A SEMIGROUP OF THETA-CURVES IN 3-MANIFOLDS

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ABSTRACT. We establish an existence and uniqueness theorem for prime decompositions of theta-curves in 3-manifolds.


Key words and phrases. 3-manifold, theta-graph, prime decomposition.

1. Introduction

This paper is concerned with the existence and uniqueness of prime decompositions of theta-graphs in 3-manifolds. A theta-graph is a graph formed by two ordered vertices, called the leg and the head, and three oriented edges leading from the leg to the head and labeled with the symbols \{-, +, 0\} (different edges should have different labels.) Theta-graphs embedded in 3-manifolds are called theta-curves; their study is parallel to the study of knots (i.e., knotted circles) in 3-manifolds. More precisely, a theta-curve is a pair \((M, \Theta)\), where \(M\) is a compact connected oriented 3-manifold and \(\Theta\) is a theta-graph embedded in \(\text{Int} M\). By abuse of language, we will call \(\Theta\) a theta-curve in \(M\). For example, every 3-manifold \(M\) contains a unique (up to isotopy) flat theta-curve that lies in a 2-disc embedded into \(M\). The flat theta-curve in \(S^3\) is called the trivial theta-curve. Further examples of theta-curves can be obtained by tying knots on the edges of flat theta-curves (see below for details). The resulting theta-curves are said to be knot-like.

By homeomorphisms of theta-curves we mean homeomorphisms of pairs preserving orientation in the ambient 3-manifolds and the orientation and the labels of the edges of theta-curves. The set of homeomorphism classes of theta-curves is denoted \(\mathcal{T}\). We define a vertex multiplication in \(\mathcal{T}\), see \(\text{[Wol]}\) for the case of theta-curves in \(S^3\). Given theta-curves \((M_i, \Theta_i)\) with \(i = 1, 2\), pick regular neighborhoods \(B_1 \subset M_1\) and \(B_2 \subset M_2\) of the head of \(\Theta_1\) and the leg of \(\Theta_2\), respectively. Glue \(M_1 \setminus \text{Int} B_1\) and \(M_2 \setminus \text{Int} B_2\) along an orientation-reversing homeomorphism \(\partial B_1 \to \partial B_2\) that carries the only intersection point of \(\partial B_1\) with the \(i\)-labeled edge of \(\Theta_1\) to the intersection point of \(\partial B_2\) with the \(i\)-labeled edge of \(\Theta_2\) for \(i \in \{-, 0, +\}\). The union \(\Theta\) of \(\Theta_1 \cap (M_1 \setminus \text{Int} B_1)\) and \(\Theta_2 \cap (M_2 \setminus \text{Int} B_2)\) is a theta-curve in \(M = M_1 \neq M_2\). The

Received May 6, 2010.

The first named author was supported in part by the RFBR grant 11-01-998 and the Program of Basic Research of Mathematical Branch of RAS. The second named author was supported in part by the NSF grant DMS-0904262.

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theta-curve \((M, \Theta)\) is called the vertex product of \(\theta_1 = (M_1, \Theta_1), \theta_2 = (M_2, \Theta_2)\) and denoted \(\theta_1 \circ \theta_2\).

The vertex multiplication is associative and turns \(T\) into a semigroup. The unit of \(T\) is the trivial theta-curve. The semigroup \(T\) is non-commutative but has a big center: it follows from the definitions that all knot-like theta-curves lie in the center of \(T\). Note also that a product theta-curve \(\theta_1 \circ \theta_2\) is trivial if and only if both \(\theta_1\) and \(\theta_2\) are trivial, see [Mot], [Wol].

We call a theta-curve prime if it is non-trivial and does not expand as a product of two non-trivial theta-curves. The following theorem is the main result of this paper.

**Theorem 1.** Let \(\theta = (M, \Theta)\) be a non-trivial theta-curve such that all 2-spheres in \(M\) are separating. Then:

1. \(\theta\) expands as a product \(\theta = \theta_1 \circ \theta_2 \circ \cdots \circ \theta_n\) for a finite sequence \(\theta_1, \ldots, \theta_n\) of prime theta-curves.
2. This expansion is unique up to relations of type \(\theta' \circ \theta'' = \theta'' \circ \theta'\), where \(\theta'\) or \(\theta''\) is knot-like.

