the category of finite $G$-orbits to the category of topological spaces. If $X$ is a $G$-CW complex, then this is a functor from $O^\text{opp}_G$ to the category of CW complexes and cellular maps between; if the $G$-CW complex $X$ is finite, then so are all the $X^H$.

Now, for any CW complex $X$, we have a natural cellular chain complex $C_*(X, \mathbb{Z})$, and this construction is functorial with respect to cellular maps. Therefore for any
$G$-CW complex, we have a natural functor

$$[G/H] \mapsto C_*^G(X^H, \mathbb{Z})$$

from the category $O_G^{opp}$ to the category of complexes of abelian groups. Let us denote the corresponding complex of functors in $\text{Fun}(O_G^{opp}, \mathbb{Z}\text{-mod})$ by $C_*^G(X, \mathbb{Z})$. This complex was constructed and studied by Bredon [Br]; in his terminology, functors from $O_G^{opp}$ to abelian groups are called coefficient systems, so that $C_*^G(X, \mathbb{Z})$ is a complex of coefficient systems. We note that up to a quasiisomorphism, it does not depend on the particular CW model for an object $X \in \text{Hom}(G)$. The embedding of the fixed point $\text{pt} \to X$ induces an injective map $\mathbb{Z} \cong C_*^G(\text{pt}, \mathbb{Z}) \to C_*^G(X, \mathbb{Z})$; let $\tilde{C}_*^G(X, \mathbb{Z}) = C_*^G(X, \mathbb{Z}) / \mathbb{Z}$ be the quotient. Again, up to a quasiisomorphism, it only depends on the class of $X$ in $\text{Hom}(G)$.

**Definition 8.1.** The complex $\tilde{C}_*^G(X, \mathbb{Z})$ is called the naive reduced $G$-equivariant cellular chain complex of the CW complex $X$. The corresponding object $\tilde{C}_*^G(X, \mathbb{Z}) \in D(O_G^{opp}, \mathbb{Z}\text{-mod})$ is called the naive reduced $G$-equivariant homology of $X$ considered as an object in $\text{Hom}(G)$.

**Lemma 8.2.** For any two objects $X, Y \in \text{Hom}(G)$, we have a natural quasiisomorphism

$$\tilde{C}_*^G(X \wedge Y, \mathbb{Z}) \cong \tilde{C}_*^G(X, \mathbb{Z}) \otimes \tilde{C}_*^G(Y, \mathbb{Z}),$$

where the tensor product in the right-hand side is the pointwise tensor product in the category $D(O_G^{opp}, \mathbb{Z}\text{-mod})$.

**Proof.** The corresponding statement “without the group $G$” is completely standard. Since the tensor product in $D(O_G^{opp}, \mathbb{Z}\text{-mod})$ is pointwise, it suffices to show that $(X \wedge Y)^H = X^H \wedge Y^H$; this is obvious from (8.1). \qed

**8.2. Stabilization.** Recall that for the usual finite pointed CW complexes, the Spanier–Whitehead stable homotopy category $\text{SW}$ is defined as follows: object are again finite pointed CW complexes, maps from $X$ to $Y$ are given by

$$\text{Hom}(X, Y) = \lim_{\leftarrow i} [\Sigma^i X, \Sigma^i Y],$$

where $[-,-]$ means the set of homotopy classes of maps, and $\Sigma$ is the suspension functor, so that $\Sigma^i X = S^i \wedge X$, where $S^i$ is the $i$-sphere.

By definition, the suspension functor $\Sigma$ is fully faithful on the category $\text{SW}$; in fact the definition is designed exactly to achieve this. The triangulated category of spectra, denoted $\text{StHom}$, is obtained by formally inverting the suspension, thus making it not only fully faithful but an autoequivalence. The passage from $\text{SW}$ to $\text{StHom}$ involves a somewhat delicate limiting procedure due to Boardman; I do not feel qualified to re-count it here, and refer the reader to any of the standard textbooks (e.g. [Ad]) for the definition of $\text{StHom}$ and further discussion. We have a natural fully faithful embedding $\text{SW} \subset \text{StHom}$ which is compatible with the suspension and is usually denoted by $\Sigma^\infty$. The category $\text{sthom}$ of finite CW spectra is given by

