RATIONAL TANGLES AND THE MODULAR GROUP

FRANCESCA AICARDI

Dedicated to V.I. Arnold, with endless gratitude

Abstract. There is a natural way to define an isomorphism between the group of transformations of isotopy classes of rational tangles and the modular group. This isomorphism allows to give a simple proof of the Conway theorem, stating the one-to-one correspondence between isotopy classes of rational tangles and rational numbers. Two other simple ways to define this isomorphism, one of which suggested by Arnold, are also shown.


Key words and phrases. Tangles, rational tangles, modular group, continued fractions, braids group, spherical braids group.

Introduction

A tangle with four ends (here called simply tangle) is an embedding of two closed segments in a ball such that their endpoints are four distinct points of the bounding sphere and the image of the interior of the segments lie at the interior of the ball.

Figure 1. Two tangles

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Two tangles are said *isotopic* if one can be deformed continuously to the other in the set of tangles with fixed endpoints.

A tangle is said *rational* if can be deformed continuously, in the set of tangles with non-fixed endpoints, to a tangle consisting of two unlinked and unknotted segments.

**Examples.** The tangles in Figure 1 are not rational, since the white strand of the left tangle is knotted, and the two strands of the tangle at right cannot be unlinked if we allow the endpoints of the strands to move on the bounding sphere. See Figure 2 for an example of rational tangle.

The main result on rational tangles is a theorem by Conway (announced in [2]) stating that it is possible to associate with every rational tangle one and only one rational number, so that two rational tangles are isotopy equivalent if and only if they are represented by the same rational number.

The idea of the elementary and self-consistent proof of this theorem given in [3] is that any rational tangle is isotopy equivalent to exactly one canonical rational tangle (alternating). To such a tangle one associates exactly one continued fraction, and hence a rational number; conversely, from a rational number, via its continued fraction, the canonical representative of exactly one class of rational tangles is constructed.

There exist other proofs of this theorem in literature, see for instance another combinatorial proof in [4] and topological proofs in [1] and [5]. See [4] also for a wide list of references.

We show here that the Conway theorem can be obtained in a simple way by considering the group of transformations of isotopy classes of rational tangles, called $\mathcal{T}$, (see Section 2), which is isomorphic to the modular group.

In fact, this isomorphism is obtained by associating to two basic moves (and their inverses) changing the isotopy classes of rational tangles, two elements (and their inverses) which generate the modular group (see Section 2).

Every rational tangle, in its standard representation, can be transformed into another rational tangle by a (not unique) series of basic moves. This allows to associate constructively to every transformation between rational tangles in standard representation an element of the modular group, namely the element obtained as product of the generators corresponding to the basic moves in exactly the same order they were applied. Similarly, to every word in the two generators and their inverses there correspond exactly one sequence of basic moves.

Observe that there are infinitely many ways to transform, via basic moves, a given rational tangle into another given rational tangle. Because of the isomorphism, each series of moves realizing the transformation gives the same element of the modular group, namely the element obtained as product of the generators corresponding to the basic moves in exactly the same order they were applied. For instance, this element will be the identity if and only if the rational tangles are isotopic.

The set of isotopy classes of rational tangles is thus obtained as the orbit of $\mathcal{T}$ on a basic rational tangle, namely a 'trivial' tangle (with untwisted strands). In Section 3 we prove that every isotopy class of rational tangle is represented by an
element of \( T \), defined up to multiplication on the right by a power of one generator, and vice versa. From this fact the Conway theorem follows, since there is a way to associate to an element of \( T \), modulo such a multiplication on the right, exactly one rational number and vice-versa.

The proposition that every rational tangle is isotopy equivalent to an alternating rational tangle is here obtained as a corollary of the isomorphism theorem, using the fact that every element of the modular group can be essentially written as product of only positive powers (or only negative powers) of the two basic elements we have considered (Section 4). In this section, again using the isomorphism theorem and the properties of continued fractions, we obtain how to associate with a rational number a unique alternating rational tangle and vice versa.

The isomorphism we have introduced (via the basic elements generating the modular group) is helpful to prove the Conway theorem and other properties of rational tangles. We show however (Section 5) that two non-constructive proofs of this isomorphisms can be obtained in two other simple ways, both involving the spherical braid group with four strands. The second of such proofs was indicated by V.I. Arnold.

We would like to underline the fact that this paper is based entirely on elementary facts. We have provided our statements with simple proofs, without claiming that the same statements are new (except for the main theorem on the isomorphism, that is perhaps — and in that case surprisingly — new).

1. Representations of Rational Tangles

Let our sphere be centered at the origin of the three space with Cartesian coordinates \( X \), \( Y \) and \( Z \), and the endpoints of the strands lie in the plane \( Z = 0 \).

We depict a tangle projected to the \( XY \)-plane, which contains the endpoints of the strands. Without loss of generality, we put these endpoints at the intersection of the sphere with the main diagonals of the plane (Fig. 2).

![Figure 2. Isotopic r-tangles](image)

For short we write \textit{r-tangle} for rational tangle, and we denote by \( R \) the space of rational tangles.

