SYMPLECTIC REFLECTION ALGEBRAS AND AFFINE LIE ALGEBRAS

PAVEL ETINGOF

To the memory of Israel Moiseevich Gelfand

ABSTRACT. The goal of this paper is to present some results and (more importantly) state a number of conjectures suggesting that the representation theory of symplectic reflection algebras for wreath products categorifies certain structures in the representation theory of affine Lie algebras (namely, decompositions of the restriction of the basic representation to finite dimensional and affine subalgebras). These conjectures arose from the insight due to R. Bezrukavnikov and A. Okounkov on the link between quantum connections for Hilbert schemes of resolutions of Kleinian singularities and representations of symplectic reflection algebras.


Key words and phrases. Symplectic reflection algebra, affine Lie algebra, basic representation, root, weight.

1. Introduction

The goal of this paper is to present some results and (more importantly) state a number of conjectures suggesting that the representation theory of symplectic reflection algebras for wreath products categorifies certain structures in the representation theory of affine Lie algebras (namely, decompositions of the restriction of the basic representation to finite dimensional and affine subalgebras). These conjectures arose from the insight due to R. Bezrukavnikov and A. Okounkov on the link between quantum connections for Hilbert schemes of resolutions of Kleinian singularities and representations of symplectic reflection algebras, and took a much more definite shape after my conversations with I. Losev.

The paper is based on my talk at the conference “Double affine Hecke algebras and algebraic geometry” (MIT, May 18, 2010). I’d like to note that it is not a complete study of the subject, but rather an attempt to give an outline for further investigation; at many places, it has speculative nature, and the picture it presents is largely conjectural.
The plan of the paper is as follows. In Section 2, we review the preliminaries on the main objects of study — symplectic reflection algebras for wreath product groups $\Gamma_n = S_n \rtimes \Gamma^n$, $\Gamma \subset \text{SL}_2(\mathbb{C})$, and affine Lie algebras. In Section 3, we define a filtration $F_\bullet$ on the Grothendieck group of the category of representations of $\Gamma_n$ coming from symplectic reflection algebras. If $\Gamma$ is cyclic, we define another filtration $F_\bullet$ on the same group, and conjecture that they have the same Poincaré polynomials. In Section 4, we use Ext groups to define the inner product on the Grothendieck group of the category of finite dimensional representations of the symplectic reflection algebra, as well as its $q$-deformation, and conjecture that the inner product is positive definite, and that the $q$-deformation degenerates at roots of unity. In Section 5, we present results and conjectures on singular and aspherical hyperplanes for symplectic reflection algebras for wreath products, and show that the aspherical locus coincides with the locus of infinite homological dimension for the spherical subalgebra. Finally, in Section 6 we present the main conjectures, which interpret the homogeneous components of the associated graded spaces for the above filtrations in terms of the decomposition of the basic representation of the affine Lie algebra corresponding to $\Gamma$ via the McKay correspondence (tensored with one copy of the Fock space) to a finite dimensional or affine subalgebra. In this section, we also explain the connection of our conjectures with the work of Bezrukavnikov and Okounkov, with the work of Gordon–Martino and Shan, and give motivation for the conjectures on the inner products (they follow from the conjectural interpretation of the Ext inner product as the Shapovalov form on the basic representation). We also present evidence for our conjectures, by discussing a number of cases when they can be deduced from the results available in the literature.

Acknowledgements. I would never have thought of these things without the encouragement and the vision of R. Bezrukavnikov and A. Okounkov. Also, I am very grateful to R. Bezrukavnikov and I. Losev for numerous discussions, without which I would have gotten nowhere.

2. Symplectic Reflection Algebras, Gan–Ginzburg Algebras, and Affine Lie Algebras

2.1. Symplectic reflection algebras for wreath products. Let $\Gamma \subset \text{SL}_2(\mathbb{C})$ be a finite subgroup. For $n \geq 1$, let $\Gamma_n$ be the wreath product $S_n \rtimes \Gamma^n$. Let $V = \mathbb{C}^2$ be the tautological representation of $\Gamma$, and $(\ , \ )$ be a symplectic form on $V$. Let $V_n = V \otimes \mathbb{C}^n$ the corresponding representation of $\Gamma_n$. Note that the form $(\ , \ )$ gives rise to a symplectic structure on $V_n$ invariant under $\Gamma_n$.

Let $k \in \mathbb{C}$, and $c: \Gamma \to \mathbb{C}$ be a conjugation-invariant function. Thus, $c = (c_0, \ldots, c_r)$, where $c_i$ is the value of $c$ on the $i$-th conjugacy class $C_i$ of $\Gamma$ (where $C_0 = \{1\}$).

For $v \in V$, let $v_i = (0, \ldots, v_i, \ldots, 0) \in V_n$ (where $v$ stands in the $i$-th place), and similarly for $\gamma \in \Gamma$ let $\gamma_i = (1, \ldots, \gamma, \ldots, 1) \in \Gamma_n$. Let $s_{ij} = (ij) \in S_n$. 
Definition 2.1 [EG]. The symplectic reflection algebra $H_{c,k}(\Gamma_n)$ is the quotient of $\mathbb{C}\Gamma_n \ltimes TV_n$ by the relations
\[
[u_i, v_j] = k \sum_{\gamma \in \Gamma} (\gamma u, v)s_{ij}\gamma_i\gamma_j^{-1}, \quad i \neq j;
\]
\[
[u_i, v_i] = (u, v) \left( \sum_{\gamma \in \Gamma} c_\gamma\gamma_i - k \sum_{j \neq i} s_{ij}\gamma_i\gamma_j^{-1} \right),
\]
for all $u, v \in V$.

Remark 2.2. 1. $H_{0,0}(\Gamma_n) = \mathbb{C}\Gamma_n \ltimes SV_n$.