For \(M = S^3\), this theorem is due to Motohashi [Mot]. A similar theorem for knots in 3-manifolds is also true; we obtain it at the end of the paper as a corollary of Theorem 1. A straightforward generalization of Theorem 1 to graphs with two 4-valent vertices is impossible because the mapping class group of the 2-sphere with 4 ordered punctures is non-trivial.

A study of prime decompositions is a traditional area of 3-dimensional topology. We refer to [Mil] and [Sch] for prime decompositions of 3-manifolds and knots in \(S^3\), to [Miy] for prime decompositions of knots in 3-manifolds, and to [Pot], [HM] for prime decompositions of orbifolds and knotted graphs in 3-manifolds. Our interest in prime decompositions of theta-curves is due to a connection to so-called knotoids recently introduced by the second named author [Tur].

Let us explain the reasons for our assumption on the 2-spheres in Theorem 1. If \(M\) contains a non-separating 2-sphere \(S\), then there is a theta-curve \(\Theta \subset M\) meeting \(S\) transversely in one point of an edge of \(\Theta\). Using \(S\) it is easy to see that tying any local knot on this edge one obtains a theta-curve isotopic to \(\Theta\). This yields an infinite family of knot-like theta-curves that are factors of \((M, \Theta)\) and compromises the existence and uniqueness of prime decompositions of \((M, \Theta)\).

To prove Theorem 1 we follow the general scheme introduced in [HM]. This scheme has been successfully applied to prove the existence and uniqueness of prime decompositions in many similar geometric situations.

## 2. Preliminaries on Knots

**Definition 1.** A knot is a pair \((Q, K)\) where \(Q\) is a compact connected oriented 3-manifold and \(K\) is an oriented simple closed curve in \(\text{Int } Q\). Two knots \((Q, K), (Q', K')\) are equivalent, if there is a homeomorphism \((Q, K) \to (Q', K')\) preserving orientations of both \(Q\) and \(K\).
We call a knot \((Q, K)\) flat if \(K\) bounds an embedded disc in \(Q\). A knot \((Q, K)\) is trivial if \(Q = S^3\) and \(K\) is flat. We emphasize that all knots in \(Q \neq S^3\) are non-trivial. Denote by \(\mathcal{K}\) the set of all equivalence classes of knots. We equip \(\mathcal{K}\) with a binary operation \# (connected sum) as follows. Let \(K_i\) be a trivial knot. Denote by \(\tau_i\) the set of all equivalence classes of knots. We equip \(\mathcal{K}\) with a binary operation \# (connected sum) as follows. Let \(K_i = (Q_i, K_i) \in \mathcal{K}\) for \(i = 1, 2\). Choose a closed 3-ball \(B_i \subset Q_i\) such that \(l_i = B_i \cap K_i\) is an unknotted arc in \(B_i\). Let \(h: \left( B_1, l_1 \right) \to \left( B_2, l_2 \right)\) be a homeomorphism which reverses orientations of both the ball and the arc. Glue \(Q_1 \setminus \text{Int} B_1 \) and \(Q_2 \setminus \text{Int} B_2\) along \(h|_{\partial B_i} : (\partial B_1, \partial l_1) \to (\partial B_2, \partial l_2)\). The resulting knot \((Q_1 \# Q_2, K_1 \# K_2)\) does not depend on the choice of \(B_1, B_2,\) and \(h\). This knot is called the connected sum of \(k_1, k_2\) and denoted \(k_1 \# k_2\).

The operation \# is commutative, associative, and has a neutral element represented by the trivial knot. A classical argument due to Fox \cite{Fox} shows that the knot \(k = k_1 \# k_2\) is trivial if and only if both \(k_1\) and \(k_2\) are trivial. Namely, if \(k\) is trivial, then \(#_{i=1} k_i = k\) and \(k_1 \# (#_{i=2} k_i) = k_1\). Therefore \(k_1 = k\) is trivial.