We denote by \( \Gamma^{||} \) and \( \Gamma^= \) the simplest r-tangles (see Fig. 3) and by 1, 2, 3, 4 the endpoints of the strands, starting with point 1 (\( X > 0, Y > 0 \)) and going to 4 in the clockwise direction.
By the constructive definition of r-tangles in [3], an r-tangle is obtained from $\Gamma\parallel$ or $\Gamma=$ by a series of moves, consisting in twisting pairs of adjacent endpoints. We denote by $X_i^+$ and $X_i^−$, $i = 1, 2, 3, 4$, the positive and the negative twists, as shown in Fig. 4.

Given an r-tangle $\Gamma$, the r-tangle obtained by applying the move $X_i^\sigma$ ($\sigma = +$ or $−$) to $\Gamma$ is denoted by $X_i^\sigma \Gamma$. Thus any expression

$$X_{i_1}^{\sigma_1}X_{i_2}^{\sigma_2} \cdots X_{i_n}^{\sigma_n} \Gamma_0,$$

where $\Gamma_0 = \Gamma\parallel$ or $\Gamma=$, represents the r-tangle obtained from $\Gamma_0$ by applying the moves $X_i^\pm$ in the order from $X_{i_n}^{\sigma_n}$ to $X_{i_1}^{\sigma_1}$.

**Remark 1.1.** Observe that $\Gamma\parallel$ and $\Gamma=$ satisfy

$$X_4^+ \Gamma\parallel = X_3^+ \Gamma\parallel = \Gamma\parallel, \quad X_4^- \Gamma= = X_4^+ \Gamma= = \Gamma=.$$
Any r-tangle, written in terms of the $X_i^\pm$, can be therefore represented such that each crossing corresponds to one of the moves $X_i^\pm$ and its sign is positive or negative according to the scheme of Figure 5. We call such representations standard representations.

![Figure 5. Positive and negative crossings in standard representations of r-tangles](image)

Isotopic r-tangles may have different standard representations.

**Example.** The representations of isotopic r-tangles in the middle and on the right of Figure 2 are standard, and they are respectively:

\[ X_3^+X_4^-X_4^+X_3^-\Gamma^-, \quad X_4^+X_3^-X_4^+\Gamma^-. \]

We recall that an r-tangle in a standard representation, where the crossings are either all positive or all negative, is said alternating.

### 2. The Group of Transformations of Isotopy Classes of r-Tangles

The following lemma is known (see for instance [3], page 305). It is essential for simplifying the set of moves, and we give here a proof in order that this paper be self-consistent.

**Lemma 2.1.** Every r-tangle $\Gamma$ is isotopic to the tangle $\hat{\Gamma}$ obtained from $\Gamma$ by a rotation by $\pi$ about the $Y$-axis and to the tangle $\Gamma'$ obtained from $\Gamma$ by a rotation by $\pi$ about the $X$-axis.

**Proof.** The basic tangles $\Gamma\parallel$ and $\Gamma\|$ are invariant under these rotations. The tangles with only one crossing, obtained as $X_3^+\Gamma^-$, $X_4^+\Gamma^-$ or as $X_3^\parallel\Gamma\parallel$, $X_4^\parallel\Gamma\parallel$ (see Remark 1.1), are also invariant under such rotations. Given a tangle $\Gamma$, we write its expression (1) in terms of $n$ moves. Therefore we have

\[ \Gamma = X_1^{\sigma_i} \Phi, \]

where the r-tangle $\Phi$ is expressed in terms of $n - 1$ moves. Let us suppose that the isotopy class of $\Phi$ is invariant under the considered rotations, that is,

\[ \hat{\Phi} \sim \Phi \sim \Phi. \]

The following figure shows that $\Gamma$ is isotopy equivalent, by consequence, to $\Gamma'$ and to $\hat{\Gamma}$, when $\Gamma = X_1^+ \Phi$. Indeed:

\[ \Gamma' = X_1^+ \Phi \sim X_1^+ \Phi = \Gamma, \]

and

\[ \hat{\Gamma} = X_3^+ \hat{\Phi} \sim X_3^+ \hat{\Phi} \sim X_1^+ \Phi = \Gamma. \]
Similarly, if $\Gamma = X_2^+ \Phi$:

$$\Gamma) = X_2^+ \Phi) \sim X_2^+ \hat{\Phi} \sim X_2^+ \Phi = \Gamma,$$

and

$$\hat{\Gamma} = X_2^+ \hat{\Phi} \sim X_2^+ \Phi = \Gamma.$$  

The cases with negative twists, as well as the cases with $\Gamma = X_3^\pm \Phi$ and $\Gamma = X_4^\pm \Phi$ are analogous. The lemma follows.