2. If $\Gamma \neq 1$, $H_{c,k}(\Gamma_n)$ is the universal filtered deformation of $H_{0,0}(\Gamma_n)$.

3. For $n = 1$, the algebra $H_{c,k}(\Gamma_n)$ is independent of $k$; we will denote it by $H_c(\Gamma)$.

4. One has $H_{c,0}(\Gamma_n) = \mathbb{C}S_n \ltimes H_c(\Gamma)^{\otimes n}$.

Example 2.3. Let $\Gamma = 1$, and $c_0 = t$. Then we get the algebra $H_{t,k}(S_n)$, which is the quotient of $\mathbb{C}S_n \ltimes TV_n$ by the relations
\[
[x_i, x_j] = 0, \quad [y_i, y_j] = 0,
\]
\[
[y_i, x_j] = ks_{ij}, \quad [y_i, x_i] = t - k \sum_{j \neq i} s_{ij},
\]
where $\{x, y\}$ is the standard basis of $V = \mathbb{C}^2$ (such that $(y, x) = 1$). This is the rational Cherednik algebra for $S_n$ with parameters $t, k$.

Below we will need the following proposition.

Proposition 2.4 (I. Losev). Let $E$ be a finite dimensional representation of $\Gamma_n$. Then the set $Z_E$ of values of $c, k$ with $c_0 = 1$ such that $E$ extends to a representation of $H_{c,k}(\Gamma_n)$ is Zariski closed in $\mathbb{A}^{r+1}$.

Proof. Let $Y = \text{Hom}_{\Gamma_n}(V_n, \text{End}(E))$, and let $\tilde{Z}_E \subset Y \times \mathbb{A}^{r+2}$ be the set of maps that give a representation of $H_{c,k}(\Gamma_n)$ (without the assumption that $c_0 = 1$). Let $\tilde{Z}_E$ be the closure of the intersection of $\tilde{Z}_E$ with the set $c_0 \neq 0$. We have an action of the reductive group $\text{Aut}_{\Gamma_n}(E)$ on $\tilde{Z}_E$, and a map
\[
\phi: \tilde{Z}_E/\text{Aut}_{\Gamma_n}(E) \to \mathbb{A}^{r+2},
\]
from the categorical quotient to the space of parameters. The set $Z_E$ is the intersection of $\text{Im} \phi$ with the hyperplane $c_0 = 1$, so it suffices to show that $\text{Im} \phi$ is closed. This follows from the following lemma.

Lemma 2.5. $\phi$ is a finite morphism.

Proof. Let $X$ be an affine variety with a contracting $\mathbb{C}^*$-action. It is well known that if $f_1, \ldots, f_n$ are homogeneous regular functions on $X$ of positive degrees such that the equations $f_1 = 0, \ldots, f_n = 0$ cut out the point $0 \in X$ (set-theoretically), then the map $X \to \mathbb{A}^n$ induced by $f_1, \ldots, f_n$ is finite.

We apply this to $X = \tilde{Z}_E/\text{Aut}_{\Gamma_n}(E)$ and $f_i$ being the coordinates $c_j, k$ on $\mathbb{A}^{r+2}$. Our job is just to check that the zero set $X_0$ of the equations $c_j = 0, k = 0$ on $X$ is just a single point, i.e., the representation $E$ of $\Gamma_n$ on which $V_n$ acts by zero.
To see this, note that $X_0$ consists of semisimplifications of representations $W$ of $H_{0,0}(\Gamma_n)$ which are isomorphic to $E$ as $\Gamma_n$-representations and which are obtained as degenerations of representations of $H_{c,k}(\Gamma_n)$ with $c_0 \neq 0$.

We claim that such a representation $W$ must be supported at $0 \in V_n$ as an $SV_n$-module. Indeed, let $\psi: H_{c,k}(\Gamma_n) \to \End(W)[[h]]$ be a formal 1-parameter deformation of $W$, with $k(0) = 0$, $c(0) = 0$, but $c_0(h) \neq 0$. Since $W$ lies in the closure of $\hat{Z}_E \cap \{c_0 \neq 0\}$, such a deformation must exist. Let $B = \Im \psi$, and $B_0 = B/hB$. Since $B$ is a $\mathbb{C}[h]$-free quotient algebra of $H_{c,k}(\Gamma_n)$, $B_0$ is a finite dimensional Poisson module over the center $(SV_n)^{\Gamma_n}$ of $H_{0,0}(\Gamma_n)$. Hence, $B_0$ is supported at the zero-dimensional symplectic leaves of $V_n/\Gamma_n$ in the Poisson structure induced by the symplectic structure on $V_n$. But the only such symplectic leaf is $0 \in V_n/\Gamma_n$. Thus, $B_0$ is supported at $0$ as an $SV_n$-module. Now, the image $B'_0$ of $H_{0,0}(\Gamma_n)$ in $\End(W)$ is a quotient of $B_0$, so it is also supported at $0$. Hence, so is $W$.

This implies that in the semisimplification of $W$, the space $V_n$ must act by zero, which proves the lemma.

The proposition is proved. □

Remark 2.6. It is clear that Proposition 2.4 holds, with the same proof, for symplectic reflection algebra attached to any finite subgroup of $Sp(2n)$.

2.2. The Gan–Ginzburg algebras. For any quiver $Q$, Gan and Ginzburg defined algebras $A_{n,\lambda,k}(Q)$, parametrized by $n \in \mathbb{N}$, $k \in \mathbb{C}$, and a complex function $\lambda$ on the set $I$ of vertices of $Q$ ([GG]). We refer the reader to [GG] for the precise definition; let us just note that if $n = 1$ then this algebra does not depend on $k$, and is the deformed preprojective algebra $\Pi_{1}(Q)$ defined by Crawley-Boevey and Holland [CBH], and that $A_{n,\lambda,0}(Q) = CS_n \ltimes \Pi_{1}(Q)^{\otimes n}$.

It turns out that if $Q = Q_{\Gamma}$ is the affine quiver of ADE type corresponding to $\Gamma$ via McKay’s correspondence, then the algebra $A_{n,\lambda,k}(Q)$ is Morita equivalent to $H_{c,k}(\Gamma_n)$ under a certain correspondence between $\lambda$ and $c$. Namely, recall that McKay’s correspondence provides a labeling of the irreducible characters $\chi_i$ of $\Gamma$ by vertices $i \in I$ of $Q$.