3. From Knots to Theta-Curves

Given a knot \(k = (Q, K)\) and a label \(i \in \{-, 0, +\}\), we define a theta-curve in \(Q\) as follows. Pick a disc \(D \subset Q\) meeting \(K\) along an arc \(l' = D \cap K = \partial D \cap K\). The complementary arc \(l = K \setminus \text{Int} l'\) of \(K\) receives the label \(i\) and the orientation induced by that of \(K\), the arcs \(l'\) and \(l'' = \partial D \setminus l'\) receive the remaining labels. Then \(\Theta_K = l \cup l' \cup l''\) is a theta-curve in \(Q\) (cf. Figure 1). The homeomorphism class of the theta-curve \((Q, \Theta_K)\) does not depend on the choice of \(D\) because any two such disks are isotopic. This class is denoted \(\tau_i(k)\).

![Figure 1. The map \(\tau_i\)](image)

It is easy to see from the definitions that the map \(\mathcal{K} \to \mathcal{T}, k \mapsto \tau_i(k)\) is a semigroup homomorphism. This homomorphism is injective because its composition with the map \(\mathcal{T} \to \mathcal{K}\) removing the \(j\)-labeled edge (for \(j \neq i\)) is the identity.

A theta-curve is knot-like if it lies in the image of one of \(\tau_i\) for \(i \in \{-, 0, +\}\). As was mentioned above, the knot-like theta-curves commute with all theta-curves, i.e., lie in the center of \(\mathcal{T}\).

Given a knot \(k = (Q, K)\), a theta-curve \(\theta = (M, \Theta)\), and a label \(i \in \{-, 0, +\}\), we define the knot insertion of \(k\) into \(\theta\) to be the theta-curve \(\tau_i(k) \circ \theta = \theta \circ \tau_i(k)\). To construct this theta-curve geometrically, pick a 3-ball \(B \subset M\) such that \(B \cap \Theta\) is an unknotted arc in the \(i\)-labeled edge of \(\theta\). The theta-curve \(\tau_i(k) \circ \theta = \theta \circ \tau_i(k)\)
is obtained by cutting off \((B, B \cap \Theta)\) from \((M, \Theta)\) and coherent filling the resulting hole by \((Q, K)\). For \(Q = S^3\), this is the standard tying of local knots on the edges of \(\theta\).

In analogy with \(\tau_i\), we define a homomorphism \(\tau : \mathcal{M} \to \mathcal{T}\), where \(\mathcal{M}\) is the semigroup of compact connected oriented 3-manifolds with respect to connected summation. If \(M \in \mathcal{M}\), then \(\tau(M)\) is \(M\) with a flat theta-curve inside. This suggests a notion of a manifold insertion. The insertion of \(M \in \mathcal{M}\) into a theta-curve \(\theta = (Q, \Theta)\) yields the theta-curve \(\tau(M) \circ \theta = \theta \circ \tau(M)\) obtained by replacing a ball in \(Q \setminus \Theta\) by a copy of punctured \(M\). The same theta-curve can be obtained by inserting a flat knot in \(M\) into \(\theta\).

4. Prime Theta-Curves and Knots

**Lemma 1.** Let \(k = (Q, K)\) be a knot, \(i \in \{-, 0, +\}\), and \(\tau_i(k) = (Q, \Theta_K)\) the corresponding theta-curve. Let \(D\) be a disc in \(Q\) such that \(\partial D\) is the union of two edges of \(\Theta_K\) with labels distinct from \(i\). If \(S \subset Q\) is a 2-sphere meeting each edge of \(\Theta_K\) in one point, then there is a self-homeomorphism of \(Q\) which keeps \(\Theta_K\) fixed and carries \(S\) to a 2-sphere \(S'\) such that \(S' \cap D\) is a single arc.

**Proof.** The set \(S \cap D\) consists of an arc \(\alpha\) joining two points of \(S \cap \Theta_K\) and possibly of several circles. The innermost circle argument yields a disc \(A \subset S\) such that \(A \cap D = \partial A\). The circle \(\partial A\) bounds a disc \(A' \subset D\). Then the disc \((D \setminus A') \cup A\) is isotopic to a disc in \(Q\) which spans the same edges of \(\Theta_K\) and crosses \(S\) along \(\alpha\) and fewer circles. Continuing by induction, we obtain a spanning disc \(D'\) such that \(D' \cap S = \alpha\). There is a homeomorphism \(h: Q \to Q\) that keeps \(\Theta_K\) pointwise and carries \(D'\) to \(D\). Then \(S' = h(S)\) is a required sphere. \(\square\)

**Lemma 2.** A knot \(k = (Q, K)\) is prime if and only if the theta-curve \(\tau_i(k) = (Q, \Theta_K)\) is prime for some (and hence for any) \(i \in \{-, 0, +\}\). The 3-manifold \(Q\) is prime if and only if the flat theta-curve \(\tau(Q)\) is prime.

