TOTALLY NONFREE ACTIONS AND THE INFINITE SYMMETRIC GROUP

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To the memory of my beloved Lyotya

ABSTRACT. We consider totally nonfree (TNF) actions of groups and the corresponding adjoint invariant (AD) measures on lattices of the subgroups of the given group. The main result is the description of all adjoint-invariant and TNF measures on the lattice of subgroups of the infinite symmetric group $S_{\mathbb{N}}$. The problem is closely related to the theory of characters and factor representations of groups.

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1. INTRODUCTION

The main result of this paper is a precise description of all adjoint invariant ergodic probability Borel measures on the lattice of all subgroups of the infinite symmetric group $S_{\mathbb{N}}$. The reason why problems of this type are of importance can be briefly formulated as follows: the adjoint action of a group on the lattice of subgroups with an adjoint-invariant probability measure produces, in a certain way, a nontrivial character of the group and, consequently, determines a special factor representation of the group.

In the case of the infinite symmetric group, it turns out that this method yields *all characters of $S_{\mathbb{N}}$*. This phenomenon was first observed in [16] as a particular fact related to a certain model of factor representations of the infinite symmetric group. The list of characters itself is well-known; E. Thoma posed the problem and gave its first solution in [10]; another new proof of this theorem, which used the ergodic approach and approximation, was suggested in [15]. This proof was based on the ideas of the dynamical approach and the approximation of the characters of infinite group with the characters of finite symmetric group. The same dynamical approach, namely, the so-called grouppoid model, gives a realization of the corresponding factor-representations of type II_1. What is more important for us in this paper is that the value of an indecomposable character at a given element of the group

is equal (up to a certain factor) to the measure of the set of fixed points of this element for some special action. The most important thing is not only that the space of parameters for the AD measures and for the Thoma characters coincide, but that the representation with the character χ_α is obtained from the measure ν_α via the construction suggested in the paper; we will discuss this in more detail in another paper.

But what kind of actions can appear in this construction? This is what we define in this paper: it turns out that these are the so-called *totally nonfree* (*TNF*) actions, so it is important to describe such actions for a given group.

In this paper we go in the opposite direction: we start with definition and studying of the class of TNF actions of a group. From the point of view of ergodic theory, this kind of actions is of great interest, and, as far as we know, it has not yet been systematically investigated. Due to the lack of space, we decided to separate the discussion of the link between the questions considered here and the theory of representations, characters, and factors; these topics will be treated in another article.

In the first part of the paper (the second section) we introduce the main notions and fix definitions related to nonfree and totally nonfree actions. We develop a systematic approach to nonfree actions. Although we are mainly interested in totally nonfree actions, we also consider intermediate cases and the reduction of a general action to a TNF action. The main open problem that arises in this context concerns the *existence and the list of TNF-actions for a given group*. We use the language of the *lattice of subgroups and the adjoint action* of the original group on this lattice. The previous question is equivalent to that of the existence or nonexistence of continuous adjoint-invariant measures. The universal example of a TNF action is the adjoint action of the group on the lattice of its subgroups with a TNF measure, i.e., a measure concentrated on the set of self-normalizing subgroups (i.e., subgroups coinciding with their normalizers). An important fact asserts that this measure is a complete metric invariant of the action. In general, there are other adjoint-invariant measures that are not TNF; for example, the so-called RTNF-measures which also produced TNF action. We describe the so-called canonical skew-product of any action and sequence (which may be infinite of even transfinite) of reduced actions.

All these facts heavily depend on the properties of the group. We consider here only countable groups. Of course, for the problem to be interesting, the group should have a continuum of subgroups. It is clear that for many groups, e.g. for groups close to commutative, such a measure does not exist. But for some "large" groups we have many TNF measures (or many TNF actions) which, fortunately, can be listed up to isomorphism (in contrast to the usual situation in ergodic theory). It is natural to consider adjoint measures on the lattice of subgroups as "random subgroups"; the notion of a random subgroup was considered in [2], [1], [3] but with different kind of applications. The author believes that this question is also of interest within ergodic theory itself.

In the second part of the paper (the third section) we study the case of the infinite symmetric group. We use a fundamental classical fact about its subgroups; namely, the infinite symmetric group has no primitive subgroups except the alternating

subgroup and the whole group itself. This follows from a classical theorem due to C. Jordan (H. Wielandt was perhaps the first to observe this, see [6]). We successively exclude from consideration all other subgroups that cannot lie in the support of an adjoint-invariant measure and reduce the question to a de Finetti-like problem and to Kingman's theorem about random partitions of the naturals. The description of adjoint-invariant measures on the group $S_{\mathbb{N}}$ relies on the *new important generalization of the classical notion of Young subgroup,— namely, a signed Young subgroup; it is natural to understand a random signed Young subgroup exactly as a random subgroup whose distribution is an adjoint-invariant measure on* $L(S_{\mathbb{N}})$. The list of parameters α for these measures is exactly the list of Thoma's parameters. We briefly compare our formula with that of Thoma at the end of the paper; in a sense, our list of adjoint-invariant measures can be regarded as an explanation for the list of characters. We will return to this question elsewhere.

The direct proof of the fact that all characters of the infinite symmetric group can be obtained from TNF measures by the described construction gives us a new proof of the list of the characters of this group. The conception presented here is partially based on the paper [16], but the general approach and link to the theory of characters is new, it was announced in the authors' talk in the Henri Poincare institute [13]. A short announcement of the approach can be found in [14].

Some papers on the different topics are tightly related to our topic: the papers by R. Grigorchuk and his colleagues [7], [5] contain examples of TNF actions of groups acting on trees. Also, L. Bowen [3] found such examples of nontrivial AD-measures for the free group. The papers [2], [1] are devoted to IRS (invariant random subgroups) or AD-measures on the lattices in our terms but the goals are different.

As far as the author knows, our explicit description of all AD-measures and TNF-actions for the group $S_{\mathbb{N}}$, which we present here, is the first result of his type. Perhaps these methods can be also applied to other groups similar to $S_{\mathbb{N}}$, such as the group of rational interval exchange, $U(\infty)$ (the infinite unitary group), $GL(F_q, \infty)$, etc. It turns out that our answer is even more tightly connected to group-theoretic structure and to the theory of characters than it could be assumed before; we will apply it to the theory of characters and factor-representations of $S_{\mathbb{N}}$ and other groups in a subsequent paper.

Professors M. Abert, L. Bowen, Y. Glasner, R. Grigorchuk, Y. Guivarch, N. Gordeev, W. Knapp, T. Nagnibeda-Smirnova, G. Olshansky, M. Zischang gave me the important references on the subject.

2. Main Definitions. Totally Nonfree Actions

2.1. Fixed points, stabilizers, and sub-σ-fields. Let $(X, \mathfrak{A}, \mu)$ be a Lebesgue space with a probability measure μ defined on a σ-field $\mathfrak{A}$ of classes of mod 0 coinciding measurable sets, and let a countable group G act on this space by μ-preserving transformations. We will consider only *effective actions*, which means that only the identity $e \in G$ of the group acts as the identity transformation mod 0. Because of that we denote by the same letter elements of the group ($g \in G$) and the corresponding automorphisms ($g \colon x \mapsto gx$) of the space (X, μ).

For each element $g \in G$, we define a measurable set X_g called the *set of fixed points* of g:

$$X_g = \{x \in X, \ gx = x\}.$$

Consider the map

$$\Phi\colon G \to \mathfrak{A}; \quad \Phi\colon g \mapsto X_g.$$

Definition 1. The fixed point σ-field corresponding to the action of G under consideration is the sub-σ-field $\mathfrak{A}_G$ of the σ-field $\mathfrak{A}$ generated by the family of all sets X_g, $g \in G$.