Theorem 2.7 [GG]. If

$$\lambda(i) = \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} \chi_i(\gamma)c_{\gamma}$$

then the Gan–Ginzburg algebra $A_{n,\lambda,k}(Q)$ is naturally Morita equivalent to $H_{c,k}(\Gamma_n)$.

Remark 2.8. 1. The inverse transformation to (1) is

$$c_{\gamma} = \sum_{i \in I} \chi_i(\gamma^{-1})\lambda(i).$$

In particular, $c_0 = \sum_{i} \chi_i(1)\lambda(i)$.

2. If $\Gamma$ is cyclic, the Morita equivalence of Theorem 2.7 is actually an isomorphism.
In these notes, we will be interested in representation theory questions for the algebra \( H_{c,k}(\Gamma_n) \). These questions will always be Morita invariant, so they will be equivalent to the same questions about the Gan–Ginzburg algebra \( A_{n,\lambda,k}(Q) \); i.e., it does not matter which algebra to use.

### 2.3. Affine Lie algebras

It turns out that \( \lambda \) is a more convenient parameter than \( c \). Namely, \( \lambda \) may be interpreted in terms of affine Lie algebras. Before explaining this interpretation, let us review the basics on affine Lie algebras (cf. [K]).

Let \( \mathfrak{g} \) be the finite dimensional simply laced simple Lie algebra corresponding to the affine Dynkin diagram \( Q_\Gamma \) (agreeing that \( \mathfrak{g} = 0 \) for \( \Gamma = 1 \)), let \( \mathfrak{h} \) be its Cartan subalgebra, let \( \tilde{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K \) be the affinization of \( \mathfrak{g} \), and let \( \tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \) be the Cartan subalgebra of the affine Lie algebra \( \tilde{\mathfrak{g}} \). Define the extended affine Lie algebra \( \tilde{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}D \), where \( \{D, b(t)\} = t\beta'(t) \). Its Cartan subalgebra is \( \tilde{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}D \). The space \( \tilde{\mathfrak{h}} \) carries a nondegenerate symmetric bilinear form obtained by extending the form on \( \mathfrak{h} \) to \( \{D, D\} = 0 \), \( \{D, \mathfrak{h}\} = 0 \), \( \{D, K\} = 1 \). This form defines a form on the dual space \( \tilde{\mathfrak{h}}^* \).

Let \( \omega_i \in \mathfrak{h}^* \) be the fundamental weights of \( \tilde{\mathfrak{g}} \) (so we have \( \omega_i(D) = 0 \)), let \( \alpha_i \in \tilde{\mathfrak{h}}^* \) be its simple positive roots, and let \( \delta = \sum \chi_i(1)\alpha_i \) be the basic imaginary root. Then \( \omega_i \) and \( \delta \) form a basis of \( \tilde{\mathfrak{h}}^* \).

Now we can interpret \( \lambda \) as a weight for \( \tilde{\mathfrak{g}} \), i.e., \( \lambda \in \tilde{\mathfrak{h}}^* \). Namely,

\[
\lambda = \sum_{i \in I} \lambda(i)\omega_i
\]

(so \( \lambda(i) = (\lambda, \alpha_i) \)).

We will be interested in the “quantum” case \( c_0 \neq 0 \), i.e., when the center of our algebras is trivial. Since the parameters can be simultaneously renormalized, we may assume that \( c_0 = 1 \). In terms of \( \lambda \), this condition is written as \((\lambda, \delta) = 1\).

### 3. Filtrations on \( K_0(H_{c,k}(\Gamma_n)) \)

Now fix \( c \) and \( k \) and consider the group \( K_0(H_{c,k}(\Gamma_n)) \) (formed by finite projective modules modulo stable equivalence). Since \( \text{gr}(H_{c,k}(\Gamma_n)) = H_{0,0}(\Gamma_n) = \mathbb{C}\Gamma_n \rtimes SV_n \), by a standard theorem in algebraic K-theory we have a natural isomorphism

\[
\psi: K_0(\text{Rep} \Gamma_n) \rightarrow K_0(H_{c,k}(\Gamma_n)),
\]

where for a finite group \( G \), \( \text{Rep}G \) denotes the category of finite dimensional complex representations of \( G \) (this isomorphism sends \( \tau \in \text{Rep} \Gamma_n \) to the projective \( H_{c,k}(\Gamma_n) \)-module \( \psi(\tau) := \text{Ind}_{\Gamma_n}^{\Gamma_n}(\tau) \)). Also, we see that \( H_{c,k}(\Gamma_n) \) has finite dimensional weight, which implies that every finitely generated \( H_{c,k}(\Gamma_n) \)-module \( M \) gives a class \([M] \in K_0(H_{c,k}(\Gamma_n))\).

Now we would like to define an increasing filtration on \( K_0(H_{c,k}(\Gamma_n)) \): \( F_0 \subset F_1 \subset \cdots \subset F_n = K_0 \).

To do so, for any finitely generated \( H_{c,k}(\Gamma_n) \)-module \( M \) let \( \text{Ann}(M) \) denote its annihilator, and consider the corresponding graded ideal \( \text{gr}(\text{Ann}(M)) \subset \mathbb{C}\Gamma_n \rtimes SV_n \). Define the \textit{annihilator variety} \( AV(M) \) to be the zero set of the intersection \( \text{gr}(\text{Ann}(M)) \cap SV_n \) in \( V_n^* \). This is a \( \Gamma_n \)-invariant subset, and since \( \text{gr}(\text{Ann}(M)) \cap \)}
\((SV_n)^\Gamma_n\) is a Poisson ideal, it is a union of strata of the (finite) stratification of \(V_n\) by stabilizers of points in \(\Gamma_n\). Note that all these strata are symplectic and hence even-dimensional.

Now for \(0 \leq i \leq n\) define \(F_i K_0(\calH_{c,k}(\Gamma_n))\) to be the span of classes \([M]_i\) of modules \(M\) such that \(AV(M)\) has dimension at most \(2i\). By transport of structure using \(\psi\), this gives us a filtration on \(K_0(\Rep \Gamma_n)\), which we denote by \(F_{\psi}^{\bullet, k}\).

