ORBIFOLD EULER CHARACTERISTICS FOR DUAL INVERTIBLE POLYNOMIALS

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Abstract. To construct mirror symmetric Landau–Ginzburg models, P. Berglund, T. Hübsch and M. Henningson considered a pair \((f, G)\) consisting of an invertible polynomial \(f\) and an abelian group \(G\) of its symmetries together with a dual pair \((\tilde{f}, \tilde{G})\). Here we study the reduced orbifold Euler characteristics of the Milnor fibers of \(f\) and \(\tilde{f}\) with the actions of the groups \(G\) and \(\tilde{G}\) respectively and show that they coincide up to a sign.

Key words and phrases. Invertible polynomials, group actions, orbifold Euler characteristic.

Introduction

P. Berglund and T. Hübsch [4] proposed a method to construct some mirror symmetric pairs of manifolds. Their construction involves a polynomial \(f\) of a special form, a so called (non-degenerate) invertible one, and its Berglund–Hübsch transpose \(\tilde{f}\). In their paper, these polynomials appeared as potentials of Landau–Ginzburg models. The construction of [4] was generalized in [3] to orbifold Landau–Ginzburg models described by pairs \((f, G)\), where \(f\) is a (non-degenerate) invertible polynomial and \(G\) is a (finite) abelian group of symmetries of \(f\). For a pair \((f, G)\) one defines the dual pair \((\tilde{f}, \tilde{G})\). In [3], [10], there were described some symmetries between invariants of the pairs \((f, G)\) and \((\tilde{f}, \tilde{G})\) corresponding to the orbifolds defined by the equations \(f = 0\) and \(\tilde{f} = 0\) in weighted projective spaces. Moreover, in [7], [8], [14], it was observed that the singularities defined by \(f\) and \(\tilde{f}\) have some duality properties.

One can consider the Milnor fiber \(V_f\) of an invertible polynomial \(f\) with the action of a group of symmetries of \(f\). The monodromy transformation of the Milnor fiber is also induced by a symmetry transformation of \(f\).

In [7], there was defined an equivariant version of the monodromy zeta function and an equivariant version of the so called Saito duality. The latter one is an

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analogue of the Fourier transform from the Burnside ring of a finite abelian group $G$ to the Burnside ring of its group of characters $G^*$. It was shown that the equivariant monodromy zeta functions of Berglund–Hülsch dual polynomials $f$ and $\tilde{f}$ with respect to their maximal abelian symmetry groups $G_f$ and $\tilde{G}_f$ are, up to a sign, Saito dual to each other. This result and its proof inspired the considerations of this paper.

In [5], [6], there was introduced the concept of the orbifold Euler characteristic. Here we consider the notion of reduced orbifold Euler characteristic. For a pair $(f, G_f)$ as above, we consider the reduced orbifold Euler characteristic $\chi(V_f, G_f)$ of the Milnor fiber $V_f$ of $f$ with the $G$-action on it. For a non-degenerate $f$ (i.e., for $f$ with an isolated critical point at the origin), it can be regarded as an orbifold Milnor number of the pair $(f, G_f)$. We show that the reduced orbifold Euler characteristics $\chi(V_f, G_f)$ and $\chi(V_{\tilde{f}}, \tilde{G}_f)$ are equal up to a sign. This gives additional symmetries between the dual pairs $(f, G_f)$ and $(\tilde{f}, \tilde{G}_f)$.

1. Invertible Polynomials

A quasihomogeneous polynomial $f$ in $n$ variables is called invertible (see [11]) if the number of monomials in it coincides with the number of variables $n$, i.e., if it is of the form

$$f(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}}$$

for some coefficients $a_i \in \mathbb{C}^*$ and for a matrix $E = (E_{ij})$ with non-negative integer entries and with $\det E \neq 0$. Without loss of generality one may assume that $a_i = 1$ for $i = 1, \ldots, n$ (this can be achieved by a rescaling of the variables $x_j$) and that $\det E > 0$. An invertible quasihomogeneous polynomial $f$ is non-degenerate if it has (at most) an isolated critical point at the origin in $\mathbb{C}^n$.

The Berglund–Hülsch transpose $\tilde{f}$ of the invertible polynomial (1) is defined by

$$\tilde{f}(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i \prod_{j=1}^{n} x_j^{E_{ij}},$$

i.e., it corresponds to the transpose $E^T$ of the matrix $E$. If the invertible polynomial $f$ is non-degenerate, then $\tilde{f}$ is non-degenerate as well.

Definition. The (diagonal) symmetry group of the invertible polynomial $f$ is the group $G_f = \{(\lambda_1, \ldots, \lambda_n) \in (\mathbb{C}^*)^n : f(\lambda_1 x_1, \ldots, \lambda_n x_n) = f(x_1, \ldots, x_n)\}$, i.e., the group of diagonal linear transformations of $\mathbb{C}^n$ preserving $f$.