The sets X_g are well defined for arbitrary actions of countable groups and, more generally, for *pointwise, or measurable* actions of arbitrary groups.[1] It is worth mentioning that the above definition of the σ-field $\mathfrak{A}_G$ applies to *continuous actions of arbitrary groups* since the set of fixed points for a given automorphism is well defined with respect to mod 0: if $g_1 = g_2$ mod 0 (as the automorphisms of the space (X, μ)), then $X_{g_1} = X_{g_2}$ mod 0.

Remark 1. An action of a group G is called free if $\mu X_g = 0$ for all $g \neq \mathrm{Id}$, $g \in G$, or, in short, if the σ-field $\mathfrak{A}_G$ is trivial (the trivial σ-field will be denoted by $\mathfrak{N}$).

For pointwise actions, we can define the *stationary subgroup*, or *stabilizer*, of a point $x \in X$:

$$G_x = \{g \in G \colon gx = x\}.$$

It is clear that if $y = hx$ with $y, x \in X_0$, $h \in G$, then $G_y = h^{-1} G_x h$. In general, this notion is not well defined for uncountable groups; more exactly, it can be defined only if one can introduce the notion of the orbit partition.

Now we are going to define another sub-σ-field of the σ-field $\mathfrak{A}$ in the space X. We start with the following definition.

Definition 2. Consider the partition ξ_G of the space X into the classes of points having the same stabilizer. We call it the iso-stable partition of the triple (X, G, μ).

The iso-stable partition ξ_G is measurable since it is the limit, over an increasing sequence of finite subsets $K_n \subset G$, $\bigcup_n K_n = G$, of measurable partitions $\xi_G^{K_n}$: $\xi_G = \lim_n \xi_G^{K_n}$, where two points $x, y \in X$ belong to the same block of $\xi_G^{K_n}$ if and only if $K \subset G_x$, $K \subset G_y$. The partition ξ_G is obviously G-invariant since an element of ξ_G consists of all points that have the same stabilizer.

Definition 3. Let $\mathfrak{A}^G$ be the sub-σ-field of $\mathfrak{A}$ that consists of all sets measurable with respect to the iso-stable partition ξ_G. In the quotient space X/ξ_G, we have a well-defined action of the group G with invariant quotient (projection) measure μ_{ξ_G}; the action of G on $(X/\xi_G, \mu_{\xi_G})$ will be called the reduced action.

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[1]Recall that an action of a group G is called pointwise (or measurable) if there is a measurable set of full measure on which the action of G is defined; an action is called continuous (in Rokhlin's terminology; another term is mod 0-action) if a homomorphism $G \to \mathrm{Aut}(X, \mu)$ is defined, where $\mathrm{Aut}(X, \mu)$ is the group of all classes of measure-preserving transformations of (X, μ). For countable groups, as well for locally compact groups, these two notions are equivalent.

Proposition 1. *Assume that there is a pointwise action of a group G on a space (X, μ) with an invariant measure μ. Both sub-σ-fields $\mathfrak{A}_G$ and $\mathfrak{A}^G$ are G-invariant, and the following inclusion holds:*

$$\mathfrak{A}_G \subset \mathfrak{A}^G.$$

For a countable group G, both sub-σ-fields coincide:

$$\mathfrak{A}_G = \mathfrak{A}^G \equiv \mathfrak{A}(G).$$

Proof. The first claim is trivial: two points that cannot be separated by their fixed point sets have the same stabilizer. By definition, the σ-field $\mathfrak{A}_G$ is generated by the family of sets X_g, $g \in G$. But, since the group G is countable, a basis of the σ-field $\mathfrak{A}^G$ consists of the sets

$$Y_K = \bigcap_{g \in K} X_g,$$

where $K \subset \mathrm{Stab}(x) \subset G$ is an arbitrary finite set. Thus, the family X_g, $g \in G$, generates both σ-fields in question. $\qquad \square$

For continuous groups, the sub-σ-fields in question do not coincide in general. For instance, considering the action of the orthogonal group $SO(3)$ on the projective plane $\mathbb{RP}_2$ we see that in this case $\mathfrak{A}_G \subsetneq \mathfrak{A}^G$. Indeed, each set of fixed points has zero measure, whence $\mathfrak{A}_G = \mathfrak{N}$ (where $\mathfrak{N}$ is the trivial σ-field), while $\mathfrak{A}^G = \mathfrak{A}$ since rotations separate the points of $\mathbb{RP}_2$.

2.2. The lattice of subgroups and the adjoint action. Denote by $L(G)$ the set of all subgroups of a locally compact group G and equip it with the natural weak topology[2] and the corresponding Borel structure. For a countable group, the space $L(G)$ equipped with this topology is a compact (Cantor) space. The *adjoint action* of the group G on $L(G)$ is defined as follows. Let $g \in G$, $H \in L(G)$; then

$$\mathrm{Ad}(g)H = gHg^{-1}.$$

We will study the dynamical system $(L(G), \mathrm{Ad}(G))$ from the point of view of ergodic theory; namely, we will consider $\mathrm{Ad}(G)$-invariant Borel measures. The key problem is the existence of continuous (nonatomic) invariant measures.

Problem 1. For what groups do there exist continuous $\mathrm{Ad}(G)$-invariant Borel probability measures? Describe all such measures for a given group.

We will solve this problem for the infinite symmetric group. Of course, the theory we develop here is interesting for countable groups that have uncountably many subgroups.

It is known (see [3]) that a non-Abelian free group has a lot of such measures, but one has no general description of these measures. In [7], actions of groups on trees and more general graphs were considered, and it was verified that these actions are TNF.

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[2] A neighborhood of a subgroup in the weak topology is the set of subgroups that have the same intersection with a given compact subset of G. For a discrete group, $L(G)$ is a subspace of the compact space of all subsets of G.

A natural point of view on $\mathrm{Ad}(G)$-invariant measures is to regard them as *random subgroups of G*; more precisely, each $\mathrm{Ad}(G)$-invariant measure determines a statistics on the set of subgroups, or a random subgroup. The invariance under conjugations is a natural condition for applications. One may refine this condition and consider random subgroups with additional properties (e.g., TNF measures, or $\mathrm{Ad}(G)$-invariant measures on the set of self-normalizing subgroups, see below). In a recent paper [1], a problem related to random subgroups arises for a different reason.

The lattice structure on the space of subgroups $L(G)$ is a very popular object of algebraic studies (see, e.g., [9]); we will not use it. It is worth mentioning that an important and completely open question concerns the existence of *σ-finite invariant continuous measures on $L(G)$*. As far as we know, ergodic aspects of the natural dynamical system $(L(G), \mathrm{Ad}(G), \nu)$, where ν is an $\mathrm{Ad}(G)$-invariant measure, have not been seriously studied.

Let us now connect these dynamical systems $(L(G), \mathrm{Ad}(G), \nu)$ with nonfree actions of the group G. Namely, we can identify the quotient space with respect to the iso-stable partition ξ_G with $L(G)$.

Definition 4. Given an action of a group G on a Lebesgue space (X, μ), consider the map

$$\Psi\colon X \to L(G), \quad \Psi(x) = G_x.$$

It is a measurable homomorphism of the triple (X, G, μ) to the triple $(L(G), \mathrm{Ad}(G), \Psi_*\mu)$, where $\Psi_*\mu$ is an $\mathrm{Ad}(G)$-invariant Borel measure on $L(G)$, the image of the measure μ under Ψ:

$$\Psi_*(\mu)(B) = \mu\{x\colon G_x \in B \subset L(G)\}.$$

We will call $\Psi_*\mu$ the characteristic measure of the action (X, G, μ).

From definitions it is clear that Ψ is an *isomorphism* mod 0 between the reduced action of the group G on $(X/\xi_G, \mu_{\xi_G})$ and the adjoint action on $(L(G), \Psi_*\mu)$.