**Example 3.1.** Let \(n = 1\). Then the only nontrivial piece of the filtration \(F_{\bullet}^i\) is \(F_0\), which is the span of the classes of representations with zero-dimensional annihilator variety (i.e., finite dimensional representations). We may assume that \(\Gamma \neq 1\), since otherwise there is no finite dimensional representations and the filtration \(F_{\bullet}^i\) is trivial. Let \(\tau \geq 1\) be the number of nontrivial conjugacy classes of \(\Gamma\). Let \(\lambda = \omega_0\). In this case, simple finite dimensional modules \(L_i\) are the irreducible nontrivial \(\Gamma\)-modules \(\chi_i\), \(i = 1, \ldots, \tau\), with the zero action of \(V\) (see [CBH]). Let \(e_j\) be primitive idempotents of the representations \(\chi_j\), \(P_j = \calH_{c,k}(\Gamma) e_j\) be the corresponding projective modules. We have \(\gr P_j = SV \otimes \chi_j\). Consider the Koszul resolution of the \(\calC \ltimes SV\)-module \(\chi_j\) with the trivial action of \(V\):

\[
0 \to SV \otimes \chi_j \to SV \otimes V \otimes \chi_j \to SV \otimes \chi_j \to \chi_j \to 0.
\]

Since by McKay’s correspondence \(V \otimes \chi_j = \bigoplus_{i-j} \chi_i\) (where \(i - j\) means that \(i\) is connected to \(j\) in the quiver) this resolution can be written as

\[
0 \to SV \otimes \chi_j \to \bigoplus_{i-j} SV \otimes \chi_i \to SV \otimes \chi_j \to \chi_j \to 0.
\]

Lifting this resolution to the filtered situation, we get the resolution

\[
0 \to P_j \to \bigoplus_{j-i} P_i \to P_j \to L_j \to 0.
\]

This means that in the Grothendieck group, we have

\[
L_i = \sum a_{ij} P_j,
\]

and \(A = (a_{ij})\) is the Cartan matrix of the quiver \(Q_G\).

Thus, we see that \(F_0\) is spanned by \(\sum_{j \in I} a_{ij} P_j\) for all \(i \in I\), \(i \neq 0\). This implies that \(F_1/F_0 = \mathbb{Z} \oplus C_T\), where \(C_T\) is the center of the corresponding simply connected simple Lie group \(G\).

If \(\Gamma\) is a cyclic group, then we can define another filtration on \(K_0(\Rep \Gamma_n)\), which we will denote by \(F_{\bullet}^i\). Namely, in this case \(\calH_{c,k}(\Gamma_n)\) is a rational Cherednik algebra, and we can define the category \(\calO_{c,k}(\Gamma_n)\) of finitely generated modules over this algebra which are locally nilpotent under the action of the polynomial subalgebra \(\mathbb{C}[y_1, \ldots, y_n] \subset \calH_{c,k}(\Gamma_n)\), where \(x, y\) is a symplectic basis of \(V\) which is also an eigenbasis for \(\Gamma\). We have an isomorphism \(\eta: K_0(\Rep \Gamma_n) \to K_0(\calO_{c,k}(\Gamma_n))\) which sends \(\tau \in \Rep \Gamma_n\) to the indecomposable projective object \(P_{c,k}(\tau)\) covering the simple module \(L_{c,k}(\tau)\) with lowest weight \(\tau\). Then we can define \(F_{i} K_0(\calO_{c,k}(\Gamma_n))\) to be the span of the classes of modules whose support as \(\mathbb{C}[x_1, \ldots, x_n]\)-modules has dimension at most \(i\), and define \(F_{\psi}^{i,k} K_0(\Rep \Gamma_n)\) via transport of structure by \(\eta\).
Problem 3.2. Describe the filtrations \(F_{c,k}^\bullet\) and \(F_{c,k}^\bullet\). In particular, find their Poincaré polynomials as functions of \(c, k\).

Conjecture 3.3. The filtrations \(F_{c,k}^\bullet\) and \(F_{c,k}^\bullet\) have the same Poincaré polynomial.

Example 3.4. Let \(\Gamma = \mathbb{Z}_\ell\) (the cyclic group of order \(\ell > 1\)), and \(n = 1\). Take \(\lambda = \omega_0\) (so \((\lambda, \alpha_i) = 0\) for all \(i > 0\)). In this case, denote Verma modules in category \(\mathcal{O}\) by \(M_1, \ldots, M_\ell\) (they are labeled by the characters of \(\Gamma\)). We have inclusions \(M_\ell \subset \cdots \subset M_1\), and the simple modules are the 1-dimensional modules \(L_i = M_i/M_{i+1}\) for \(i < \ell\), and \(L_\ell = M_\ell\). Thus, in the Grothendieck group of category \(\mathcal{O}\) we have

\[M_i = \sum_{j \geq i} L_j.\]

Hence, by BGG duality for the projective objects we have

\[P_k = \sum_{i \leq k} M_i.\]

So \(M_k = P_k - P_{k-1}\) for \(k > 1\), and \(M_1 = P_1\). Hence for \(1 < i < \ell\), \(L_i = 2P_i - P_{i-1} - P_{i+1}\), while \(L_1 = 2P_1 - P_2\). Thus we see that if \(\chi_1, \chi_2, \ldots, \chi_\ell\) is the standard basis of \(\text{Rep} \, \Gamma\) (where \(\chi_\ell\) is the trivial character), then \(F_0K_0(\text{Rep} \, \Gamma)\) is the subgroup of rank \(\ell - 1\) spanned by \(2\chi_i - \chi_{i-1} - \chi_{i+1}\) for \(i < \ell\) (where \(\chi_0\) should be dropped).

On the other hand, as explained in Example 3.1, we have

\[L_i = 2P_i - P_{i-1} - P_{i+1},\]

where the indexing is understood cyclically. So \(F_0K_0(\text{Rep} \, \Gamma)\) is the subgroup of rank \(\ell - 1\) spanned by \(2\chi_i - \chi_{i-1} - \chi_{i+1}\) for \(i < \ell\) (where \(\chi_0\) should be interpreted as \(\chi_\ell\)).