$$\text{sthom} = \bigcup_i \Sigma^{-i}(\Sigma^\infty(\text{SW})) \subset \text{StHom}.$$
This is the smallest full triangulated subcategory in $\text{StHom}$ containing $\text{SW}$. Sometimes, e.g. in [BBD, 1.1.5], it is this category which is called “the stable homotopy category”. It can be equivalently described as the category of pairs $(X, n)$ of a finite CW complex $X$ and an integer $n$, with maps from $(X, n)$ to $(Y, m)$ given by

$$\lim_{l \geq \max(n, m)} [\Sigma^{l-n}X, \Sigma^{l-m}Y].$$

We should note here that the category $\text{sthom}$ is certainly not sufficient for many applications in topology, since almost all spectra representing interesting generalized homology theories are not finite CW spectra.

The cellular chain complex $\tilde{C}^q(-, \mathbb{Z})$ is compatible with suspensions, thus descends to the homology functor from $\text{sthom}$ to the derived category $\mathcal{D}(\mathbb{Z}\text{-mod})$ which we denote by $h: \text{sthom} \to \mathcal{D}(\mathbb{Z}\text{-mod})$; explicitly, $h((X, n))$ is represented by the complex $\tilde{C}^q(X)[n]$.

For $G$-spaces, one can repeat the Spanier–Whitehead construction literally; this results in a “naive $G$-equivariant Spanier–Whitehead category” $\text{SW}^\text{naive}(G)$: objects are $G$-spaces $X \in \text{Hom}(G)$, maps are given by the same formula (8.2) as in the nonequivariant case.

However, there is a more interesting and more natural option. Namely, the $i$-sphere $S^i$ is the one-point compactification of the $i$-dimensional real vector space $V$; in the equivariant world, this vector space should be allowed to carry a non-trivial $G$-action. To make sense of the direct limit in (8.2), one fixes a so-called complete $G$-universe, that is, an infinite dimensional representation $U$ of the group $G$ which has an invariant inner product and is “large enough” in the sense that every finite-dimensional inner-product representation $V$ of the group $G$ occurs in $U$ countably many times. Then the limit in (8.2) should be taken over all inner product $G$-invariant subspaces $V \subset U$:

$$[X, Y]_U = \lim_{V \subset U} [\Sigma^V X, \Sigma^V Y],$$

where $\Sigma^V$ is the $V$-suspension functor given by $\Sigma^V(X) = S^V \wedge X$, and $S^V \in \text{Hom}(G)$ is the one-point compactification of $V$, with base point at infinity. This construction works for an arbitrary compact topological group $G$. However, when $G$ is a finite group with discrete topology, there is an obvious preferred choice of the universe: we can take $U = R^{\oplus \infty}$, the sum of a countable number of copies of the regular representation $R = \mathbb{R}[G]$. This results in the following definition.

**Definition 8.3.** The “genuine” $G$-equivariant Spanier–Whitehead category $\text{SW}(G)$ has finite pointed $G$-CW complexes as objects, and maps from $X$ to $Y$ are given by

$$\text{Hom}(X, Y) = \lim_{i} [\Sigma^i R X, \Sigma^i R Y],$$

where $[-, -]$ stands for the set of morphisms in $\text{Hom}(G)$, $R = \mathbb{R}[G]$ is the regular representation of the group $G$, and $\Sigma^i R$ by abuse of notation denotes $(\Sigma^i R)^i$.
Among other representations of $G$, $R$ contains the trivial one, so that passing to the limit in Definition 8.3 includes taking the limit (8.2), and we have a natural tautological functor

$$i!: SW^\text{naive}(G) \to SW(G)$$

(8.3)

which is identical on objects. However, this is most certainly not identical on morphisms, thus not an equivalence.