**Proof.** Recall that a knot (resp., a theta-curve) is prime if it is non-trivial and does not split as a connected sum (resp., a product) of two non-trivial knots (resp., theta-curves). Since \(\tau_i\) is an injective homomorphism, if \(\tau_i(k)\) is prime then so is \(k\). Suppose that \(\tau_i(k)\) is not prime. Then there is a sphere \(S \subset Q\) meeting each edge of \(\Theta_K\) in one point and dividing \((Q, \Theta_K)\) into two pieces \((Q_j, Q_j \cap \Theta_K), j = 1, 2\), not homeomorphic to a 3-ball with three radii. Denote by \(D\) a disc spanning the edges of \(\Theta_K\) with labels distinct from \(i\). By Lemma 1 we may assume that \(S \cap D\) is an arc dividing \(D\) into two subdiscs. We conclude that after deleting one of the edges spanned by \(D\), i.e., after returning to \(k = (Q, K)\), the pieces \((Q_j, Q_j \cap K)\) remain non-trivial, i.e., are not homeomorphic to a 3-ball with two radii. We conclude that \(k\) splits as a connected sum of two non-trivial knots.

The second claim of the lemma is obtained by applying the first claim to the flat knot \(K_0 \subset Q\). It is clear that \(Q\) is prime if and only if the knot \((Q, K_0)\) is prime. The latter holds if and only if the theta-curve \(\tau_i(K_0) = \tau(Q)\) is prime. \(\square\)
5. Spherical Reductions

Spherical reductions are operations inverse to taking products of theta-curves and inserting knots. Denote by $\mathcal{U}$ the set of all pairs $(M, G)$, where $M$ is a compact connected oriented 3-manifold and $G \subset M$ is either a theta-graph, or a knot labeled by $i \in \{-, 0, +\}$, or the empty set. The pairs are considered up to homeomorphisms preserving all orientations and labels. In other words, $\mathcal{U} = T \sqcup \mathcal{K}_- \sqcup \mathcal{K}_0 \sqcup \mathcal{K}_+ \sqcup \mathcal{M}$, where $\mathcal{K}_i$ is the set of $i$-labeled knots.

**Definition 2.** Let $(M, G) \in \mathcal{U}$. A separating sphere $S$ in $M$ is **admissible** if it is in general position with respect to $G$ and $S \cap G$ is either empty or consists of 2 or 3 points.

**Definition 3.** Given an admissible sphere $S$ in $(M, G) \in \mathcal{U}$, we cut $(M, G)$ along $S$ and add cones over two copies of $(S, S \cap G)$ on the boundaries of the resulting two pieces of $(M, G)$. This gives two pairs $(M_j, G_j) \in \mathcal{U}$, $j = 1, 2$, where the orientations and labels of the edges of $G_j$ are inherited from those of $G$. We say that these pairs are obtained by **spherical reduction** of $(M, G)$ along $S$.

The sphere $S$ as above and the reduction along $S$ are **inessential** if $S$ bounds a 3-ball $B \subset M$ such that $B \cap G$ is either empty, or a proper unknotted arc, or consists of three radii of $B$. The reduction along an inessential sphere produces a copy of $(M, G)$ and a trivial pair which is either $(S^3, \varnothing)$, or a trivial knot, or a trivial theta-curve.

6. Mediator Spheres

**Definition 4.** Let $M$ be a 3-manifold and $S_1, S_2, S_3$ three mutually transversal spheres in $M$. We call $S_3$ a sphere-mediator for $S_1, S_2$, if both numbers $\#(S_3 \cap S_1)$ and $\#(S_3 \cap S_2)$ are strictly smaller than $\#(S_2 \cap S_1)$. Here $\#$ denotes the number of circles.