We see that while the Poincaré polynomials of both filtrations are the same (and equal \(1 + (\ell - 1)t\)), the filtrations are not quite the same and even are not isomorphic (although they, of course, become isomorphic after tensoring with \(\mathbb{Q}\)). Indeed, the quotient group for the filtration \(F_\bullet\) is free (i.e., \(\mathbb{Z}\)), while for the filtration \(F_\bullet\) it is not free (namely, it is \(\mathbb{Z} \oplus \mathbb{Z}_\ell\)).

In a similar manner one checks the coincidence of the Poincaré polynomials for any value of \(\lambda\) (or \(c\)) and \(n = 1\); namely, both polynomials are equal to \(\ell + m(t-1)\), where \(m\) is the dimension of the span of the roots \(\alpha\) of \(\mathfrak{g}\) such that \((\lambda, \alpha)\) is an integer. This implies that the Poincaré polynomials also coincide for \(H_{c,0}(\Gamma_n)\) for \(n > 1\).

4. The Inner Product on the Grothendieck Group of the Category of Finite Dimensional Modules

4.1. The inner product \((\ , \)\) It is known (see e.g. [ES]) that \(H_{c,k}(\Gamma_n)\) has finitely many irreducible finite dimensional modules. So we can define the finitely generated free abelian group \(K_0(H_{c,k}(\Gamma_n)\text{-mod}_f)\) (where the subscript \(f\) denotes finite dimensional modules). We have a natural pairing

\[B: K_0(H_{c,k}(\Gamma_n)\text{-mod}) \times K_0(H_{c,k}(\Gamma_n)\text{-mod}_f) \to \mathbb{Z}\]
given by
\[ B(M, N) = \dim \text{RHom}(M, N), \]
where the dimension is taken in the supersense. Note that this makes sense since \( H_{c,k} \) has finite homological dimension. Also, the same formula, \( (M, N) = \dim \text{RHom}(M, N) \), defines an inner product on \( K_0(H_{c,k}(\Gamma_n)-\text{mod}) \).

**Conjecture 4.1.** The inner product \((\ , \ )\) is symmetric and positive definite (in particular, nondegenerate).

The motivation for Conjecture 4.1 is explained in Section 6.4. Conjecture 4.1 would imply

**Conjecture 4.2.** The natural map
\[ \zeta: K_0(H_{c,k}(\Gamma_n)-\text{mod}) \to F_0K_0(H_{c,k}(\Gamma_n)-\text{mod}) \]
is injective (hence an isomorphism).

Indeed, \( B(\zeta(M), N) = (M, N) \).

**Example 4.3.** Let \( n = 1 \). In this case, from (2) we see that \( (L_i, L_j) = a_{ij} \), and Conjecture 4.1 follows.

In a similar way, using the results of [CBH], one can show that Conjecture 4.1 holds for any \( \lambda \) and \( n = 1 \). This implies that it holds for \( H_{c,0}(\Gamma_n) \) for \( n > 1 \).

**Proposition 4.4.** \(^1\) Let \( \Gamma \) be a cyclic group. Then for any \( H_{c,k}(\Gamma_n) \)-modules \( M, N \) from category \( O \), one has
\[ \text{Ext}^i_O(M, N) \cong \text{Ext}^i(M, N), \]
where the subscript \( O \) means that \( \text{Ext} \) is taken in category \( O \).

**Proof.** Fix a resolution of \( M \) by finitely generated free modules over \( H := H_{c,k}(\Gamma_n) \):
\[ \cdots \to F_1 \to F_0 \to M \to 0. \]
Then \( \text{Ext}^i(M, N) \) is the cohomology of the corresponding complex
\[ \text{Hom}_H(F_0, N) \to \text{Hom}_H(F_1, N) \to \cdots. \]  
(3)
Let \( \hat{H} \) be the completion of \( H \) near 0 as a right \( \mathbb{C}[y_1, \ldots, y_n] \)-module:
\[ \hat{H} := H \otimes_{\mathbb{C}[y_1, \ldots, y_n]} \mathbb{C}[[y_1, \ldots, y_n]] := \varprojlim H/Hm^r, \]
where \( m \) is the augmentation ideal in \( \mathbb{C}[y_1, \ldots, y_n] \). Then the \( H \)-action on \( M \) and \( N \) extends to a continuous action of the completed algebra \( \hat{H} \). Set \( \hat{F}_i = \hat{H} \otimes H F_i \). Then complex (3) coincides with the complex
\[ \text{Hom}_{\hat{H}}(\hat{F}_0, N) \to \text{Hom}_{\hat{H}}(\hat{F}_1, N) \to \cdots, \]  
(4)
**Lemma 4.5.** The sequence
\[ \cdots \to \hat{F}_1 \to \hat{F}_0 \to M \to 0 \]
is exact, i.e., it is a resolution of \( M \) as an \( \hat{H} \)-module.

\(^1\)The author is grateful to I. Losev, who caught an error in the original proof of this proposition.
Proof. Step 1. One has $\hat{H} \otimes_H M = M$. Indeed, we have natural maps $M \to \hat{H} \otimes_H M \to M$ (the second map is the action of $\hat{H}$ on $M$), the composition is the identity, and the first map is surjective.2

Thus, it suffices to show that $\text{Tor}_i^H(\hat{H}, M) = 0$ for $i > 0$.

Step 2. One has $\text{Tor}_i^H(\hat{H}, M) = 0$ for $i > 0$ if $M = M(\tau)$ is a Verma module.

Indeed, in this case we can take $F_\bullet$ to be the Koszul resolution: $F_j = H \otimes \bigwedge^j (y_1, \ldots, y_n) \otimes \tau$, with the differential being the Koszul differential written in terms of right multiplication by $y_i$. Then $\hat{F}_j = \hat{H} \otimes \bigwedge^j (y_1, \ldots, y_n) \otimes \tau$ with differential defined in the same way, so the exactness of $\hat{F}_\bullet$ follows from the following claim.