From the lemma above we deduce evidently also the following

**Proposition 2.2.** For every $r$-tangle $\Gamma \in \mathcal{R}$:

$$X_2^\pm \Gamma \sim X_3^\pm \Gamma, \quad X_2^\pm \Gamma \sim X_4^\pm \Gamma.$$  

Therefore, in order to classify the $r$-tangles up to isotopies, it is possible to consider only two basic moves that change the isotopy class of an $r$-tangle. We call them $A$ and $B$, and precisely:

$$A := X_3^+, \quad B := X_2^+.$$  

Their inverses are denoted, respectively, $A^{-1}$ and $B^{-1}$ and satisfy $A^{-1} = X_3^-$, $B^{-1} = X_2^-$.  

**Examples.** The $r$-tangles in the middle and on the right of Figure 2 have the following representation:

$$A^1 B^1 B^{-2} A^2 \Gamma =, \quad A^1 B^{-1} A^2 \Gamma =.$$  

Note that the moves $A$ and $B$ keep invariant the tangle endpoint numbered by 1 lying in the first quadrant of the $XY$-plane.

There is another move keeping invariant this endpoint. It is the twist of the endpoints 2 and 4 by a rotation by $\pi$ about the diagonal $X = Y$, as shown in Fig. 6. We denote this move by $R$.

We now define a group of moves, generated by $A$, $B$, $A^{-1}$ and $B^{-1}$, and we denote it by $\mathcal{J}$. 
The moves $A$, $B$ and $R$ obtained by rotating by $\pi$ one half of the sphere containing the end-points of the r-tangle $\Gamma$.

Two words $W$ and $Q$ represents the same element if and only if, for every $\Gamma \in \mathcal{R}$, $Q \Gamma \sim W \Gamma$.

For instance, the equation $AA^{-1} = E$ means that, for any r-tangle $\Gamma$, $AA^{-1} \Gamma \sim E \Gamma = \Gamma$. This equivalence is indeed obtained by the Reidemeister move which eliminates two crossings.

A sequence of $k$ consecutive $A$ ($B$) moves will be denoted by $A^k$ ($B^k$) and its inverse by $A^{-k}$ ($B^{-k}$).

Any word $W = T_1T_2\cdots T_n$ will be thus written as

$$W = A^{a_1}B^{a_2}A^{a_3}B^{a_4}\cdots \quad \text{or} \quad W = B^{a_1}A^{a_2}B^{a_3}A^{a_4}\cdots, \quad (2)$$

where $a_i$ are nonzero integers. The word $W$ has the inverse $W^{-1} = \cdots B^{-a_4}A^{-a_3}B^{-a_2}A^{-a_1}$ or $W = \cdots A^{-a_4}B^{-a_3}A^{-a_2}B^{-a_1}$ satisfying evidently $Q^{-1}Q = QQ^{-1} = E$. 
We define now a homomorphism from $T$ to $\text{PSL}(2, \mathbb{Z})$.

Let $A$ and $B$ be the following elements of $\text{PSL}(2, \mathbb{Z})$, which are defined up to multiplications by $-1$:

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}. \quad (3)$$

**Definition.** The map $\mu: T \mapsto \text{PSL}(2, \mathbb{Z})$ is defined by $\mu(A) = A$, $\mu(B) = B$, $\mu(A^{-1}) = A^{-1}$, $\mu(B^{-1}) = B^{-1}$, and $\mu(E) = E$, $E$ being the identity $(2 \times 2)$-matrix, up to multiplication by $-1$. Moreover, $\mu$ sends any sequence of moves $A$ and $B$ and their inverses to the operator of $\text{PSL}(2, \mathbb{Z})$ obtained as the corresponding product of $A$, $B$ and their inverses.

Recall that the standard basis for $\text{PSL}(2, \mathbb{Z})$ is $\{A, S\}$, where $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, up to multiplication by $-1$, and satisfies $S^2 = E$. \quad (4)

Observe that $S$ is written in terms of the basis $\{A, B, A^{-1}, B^{-1}\}$ as $S = AB^{-1}A = B^{-1}AB^{-1}$. \quad (5)

**Lemma 2.3.** Every operator $Q \in \text{PSL}(2, \mathbb{Z})$ can be written as $VT$, where $T$ is either a word in sole $A$, $B$ or a word in sole $A^{-1}$, $B^{-1}$, and $V = E$ or $V = S$.

**Proof of the lemma.** It follows from the following relations holding in $\text{PSL}(2, \mathbb{Z})$, which are derived from (5):

$$SA = B^{-1}S, \quad SB = A^{-1}S, \quad SA^{-1} = BS, \quad SB^{-1} = AS. \quad (6)$$

We are now ready to prove that the homomorphism $\mu$ is an isomorphism.

**Theorem 2.4.** The group $T$ is isomorphic to the modular group$^1$.

**Proof.** To prove that $\mu$ is an isomorphism, we have to prove that, for any two different words $W$ and $W'$ in $T$,

$$\mu(W) = \mu(W') \iff \forall \Gamma \in \mathbb{R} \quad W\Gamma \sim W'\Gamma. \quad (7)$$

**Proof of (7)$\Rightarrow$.** To every element of $\text{PSL}(2, \mathbb{Z})$ there correspond different words in the generators $A$, $B$, $A^{-1}$ and $B^{-1}$, since they are not independent. We have therefore to prove that whenever $\mu(W) = \mu(W')$, $W\Gamma \sim W'\Gamma$ for every r-tangle $\Gamma$. This is true if the relations 5 and 4 are the image by $\mu$ of relations holding in $T$.