Proposition 2. *The characteristic measure $\Psi_*\mu$ is a metric invariant of measure-preserving actions in the following sense*: *if two measure-preserving actions of a countable group G on spaces (X^i, μ^i), $i = 1, 2$, are metrically isomorphic, then the corresponding measures $\Psi_*\mu^i$, $i = 1, 2$, on $L(G)$ coincide.*

Proof. It suffices to observe that every isomorphism between two actions of G must send the set of points with a given stabilizer for one action to the same set for the other action. □

The map Ψ is nothing but the factorization of the space (X, μ) with respect to the iso-stable partition ξ_G, which identifies the quotient space X/ξ_G with the image $L(G)$. The quotient measure μ_{ξ_G} tautologically coincides with the characteristic measure $\Psi_*\mu$.

For a free action, Ψ is a constant map and the characteristic measure is the δ-measure at the identity subgroup $\{e\} \in L(G)$. If the action of the group is effective then $\bigcap_x G_x = \{e\}$.

2.3. Totally nonfree (TNF) actions

Definition-Theorem 1. *A measure-preserving action of a countable group G on a space (X, μ) is called totally nonfree* (TNF) *if one of the following equivalent conditions holds*:

1. *The σ-field $\mathfrak{A}_G$ $(= \mathfrak{A}^G = \mathfrak{A}(G))$ generated by all sets of fixed points coincides with the whole σ-field $\mathfrak{A}$ of all measurable subsets of X. Equivalently, the iso-stable partition ξ_G coincides* mod 0 *with the partition into separate points.*

2. *The map $\Psi\colon X \to X/\xi_G \simeq L(G)$ is a* mod 0 *isomorphism of the action of G on (X, μ) and the adjoint action on $(L(G), \Psi_*\mu)$.*

If an action is TNF, then we say that its characteristic measure is a TNF measure on $L(G)$.

The equivalence of the above two conditions directly follows from the definitions of the previous section. It is also clear that the definitions are correct with respect to changing the actions on sets of zero measure.

TNF actions are an opposite extreme to free actions.

The characteristic measure of an ergodic TNF-action is a complete metric invariant, therefore, the metric classification of TNF actions of a countable group G reduces to the calculation of the characteristic measures $\Psi_*\mu$ on the lattice $L(G)$. Thus, the classification problem for TNF actions is in a sense smooth (tame), in contrast to the general isomorphism problem in ergodic theory.

Definition 5. The *normalizer* of a subgroup $\Lambda \subset G$ is the subgroup $N(\Lambda) = \{g \in G\colon g\Lambda g^{-1} = \Lambda\}$. A subgroup $H \subset G$ for which $N(H) = H$ is called *self-normalizing*.[3] Denote the set of all self-normalizing subgroups of G by $LN(G)$.

The following claim is obvious.

Proposition 3. *A transitive action of a group G (the left action of G on a homogeneous space G/H) is totally nonfree if and only if the stabilizer (i.e., H) is a self-normalizing subgroup $(N(H) = H$, or $H \in LN(G))$.*

Indeed, any two points belong to the same orbit (for any $x, y \in X$, there exists $h \in G$ such that $y = hx$); if they have the same stabilizer G_x, then $h \in N(G_x)$, where $N(G_x) = \{h \in G\colon h^{-1}G_x h = G_x\}$ is the normalizer of G_x. Consequently, either $N(G_x) \neq G_x$ and the action is not TNF, or $N(G_x) = G_x$ and the action is TNF.

A similar situation holds for general actions.

Proposition 4. 1. *If a measure-preserving action of a group G on a space (X, μ) is TNF, then for almost all $x \in X$ the stabilizers G_x are self-normalizing: $N(G_x) = G_x$. Equivalently, $\mu\{x\colon G_x \in LN(G)\} = 1$. In other words, the characteristic measure of a TNF-action is concentrated on the set $LN(G)$ of self-normalizing subgroups.*

2. *The adjoint action of the group G on the measure space $(L(G), \nu)$ is TNF if and only if ν-almost all subgroups $H \in L(G)$ have self-normalizing normalizator: $N(N(H)) \equiv N^2(H) = N(H)$. In particular, the adjoint action of the group on the lattice $(L(G), \nu)$ with an $\mathrm{Ad}(G)$-invariant TNF-measure ν is a TNF action.*

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[3]It is more natural to call such a subgroup *abnormal*, or *anormal*.

Proof. 1. Assume that the action is TNF but at the same time there exists a measurable set A of positive μ-measure such that the stabilizer G_x of every point $x \in A$ is not self-normalizing. Then there exists a point $hx \in A$ with $h \in G$, $h \notin G_x$, such that $hx \neq x$ but $hG_xh^{-1} = G_x$; consequently, x and hx have the same stabilizer, which contradicts the TNF property.

2. For the adjoint action of the group G on $L(G)$ the stabilizer $G_H = N(H)$, so the condition $N(G_H) = G_H$ is equivalent to the condition $N(H) = N^2(H)$ for ν-almost all H; by virtue of the previous sttement we have a TNF-action. $\square$

Remark 2. 1. As we saw, the adjoint action on $L(G)$, ν could be TNF not only for TNF-measures but for ν with the property $\nu\{H \colon N^2(H) = H\} = 1$. *We will say that such measures are RTNF-measures.* In other words, TNF adjoint action takes place for measures ν which are concentrated on $LN^2(G)$: $\nu(LN^2(G)) = 1$. We will see that the infinite symmetric group provides examples of such measures.

2. The condition $\mu\{x \colon G_x \in LN(G)\} = 1$ is only necessary but not sufficient for the action to be TNF, because the stabilizers of two points x, y that belong to different orbits can be the same self-normalizing subgroup: $G_x = G_y \in LN(G)$.

Now we can specify Problem 1 formulated above and reduce the description of TNF actions of a group to the following question.

Problem 2. Given a group G, describe all ergodic continuous (nonatomic) probability TNF (correspondingly RTNF) measures on the lattice $L(G)$ of its subgroups. Or, equivalently, describe all ergodic continuous (nonatomic) Borel probability $\mathrm{Ad}(G)$-invariant measures on the subset $LN(G)$ (correspondingly on $LN^2(G)$) of the whole space $L(G)$.

We will see different answers to Problems 1 and 2 for the infinite symmetric group.

Remark that for an ergodic TNF-measure ν the zero-one law with respect to $LN(G)$ holds: either $\nu(LN(G)) = 0$ or $\nu(LN(G)) = 1$. It is interesting to characterize the TNF actions of a given group G from the point of view of ergodic theory and that of representation theory: what kind of ergodic properties can TNF actions have, what kind of factor representations can arise for TNF actions, etc.

It is interesting also to describe other classes of $\mathrm{Ad}(G)$-invariant measures depending on the property of subgroup of full measure (or to describe random subgroups of various algebraic types).

2.4. The canonical skew product, and reduced (RTNF) action. Now we consider general actions and describe the canonical reduction which leads to TNF actions. First we consider the factorization with respect to the iso-stable partition ξ_G in order to define the the *first canonical skew product structure for general actions of groups.*

Consider a measure-preserving action of a countable group G on a Lebesgue space (X, μ) and the G-invariant iso-stable partition ξ_G (see Section 1.1). The reduced action (quotient action) of G on the space $(X/\xi_G, \mu_{\xi_G})$ (see Definition 3) is isomorphic to the adjoint action of G on the space $L(G)$ equipped with the characteristic measure $\Psi_*\mu$. We regard the triple $(L(G), \Psi_*\mu, \mathrm{Ad}(G))$ as the base of the canonical skew product structure for the action (X, μ, G).

Recall that a skew product is defined if we have an action of the group on the base and a 1-cocycle on the base with values in the group of automorphisms of the typical fiber (Y, α). For a free action of G this skew product is trivial, the base consists of a single point, and the fiber is $(Y, \alpha) \equiv (X, \mu)$. In the case of a TNF action, Y is a one-point space and the base coincides with the whole space $(X, \mu) \equiv (L(G), \Psi_*\mu)$.