Claim. Let $E$ be any vector space, and let $C_\bullet = E[[y_1, \ldots, y_n]] \otimes \bigwedge^\bullet (y_1, \ldots, y_n)$, equipped with the Koszul differential. Then $C_\bullet$ is exact in all nonzero degrees, and its zeroth homology is $E$.

Step 3. One has $\text{Tor}_i^H(\hat{H}, M) = 0$ for $i > 0$, for any $M \in \mathcal{O}$.

This is shown by induction in $i$. For brevity, write $T_i(M)$ for $\text{Tor}_i^H(\hat{H}, M)$. For each particular $i$, by the long exact sequence of homology, it suffices to prove that $T_i(M) = 0$ for irreducible $M$ (as any module in $\mathcal{O}$ has finite length).

The base of induction ($i = 1$) also follows from the long exact sequence of homology. Indeed, let $M = L(\tau)$ be irreducible, and $0 \to K \to M(\tau) \to M \to 0$ be a short exact sequence. By Steps 1 and 2, a portion of the corresponding long exact sequence looks like:

$$0 = T_1(M(\tau)) \to T_1(M) \to K \to M(\tau) \to M \to 0,$$

and the map $K \to M(\tau)$ is the same as in the short exact sequence, so $T_1(M) = 0$.

It remains to fulfill the induction step. Assume that the statement is known for $i = m$ and let us prove it for $i = m + 1$. Again assuming $M$ is irreducible and considering the above short exact sequence, we get from the corresponding long exact sequence (using Step 2 and the induction assumption):

$$0 = T_{m+1}(M(\tau)) \to T_{m+1}(M) \to T_m(K) = 0,$$

so $T_{m+1}(M) = 0$, as desired.

The lemma is proved.

By Lemma 4.5, the sequence

$$\cdots \to \hat{F}_1 \to \hat{F}_0 \to M \to 0$$

is a resolution of $M$ in the pro-completion of category $\mathcal{O}$. Since category $\mathcal{O}$ has enough projectives ([GGOR]), this resolution is quasiisomorphic to a resolution inside category $\mathcal{O}$. Thus, complex (4) computes $\text{Ext}_\mathcal{O}^\bullet(M, N)$, as desired.

Corollary 4.6. Conjecture 4.1 (and hence 4.2) holds for the case of cyclic $\Gamma$ (and arbitrary $n, c, k$).

---

2Indefinitely, if $\sum_{i \geq 0} a_i, a_i \in H$, is a convergent series in $\hat{H}$ then for any $v \in M$, there is $i_0$ such that for $i \geq i_0$ one has $a_i v = 0$, so $(\sum_{i \geq 0} a_i) \otimes v = \sum_{i=0}^{i_0} a_i \otimes v$, which is the image of $\sum_{i=0}^{i_0} a_i v \in M$. 
Proof. Let $L_i, i = 1, \ldots, p$, be the irreducible modules in $O$. Let $P_i$ be the projective covers of $L_i$, and $M_i$ the standard modules. Then by the BGG reciprocity

$$[P_i, L_j] = \sum_s [P_i : M_s][M_s : L_j] = \sum_s [M_s : L_i][M_s : L_j].$$

So if $N$ is the matrix of multiplicities $[M_s : L_i]$ then we see that the Cartan matrix $C = ([P_i, L_j])$ is given by the formula $C = N^T N$. Thus $C^{-1}$ is symmetric and positive definite. On the other hand, using Proposition 4.4, it is clear that $(L_i, L_j) = (C^{-1})_{ij}$ if $L_i, L_j$ are finite dimensional. So the Gram matrix of the form $(,)$ in the basis $L_i$ is a principal submatrix of the matrix $C^{-1}$. This implies Conjecture 4.1. □

4.2. The $q$-deformed inner product. One can also consider the $q$-deformation of the inner product $(,)$:

$$(M, N)_q = \sum (-q)^j \dim \text{Ext}^j(M, N).$$

Conjecture 4.7. If $q$ is not a root of unity, then the form $(,)_q$ is nondegenerate.

The motivation for Conjecture 4.7 is explained in Section 6.4.

Example 4.8. Let $\Gamma \neq 1, n = 1, \lambda = \omega_0$. Then by (2), the matrix of $(,)_q$ in the basis of simple modules is the $q$-deformed Cartan matrix $A_q$: $(a_q)_{ij} = 1 + q^2$, $(a_q)_{ij} = -q$ if $i$ is connected to $j$, and zero otherwise. It is well known from the work of Lusztig and Kostant that $\det(A_q)$ is a ratio of products of binomials of the form $1 - q^l$. This implies Conjecture 4.7 in this case. Similarly one handles the case of general $\lambda$ and the case of $H_{c,0}(\Gamma_n), n > 1$.

Remark 4.9. After this paper appeared online, Conjecture 4.7 for cyclic groups $\Gamma$ under some restrictions on the parameters was proved by Gordon and Losev [GL].

5. The Singular and Aspherical Hyperplanes

5.1. Singular hyperplanes. I. Losev [Lo, Theorem 1.4.2] showed that the algebra $H_{c,k}(\Gamma_n)$ is simple outside of a countable collection of hyperplanes in the space of parameters $c, k$. The following conjecture makes this statement more precise.

Conjecture 5.1. The algebra $H_{c,k}(\Gamma_n)$ is simple if and only if $(\lambda, k)$ does not belong to the hyperplanes

$$E_{m,N}: km + N = 0, \ m \in \mathbb{Z}, \ 2 \leq m \leq n, \ N \in \mathbb{Z}, \ \gcd(m, N) = 1,$$

and the hyperplanes

$$H_{a,m,N}: (\lambda, a) + km + N = 0,$$

where $m$ is an integer with $|m| \leq n - 1$, $\alpha$ is a root of $g$, and $N \in \mathbb{Z}_{\geq 0}$.