**Lemma 3.** Let $(M, G) \in \mathcal{U}$ and $S_1, S_2$ be admissible essential mutually transversal spheres in $M$ such that $S_1 \cap S_2 \neq \varnothing$. If all 2-spheres in $M$ are separating, then there is an essential sphere-mediator $S_3$ for $S_1, S_2$.

**Proof.** Using an innermost circle argument, we can find two discs in $S_1$ intersecting $S_2$ solely along their boundaries. Since $S_1$ meets $G$ in $\leq 3$ points, one of the discs, $D$, meets $G$ in $\leq 1$ point. The circle $\partial D$ splits $S_2$ into two discs $D', D''$ such that $S' = D' \cup D$ and $S'' = D'' \cup D$ are embedded spheres in $M$. Since all spheres in $M$ are separating, $S'$ and $S''$ bound manifolds $W', W'' \subset M$ respectively, so that $W' \cap W'' = D$ and $\partial (W' \cup W'') = S_2$. Let $X$ be the closure of $M \setminus (W' \cup W'')$, see Figure 2.

**Case 1:** $D \cap G = \varnothing$. Since the intersection of $G$ with a separating sphere cannot consist of one point and $G$ meets $S_2$ in $\leq 3$ points, at least one of the spheres $S', S''$, say, $S'$, does not meet $G$. If $S' = \partial W'$ is essential, then pushing it slightly inside $W'$ we obtain a sphere-mediator for $S_1, S_2$. Indeed, the latter sphere is disjoint from $S_2$ and meets $S_1$ in fewer circles.

If $S'$ is inessential, then it bounds a 3-ball in $M$ disjoint from $G$. This ball is either $W'$ or $W'' \cup X$. The second option is impossible, since $W'' \cup X$ contains the
Figure 2. The spheres $S_1$ and $S_2$

esential sphere $S_2$. Hence $W'$ is a 3-ball, and we can use it to isotope $D'$ to the
other side of $S_1$ and thus transform $S_2$ into a parallel copy of $S''$ disjoint from $S_2$.
This copy of $S''$ is a sphere-mediator for $S_1, S_2$: it intersects $S_1$ in fewer circles
than $S_2$ and is essential, since $S_2$ is essential.

Case 2: $D \cap G$ is a one-point set. An argument as above shows that one of the
spheres $S', S''$, say $S'$, meets $G$ in two points. If $S'$ is essential, then after a small
isotopy it can be taken as a sphere-mediator. If $S'$ is inessential, then $W'$ is a 3-ball
and $W' \cap G$ is an unknotted arc. As above, we can use $W'$ to isotope $D'$ to the
other side of $S_1$ and thus transform $S_2$ into a sphere-mediator for $S_1, S_2$. □

7. Digression into Theory of Roots

Let $\Gamma$ be an oriented graph. The set of vertices of $\Gamma$ will be denoted $V(\Gamma)$. By
a path in $\Gamma$ from a vertex $V$ to a vertex $W$ we mean a sequence of coherently
oriented edges $V \to V_1 \to V_2 \to \ldots \to V_n \to W$, where $V_1, \ldots, V_n \in V(\Gamma)$. A vertex $W$ of $\Gamma$
is a subordinate of a vertex $V$, if either $V = W$ or there is a path from $V$ to $W$ in $\Gamma$.
A vertex $W$ is a root of $V$, if $W$ is a subordinate of $V$ and $W$ has no outgoing
edges.

We say that $\Gamma$ has property (F) if for any vertex $V \in V(\Gamma)$ there is an integer
$C \geq 0$ such that any path in $\Gamma$ starting at $V$ consists of no more than $C$ edges. It
is obvious that if $\Gamma$ has property (F) then every vertex of $\Gamma$ has a root. To study
the uniqueness of the root, we need the following notion.

Definition 5. Two edges $e$ and $d$ of $\Gamma$ are equivalent if there is a sequence of edges
$e = e_1, e_2, \ldots, e_n = d$ of $\Gamma$ with the same initial vertex such that the terminal
vertices of $e_i$ and $e_{i+1}$ have a common root for all $i = 1, \ldots, n - 1$.