For an invertible polynomial $f(x_1, \ldots, x_n) = \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{E_{ij}}$ the symmetry group $G_f$ is finite. Its order $|G_f|$ is equal to $d = \det E$ (see [11], [8]). The (natural) monodromy transformation of $f$ is induced by an element of the symmetry group $G_f$.

For a finite abelian group $G$, let $G^* = \text{Hom}(G, \mathbb{C}^*)$ be its group of characters. (As abelian groups $G$ and $G^*$ are isomorphic, but not in a canonical way.) One can
show (see e.g. [3], [7], [8]) that the symmetry group \( G_f \) of the Berglund–Hübsch transpose \( \tilde{f} \) of an invertible polynomial \( f \) is canonically isomorphic to \( G_f \).

For a subgroup \( H \subset G \) its dual (with respect to \( G \)) \( \tilde{H} \subset G^* \) is the kernel of the natural map \( i^*: G^* \to H^* \) induced by the inclusion \( i: H \hookrightarrow G \) (see [3], [8]). One has \(|H| \cdot |\tilde{H}| = |G| \).

2. Reduced Orbifold Euler Characteristic

For a “relatively good” topological space \( X \), say, a union of cells in a finite \( CW \)-complex or a quasi-projective complex or real analytic variety, its Euler characteristic is
\[
\chi(X) = \sum_i (-1)^i \dim H^i(X; \mathbb{R}),
\]
where \( H^i(X; \mathbb{R}) \) are the cohomology groups with compact support. (This Euler characteristic is additive. For quasi-projective complex analytic varieties it coincides with the one defined through the usual cohomology groups.)

Let \( G \) be a finite group acting on \( X \). For a subgroup \( H \subset G \), let \( X^H \subset X \) be the set of fixed points of the subgroup \( H \): \( X^H = \{ x \in X : gx = x \text{ for all } g \in H \} \).

The orbifold Euler characteristic of the pair \((X, G)\) is defined by
\[
\chi(X, G) = \frac{1}{|G|} \sum_{(g, h) : gh = hg} \chi(X^{(g, h)}),
\]
where \(<g, h>\) is the subgroup of \( G \) generated by \( g \) and \( h \) (see [1], [2], [5], [6], [9], [12], [13]). The orbifold Euler characteristic is an additive function on the Grothendieck ring of quasi-projective \( G \)-varieties.

Now let \( G \) be abelian. We will use the notion of the reduced orbifold Euler characteristic of a \( G \)-space \( X \). The usual reduced (modulo a point) Euler characteristic is \( \overline{\chi}(X) = \chi(X) - 1 = \chi(X) - \chi(pt) \). In the \( G \)-equivariant setting the role of the point is played by the one point set with the trivial \( G \)-action — the unit in the Grothendieck ring of \( G \)-varieties. Its orbifold Euler characteristic is equal to \(|G| - \) the number of elements of \( G \) (for an abelian \( G \)). Therefore the reduced orbifold Euler characteristic of a \( G \)-set \( X \) will be defined as
\[
\overline{\chi}(X, G) = \chi(X, G) - |G|.
\]

**Remark.** For a germ \( F: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0) \) with an isolated critical point at the origin, its Milnor number is equal to \((-1)^{n-1} \overline{\chi}(V_F) \), where \( V_F \) is the Milnor fiber. Therefore for a non-degenerate invertible polynomial \( F \) with a group of symmetries \( G \), the reduced orbifold Euler characteristic \( \overline{\chi}(V_f, G) \) of the Milnor fiber \( V_f \) of \( f \) multiplied by \((-1)^{n-1} \) can be regarded as an orbifold Milnor number of \((f, G)\).

3. Orbifold Euler Characteristic for Invertible Polynomials

Let \( f(x_1, \ldots, x_n) \) be an invertible polynomial and let \( G \) be a subgroup of the group \( G_f \) of symmetries of \( f \). Let \( (\tilde{f}, \tilde{G}) \) be the Berglund–Henningson dual of the pair \((f, G)\), i.e., \( \tilde{G} \) is the subgroup of \( G_{\tilde{f}} = G_f^* \) dual to \( G \). Let \( V_f = \{ x \in \mathbb{C}^n : \)
\[ f(x) = 1 \] be the Milnor fiber of the polynomial \( f \). This is a complex analytic manifold with a \( G \)-action.

**Theorem 1.** One has
\[
\bar{\chi}(V_f, \tilde{G}) = (-1)^n \tilde{\chi}(V_f, G).
\]

**Proof.** For a subset \( I \subseteq I_0 = \{1, 2, \ldots, n\} \), let
\[
(C^*)^I := \{(x_1, \ldots, x_n) \in \mathbb{C}^n : x_i \neq 0 \text{ for } i \in I, x_i = 0 \text{ for } i \notin I\}
\]
be the corresponding coordinate torus. One has \( V_f = \bigsqcup_{I \subseteq I_0} V_f \cap (C^*)^I \). Each \( V_f \cap (C^*)^I \) is invariant with respect to the \( G \)-action. Therefore
\[
\bar{\chi}(V_f, G) = \sum_{I \subseteq I_0} \bar{\chi}(V_f \cap (C^*)^I, G).
\]

Let \( G_f^I \subset G_f \) and \( G_f^I \subset G_f^* = G_f^* \) be the isotropy subgroups of the actions of the symmetry groups \( G_f^I \) and \( G_f^* \) on the torus \( (C^*)^I \) respectively. (All points of the torus \( (C^*)^I \) have one and the same isotropy subgroup.)