To pass to the spectra, one again inverts the suspension, either in the genuine category $SW(G)$ or in the naive category $SW^\text{naive}(G)$; this results in the so-called category of naive $G$-spectra $StHom^\text{naive}(G)$ and the category of genuine $G$-spectra $StHom(G)$. Both are triangulated categories. For the precise definitions, I refer the reader to the book [LMS], the standard reference on the subject. The suspension functor $\Sigma$ is an autoequivalence on $StHom^\text{naive}(G)$, and all the suspension functors $\Sigma^V, V \subset R^{\oplus \infty}$, are autoequivalences on $StHom(G)$. We again have full embeddings $\Sigma^\infty: SW^\text{naive} \subset StHom^\text{naive}(G)$, $\Sigma^\infty: SW \subset StHom(G)$, and we can define the triangulated categories of finite $G$-spectra by

$$sthom^\text{naive}(G) = \bigcup_i \Sigma^{-i}(\Sigma^\infty(\text{SW}(G))) \subset StHom^\text{naive}(G)$$

and

$$sthom(G) = \bigcup_i \Sigma^{-iR}(\Sigma^\infty(\text{SW}(G))) \subset StHom(G),$$

where $\Sigma^{-iR}$ stands for $(\Sigma^R)^{-i}$. As in the non-equivariant case, one can equivalently describe these categories as the categories of pairs $(X, n)$, $X \in \text{Hom}(G)$, $n \in \mathbb{Z}$, where $(X, n)$ corresponds to $\Sigma^{-i}(X) \in StHom^\text{naive}(G)$ in the naive case, and to $\Sigma^{-iR}(X) \in StHom(G)$ in the genuine case.

8.3. Equivariant homology. The Bredon equivariant homology functor

$$\tilde{C}^G_*(-, \mathbb{Z}): \text{Hom}(G) \to \mathcal{D}(O_G^{op}, \text{Ab})$$

of Definition 8.1 obviously extends to the naive category $SW^\text{naive}(G)$, and then further to the category of finite spectra, so that we have a natural functor

$$sthom^\text{naive}(G) \to \mathcal{D}(O_G^{op}, \text{Ab})$$

(8.4)

which we denote by $h^G_{\text{naive}}(-)$. However, a moment’s reflection shows that it does not extend to the genuine category $sthom(G)$. This is where the Mackey functors come in.

Definition 8.4. The $G$-equivariant homology $h^G(\Sigma^\infty X, \mathbb{Z})$ of a finite pointed $G$-CW complex $X \in \text{Hom}(G)$ is the induced derived Mackey functor

$$h^G(X, \mathbb{Z}) = q^\text{opp}_!\tilde{C}^G_*(X, \mathbb{Z}) \in \mathcal{D}M(G),$$

where $q^\text{opp}_!: \mathcal{D}(O_G^{op}, \text{Z-mod}) \to \mathcal{D}S(O_G, \text{Z-mod}) = \mathcal{D}M(G)$ is the induction functor of Section 5.3.

Lemma 8.5. For any two finite pointed $G$-CW complexes $X, Y$, we have a natural isomorphism

$$h^G(X \wedge Y, \mathbb{Z}) \cong h^G(X, \mathbb{Z}) \otimes h^G(Y, \mathbb{Z}) \in \mathcal{D}M(G).$$
Proof. A combination of Lemma 8.2 and (6.16) of Lemma 6.20.

Proposition 8.6. For any finite-dimensional representation \( V \) of the group \( G \), the object
\[
h^G(S^V, \mathbb{Z}) \in \mathcal{DM}(G)
\]
is invertible in the sense of Section 7.4.