We say that $\Gamma$ has property (EE) if any edges of $\Gamma$ with common initial vertex
are equivalent. The following theorem is a version of the classical Diamond Lemma
due to Newman [New].

Theorem 2 [HM]. If $\Gamma$ has properties (F) and (EE), then every vertex of $\Gamma$ has a
unique root. □
Note that in [HM] the role of the property (F) is played by the property (CF), which says that there is a map \( c: V(\Gamma) \to \{0, 1, 2, \ldots\} \) such that \( c(V) > c(W) \) for every edge \( \overrightarrow{VW} \) of \( \Gamma \). The property (F) implies (CF): an appropriate map \( c \) is defined as follows: for any vertex \( V \) of \( \Gamma \), \( c(V) \) is the maximal number of edges in a path in \( \Gamma \) starting at \( V \).

Recall from Section 5 the set \( U \) whose elements are (homeomorphism classes of) theta-curves, labeled knots, and 3-manifolds. We construct an oriented graph \( \Gamma \) as follows. A vertex of \( \Gamma \) is a finite sequence of elements of \( U \) (possibly with repetitions) considered up to the following transformations: (i) permutations that change the position of labeled knots and 3-manifolds in the sequence but keep the order of theta-curves; (ii) permutations of two consecutive terms of a sequence \( \theta', \theta'' \) allowed when both terms \( \theta', \theta'' \) are theta-curves and at least one of them is knot-like; (iii) insertion or deletion trivial theta-curves, trivial labeled knots, and copies of \( S^3 \).

We now define the edges of \( \Gamma \). Let a vertex \( V \) of \( \Gamma \) be represented by a sequence \( u_1, \ldots, u_n \in U \) and let \( i \in \{1, \ldots, n\} \). Suppose that \( (M_1, G_1), (M_2, G_2) \) are obtained from \( u_i = (M, G) \) by an essential spherical reduction along a sphere \( S \subset M \). If \( G \) is a theta-curve, we choose the numeration so that \( (M_1, G_1) \) contains the leg of \( G \) and \( (M_2, G_2) \) contains the head of \( G \). If \( G \) is a knot or an empty set, then the numeration is arbitrary. Let \( W \) be the vertex of \( \Gamma \) represented by the sequence \( u_1, \ldots, u_{i-1}, (M_1, G_1), (M_2, G_2), u_{i+1}, \ldots, u_n \). We say that \( W \) is obtained from \( V \) by essential spherical reduction along \( S \). Two vertices \( V, W \) of \( \Gamma \) are joined by an edge \( \overrightarrow{VW} \) if \( W \) can be obtained from \( V \) in this way.

**Lemma 4.** \( \Gamma \) has property (F).

**Proof.** This is a special case of Lemma 6 of [HM]. \( \square \)

**Definition 6.** For each \( u \in U \), we define a subgraph \( \Gamma_u \) of \( \Gamma \) as follows. The vertices of \( \Gamma_u \) are all vertices of \( \Gamma \) subordinate to the vertex of \( \Gamma \) represented by the 1-term sequence \( u \). The edges of \( \Gamma_u \) are all the edges of \( \Gamma \) with both endpoints in \( \Gamma_u \).

**Lemma 5.** Let \( u = (M, G) \) be a theta-curve or a labeled knot such that all 2-spheres in \( M \) are separating. Then \( \Gamma_u \) has property (EE).

**Proof.** Let \( V = (u_1, \ldots, u_n) \) be a vertex of \( \Gamma_u \). Suppose that edges \( \overrightarrow{VW_1}, \overrightarrow{VW_2} \) of \( \Gamma_u \) correspond to reductions along essential spheres \( S_1, S_2 \). These spheres lie in the ambient 3-manifolds of \( u_p, u_q \) for some \( p, q \in \{1, \ldots, n\} \). If \( p \neq q \), then \( S_1, S_2 \) survive the reduction along each other. Thus we may consider \( S_1 \) as a sphere in (a term of) \( W_2 \) and \( S_2 \) as a sphere in (a term of) \( W_1 \). Both these spheres are essential and the reductions of \( W_2 \) along \( S_1 \) and of \( W_1 \) along \( S_2 \) yield the same vertex, \( W \), of \( \Gamma_u \). Any root of \( W \) is a common root of \( W_1 \) and \( W_2 \), and therefore the edges \( \overrightarrow{VW_1}, \overrightarrow{VW_2} \) are equivalent.