Conjecture 5.1 holds for $n = 1$ (by [CBH]), and also for any $n$ in the case of cyclic $\Gamma$. Indeed, in this case, the simplicity of $H_{c,k}$ is equivalent to the semisimplicity of the corresponding cyclotomic Hecke algebra (due to R. Vale; see [BC], Theorem 6.6), which is known to be semisimple exactly away from the above hyperplanes.
5.2. The aspherical locus. It is also interesting to consider the aspherical locus. Namely, let \( e_{\Gamma_n} = \frac{1}{|\Gamma_n|} \sum_{g \in \Gamma_n} g \) be the averaging element for \( \Gamma_n \).

**Definition 5.2.** The aspherical locus \( A(\Gamma_n) \) is the set of \((\lambda, k)\) such that there exists a nonzero \( H^c,k(\Gamma_n)\)-module \( M \) with \( e_{\Gamma_n} M = 0 \) (i.e., equivalently, the two-sided ideal generated by \( e_{\Gamma_n} \) in \( H^c,k(\Gamma_n) \) is a proper ideal).

**Conjecture 5.3.** The aspherical locus is the union of the hyperplanes \( E_{m,N} \) for \( 1 \leq N \leq m - 1 \) and the hyperplanes \( H_{\alpha,m,N} \) for \(|m| \leq n - 1 \) and

\[
0 \leq N \leq \sqrt{n + \frac{m^2}{4} + \frac{m}{2} - 1}.
\]

For cyclic \( \Gamma \), this conjecture is proved in [DG]. It is also easy to prove it for \( n = 1 \) (in this case it follows from [CBH]). Also, for any \( \Gamma, n \) we have

**Theorem 5.4.** The hyperplanes specified in Conjecture 5.3 are contained in the aspherical locus.

**Proof.** Let us start with the hyperplanes \( k = -N/m \). It follows from [Lo], Theorem 1.2.1, that it suffices to show that a point on this hyperplane gives rise to an aspherical point for some parabolic subgroup \( W \subset \Gamma_n \). Take the parabolic subgroup \( S_n \subset \Gamma_n \) (stabilizer of a point \((v, \ldots, v)\), where \( 0 \neq v \in V \)). It is well known that \( k = -N/m \) with \( N, m \) as in the theorem are aspherical for \( S_n \). So we are done in this case.

Consider now the hyperplanes \( H_{\alpha,m,N} \). Let us first show that the generic point of each hyperplane is contained in the aspherical locus. By [Lo], to this end, it suffices to show that for a generic point of each hyperplane there is a finite dimensional representation of \( H^c,k(\Gamma_q) \) which is killed by \( e_{\Gamma_q} \) for some \( q \leq n \). Such a representation was constructed in the paper [EM]. Namely, for a positive real affine root \( \tilde{\alpha} \) along the hyperplane

\[
(\lambda, \tilde{\alpha}) + k(b - a) = 0
\]

there is a representation \( U \) of \( H^c,k(\Gamma_q) \) which, as a \( \Gamma_q \)-module, looks like \( \pi_{a,b} \otimes Y_{\Gamma_q} \), where \( Y \) is an irreducible representation of \( \Gamma_q \), and \( \pi_{a,b} \) is the irreducible representation of \( S_q \) whose Young diagram is a rectangle of width \( a \) and height \( b \) (so \( q = ab \)). The space \( e_{\Gamma_q} U \) of \( \Gamma_q \)-invariants in \( U \) is \( e_{\Gamma_q} U = (\pi_{a,b} \otimes (Y^T)_{\Gamma_q})_{\deltaY} \).

Let \( \tilde{\alpha} = \alpha + N\delta \). Then \( N = \dim YT \). It is well known that \( e_{\Gamma_q} U = 0 \) if and only if \( N \leq b - 1 \). Also, the hyperplane equation now looks like

\[
(\lambda, \alpha) + k(b - a) + N = 0.
\]

The number \( m = b - a \) can take values between \( 1 - n \) and \( n - 1 \), i.e., \(|m| \leq n - 1 \). However, we have a restriction that the area of the rectangle is \( q: 1 \leq ab = b(b - m) = q \leq n \).

The larger root of the equation \( b(b - m) = n \) with respect to \( b \) is

\[
b_+ = \sqrt{n + \frac{m^2}{4} + \frac{m}{2}},
\]

and the smaller root of this equation is negative. Therefore, if

\[
0 \leq N \leq b_+ - 1,
\]
then one can take \( b = [b_+] \), and we have \( 1 \leq b \leq b_+ \), and also \( b - m \geq 1 \) (as \( n \geq |m| + 1 \)), so we obtain an aspherical representation. But this is exactly the condition in the theorem.

Finally, we should separately consider the case \( N = 0 \) and \( \alpha < 0 \), i.e., \( (\lambda, \alpha) + km = 0 \) (as in this case \( \bar{\alpha} \) is not a positive root). But in this case we can replace \( \alpha \) with \(-\alpha\) and \( m \) with \(-m\).

To pass from the generic point on the hyperplane to an arbitrary point, one can use Proposition 2.4.

The theorem is proved. \( \Box \)

5.3. Aspherical locus and homological dimension. One of the reasons aspherical values are interesting is the following theorem, connecting them to homological dimension.

Let \( V \) be a finite dimensional symplectic vector space, and \( G \subset Sp(V) \) a finite subgroup. Let \( t \in \mathbb{C}, \ c \) be a conjugation-invariant \( \mathbb{C} \)-valued function on the set of symplectic reflections in \( G \), and let \( H = H_{t,c}(G, V) \) be the symplectic reflection algebra attached to \( G, V, t, c \) (see [EG]). Let \( e = e_G \) be the symmetrizing idempotent of \( G \), and \( eH_e \) be the corresponding spherical subalgebra.

The proof of the following theorem was explained to me by R. Bezrukavnikov.

Theorem 5.5. The algebra \( eH_e \) has finite homological dimension if and only if \( HeH = H \) (in which case \( eH_e \) is Morita equivalent to \( H \), so its homological dimension is \( \dim V \)).

In other words, \((t, c)\) belongs to the aspherical locus if and only if \( eH_e \) has infinite homological dimension.