We define therefore $S := AB^{-1}A \in T$, and we prove:

(i) $B^{-1}AB^{-1} = S$;

(ii) $S = SS^{-1}$.

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$^1$In this paper the term modular group means the projective special linear group $\text{PSL}(2, \mathbb{Z})$, because of the isomorphisms between these groups.
Proof of (i). The next figure shows that for any tangle $\Gamma$,
\[
AB^{-1}\Gamma \sim R\Gamma, \quad B^{-1}AB^{-1}\Gamma \sim R\Gamma.
\]
Hence item (i) is satisfied by $S = R$.

\begin{tikzpicture}
\node{\includegraphics[width=0.8\textwidth]{figure.png}};
\end{tikzpicture}

Proof of (ii).

The tangles $R\Gamma$ and $R^{-1}\Gamma$ are isotopic to the tangle obtained by rotations by $\pm \pi$ of the whole ball containing $\Gamma$ about the diagonal $Y = X$. Observe that such rotations, unlike the vertical and horizontal rotations by $\pi$ considered in Lemma 2.1, do not preserve the isotopy class of $\Gamma$. However, from $R\Gamma \sim R^{-1}\Gamma$ we conclude $R^2\Gamma \sim R^{-2}\Gamma \sim \Gamma$, that is, $R^2 = E$ (a rotation by $2\pi$ is the identity).

Proof of (7) $\Leftarrow$. To conclude the proof of the theorem, we have to exclude that there exist some words $W$ and $W'$ in $T$ such that, for every $r$-tangle $\Gamma$, the isotopy classes of $W\Gamma$ and $W'\Gamma$ coincide, but $\mu(W) \neq \mu(W')$. In this case there should be a move $Q$, namely $Q = W^{-1}W'$, which is isotopy equivalent to the identity, and satisfies $\mu(Q) \neq E$.

Using Lemma 2.3, we write the operator $Q := \mu(Q)$ in the form $Q = VT$. $T$ is a word either in $A, B$ or in $A^{-1}, B^{-1}$. Suppose that the word $T$ ends with $A^a$,
for some nonzero $a \in \mathbb{Z}$. Consider the r-tangle $Q' \Gamma_0$, where $Q' = VT$, $T$ being the word in the moves $A$ and $B$ obtained translating $A$ to $A$ and $B$ to $B$ from the word $T$, and $V$ is equal to $R$ or to $E$, according to $V$. The crossings of $TT'$ are either all positive or all negative, and, since the move $R$ is equivalent to a rotation changing the sign of all crossings, the r-tangle $Q' \Gamma_0$ is alternating. It is therefore evident that $Q' \Gamma_0 \sim \Gamma_0$. But we have $Q' \Gamma_0 \sim Q' \Gamma_0$, since $\mu(Q') = \mu(Q)$. Therefore $Q' \Gamma_0 \sim \Gamma_0$. This contradicts the hypothesis that the move $Q$ is isotopy equivalent to the identity. If $T$ ends with $B$ for some nonzero $a \in \mathbb{Z}$, then we consider the analog move $Q'$ and the tangle $Q' \Gamma_0$, getting the relations $Q' \Gamma_0 \sim Q' \Gamma_0$, whence $Q' \Gamma_0 \sim \Gamma_0$, contradicting the hypothesis. This concludes the proof of the theorem. □

Let us denote $\Gamma_0$ the tangle $\Gamma_0$. Observe that

$$\Gamma_0 \sim B^{-1} A \Gamma_0,$$

or

$$\Gamma_0 \sim A^{-1} B \Gamma_0. \tag{8}$$

We obtain the following corollary:

**Corollary 2.5.** The space of the isotopy classes of rational tangles coincides with the orbit of $\Gamma_0$ under the action of the group $T$.

**Proof.** Every r-tangle is isotopy equivalent to an r-tangle reached from $\Gamma_0$ or $\Gamma_0$ by a series of twists $X_i$. By Corollary 2.2, to generate all the isotopy classes in $\mathcal{R}$ the moves $A$, $B$ and their inverses are sufficient. Also $\Gamma_0$ can be obtained from $\Gamma_0$ by $A$, $B$ and their inverses, by (8). Since every word in $A$, $B$ and their inverses is an element of $T$, and all classes are representable starting from $\Gamma_0$, the corollary follows. □

### 3. Rational Tangles and Rational Numbers

In this section we prove that every class of r-tangles different from that of $\Gamma_0$ is uniquely represented by a rational number, and, vice versa, that every rational number represents uniquely a class of rational tangles, i.e., we give a proof of the Conway theorem.

We denote by $\tilde{\Gamma}$ the isotopy class of the r-tangle $\Gamma$, and by $\tilde{\mathcal{R}}$ the space of the isotopy classes of r-tangles.