It remains to consider the case where both spheres \( S_1, S_2 \) lie in the ambient 3-manifold \( M_p \) of the same term \( u_p = (M_p, G_p) \) of \( V \). Note that \( M_p \) is a submanifold of \( M \) and therefore all 2-spheres in \( M_p \) are separating. We prove the equivalence of the edges \( \overrightarrow{VW_1}, \overrightarrow{VW_2} \) by induction on the number \( m \) of circles in \( S_1 \cap S_2 \).
Base of induction. Let $m = 0$, i.e., $S_1$, $S_2$ are disjoint. Then each of these spheres survives the reduction along the other. Thus we may consider $S_1$ as a sphere in (a term of) $W_2$ and $S_2$ as a sphere in (a term of) $W_1$. Consider the vertices $W_3$, $W'_3$ of $\Gamma_u$ obtained by reducing $W_2$ along $S_1$ and $W_1$ along $S_2$. Let us prove that $W_3 = W'_3$, see the diagram on the left-hand side of Figure 3. Assume first that each sphere $S_1$, $S_2$ meets $G_p$ in three points. Then $G_p$ is a theta-curve and the reductions of $(M_p, G_p)$ along $S_1$, $S_2$ give three nontrivial theta-curves $\theta_i = (Q_i, \Theta_i)$, $1 \leq i \leq 3$, where $\Theta_1$ and $\Theta_3$ contain the leg and the head of $G_p$, respectively. It follows that both $W_3$ and $W'_3$ are obtained from $V$ by replacing the term $(M_p, G_p)$ with 3 terms $\theta_1, \theta_2, \theta_3$, see Figure 3. The other cases where at least one of the spheres $S_1$, $S_2$ meets $G_p$ in 2 or 0 points are treated similarly.

![Diagram](image_url)

**Figure 3.** (A) Reductions along disjoint spheres. (B) The ordering of $\theta_i$ is natural

We claim that any root $R$ of $W_3$ is a common root of $W_1$ and $W_2$. Indeed, if $S_1$ is essential in $W_2$, then $R$ is a root of $W_2$ by the definition of a root. If $S_1$ is inessential in $W_2$, the reduction along it results in adding to $W_2$ either $S^3$, or a trivial knot, or a trivial theta-curve. Then $W_2 = W_3$ by the definition of a vertex of $\Gamma$. Therefore $R$ is a root of $W_2$. Similarly, $R$ is a root of $W_1$. Therefore, the edges $\overrightarrow{VW_1}$, $\overrightarrow{VW_2}$ are equivalent.

**Inductive step.** Let $\#(S_1 \cap S_2) = m + 1$. It follows from Lemma 3 that there is an essential sphere-mediator $S_3$ such that it intersects $S_1$ and $S_2$ in a smaller number of circles. By the inductive assumption we know that the corresponding edge $\overrightarrow{VW_3}$ is equivalent to $\overrightarrow{VW_1}$ and $\overrightarrow{VW_2}$. It follows that $\overrightarrow{VW_1}$ and $\overrightarrow{VW_2}$ are also equivalent.

**Corollary 1.** Any vertex of $\Gamma_u$ has a unique root.

This follows from Theorem 2 and Lemmas 4 and 5.

**8. Proof of Theorem 1**

We apply to $\theta$ consecutive reductions along essential spheres meeting the corresponding theta-curves in three points. After $m \geq 1$ reductions we obtain a sequence of $m + 1$ theta-curves. Since $\Gamma$ has property (F), for some $m$ there will be no essential spheres meeting the corresponding theta-curves in three points. This means
that all the theta-curves obtained after \( m \) reductions are prime. We obtain thus a
sequence of prime theta-curves whose product is equal to \( \theta \). This proves the first
claim of the theorem.