Proof. Since \( h^G(S^V, \mathbb{Z}) = q_!^{opp} \tilde{C}^G(S^V, \mathbb{Z}) \), we can use the criterion of Proposition 7.11. To check the condition (i), note that for any subgroup \( H \subset G \), we have
\[
\tilde{C}^G(S^V, \mathbb{Z})([G/H]) \cong \tilde{C}^G((S^V)^H, \mathbb{Z}),
\]
and since \( (S^V)^H = S^{V^H} \) is a sphere, its reduced homology is \( \mathbb{Z}[\dim V^H] \). To check the condition (ii), fix two subgroups \( H_1 \subset H_2 \subset G \), let \( S_i = (S^V)^{H_i} = S^{V^{H_i}} \), \( i = 1, 2 \), let \( f: [G/H_1] \to [G/H_2] \) be the quotient map, let \( f: S_2 \to S_1 \) be the corresponding cellular embedding, and let \( \tilde{W} = \text{Aut}(f) \subset \text{Aut}[G/H_1] \). Since for any proper subgroup \( W' \subset W \), we have a factorization
\[
[G/H_1] \to [G/H_1]/W' \to [G/H_2]
\]
of the map \( f \) through some \( G \)-orbit \( [G/H_1]/W' \), every proper subgroup \( W' \subset W \) appears as \( \text{Aut}(\alpha) \) in (7.6). The group \( W \) itself appears only if the map \( [G/H_1]/W \to [G/H_2] \) is not an isomorphism, or in other words, \( S_2 \subset S_1^W \) is a proper inclusion. We have to show that the relative homology object
\[
\tilde{C}^G(S^V, \mathbb{Z}) \in \mathcal{D}^b(W, \mathbb{Ab})
\]
lies in the subcategory \( \mathcal{D}_{\{\text{Aut}(\alpha)\}}(W, \mathbb{Ab}) \subset \mathcal{D}^b(W, \mathbb{Ab}) \). If \( S_2 \neq S_1^W \), \( W \) appears in the family \( \{\text{Aut}(\alpha)\} \), so that \( \mathcal{D}_{\{\text{Aut}(\alpha)\}}(W, \mathbb{Ab}) = \mathcal{D}^b(W, \mathbb{Ab}) \) and there is nothing to prove; thus we may assume \( S_2 = S_1^W \). But since by definition, the stabilizer of any non-trivial cell in the quotient \( S_1/S_1^W \) is a proper subgroup \( W \), the claim then immediately follows from Lemma 7.9. \( \square \)

Corollary 8.7. The \( G \)-equivariant homology of Definition 8.4 extends to a well-defined functor
\[
h^G(-, \mathbb{Z}): \text{sthom}(G) \to \mathcal{DM}(G).
\]

Proof. For any finite-dimensional \( G \)-module \( V \), let \( \sigma^V: \mathcal{DM}(G) \to \mathcal{DM}(G) \) be the functor given by
\[
\sigma^V(M) = M \otimes h^G(S^V, \mathbb{Z}).
\]
Then by Lemma 8.5, we have \( h^G(\Sigma^V X, \mathbb{Z}) \cong \sigma^V(h^G(X, \mathbb{Z})) \) for any finite \( G \)-CW complex \( X \in \text{Hom}(G) \), and by Proposition 8.6, \( \sigma^V \) is an autoequivalence. By definition, every object \( X' \in \text{sthom}^G \) is of the form \( X' = (\Sigma^V)^{-1}(\Sigma^\infty(X)) \) for some such \( V \) and \( X' \); the desired extension is then given by
\[
h^G(X', \mathbb{Z}) = (\sigma^V)^{-1}(h^G(X, \mathbb{Z})),
\]
and by Lemma 8.5, this does not depend on the choice of the identification \( X' = (\Sigma^V)^{-1}(\Sigma^\infty(X)) \). \( \square \)
8.4. A dictionary. Corollary 8.7 allows one to compare notions from the $G$-equivariant stable homotopy theory with those of the theory of Mackey functor. We finish the section, and indeed the whole paper, with a short dictionary saying what should correspond to what. All the material on equivariant stable homotopy is taken from [LMS]. Personally, I also find very useful the brief introduction to [HM].

First of all, the tautological functor (8.3) obviously extends to finite spectra, and by definition, we have a commutative diagram of triangulated categories and triangulated functors

$$
\begin{align*}
\sthom^{\text{naive}}(G) & \xrightarrow{i} \sthom(G) \\
\mathcal{D}(O_G^{\text{opp}}, \mathbb{Z}) & \xrightarrow{\delta^{\text{opp}}} \mathcal{DM}(G).
\end{align*}
$$

(8.5)

The smash product on the category Hom($G$) extends to the categories of finite spectra $\sthom^{\text{naive}}(G)$ and $\sthom(G)$, so that they become symmetric tensor triangulated categories. Unfortunately, the product does not combine well with the stabilization procedure of (8.2), so that this extension is very non-trivial already in the non-equivariant case, see e.g. [Ad]. Although in the last fifteen years, new and more satisfactory approaches appeared (e.g. [EKMM] or [HSS]), still, none of them can be recounted in a few pages. Nevertheless, whatever construction one uses, our equivariant homology functor $h^G$ ought to be compatible with the smash products.