Let $\tilde{\mathcal{Q}} := \{(p, q) \in \mathbb{Z}^2 \setminus (0, 0) \) \text{ modulo } (p, q) \sim (rp, rq), \forall r \in \mathbb{Z} \setminus 0\}.$

Let $v_0 = (0, 1)$ and $v_\infty = (1, 0)$. Observe that $(0, r)$, for all nonzero $r \in \mathbb{Z}$, represent the same element as $v_0$ in $\tilde{\mathcal{Q}}$, and $(r, 0)$, for all nonzero $r \in \mathbb{Z}$, represent the same element as $v_\infty$.

**Definition.** We define the map $\rho: \tilde{\mathcal{R}} \rightarrow \tilde{\mathcal{Q}}$ by the following equations:

$$\rho(\Gamma_0) = v_0;$$

if $\Gamma = Q \Gamma_0$, then

$$\rho(\tilde{\Gamma}) = \mu(Q)v_0.$$
Theorem 3.1. The map \( \rho \) associates with every isotopy class of \( r \)-tangles one and only one element of \( \tilde{Q} \).

Before proving this theorem, we make some observations.

Remark 3.2. The \( r \)-tangles \( \Gamma_0 \) and \( \Gamma_{||} \) possess the following invariances: for every \( m \in \mathbb{Z} \)
\[
B^m \Gamma_0 \sim \Gamma_0, \quad A^m \Gamma_{||} \sim \Gamma_{||}.
\]
No other element of \( \mathcal{T} \) keeps invariant \( \Gamma_0 \) nor \( \Gamma_{||} \).

Observe that, from Eq. 8 and Remark 3.2 it follows also that
\[
R \Gamma_0 \sim \Gamma_{||}, \quad R \Gamma_{||} \sim \Gamma_0.
\]
(9)

Lemma 3.3. There is no element of \( T \) keeping invariant a class of \( \tilde{R} \) different from \( \tilde{\Gamma}_0 \) or \( \tilde{\Gamma}_{||} \).

Proof. Suppose the equation \( W \Gamma \sim \Gamma \) be fulfilled by some \( r \)-tangle \( \Gamma \) and some word \( W \neq E \). Let \( \Gamma = Q \Gamma_0 \) for some \( Q \in \mathcal{T} \). By hypothesis we have \( WQ \Gamma_0 \sim Q \Gamma_0 \), and, by consequence, \( Q^{-1}WQ \Gamma_0 \sim \Gamma_0 \). By Remark 3.2 we obtain that \( Q^{-1}WQ = B^k \), for some integer \( k \). The last equation is fulfilled either if both \( W \) and \( Q \) are powers of \( B \), and hence \( \Gamma \sim \Gamma_0 \), and \( W = B^k \), or if \( Q = A^jR \), and hence, by Eq. 9, \( \Gamma \sim \Gamma_{||} \), and \( W = A^{-k} \), being \( RA^{-j}A^{-k}A^jR = B^k \).

\[ \blacksquare \]

Lemma 3.4. Every class in \( \tilde{R} \) is uniquely represented by an element of \( \mathcal{T} \), modulo multiplication from the right by \( B^m \), \( m \in \mathbb{Z} \).

Proof. Every class of \( r \)-tangles can be represented as \( Q \Gamma_0 \), for some \( Q \in \mathcal{T} \), by Corollary 2.5. With the \( r \)-tangle \( \Gamma = Q \Gamma_0 \) we associate the element \( Q \in \mathcal{T} \). By Remark 3.2, \( \tilde{\Gamma} = QB^m \Gamma_0 \), therefore if we associate \( Q \) to \( \Gamma \), we may as well associate \( QB^m \), for all \( m \in \mathbb{Z} \). On the other hand, if \( Q' \neq QB^m \) for some \( m \), then \( Q' \Gamma_0 \sim Q \Gamma_0 \), by the same remark.

\[ \blacksquare \]

Proof of Theorem 3.1. We observe first that the map \( \rho \) respects the invariances of Remark 3.2.

Indeed, \( \rho(B^m \Gamma_0) = B^m v_0 = \left( \begin{array}{cc} 1 & 0 \\ m & 1 \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = \left( \begin{array}{c} 0 \\ m \end{array} \right) = mv_0 \sim v_0 = \rho(\tilde{\Gamma}_0) \).

Moreover, using Eq. 8,
\[
\rho(\tilde{\Gamma}_{||}) = B^{-1}A v_0 = \left( \begin{array}{rr} 1 & 1 \\ -1 & 0 \end{array} \right) \left( \begin{array}{c} 0 \\ 1 \end{array} \right) = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = v_{\infty}.
\]
(10)

Therefore \( \rho(A^m \tilde{\Gamma}_{||}) = A^m B^{-1} A v_0 = A^m v_{\infty} = \left( \begin{array}{cc} 1 & m \\ 0 & 1 \end{array} \right) \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = \left( \begin{array}{c} m \\ 0 \end{array} \right) = mv_{\infty} \sim v_{\infty} \), that is, \( \rho(A^m \tilde{\Gamma}_{||}) = \rho(\tilde{\Gamma}_{||}) \).