Let \( \mathbb{Z}^n \) be the lattice of monomials in the variables \( x_1, \ldots, x_n \) \((k_1, \ldots, k_n) \in \mathbb{Z}^n \) corresponds to the monomial \( x_1^{k_1} \cdots x_n^{k_n} \) and let \( Z^I := \{(k_1, \ldots, k_n) \in \mathbb{Z}^n : k_i = 0 \text{ for } i \notin I\} \). For a polynomial \( F \) in the variables \( x_1, \ldots, x_n \), let supp \( F \subset \mathbb{Z}^n \) be the set of monomials (with non-zero coefficients) in \( F \).

One has
\[
\bar{\chi}(V_f, G) = \frac{1}{|G|} \sum_{I \subseteq I_0} \chi(V_f \cap (C^*)^I) \cdot |G_f^I \cap G|^2 - |G|
\]
\[
= \sum_{I \subseteq I_0} \frac{1}{|G|} \chi((V_f \cap (C^*)^I)/G_f) \cdot \frac{|G_f^I|}{|G_f^*|} \cdot |G_f^I \cap G|^2 - |G|. \tag{2}
\]

The coefficient \( \chi((V_f \cap (C^*)^I)/G_f) \) is not zero if and only if \( \chi(V_f \cap (C^*)^I) \neq 0 \). This is the case if and only if supp \( f \cap \mathbb{Z}^I \) consists of \(|I|\) points, i.e., if in \( f \) there are exactly \(|I|\) monomials in variables \( x_i \) with \( i \in I \).

Let \(|\text{supp } f \cap \mathbb{Z}^I| = |I| = k \), and let \( 0 < k < n \), i.e., \( I \) is a proper subset of \( I_0 \).

In [7] it was shown that in this case \( \chi((V_f \cap (C^*)^I)/G_f) = (-1)^{|I|-1} \). Remumbering the coordinates \( x_i \) and the monomials in \( f \) permits to assume that
\[
E = \begin{pmatrix} E_I & 0 \\ \ast & E_T \end{pmatrix},
\]
where \( E_I \) and \( E_T \) are square matrices of sizes \( k \times k \) and \((n-k) \times (n-k)\) respectively.

In this case \(|\text{supp } \tilde{f} \cap \mathbb{Z}^T| = n-k = |T|\). Moreover, according to [7, Lemma 1], one has \( G_f^T = \tilde{G}_f^I \subset G_f \) and therefore \(|G_f^I| \cdot |G_f^T| = |G_f^I| \cdot |G_f^T| = |G_f^I| \cdot |G_f^T|^2 \). The summand on the right hand side of the equation (2) corresponding to \( I \) is equal to
\[
\frac{(-1)^{k-1}}{|G|} \cdot \frac{|G_f^I|}{|G_f^I|} \cdot |G_f^I \cap G|^2.
\]
The subgroup $G_f^\pi \cap \tilde{G}$ of $G_f$ is dual to the subgroup $G_f^\pi + G_f$ of $G_f$. Therefore

$$|G_f^\pi \cap \tilde{G}| = \frac{|G_f||G_f^\pi \cap G|}{|G_f^\pi||G|}.$$ 

Thus the summand on the right hand side of the analogue of the equation (2) for $\chi(V_f, \tilde{G})$ corresponding to $I$ is equal to

$$\frac{(-1)^{n-k-1}}{|G|} \cdot \frac{|G_f^\pi|}{|G_f^\pi||\tilde{G}|} \cdot |G_f^\pi \cap \tilde{G}|^2 = (-1)^{n-k-1} \frac{|G_f||G_f^\pi \cap G|^2}{|G_f^\pi||G|}.$$ 

One can see that it differs from the corresponding summand in $\chi(V_f, G)$ only by the sign $(-1)^n$. The isotropy subgroup $G_I^0_f$ is trivial and therefore the summand in $\chi(V_f, G)$ corresponding to $I = I_0$ is equal to

$$\frac{(-1)^{n-1}}{|G|} \cdot |G_f| = (-1)^{n-1} \cdot |\tilde{G}|,$$

i.e., to the term in $\chi(V_f, \tilde{G})$ outside of the sum multiplied by $(-1)^n$. Analogously, the summand in $\chi(V_f, \tilde{G})$ corresponding to $I = I_0$ is equal to the term in $\chi(V_f, G)$ outside of the sum multiplied by the same factor. This proves the statement. $\Box$

References


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