Definition 6. The action of the group G on the base $(X/\xi_G, \mu_{\xi_G}, G) \simeq (L(G), \Psi_*\mu, \mathrm{Ad}(G))$ will be called canonical reduced action.

The action on the space (X, μ, G) becomes a skew-product with fiber (Y, α), and we have a 1-cocycle c which is a measurable function on the product of the group and base with values in the group of measure-preserving transformations of the fiber (Y, α): $c\colon G \times L(G) \to \mathrm{Aut}\, Y$. Recall that the general condition on a cocycle c on the space $L(G)$ with the adjoint action of the group G and an arbitrary group of coefficients has the form

$$c(g_1 g_2, H) = c(g_1, H)c(g_2, g_1 H g_1^{-1}), \quad c(e, H) = \mathrm{Id},$$

where $g \in G$, $H \in L(G)$, and $c(\cdot, \cdot)$ is a measurable function on $G \times L(G)$ with values in some group. From this we can conclude that for a fixed H, the restriction of the map $g \mapsto c(g, H)$ to $g \in H$ is a homomorphism of the group H. But our cocycle that defines the canonical skew product has a stronger property.

Proposition 5. *If the action of G on (X, μ) is ergodic, then the above construction defines a decomposition of the space (X, μ) into the direct product $(X \approx L(G) \times Y; \mu_{\xi_G} \times \alpha)$, where (Y, α) is the typical fiber of the skew product; the action of G on the base is the adjoint action with the $\mathrm{Ad}(G)$-invariant measure $\Psi_*(\mu)$; the 1-cocycle $c(\cdot, \cdot)$ is a function on the space $G \times L(G)$ with values in the group $\mathrm{Aut}(Y, \nu)$ of measure-preserving automorphisms of the space (Y, ν).*

A necessary and sufficient condition for a cocycle c to define the canonical skew product is as follows.

1. If ν is a TNF measure, which means that action on the base is TNF, then

$$c(h, H) = \mathrm{Id}$$

for ν-almost all $H \in L(G)$ and $h \in H$, where Id is the $\mathrm{mod}\, 0$ identity map on the space Y; in other words, the homomorphism mentioned above is identical.

2. If ν is not a TNF measure, then, in addition to the previous condition, the following property holds: for ν-almost all $H \in L(G)$ and $h \in N(H) \setminus H$,

$$\alpha(\mathrm{Fix}_{c(h,H)}) = 0,$$

where $\mathrm{Fix}_\phi = \{y \in Y : \phi(y) = y\}$ is the set of fixed points of the automorphism ϕ.

The first condition means that for ν-almost all subgroups H, for all $h \in H$, almost all points of Y are fixed points of the automorphism $c(h, H)$. The second condition means that for ν-almost all subgroups H, for all $h \notin H$, the set of fixed points of $c(h, H)$ is, on the contrary, of zero measure. These conditions on a cocycle follow from the fact that, by definition, the fibers of the partition ξ coincide with the sets of points with a given stabilizer. We will not discuss details and similar questions. Unfortunately, it is difficult to verify whether there exists a cocycle

satisfying this condition for a given Ad(G)-invariant measure ν. At the same time, for the analysis of a given action it is important to use the canonical skew product and to study the corresponding Ad(G)-invariant measure on $L(G)$ and cocycle.

Now let us consider the action of the group G on the base of the canonical skew-product. We assume that this action is effective; otherwise we must apply the argument to the quotient of the group G by the kernel.

Definition-Theorem 2. *Let $\pi\colon X \to (X/\xi_G)$ be the canonical projection over iso-stable partition; the stabilizer of the projection of the point with respect to reduced action is normalizer of the stabilizer:*

$$\mathrm{Stab}_{\pi(x)} = N(\mathrm{Stab}_x).$$

The action of a group G on a space (X, μ) is called reduced totally nonfree (RTNF) *if the reduced action is a TNF action, or, equivalently,*

$$\mu\{x\colon N(G_x) \in LN(G)\} = 1 \ \Leftrightarrow \ \mu\{x\colon N^2(G_x) = N(G_x)\} = 1$$

(see the section above).

The construction implies that the adjoint action of an element $g \in \mathrm{Stab}_{\pi(x)}$ must preserve the stabilizer Stab_x; this means that $g \in N(\mathrm{Stab}_x)$ and vice versa. *A RTNF-measure on $L(G)$ is, by definition, the characteristic measure ν of a RTNF action*; it has the following property: $\nu\{H \in L(G)\colon N^2(H) = N(H)\} = 1$ (the second normalizer of a subgroup coincides with its first normalizer).

The following commutative diagram shows the first step of our classification:

$$
\begin{array}{ccc}
X & \xrightarrow{\ \Psi\ } & L(G) \\
{\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle N} \\
X/\xi_G & \xrightarrow{\ \Psi\ } & L(G).
\end{array}
$$

Here the map $\Psi\colon y \mapsto G_y$ sends a point y to its stabilizer, $\pi\colon X \to X/\xi_G$ is the canonical projection, and the map $N\colon H \mapsto N(H)$ sends a subgroup to its normalizer.

In general, the quotient of the canonical skew product is not a TNF action, because the stabilizer of a point of the quotient space is the normalizer of the stabilizer of the original point, $\mathrm{Stab}_{\Psi(x)} = N(\mathrm{Stab}_x)$, but different stabilizers may have the same normalizers and, consequently, we can obtain different points with the same stabilizers. Thus, we can apply again the map Ψ to the base $(X/\xi_G, \nu_\xi)$ and consider the second canonical skew product of the base, the third one, etc. This gives an hierarchy of Ad-invariant measures on $L(G)$.

Definition 7. We will denote by $AD(G)$ the space of all Ad(G)-invariant measures on the lattice $L(G)$ (they will be called simply "AD-measures"). Denote by $\mathcal{N} = N_*$ the operation on the set of measures on $L(G)$ corresponding to the *normalization of subgroups*: $[(\mathcal{N})\nu](F) = \nu(N^{-1}F)$, $F \subset L(G)$. It is clear that $\mathcal{N}$ sends AD-measures to AD-measures.

It follows from our definitions that if a measure ν on $L(G)$ is RTNF, then the measure $\mathcal{N}(\nu)$ is TNF; in particular, if ν is a TNF measure, then $\mathcal{N}(\nu) = \nu$. Thus the operation $\mathcal{N} \colon \{\text{RTNF measures}\} \to \{\text{TNF measures}\}$ is a projection.

We have a hierarchy of AD-measures on the lattice $L(G)$:

$$AD \supset \mathcal{N}(AD) \supset \mathcal{N}^2(AD) \supset \cdots \supset RTNF \supset TNF.$$

It is natural to assume that for some groups G, the chain of these normalizations as well as the chain of the steps of reductions above can be infinite or even transfinite. The most interesting classes of AD-measures are TNF, RTNF, and AD itself.

Remark that for an RTNF measure ν the measure $\mathcal{N}(\nu)$ is TNF measure, and, although ν is not TNF, the adjoint action of the group G on the $(L(G), \nu)$ is TNF. Indeed, by definition of RTNF, for ν-almost all subgroups H one has $N^2(H) = N(H)$, but $N(H)$ is the stabilizer of H, so ν-almost all stabilizers are self-normal. Moreover, the adjoint action of G on the space $(L(G), \nu)$ for a RTNF-measure ν is metrically isomorphic to the adjoint action of G on the space $(L(G), \mathcal{N}(\nu))$ and normalization $N \colon L(G) \to L(G)$ is that isomorphism of the spaces and actions.

2.5. Remarks. 1. *AD-measures.* The natural question is whether each ergodic AD-measure is characteristic measure for some ergodic action of the group G. We formulate the necessary and sufficient condition on an AD measure to be characteristic.