**Lemma 8.8.** For any two objects $X, Y \in \sthom(G)$, we have

$$h^G(X \wedge Y) \cong h^G(X) \otimes h^G(Y).$$

**Sketch of a proof.** To give an honest proof, we would need to use an exact definition of the product of spectra, which would be beyond the scope of the paper; instead, we show how the statement is deduced from the standard properties of the smash product. Write down $X = (\Sigma^X)^{-1} Y$, $Y = (\Sigma^W)^{-1} Y'$, $x', Y' \in \text{Hom}(G)$, $V, W \subset R^{\text{finite}}$. By Lemma 8.5, we have $h^G(X') \otimes h^G(Y')$ and $h^G(V \wedge W) \cong h^G(V') \otimes h^G(W')$, so that

$$\sigma^{V \wedge W} \cong \sigma^V \circ \sigma^W \cong \sigma^W \circ \sigma^V,$$

where $\sigma^V$ is as in the proof of Corollary 8.7. Then

$$h^G(X \wedge Y) \cong h^G((\Sigma^X)^{-1} X' \wedge (\Sigma^W)^{-1} Y') \cong h^G((\Sigma^{V \wedge W})^{-1} (X' \wedge Y')) \\
\cong (\sigma^{V \wedge W})^{-1} h^G(X') \otimes h^G(Y') \cong h^G(X) \cong h^G(Y),$$

as required.

An analogous statement for $\sthom^{\text{naive}}(G)$ is also obviously true. Moreover, the functor $i_!$ of (8.5) is also tensor, and $\delta^{\text{opp}}$ is tensor by (6.16) of Lemma 6.20, so that (8.5) is actually a diagram of tensor functors.

Next, for any subgroup $H \subset G$, the fixed points functor $X \mapsto X^H$ from Hom($G$) to the category of CW complexes preserves products; since the fixed points subset of a sphere $S^V$ is also a sphere, the fixed points functor is compatible with (8.2)
and extends to the category $\text{sthom}(G)$ of finite $G$-spectra. The result is called the geometric fixed points functor and denoted by

$$\Phi^H: \text{sthom}(G) \to \text{sthom}. $$

It is also compatible with the smash product. Under our the equivariant homology functor $h^G$, it goes to the fixed points functor on the category $\mathcal{DM}(G)$:

**Lemma 8.9.** For any subgroup $H \subset G$ and any $X \in \text{sthom}(G)$, we have a quasi-isomorphism

$$h \circ \Phi^H(X) \cong \Phi^{[G/H]} \circ h^G(X)$$

(8.6)

**Proof.** By (6.16) of Lemma 6.20, we have

$$\Phi^{[G/H]}(\sigma^V(M)) \cong \Phi^{[G/H]}(M)[\dim V^H]$$

for any $V \subset R^{\otimes \infty}$ and any $M \in \mathcal{DM}(G)$. Therefore the statement is compatible with suspensions, and it suffices to prove it for $X = \Sigma^\infty(X_0)$ for some finite $G$-CW complex $X_0 \in \text{Hom}(G)$. Then it immediately follows from (6.15) of Lemma 6.20.

Consider now the full categories of spectra $\text{StHom}^{\text{naive}}(G), \text{StHom}(G)$. The tautological functor $i_!$ of (8.3) extends to all spectra, although it becomes rather non-trivial in the process. So do the smash product and the geometric fixed point functors. Just as in the case of the finite spectra, the functors $\Phi^H, H \subset G$, and $i_!$ are tensor functors. It is natural to expect that our equivariant homology functor $h^G$ also extends to $\text{StHom}(G)$.

**Conjecture 8.10.** The equivariant homology functor $h^G$ of Corollary 8.7 and the naive homology functor $h^{G}_{\text{naive}}$ of (8.4) extend to the categories of spectra, so that (8.5) extends to a commutative diagram

$$\begin{array}{ccc}
\text{StHom}^{\text{naive}}(G) & \xrightarrow{i_!} & \text{StHom}(G) \\
\downarrow h^G_{\text{naive}} & & \downarrow h^G \\
\mathcal{D}(\mathcal{O}_G^{\text{opp}}, \mathbb{Z}\text{-mod}) & \xrightarrow{q^{\text{opp}}_!} & \mathcal{DM}(G)
\end{array}$$

of tensor triangulated functors. Moreover, for any subgroup $H \subset G$, (8.6) extends to an isomorphism

$$\Phi^{[G/H]} \circ h^G(X) \cong h \circ \Phi^H(X)$$

of functors from $\text{StHom}(G)$ to $\mathcal{D}(\mathbb{Z}\text{-mod})$.