We now prove the second claim. Consider an expansion of \( \theta \) as a product of \( n \)
prime theta-curves \( \theta_1, \ldots, \theta_n \). Let \( W \) be the sequence \( \theta_1, \ldots, \theta_n \). It may happen
that the theta-curve \( \theta_j = (Q_j, \Theta_j) \) admits an essential reduction along a sphere
\( S \subset Q_j \) meeting \( \Theta_j \) in two points. These points have to lie on the same edge \( e \) of
\( \Theta_j \) because otherwise the sphere \( S \) would be non-separating. If \( i \in \{-, 0, + \} \)
is the label of \( e \), then this spherical reduction produces a theta-curve \( \theta'_j \) and a knot
\( k_j \in \mathcal{K}_j \) such that \( \theta_j = \theta'_j \circ \tau_i(k_j) \). Since \( \theta_j \) is prime, \( \theta'_j \) is trivial. We may conclude
that \( \theta_j = \tau_i(k_j) \) is knot-like, where \( k \) is a prime knot by Lemma 2. Similarly, if
\( \theta_j = (Q_j, \Theta_j) \) admits an essential reduction along a sphere disjoint from \( \Theta_j \), then
\( \theta_j = \tau(Q) \) is also knot-like, where \( Q \) is a prime manifold.

Replacing in the sequence \( W \) all knot-like \( \theta_j \) by the corresponding knots \( k_j \), we obtain
a root of the vertex \( \theta \) of \( \Gamma \). By the uniqueness of the root (Corollary 1),
the expansion \( \theta = \prod_{i=1}^n \theta_i \) is unique up to the commutation relations of knot-like
theta-curves with all the others.

9. Corollaries

**Theorem 3.** Let \( k = (Q, K) \) be a non-trivial knot such that all 2-spheres in \( Q \) are
separating. Then \( k \) expands as a connected sum \( k = k_1 \# k_2 \# \cdots \# k_n \) of \( n \geq 1 \)
prime knots. This expansion is unique up to permutations of \( k_1, \ldots, k_n \).

**Proof.** Pick any \( i \in \{-, 0, + \} \). The claim follows from Theorem 1, the injectivity
of the semigroup homomorphism \( \tau_i : \mathcal{K} \to \mathcal{T} \), and the fact that \( k \in \mathcal{K} \) is prime if
and only if \( \tau_i(k) \) is prime (Lemma 2).

A more general version of this theorem was proved by Miyazaki [Miy].

Let \( \mathcal{U}^o = \mathcal{T}^o \cup \mathcal{K}_0^- \cup \mathcal{K}_0^+ \cup \mathcal{K}_0 \cup \mathcal{M}^o \) be a subset of \( \mathcal{U} \) consisting of the pairs
\( (M, G) \) such that all spheres in \( M \) are separating. Any element of \( \mathcal{U}^o \) expands as a product
(or connected sum) of prime elements of \( \mathcal{U}^o \). Let \( \widehat{\mathcal{T}}^o \) be the subsemigroup of
\( \mathcal{T}^o \) consisting of theta-curves having no knot-like factors. Similarly, denote by
\( \widehat{\mathcal{K}}^o \) the subsemigroup of \( \mathcal{K}^o \) consisting of knots having no 3-manifold summands.
A knot \( (Q, K) \in \mathcal{K}^o \) lies in \( \widehat{\mathcal{K}}^o \) if and only if \( Q \setminus K \) is an irreducible 3-manifold. For
\( i \in \{-, 0, + \} \) we denote by \( \widehat{\mathcal{K}}^o_i \) a copy of \( \mathcal{K}^o \) formed by \( i \)-labeled knots.

**Theorem 4.** The following holds:

1. \( \mathcal{M}^o \) and \( \widehat{\mathcal{K}}^o \) are free abelian semigroups freely generated by their prime
   elements.
2. \( \widehat{\mathcal{T}}^o \) is a free semigroup freely generated by its prime elements.
3. \( \mathcal{K}^o = \widehat{\mathcal{K}}^o \times \mathcal{M}^o \).
4. \( \mathcal{T}^o = \widehat{\mathcal{T}}^o \times C \), where \( C = \widehat{\mathcal{K}}^- \times \widehat{\mathcal{K}}^+ \times \mathcal{M}^o \) is the center of \( \mathcal{T}^o \).

**Proof.** This follows from Theorems 1 and 3.
References


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