Proposition 6. *Suppose ν is an ergodic AD-measure on $L(G)$; then ν is characteristic measure for an ergodic action of G on a space (X, μ) if and only if there exists a probability AD-measure $\bar{\nu}$ with the properties*:

(1) *the adjoint action of G on $(L(G), \bar{\nu})$ is ergodic*;
(2) $\mathcal{N}(\bar{\nu}) = \nu$.

In this case we can define $X = L(G)$, $\mu = \bar{\nu}$.

It is not clear if such a measure $\bar{\nu}$ exists for all AD-measures ν.

2. *Fibre bundle over $L(G)$.* Each subgroup H is normal subgroup in its normalizer $N(H)$, so we have a fibre bundle over $L(G)$ in which the fibre over $H \in L(G)$ is the group $N(H)/H$. This bundle is invariant under the adjoint action of the group. We will use it for the theory of characters of the group.

3. *The TNF limit of the normalizations of AD-measures.* It is natural to assume that for every measure ν from the class of AD-measures on a given group G, there exists an AD-measure $\bigcap \nu$ which is the limit in n of the sequence of successive normalizations: $\nu \mapsto \mathcal{N}_*{}^n(\nu)$, $n = 1, 2, \ldots$. This limit (for some groups, it may be transfinite) must be a TNF measure.

4. *TNF actions for continuous groups.* As we know, the σ-fields $\mathfrak{A}_G$ and $\mathfrak{A}^G$ can be different. It is natural to define a TNF action of a general group as an action for which the σ-field $\mathfrak{A}^G$ is the complete σ-field, or for which the stabilizers separate points. In this case, we again have an isomorphism between a TNF action and the adjoint action on the lattice of subgroups.

5. *The continuous version of combinatorial multi-transitivity.* The continuous counterpart of the notion of transitivity (or topological transitivity) of actions of discrete groups is that of ergodicity. What is the analog of double transitivity? A common explanation is that this is the ergodicity of the action on the Cartesian square. But I believe that this parallel is too weak. The definition of double transitivity in combinatorics can be formulated as the transitivity of the action of the stabilizer of a point x on the space $X \setminus x$. Thus we suggest a notion of multiple transitivity which is related to our consideration as follows.

Definition 8. Assume that a countable group G acts on a standard space (X, μ) with a G-invariant continuous measure. We say that the action is metrically double transitive if for μ-almost every point $x \in X$, the action of the stabilizer $G_x \subset G$ on (X, μ) is ergodic. We say that the action is metrically k-transitive if for almost every (in the sense of the measure μ^k on X^k) choice of points $x_1, x_2, \ldots, x_{k-1}$, the action of the intersection of subgroups $\bigcap_{i=1}^{k-1} G_{x_i}$ on (X, μ) is ergodic.

It is natural to consider this definition only for TNF actions of G. It will be clear that all TNF actions of the infinite symmetric group are k-transitive for an arbitrary positive integer k.

It is of interest to find all countable groups for which TNF k-transitive actions exist for any positive integer k.

This problem is perhaps related to the class of *oligomorphic groups*, which was defined by P. Cameron [4] (a subgroup G of the group $S_\mathbb{N}$ of all finite permutations of $\mathbb{N}$ is called oligomorphic if for any positive integer k, the number of orbits of the diagonal action of G in the Cartesian product $\mathbb{N}^k$ is finite).

3. The List of Random Subgroups of the Infinite Symmetric Group

3.1. Signed partitions and signed Young subgroups of symmetric groups.
We consider the countable group $S_\mathbb{N}$, that is, the infinite symmetric group of all finite permutations of the set of positive integers $\mathbb{N}$ (or of an arbitrary countable set). In this section, we will give the list of all AD-measures on the lattice $L(S_\mathbb{N})$ of subgroups of this group and, in particular, the list of TNF measures. We will use some classical facts about permutation groups and the probabilistic approach.

The lattice $L(S_\mathbb{N})$ is very large and contains very different types of subgroups. Nevertheless, the support of an AD-measure consists of subgroups of a very special kind, the so-called signed Young groups. The topology and the Borel structure on $L(S_\mathbb{N})$ are defined as usual; this is a compact (Cantor) space.

Definition 9 (Signed partitions)**.** A signed partition η of the set $\mathbb{N}$ is a finite or countable partition $\mathbb{N} = \bigcup_{B \in \mathcal{B}} B$ of $\mathbb{N}$ together with a decomposition $\mathcal{B} = \mathcal{B}^+ \cup \mathcal{B}^- \cup \mathcal{B}^0$ of the set of its blocks, where $\mathcal{B}^0$ is the set of all single-point blocks; elements of $\mathcal{B}^+$ are called positive blocks, and elements of $\mathcal{B}^-$ are called negative blocks (thus each positive or negative block contains at least two points), and we denote by B_0 the union of all single-point blocks: $B_0 = \bigcup_{\{x\} \in \mathcal{B}^0} \{x\}$.

Denote the set of all signed partitions of $\mathbb{N}$ by SPart($\mathbb{N}$).

Recall that in the theory of finite symmetric groups, the Young subgroup Y_η corresponding to an ordinary partition $\eta = \{B_1, B_2, \ldots, B_k\}$ is $\prod_{i=1}^{k} S_{B_i}$, where S_B is the symmetric group acting on B. We will define a more general notion of a *signed Young subgroup* which makes sense both for finite and infinite symmetric groups. We will use the following notation: $S^+(B)$ is the symmetric group of all finite permutations of elements of a set $B \subset \mathbb{N}$, and $S^-(B)$ is the alternating group on B.[4]

Definition 10 (Signed Young subgroups). The signed Young subgroup Y_η corresponding to a signed partition η of $\mathbb{N}$ is

$$Y_\eta = \prod_{B \in \mathcal{B}^+} S^+(B) \times \prod_{B \in \mathcal{B}^-} S^-(B).$$

Note that on the set $B_0 \subset \mathbb{N}$, the subgroup Y_η act identically, so that the partition into the orbits of Y_η coincides with η.

It is not difficult to describe the conjugacy class of Young subgroups with respect to the group of inner automorphisms: $Y_\eta \sim Y_{\eta'}$ if and only if η and η' are equivalent up to the action of $S_\mathbb{N}$. But it is more important to consider conjugacy with respect to the group of outer automorphisms. This is the group $S^\mathbb{N}$ of all permutations of $\mathbb{N}$. Denote by $r_0^\pm$ the number of infinite positive (respectively, negative) blocks, and by $r_s^\pm$ the number of finite positive (respectively, negative) blocks of length $s > 1$. Obviously, the list of numbers $\{r_0^\pm, r_1^\pm, \ldots\}$ is a complete set of invariants of the group of outer automorphisms.

3.2. Statement of the main result. Consider a sequence of positive numbers $\alpha = \{\alpha_i\}_{i \in \mathbb{Z}}$ such that

$$\alpha_i \geqslant \alpha_{i+1} \geqslant 0 \text{ for } i > 0; \quad \alpha_{i+1} \geqslant \alpha_i \geqslant 0 \text{ for } i < 0; \quad \alpha_0 \geqslant 0; \quad \sum_{i \in \mathbb{Z}} \alpha_i = 1.$$

Consider a sequence of $\mathbb{Z}$-valued independent random variables ξ_n, $n \in \mathbb{N}$ with the distribution

$$\mathrm{Prob}\{\xi_n = v\} = \alpha_v \quad \text{for all } n \in \mathbb{N}, \ v \in \mathbb{Z}.$$

Thus we have defined a Bernoulli measure μ_α on the space of integer sequences

$$\mathbb{Z}^\mathbb{N} = \{\xi = \{\xi_n\}_{n \in \mathbb{N}} \colon \xi_n \in \mathbb{Z}\}.$$

Definition 11 (A random signed Young subgroup and the measures ν_α). Fix a sequence $\alpha = \{\alpha_i, \ i \in \mathbb{Z}\}$ and the corresponding Bernoulli measure μ_α; for each realization of the random sequence $\{\xi_n\}$, $n \in \mathbb{N}$ with the distribution μ_α, define a random signed partition $\eta(\xi)$ of $\mathbb{N}$ as follows:

$$\eta(\xi) = \{B_i \subset \mathbb{N}, \ i \in \mathbb{Z}\}, \qquad B_i := \{n \in \mathbb{N} \colon \xi_n = i\}.$$

Here, $\mathcal{B}^+ = \{B_i, \ i > 0\}; \mathcal{B}^- = \{B_i, \ i < 0\}$ and B_0 is understood as the union of one-point blocks. The correspondence $\xi \mapsto \eta(\xi)$ defines a probability measure on the set $\mathrm{SPart}(\mathbb{N})$ of signed partitions, or a random signed partition, the image of the

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[4]Traditionally, the alternating group is denoted by A_n; V. I. Arnold was very enthusiastic about the idea to denote it by S_n^- in order not to confuse it with the Lie algebra A_n; I agree with this idea.