We have to prove that for every pair of classes \( \tilde{\Gamma} \) and \( \tilde{\Gamma}' \) in \( \tilde{R} \)
\[
\tilde{\Gamma} = \tilde{\Gamma}' \iff \rho(\tilde{\Gamma}) = \rho(\tilde{\Gamma}').
\]
(11)
Proof of (11)⇒. By Lemma 3.4, if $\tilde{\Gamma} = \tilde{\Gamma}'$ then $\Gamma = Q\Gamma_0$ and $\Gamma' = Q'\Gamma_0$ with $Q' = QB^n$. If $\mu(Q) = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$, we obtain $\mu(Q') = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \left(\begin{array}{cc} 1 & 0 \\ m & 1 \end{array}\right) = \left(\begin{array}{cc} a + mb & b \\ c + md & d \end{array}\right)$.
Therefore $\rho(\tilde{\Gamma}) = \mu(Q)v_0 = (b, d)$ and $\rho(\tilde{\Gamma}') = \mu(Q')v_0 = (b, d)$.

Proof of (11)⇐. Let $\Gamma \sim Q\Gamma_0$, $\Gamma' \sim Q'\Gamma_0$, $\mu(Q) = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right)$, $\mu(Q') = \left(\begin{array}{cc} a' & b' \\ c' & d' \end{array}\right)$.

We suppose that $\rho(\tilde{\Gamma}) = \rho(\tilde{\Gamma}')$, that is, $(b, d) \sim (b', d') \in \hat{\mathbb{Q}}$. This equality implies $b = b'$ and $d = d'$, because $b$, $d$ as well as $b'$, $d'$ are relatively prime, constituting a column of a discriminant one matrix. The matrices $Q := \mu(Q)$ and $Q' = \mu(Q')$ may differ for the first column, and they have the same unit discriminant. We obtain therefore $a' = a + bk$ and $c' = c + dk$ for some integer $k$, and this implies $Q' = QB^k$. We thus obtain $\Gamma' \sim Q'\Gamma_0 \sim QB^k\Gamma_0 \sim Q\Gamma_0$, and hence $\Gamma' \sim \Gamma$. □

Consider the map $t$ sending every element of $\hat{\mathbb{Q}}$ with $q \neq 0$ to a rational number: $t: (p, q) \mapsto p/q$. Every rational number is the image by $t$ of one and only one element of $\hat{\mathbb{Q}}$. We will write $p/q = t(v)$, where $v = (p, q) \in \hat{\mathbb{Q}}$. Evidently $mv = (mp, mq) \sim v$ in $\hat{\mathbb{Q}}$.

Theorem 3.1 has the following corollary:

Corollary 3.5. The map $\rho \circ t: \hat{\mathcal{R}} \rightarrow \mathbb{Q}$ associates with every isotopy class of r-tangles different from $\Gamma^{||}$ one and only one rational number.

Proof. By Theorem 3.1, the map $\rho$ associates with every isotopy class of r-tangles an element of $\hat{\mathbb{Q}}$. The map $t$ associates with every element of $\hat{\mathbb{Q}}$, $q \neq 0$, a rational number. The element $(r, 0) \in \hat{\mathbb{Q}}$ is the image by $\rho$ of the class of $\Gamma^{||}$, by Eq. 10. The corollary follows. □

Corollary 3.6. The map $(\rho \circ t)^{-1}: \mathbb{Q} \rightarrow \hat{\mathcal{R}}$ associates with every rational number one and only one isotopy class of r-tangles.

Proof. With the rational number $p/q$, $t^{-1}$ associates $v = (p, q) \in \hat{\mathbb{Q}}$. Since $q \neq 0$, and $p$ and $q$ are coprime, the pair $(p, q)$ defines a matrix $Q \in \text{PSL}(2, \mathbb{Z})$ such that $Qv_0 = v$, up to a right factor equal to $B^m$, for some $m \in \mathbb{Z}$ as we have seen in the proof of Eq. 11. Therefore also the element $Q \in \mathcal{F}$ such that $\mu(Q) = Q$ is defined up to a right factor equal to $B^m$, for some $m \in \mathbb{Z}$. But all elements $QB^m$ define the same class $Q\Gamma_0$, therefore the image of $v$ by $\rho^{-1}$ is well defined and unique by Remark 3.2. □

4. Alternating Rational Tangles and Continued Fractions

In this section we obtain first that every r-tangle is isotopy equivalent to an alternating r-tangle, as a consequence of the results of the preceding section.

Corollary 4.1. A rational tangle different from $\Gamma^{||}$ is equivalent to a tangle obtained by $\Gamma_0$ either by sole positive twists or sole negative twists.