Moreover, on the level of spectra, the tautological functor $i_!$ acquires a right-adjoint $i^*: \text{StHom}(G) \to \text{StHom}^{\text{naive}}(G)$, and we can consider the diagram

$$\begin{array}{ccc}
\text{StHom}^{\text{naive}}(G) & \xleftarrow{i^*} & \text{StHom}(G) \\
\downarrow h^G_{\text{naive}} & & \downarrow h^G \\
\mathcal{D}(\mathcal{O}_G^{\text{opp}}, \mathbb{Z}\text{-mod}) & \xleftarrow{q^{\text{opp*}}} & \mathcal{DM}(G),
\end{array}$$
where $q^{opp*}$ is the restriction with respect to the embedding $q^{opp}: O^{opp}_G \to Q\Gamma_G$ — that is, the right-adjoint functor to $q^{opp}$. By base change, we have a natural map

$$h^G_{\text{naive}} \circ i^* \to q^{opp*} \circ h^G.$$  

**Conjecture 8.11.** The base change map $h^G_{\text{naive}} \circ i^* \to q^{opp*} \circ h^G$ is an isomorphism.

Using $i^*$, one can define another fixed point functor associated to a subgroup $H \subset G$. Indeed, we have an obvious fixed points functor $\text{sthom}_{\text{naive}}(G) \to \text{sthom}$, and it extends to a functor from $\text{StHom}_{\text{naive}}(G)$ to $\text{StHom}$ (isomorphic to $\Phi^H \circ i!$); composing it with $i^*$, we obtain a triangulated functor

$$\text{StHom}(G) \to \text{StHom}.$$

This functor is called the Lewis–May fixed points functor. There is no standard letter associated to it; usually it is denoted simply by $X \mapsto X^H$.

If Conjecture 8.10 and Conjecture 8.11 are known, then the following is immediate.

**Corollary 8.12.** For any $X \in \text{StHom}(G)$, we have a functorial isomorphism

$$\Psi^{[G/H]}(h^G(X)) \cong h(X^H).$$

**Proof.** By Conjecture 8.11 and Conjecture 8.10, we have

$$h(X^H) \cong h(\Phi^H (i_!i^*(X))) \cong \Phi^{[G/H]}(q^{opp}q^{opp*}h^G(X)),$$

and by Lemma 6.20, this is isomorphic to

$$q^{opp*}(h^G(X))([G/H]) \cong \Psi^{[G/H]}(h^G(X)),$$

as required. \qed

Let me conclude the paper with the following remark. As we have noted in Remark 3.9, the categories $QC(-,-)$ used in Section 3 in our definition of derived Mackey functors are naturally symmetric monoidal, so that the group completions $\Omega B|QC(-,-)$ of their classifying spaces are infinite loop spaces. Thus one can define a category $B_G$ “enriched in spectra” with objects $[G/H]$ and morphisms given by $\Omega B|Q\Omega G(-,-)$. If one does this with enough precision, one can then define the category of “enriched functors” from $B_G$ to $\text{StHom}$ in such a way that it becomes a triangulated category. As I understand, this construction is well-known in topology, and it is well-known that the resulting category is $\text{StHom}(G)$. It seems that this has not been written down (probably because the technology needed to make this precise appeared not long ago and is still rather cumbersome, and the difficulties of working out all the details outweigh the benefits of giving yet another description of a well-studied object). Be it as it may, if this sketch is assumed to work, then the comparison with the present paper becomes quite transparent: all we do is replace spectra with complexes which compute their homology, and replace “enriched functors” with $A_\infty$-functors. What I don’t know is whether our third description of $\mathcal{D}\mathcal{M}(G)$, namely the one given in Section 6, has any counterpart with spectra instead of complexes. It seems it would be very useful, since for
Mackey functors, this description is by far the most effective. On the other hand, on the level of complexes one has to work with an $A_\infty$-coalgebra $T^C$, not an $A_\infty$-algebra, and various finiteness phenomena become crucially important; it is not clear whether this can be made to work in $\text{StHom}$.

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