Bernoulli measure μ_α. The correspondence $\xi \mapsto Y_{\eta(\xi)}$ defines a measure, which we denote by ν_α, on the set of signed Young subgroups, i.e., a measure on the lattice $L(S_\mathbb{N})$ of subgroups of $S_\mathbb{N}$.

Note that all nonempty blocks of the random signed partition $\eta(\xi)$ that consist of more than one point are infinite with ν_α-probability one.

Now we describe the list of all AD and TNF measures for the group $S_\mathbb{N}$.

Theorem 1. 1. *Every measure ν_α is a Borel ergodic AD-measure on the lattice $L(S_\mathbb{N})$; every ergodic probability Borel AD-measure on this lattice coincides with the measure ν_α for some α.*

2. The measure ν_α is a RTNF-measure for all α, and it is a TNF-measure if and only if $\alpha_i = 0$ for all $i \leqslant 0$. So, the adjoint action of the group $S_\mathbb{N}$ on the lattice $L(S_\mathbb{N})$ with any AD-measure is a TNF-action.

3.3. Proofs

Proof. 1. The easy part of the proof is to check that the measures ν_α are indeed ergodic AD-measures on $L(S_\mathbb{N})$. The invariance follows from the construction, because μ_α, being a Bernoulli measure, is invariant under all permutations of indices. The symmetric, alternating, and identity subgroups of the symmetric groups $S(B)$ are normal, so they are $\mathrm{Ad}(G)$-invariant. Consequently, the measure ν_α, being the image of μ_α, is $\mathrm{Ad}(G)$-invariant. The ergodicity with respect to permutations also follows from the ergodicity of the Bernoulli measure.

2. Now suppose that we have an ergodic AD-measure on $L(S_\mathbb{N})$. We will filter out, step by step, classes of subgroups of $S_\mathbb{N}$ that cannot support any AD-measure, and will finally obtain the class of signed Young groups as the only possible class. Then we will construct all AD-measures on this class.

(a) A classical result of the theory of permutation groups asserts that the group $S_\mathbb{N}$ has no primitive subgroups except the whole group $S_\mathbb{N}$ and the alternating group.[5] This is a more or less direct corollary of the fundamental estimates obtained by C. Jordan for finite symmetric groups, which were generalized by H. Wielandt [17] (see, e.g., [6, Chapter 8]). Namely, this is a corollary of Jordan's theorem asserting that if a primitive subgroup of S_n has an element with support of size k, then $n < \beta(k)$; a sharp bound on $\beta(k)$ is still unknown.

(b) Now consider an AD-measure ν on the set of imprimitive but transitive subgroups of $S_\mathbb{N}$. Assume that such a subgroup $H \in L(S_\mathbb{N})$ has an invariant partition θ. For the action of H to be transitive, all nontrivial minimal blocks of θ must have the same length $l > 1$, which must be finite (because finite permutations cannot move one infinite block to another one).[6] Denote by $\theta(H)$ the partition of $\mathbb{N}$ into the minimal blocks for H. The map $H \to \theta(H)$ assigns to any ν-almost every imprimitive subgroup a partition into blocks of length $l(H)$; because of the $\mathrm{Ad}(G)$-ergodicity of ν, this length is the same for ν-almost all subgroups H. Thus, the map $H \mapsto \theta(H)$ sends ν to a probability measure on the set of partitions of $\mathbb{N}$

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[5]A primitive subgroup is a subgroup that has no nontrivial invariant partitions.

[6]Note that the lengths of all blocks for a given imprimitive group may be either bounded (so-called "almost primitive groups") or unbounded ("totally imprimitive subgroups"), see [6], but this difference is not important for our purposes.

with countably many blocks of the same length $l > 1$, and this measure is invariant with respect to the action of $S_{\mathbb{N}}$ on the space of such partitions. Let us show that there are no such finite measures.

Lemma 1. *There are no probability measures on the space* $\mathrm{Part}(l)$ *of all partitions of* $\mathbb{N}$ *into* (*countably many*) *blocks of length* $l > 1$ *that are invariant with respect to the group* $S_{\mathbb{N}}$.

Remark. The space $\mathrm{Part}(l)$ equipped with the weak topology is locally compact but not compact; its natural compactification consists of all partitions whose blocks have length at most l.

Proof. Consider the case $l = 2$, the same argument applies to an arbitrary l. Each partition from $\mathrm{Part}(2)$ determines a symmetric matrix (a symmetric tensor if $l > 2$) $\{a_{i,j}\}$, $a_{i,j} = a_{j,i}$, $a_{i,i} = 0$, $i, j = 1, 2, \ldots$, with only one entry in each row and each column equal to 1, all the other entries being equal to 0. Because of the $S_{\mathbb{N}}$-invariance we have a random symmetric matrix $\{a_{i,j}\}$ in which the unique element in each row which is equal to 1 must be uniformly distributed along the row. This is impossible for an infinite matrix. □

(c) We have proved that an AD-measure on the lattice $L(S_{\mathbb{N}})$ takes the value 0 on the set of all transitive subgroups; so we have reduced the analysis to intransitive subgroups. Fix such a generic intransitive subgroup $H \in L(S_{\mathbb{N}})$ and consider the maximal partition $\eta(H)$ into its transitive components. The action of the group H on each such component must be primitive, because imprimitive cases can be discarded for the same reason as in the previous part of the proof. For the same reason, it is obvious that all components of this partition must be infinite. Consequently, the action of H on each component is either the action of the whole symmetric group, or that of the alternating group (see part (a)) of the proof), or that of the identity group on the single-point blocks. We denote the blocks by B_i, $i > 0$ if the action of H is the action of the symmetric group of B_i, and by B_i, $i < 0$ if the action of H is the action of the alternating group of B_i. The action of the identity group on all single-point blocks can be regarded as the identity action on the union of such blocks B_0. Thus we have a *signed partition* $\eta(H)$ such that the action of H on each block B_i, $i > 0$, is the action of $S^+(B)$, the action of H on each block B_i, $i < 0$, is the action of $S^-(B)$, and the action on B_0 is the identity action. This means that $H \subset Y_\eta = \prod_i S^{\pm}(B_i)$ and the restriction of the action of H to B_i is the action of $S^{\pm}(B)$.

For each $i \neq 0$ denote the group

$$K_i = \{g \in S^{\pm}(B_i) \colon \exists \bar{g} \in H,\ \bar{g}|_{B_j} = \mathrm{id}\ \forall j \neq i,\ \bar{g}|_{B_i} = g\};$$

this is the subgroup of the all elements in H that act as identity on all B_j, $j \neq i$.

It is clear that K_i is a normal subgroup in $S^{\pm}(B_i)$ (because it is the kernal of a homomorphism), so K_i is either $S^{\pm}(B_i)$ or $K_i = \{\mathrm{id}\}$, and

$$\prod_{i \neq 0} K_i \subset H.$$

Thus, we need to prove that $K_i = H|_{B_i} = S^{\pm}(B_i)$ for all $i \neq 0$ (and in particular $K_i \neq \{\mathrm{id}\}$ if $i \neq 0$). There are no problem with i if $K_i = S^+(B_i) = H|_{B_i}$.