Proof. We write $\Gamma \sim Q\Gamma_0$. Hence we consider $Q = \mu(Q)$. $Q$ is different from $A^mS$ since $\Gamma \sim \Gamma^{||}$, according to Eq. 8, Remark 3.2 and Eq. 9. By Lemma 2.3 and Theorem 2.4, $Q$ can be written as $VT$, where $T$ is a word either in $A$ and
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B or in $A^{-1}$ and $B^{-1}$, and $V = R$ or $V = E$. If $V = E$, the proof is finished, since the tangle $T\Gamma$ is alternating. If $V = R$, then we apply to every generator in the word $T$ the equations obtained by the relations in $T$ corresponding to the relations (6) in $\text{PSL}(2, \mathbb{Z})$. Therefore we obtain $RT = TR$, where $T$ is obtained by $T$ exchanging $B$ with $A^{-1}$ and $A$ with $B^{-1}$. Hence $\Gamma \sim TR\Gamma_0 \sim T\Gamma$. Since $\Gamma \sim A^m \Gamma^\prime$, we can write $\Gamma \sim T\Gamma^\prime$, where $T\Gamma^\prime$ is obtained by exchanging $B$ with $A^{-1}$ and $A$ with $B^{-1}$.

4.1. Alternating rational tangles from continued fractions. We prove now that the continued fraction procedure allows to associate with every rational number one and only one alternating rational tangle.

Let $p, q$ and $a_i$ ($i = 1, \ldots, n$) represent natural numbers. A positive rational number has a unique representation by a continued fraction with positive elements $a_i$:

$$\frac{p}{q} = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \cdots + \frac{1}{a_n}}}}.$$  

As a consequence, a negative rational number has a unique representation by a continued fraction with negative elements $-a_i$:

$$-\frac{p}{q} = -a_1 - \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \cdots + \frac{1}{a_n}}}} = (-a_1) + \frac{1}{(-a_2) + \frac{1}{(-a_3) + \frac{1}{(-a_4) + \cdots + \frac{1}{(-a_n)}}}}.$$  

A continued fraction with strictly positive elements is denoted by $p/q = [a_1; a_2, \ldots, a_n]$.

Note that $a_1$ may be zero, but all $a_i$, for $i > 1$, are positive, and $a_n$ is bigger or equal to 2. Therefore, we can always write an equivalent expression for $p/q$:

$$\frac{p}{q} = [a_1, a_2, a_3, \ldots, a_n - 1, 1].$$
Example. $7/3 = [2, 3] = [2, 2, 1]$, since
\[\frac{2}{3} = 2 + \frac{1}{3} = 2 + \frac{1}{2 + \frac{1}{2}}.\]

If the fraction is negative, we will write evidently $-\frac{p}{q} = [-a_1, -a_2, -a_3, \ldots, -a_n + 1, -1]$.

In this way we may always suppose $n$ be odd, because if $n$ is even, then we decrease $a_n$ by one and we add the $(n + 1)$-th element equal to 1 (or $-1$). We call such a continued fraction odd continued fraction.

Proposition 4.2. Let the odd continued fraction of $p/q$ be
\[\left[\begin{array}{c}
[a_1; a_2, a_3, \ldots, a_n]
\end{array}\right],\] (12)
where the $a_i$ are either all strictly positive or all strictly negative but $a_1$ may be zero. Then $p/q = t(\rho(\widetilde{Q}_0))$, where $Q_0$ is the alternating $r$-tangle obtained by applying to $\Gamma_0$ the sequence of moves
\[A^{a_1} B^{a_2} A^{a_3} \cdots B^{a_n-1} A^{a_n}.\]

Proof. We define the map $\tau_Q: \mathbb{Q} \rightarrow \mathbb{Q}$ in the following way: for every $z = \frac{p}{q} \in \mathbb{Q}$
\[\tau_Q(z) = t(Q(\frac{p}{q})).\]

Evidently, $t(Q(\frac{p}{q})) = t(Q(\frac{q}{p}))$. For every $\alpha \in \mathbb{Z}$ we have:
\[\tau_{A^\alpha}(z) = z + \alpha, \quad \tau_{B^\alpha}(z) = \frac{1}{\alpha + \frac{1}{z}}.\]

If $\alpha = 0$, we obtain $\tau_{E}(z) = z$. Moreover, if $Q'' = QQ'$, then $\tau_{Q''}(z) = \tau_Q(\tau_{Q'}(z))$. Therefore, if $Q = A^{a_1} B^{a_2} A^{a_3} \cdots B^{a_n-1} A^{a_n}$ then
\[\tau_Q(0) = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4 + \cdots + \frac{1}{a_n}}}}.\]

We thus associate with the odd continued fraction (12) of $p/q$ the sequence of moves
\[Q = A^{a_1} B^{a_2} A^{a_3} \cdots B^{a_n-1} A^{a_n}\]
and the alternating tangle $Q \Gamma_0$. By construction, $p/q = t(\rho(Q \Gamma_0))$. $\square$

5. Rational Tangles and the Spherical Braid Group with Four Strands

We give here a short proof of Theorem 2.4, using the spherical braid group with four strands, that is, the group of braids whose strands have the upper and lower endpoints lying on two concentric 2-dimensional spheres.

Let $F$ be the manifold of four nonordered distinct points in $S^2$. 
Every transformation between rational tangles defines a closed path in $F$, by definition of rational tangles, and hence an element of $\pi_1(F)$.