We must consider two cases: the first case is $K_i = \{\mathrm{id}\}$ but $H|_{B_i} = S^{\pm}(B_i)$ (in this case it does not matter $H|_{B_i} = S^+$ or S^-, so $i \neq 0$), and the second case is $K_i = S^-(B_i) \neq H|_{B_i} = S^+(B_i)$. Let us consider the first case. Suppose that for some $i \neq 0$, $K_i = \{\mathrm{id}\}$ but $H|_{B_i} = S^{\pm}(B_i)$. Then there exists at least one $j \neq i$ for which

$$K_i = \{g \in S^{\pm}(B_i) \colon \exists \bar{g} \in H,\ \bar{g}|_{B_j} = \mathrm{id},\ \bar{g}|_{B_i} = g\}.$$

Indeed, the intersection could be either $\{\mathrm{id}\}$ or $S^{\pm}(B_i)$ for all j and if the intersection in the definition of K_i over all $j \neq i$ is $\{\mathrm{id}\} = K_i$, then such j exists. It means that for this j and for $h \neq \mathrm{id}$, $h \in H|_{B_i}$ there exist $h' \in H|_{B_j}$ and $g \in H$ such that $g|_{B_i} = h$, $g|_{B_j} = h'$. So we have a map from $H|_{B_i}$ to $H|_{B_j}$ which is a homomorphism, and consequently an isomorphism T that is simply a bijection between B_i and B_j. This bijection could be arbitrary because of the invariance under conjugation of the group. Thus, the action on of the group H on $B_i \cup B_j$ is as follows: if $n \in B_i$ and $Tn = m \in B_i$, then $gm = Tgn$ or $gT = Tg$ on $B_i \cup B_j$. If we restrict the action of H to $B_i \cup B_j$, we obtain that the group H acts periodically (or "simultaneously") on B_i and B_j.

Lemma 2. *There is no AD-invariant measure that is concentrated on an intransitive subgroup $H \subset S_{\mathbb{N}}$ of the following form*: *Suppose $\mathbb{N} = \mathbb{N}' \times K$ ($\mathbb{N}'$ is infinite), then H has the form*:

$$H = S_{\mathbb{N}'} \times \{\mathrm{id}_K\} \subset S_{\mathbb{N}}, \quad H = \{g \colon g = (g', \mathrm{id}_K);\ g' \in S_{\mathbb{N}'}\};$$

in another words the action of H on each set $\mathbb{N}' \times k$, $k \in K$, does not depend on k.

Proof. A random group H of this type must define a $S_{\mathbb{N}}$-invariant random partition of $\mathbb{N}$ onto $|K|$ parts and $S_{\mathbb{N}'}$-invariant random bijections between all parts. Invariant random partitions do exist (see paragraph (d)) but invariant bijection does not because of the absence of probability measure on the group $S_{\mathbb{N}'}$. $\square$

So we do not need to consider the subgroups H for which the first case takes place and consequently we already proved that $K_i = S^{\pm}(B_i)$, $i \neq 0$ ($S^{\pm}$ may refer to S^+ or S^-).

Suppose now that for some i one has $K_i \supset S^-(B_i)$ and $H|_{B_i} = S^+(B_i)$. Again find $j \neq 0$ for which $K_i = \{g \in S^+(B_i) \colon \exists \bar{g} \in H,\ \bar{g}|_{B_j} = \mathrm{id},\ \bar{g}|_{B_i} = g\}$. Because of the definition of K_i it is clear that $H|_{B_i \cup B_j} \supset S^-(B_i) \times S^{\pm}(B_j)$, and the last subgroup has index at most two in $H|_{B_i \cup B_j}$, but also we have $H|_{B_i} = S^+(B_i)$, so $H|_{B_i \cup B_j} = S^+(B_i) \times S^{\pm}(B_j)$. This means that $K_i = H|_{B_i} = S^+(B_i)$.

So we proved that $H = \prod_i H|_{B_i}$ and each $H|_{B_i} = S^{\pm}(B_i)$ for $i \neq 0$, or in another words we proved that H must be a signed Young subgroup: only signed Young subgroups can carry AD-invariant measures on the lattice $L(S_{\mathbb{N}})$.

The measures $\nu\alpha$ defined above are concentrated on signed Young subgroups by definition.

(d) Now we will prove that each random subgroup in the infinite symmetric group is indeed one of the measures ν_{α}.

We must identify the required measures with the ergodic limits with respect to conjugation of signed Young subgroups. Because of the correspondence between signed Young subgroups and signed partitions, this question is equivalent to the

description of $S_{\mathbb{N}}$-invariant measures on the set of signed partitions. The last question is similar to the classical de Finetti problem concerning $S_{\mathbb{N}}$-invariant measures on the space of all functions on $\mathbb{N}$ (see [11]). The only small difference lies in the fact that, in contrast to the classical situation, here we have three types of blocks of signed partitions instead of one type in the ordinary de Finetti theorem.

Lemma 3 (An analog of classical de Finetti's theorem; Kingman's theorem [8])**.** *Consider the space* $\mathrm{SPart}(\mathbb{N})$ *of signed partitions of* $\mathbb{N}$; *every ergodic* $S_{\mathbb{N}}$-*invariant measure on* $\mathrm{SPart}(\mathbb{N})$ *is determined by a sequence* $\alpha = \{\alpha_i\}_{i \in \mathbb{Z}}$ *such that* $\alpha_i \geqslant \alpha_{i+1} \geqslant 0$ *for* $i > 0$, $\alpha_{i+1} \geqslant \alpha_i \geqslant 0$ *for* $i < 0$, $\alpha_0 \geqslant 0$, *and* $\sum_{i \in \mathbb{Z}} \alpha_i = 1$, *as described above.*

Proof. The lemma can be proved by any of the methods people use to prove de Finetti's theorem. For completeness, we present a proof, applying our old ergodic method from [11]. In order to find all ergodic measures ν on a compact $S_{\mathbb{N}}$-space X using the pointwise ergodic theorem for the group $S_{\mathbb{N}}$ (which is in fact a theorem on the convergence of martingales), it suffices to find the weak limits of measures (when they do exist)

$$\lim \frac{1}{n!} \sum_{g \in S_n} \delta_{gx}$$

for all $x \in X$. More exactly, we need to calculate the limits

$$\lim_{n} \frac{1}{n!} \sum_{g \in S_n} f(gx)$$

for continuous functions $f \in C(X)$. In our case, it suffices to consider cylinder functions on $\mathrm{SPart}(\mathbb{N})$ which depend on finitely many blocks. Fix a signed partition η and label its blocks with integers in an arbitrary way so that positive (negative) integers correspond to positive (negative) blocks and B_0 is the union of one-point blocks. Consider the $\mathbb{Z}$-valued sequence x_n, $n \in \mathbb{N}$, defined as follows: $x_n = s$ if $n \in B_s$. Now we may say that the signed partition η is the partition corresponding to the sequence $\{x_n\}$, and each such sequence determines a signed partition. The action of $S_{\mathbb{N}}$ on the set of signed partitions and its action by permutations of coordinates of sequences obviously agree, so our problem reduces to the description of $S_{\mathbb{N}}$-invariant measures on the space of all elements of $\mathbb{Z}^{\mathbb{N}}$. But this is exactly de Finetti's problem. Start with an arbitrary sequence $\{x_n\} \in \mathbb{Z}^{\mathbb{N}}$ and calculate the limit

$$\lim_{n} \frac{1}{n!} \#\{g \in S_n \colon x_{gn} = v\} = \alpha_v$$

under the assumption that it does exist (it exists for almost all $x \in X$). Thus we obtain the one-dimensional distribution of the random (with respect to the measure ν) sequence x_n. In order to prove that this measure is a Bernoulli measure on $\mathbb{Z}^{\mathbb{N}}$, we must calculate the joint distribution of several coordinates of x_n. But because of the complete transitivity of the action of S_n, for any choice of $v_1, \ldots, v_t$ and for $n \gg t$ we have

$$\frac{\#\{g \in S_n \colon x_{gi} = v_i, \ i = 1, 2, \ldots, t, \ n \gg t\}}{n!} \approx \prod_{i=1}^{t} \alpha_{v_i},$$

which means that the random variables x_n, $n = 1, 2, \ldots$, are independent. Thus all AD-measures arise from Bernoulli measures on the space of signed partitions, i.e., $\nu = \nu_\alpha$ for some α. The assertion of the theorem for the unsigned partitions is Kingman's theorem ([8]), but our proof is different. $\qquad\square$

(e) Consider the random signed Young subgroup Y_η constructed from a sequence α with $\alpha_i = 0$ for all $i \leqslant 0$. Then all blocks B_i, $i \leqslant 0$, are empty with probability one. Then, obviously, the normalizer $N(Y_\eta)$ coincides with Y_η, since each block of η gives rise to the self-normalizing subgroup $S^+(B_i)$. Consequently, the measure ν_α is TNF. If $\alpha_i > 0$ for some $i \leqslant 0$, then the corresponding block B_i is not empty with probability one, whence $N(S_{B_i}^-) = S_{B_i}^+ \neq S_{B_i}^-$, so that Y_η is not self-normalizing. But

$$N(Y_\eta) = \prod_{i=-\infty}^{+\infty} S_{B_i}^+ .$$

Thus, $N^2(Y_\eta) = N(Y_\eta)$, so that ν_α is a RTNF measure in the terminology of Section 1. This completes the proof of Theorem 1. $\qquad\square$

Corollary 1. *The action of the group $S_\mathbb{N}$ on the measure space $(L(S_\mathbb{N}), \nu_\alpha)$ is ergodic.*

Indeed, this is a corollary of the fact that the measure ν_α is the image of the Bernoulli measure μ_α and the correspondence $\mu_\alpha \mapsto \nu_\alpha$ between measures commutes with the action of the group. The corresponding representation of the group $S_\mathbb{N}$ in the space $L^2_{\nu_\alpha}(L(S_\mathbb{N}))$ will be considered elsewhere.

Corollary 2. *There are three degenerate measures ν_α, in the following cases (in the parentheses we indicate the corresponding character and representation, see below):*

(1) *$\alpha_1 = 1$, $\alpha_i = 0$, $i \neq 1$; in this case, $\nu_\alpha = \delta_{S_\mathbb{N}}$ ($\chi(g) \equiv 1$, the identity representation);*

(2) *$\alpha_{-1} = 1$, $\alpha_i = 0$, $i \neq -1$; in this case, $\nu_\alpha = \delta_{S_\mathbb{N}^-}$ ($\chi(g) = (-1)^{\mathrm{sgn}(g)}$, the alternating representation);*

(3) *$\alpha_0 = 1$, $\alpha_i = 0$, $i \neq 0$; in this case, $\nu_\alpha = \delta_{\mathrm{Id}_{S_\mathbb{N}}}$ ($\chi(g) = \delta_e(g)$, the regular representation).*

An ergodic AD-measure ν_α is atomic only in these three cases (in which it is in fact a δ-measure); in all the other cases, ν_α is a continuous measure.

3.4. Remarks and a formula for the measures of the sets of fixed points. It makes sense to compare the language which we use here (the action on $L(G)$) with that used in [16] (the action on the Bernoulli sequences).

More concretely, consider the action of $S_\mathbb{N}$ on the space $\mathbb{Z}^\mathbb{N}$ (instead of $L(S_\mathbb{N})$) and ask for a description of TNF and RTNF measures. The answer differs a little from that for the action on the space of Young subgroups. Namely, the following result holds.

Proposition 7. *The measure μ_α on the space $X = \mathbb{Z}^\mathbb{N}$ with the action of the group $S_\mathbb{N}$ is a TNF measure if and only if all α_i, $i \neq 0$, are distinct. If $\alpha_i = \alpha_j$ for some $i \neq j$, then the action of $S_\mathbb{N}$ is RTNF but not TNF. The canonical projection $X \to X/\xi_G$ is the factorization with respect to the following equivalence relation on*

$X = \mathbb{Z}^{\mathbb{N}}$: *two elements* $\{x_n\}_{n\in\mathbb{N}}$, $\{x'_n\}_{n\in\mathbb{N}} \in X$ *are equivalent if for every* $v \in \mathbb{Z}$ *either* $\{n \in \mathbb{N}: x_n = v\} = \{n \in \mathbb{N}: x'_n = v\}$, *or there exists* $v' \in \mathbb{Z}$ *with* $\alpha_v = \alpha_{v'}$ *such that*

$$\{n \in \mathbb{N}: x_n = v\} = \{n \in \mathbb{N}: x'_n = v'\},$$

and

$$\{n \in \mathbb{N}: x_n = v'\} = \{n \in \mathbb{N}: x'_n = v\}.$$

Thus, in this case the action is RTNF but not TNF if we have multiplicity in the values of α: $\alpha_i = \alpha_j$, $i \neq j$; this is not the case for the action in $L(S_{\mathbb{N}})$.

The supports of the measures ν_α in the topological sense (i.e., the minimal closed subsets of full measure) coincide for all α that have the same number of infinite blocks. The support of ν_α for α having infinitely many infinite blocks coincides with the space of all signed Young subgroups.

In the case of the infinite symmetric group, all AD-measures are invariant under the group $S^{\mathbb{N}}$ of all permutations of $\mathbb{N}$. Conjugation with respect to this group is an extension of the usual conjugation; however, for a generic subgroup H from a set A of full ν_α-measure, its orbit under the action of $S^{\mathbb{N}}$ is much larger than A. In other words, the frequencies α_i are invariant under the usual conjugation, but not under its extension. This fact is related to the so-called Kolmogorov effect (see [12]).

Theorem 1 gives more than just the list of AD-measures on the group $S_{\mathbb{N}}$; it helps to give a new proof of Thoma's formula for indecomposable characters of this group. This will be the subject of our next article, and now we merely carry out the calculations and give a short commentary. Here we present a formula for characters in the "positive" case.[7]

Theorem 2. *For an ergodic AD-measure* ν_α,

$$\nu_\alpha(F_g) \equiv \nu_\alpha\{H: gHg^{-1} = H\} = \nu_\alpha\{H: g \in N(H)\} = \prod_{n>1} [p_n(\alpha)]^{c_n(g)},$$

where

$$p_n(\alpha) = \sum_{i \neq 0} \alpha_i^n$$

(Newton's power sum) and $c_n(g)$ *is the number of cycles of length* $n > 1$ *of a permutation* g.

In the case where $\alpha_i = 0$ for $i < 0$, this formula coincides with Thoma's formula [15] for characters of the infinite symmetric group since the measure of the set of fixed points is equal to the value of the character:

$$\chi_\alpha(g) = \nu_\alpha(\text{Fix}(g)).$$

In the general case, Thoma's formula involves super-Newton instead of Newton sums:

$$p_n(\alpha) = \sum_{i>0} \alpha_i^n + (-1)^{n-1} \sum_{i<0} \alpha_i^n = \sum_{i\neq 0} (\text{sgn}\, i)^{n-1} \alpha_i^n.$$

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[7]For convenience, we have sightly changed the notation: usually, $\alpha_i \equiv \beta_i$ for $i < 0$, and $\alpha_0 \equiv \gamma$.

The measure of the set of fixed points does not depend on the types of blocks; however, for a general parameter α the value of the character is not equal just to the measure of this set, the formula involving a certain multiplier (see [16]). We will return to this question and give a model of representations in the next article.

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