Consider any tangle $\Gamma$. An expression like

$$
\Gamma' = X_{i_1}^{\sigma_1} X_{i_2}^{\sigma_2} \cdots X_{i_n}^{\sigma_n} \Gamma,
$$

defines a transformation of the isotopy class of $\Gamma$ to another, that is, an element of $\mathcal{T}$.

Observe that the elements $X_i$, $i = 1, 2, 3, 4$, generate the spherical braid group with four strands. This defines a natural isomorphism between $\pi_1(F)$ and the group $\mathcal{T}$.

The isomorphism between the spherical braid group with four strands and $\text{PSL}(2, \mathbb{Z})$ results by the following (known) lemmas:

**Lemma 5.1.** The spherical braid group with four strands is isomorphic to the group $\tilde{B}_3 := B_3 / \langle \omega \rangle$, where $B_3$ is the group of braids with three strands, generated by $\sigma_1$ and $\sigma_2$:

![Diagram of the spherical braid group](image)

and $\langle \omega \rangle$ is the relation

$$
\sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2 = 1.
$$

**Lemma 5.2.** $\tilde{B}_3$ is isomorphic to $\text{PSL}(2, \mathbb{Z})$

Let us give the idea of proof of Lemma 5.1. Let the upper end-points of the four strands of the spherical braid be the points 1, 2, 3, and 4 shown in Fig. 8a. Any rotation of the internal sphere permuting cyclically the lower endpoints does not change the isotopy class of the spherical braid (Fig. 8b). Therefore, we may assume that a strand connects the upper endpoint 1 to a fixed lower point, say the lower endpoint 1 (Fig. 8c). Any braid can be then transformed to an element of $B_3$, by removing any linking with the first strand, as shown in Fig. 8d,e.

![Diagram of the proof](image)

The element $\sigma_1 \sigma_2 \sigma_1 \sigma_2 \sigma_1 \sigma_2$ is equivalent to a rotation of the inner sphere by $2\pi$ about the central vertical axis (containing the first strand), rotation that evidently does not change the isotopy class of the spherical braid, and that commutes with
all elements of $B_3$. Any other element of $B_3$, different from a power of this element, cannot be removed by a rotation of the sphere.

5.1. The isomorphism between $\tilde{B}_3$ and $\mathcal{T}$. There is a natural way to associate with every element of $\mathcal{T}$ an element of $\tilde{B}_3$ and vice-versa, as the following figure shows. Observe that this provides a proof of Lemma 5.2, in virtue of Theorem 2.4.

The construction can be described as follows: the endpoints 4, 3 and 2 of the r-tangle $\Gamma$ become, respectively, the upper endpoints of the strands of the braid, and the endpoints 4, 3 and 2 of the r-tangle $Q\Gamma$ become, respectively, the lower endpoints of the strands of the braid. The topological resolution of each crossing remains unchanged in the transformation, so that we obtain:

\[ A \mapsto \sigma_1, \quad B \mapsto \sigma_2^{-1}. \]  

To any word of $\mathcal{T}$ in $A$ and $B$ and their inverses there corresponds therefore the word in $B_3$ where the generators are translated according to (15) and are put in the inverse order, if the order from left to right of the generators in a word of $B_3$ is (as usually) interpreted as their order from top to bottom in the braid.

The relation $\langle \omega \rangle$ is therefore the translation of Eq. 4 by means of (5). \qed

Observe that the construction above is similar to that showed in [4], but we haven’t any ambiguity problem, since we deal with $\mathcal{T}$, and not with rational tangles.

5.2. A remark by Arnold. We conclude this paper with a remark by Arnold, indicating another way to prove Theorem 2.4.

The space $F$ of configurations of four nonordered points in $S^2$ can be considered as the base of a fibration $p: E \to F$, whose fiber is the torus, twice covering the
Riemann sphere with four ramification points. The fundamental group $\pi_1(F)$ acts in a natural way on this fibration: every element of $\pi_1(F)$ defines a map: $T^2 \to T^2$ over the same base point $f \in F$, and hence a map of $H_1(T^2)$ to itself, which is an element of the modular group.

In the following figures we illustrate this remark.

**Figure 9.** (a–b) The sphere with four ramification points, (c–d) the cylinder obtained cutting along two segments joining respectively the points 1–2 and 3–4, (e) the torus $T^2$ two times covering the cylinder.

**Figure 10.** (a) The torus $T^2$ obtained as $\mathbb{R}^2/(2\mathbb{Z})^2$: the double segments indicates the generators of $\pi_1(T^2)$. (c–d) Action on $T^2$ of the moves $A$ and $B$ (exchanging respectively the point 3 with the point 4 and the point 2 with the point 3), coinciding with the action of the operators $A, B \in \text{PSL}(2, \mathbb{Z})$ on the generators $e_1, e_2$ of $\mathbb{Z}^2$.

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References


ICTP, STRADA COSTIERA, 11, I – 34151 TRIESTE ITALY
E-mail address: faicardi@ictp.it