The proof of Theorem 5.5 is based on the following general proposition.

Proposition 5.6. Let \( A \) be an associative algebra with an idempotent \( e \). Assume that:

(i) \( A \) has finite (left) homological dimension;
(ii) \( eAe \) is a Gorenstein algebra (i.e., its dualizing module is an invertible bimodule), and \( eA \) is a Cohen–Macaulay module over \( eAe \) of dimension \( 0 \);

and

(iii) the natural map \( \phi: A \to \text{End}_{eAe}(eA) \) is an isomorphism.

Then \( eAe \) has finite homological dimension if and only if \( AeA = A \).

Proof. If \( AeA = A \) then the functor \( N \mapsto eN \) is an equivalence of categories \( A\text{-mod} \to eA\text{-mod} \) (i.e., \( A \) and \( eA \) are Morita equivalent), so the homological dimension of \( eAe \) equals the homological dimension of \( A \), i.e., is finite (so for this part of the proposition we only need condition (i)). It remains to show that if the homological dimension of \( eAe \) is finite then \( AeA = A \).

Assume that \( eAe \) has finite homological dimension. Consider the functor \( F \) from \( eAe\text{-mod} \) to \( A\text{-mod} \) defined by

\[
F(N) = \text{Hom}_{eAe}(eA, N).
\]
We will show that conditions (i) and (ii) imply that the functor $F$ is exact, i.e., $eA$ is a projective $eAe$-module. Taking into account condition (iii), this means that $eA$ defines a Morita equivalence between $eAe$ and $A$, so $AeA = A$, as desired.

To show that $F$ is exact, denote by $\omega_{eAe}$ the dualizing module for $eAe$, and note that by condition (ii),

$$\text{Ext}^i_{eAe}(eA, \omega_{eAe}) = 0, \quad i > 0,$$

which implies that for any projective $eAe$-module $P$, we have

$$\text{Ext}^i_{eAe}(eA, P) = 0, \quad i > 0$$

(as $\omega_{eAe}$ is an invertible bimodule, and hence $P$ is a direct summand in $Y \otimes \omega_{eAe}$, where $Y$ is a vector space). Now by induction in $m$, from the long exact sequence of cohomology it follows that if $N$ has a projective resolution of length $m$, then

$$\text{Ext}^i_{eAe}(eA, N) = 0, \quad i > 0.$$

But the assumption that $eAe$ has finite homological dimension implies that any $eAe$-module $N$ has a finite projective resolution. Thus, the last equality holds for any $eAe$-module $N$, which implies that the functor $F$ is exact. The proposition is proved.

Proof. (of Theorem 5.5) To deduce Theorem 5.5 from Proposition 5.6, it suffices to note that conditions (i)–(iii) of Proposition 5.6 hold for symplectic reflection algebras (see [EG, Theorem 1.5]).

Note that if $t = 0$ then $eHe$ is commutative, and Theorem 5.5 reduces to the statement that Spec($eHe$) is smooth if and only if $HeH = H$, which is proved in [EG]. Moreover, by localizing $H$, one obtains a stronger result (also from [EG]), stating that the smooth and Azumaya loci for $H$ coincide.

More generally, we have the following corollary of Proposition 5.6.

Corollary 5.7. If $A$ has finite homological dimension and the natural map $\phi: A \to \text{End}_{eAe}(eA)$ is an isomorphism, then $U_{\text{sm}} = U_{\text{Az}}$.

This corollary generalizes a result of Tikaradze [T], who proved, in particular, that $U_{\text{sm}} = U_{\text{Az}}$ for symplectic reflection algebras in positive characteristic.

6. Relation to the Representation Theory of Affine Lie Algebras

6.1. The setup. The conjectural relation of the questions in Sections 3, 4 to representations of affine Lie algebras is based on the well known fact that the graded space $\bigoplus_{n \geq 0} K_0(\text{Rep}^i_{n})C$ has the same Hilbert series as $F^{\otimes r+1}$, where $F = \mathbb{C}[x_1, x_2, \ldots]$ is the Fock space, and $r$ is the number of nontrivial conjugacy classes in $\Gamma$. 

Consider the Lie algebra $\widetilde{\mathfrak{g}} \oplus \mathbb{C}$. It is spanned by elements $bt^j, b \in \mathfrak{g}, j \in \mathbb{Z}; a_j, j \in \mathbb{Z} \setminus 0$; the scaling element $D$; and the central element $K$. The elements $a_i$ commute with $bt^j$, and we have

$$[D, a_i] = i a_i, \quad [D, bt^j] = i b t^j,$$

and

$$[a_i, a_j] = i \delta_{i-j} K, \quad [at^i, bt^j] = [a, b]t^{i+j} + i \delta_{i-j} (a, b) K,$$

where $(\ , \ )$ is the invariant inner product on $\mathfrak{g}$ normalized so that $(\alpha, \alpha) = 2$ for all roots $\alpha$.

Let $V_0$ be the basic representation of $\widetilde{\mathfrak{g}}$ at level 1 (i.e., with highest weight $\omega_0$); then $V := V_0 \otimes F$ is an irreducible representation of $\widetilde{\mathfrak{g}} \oplus \mathbb{C}$, and by the Frenkel-Kac theorem (see [K]), the space $F_{\otimes r + 1}$ can be viewed as the sum of the weight subspaces of weight $\omega_0 - n\delta$ in this representation; more precisely, one has

$$V = F_{\otimes r + 1} \otimes \mathbb{C}[\mathfrak{Q}_g],$$

where $\mathfrak{Q}_g$ is the root lattice of $\mathfrak{g}$, and for $\beta \in \mathfrak{Q}_g$, $F_{\otimes r + 1} \otimes e^\beta$ is the sum of weight subspaces of weight $\omega_0 - (n + \frac{\beta^2}{2})\delta + \beta$. So for either of the two filtrations defined in the previous sections there is a vector space isomorphism

$$\text{gr} K_0(\text{Rep} \Gamma_n) \cong V[\omega_0 - n\delta],$$

(cf. [FJW]) and the problem of finding the Poincare polynomials of the filtrations can be reformulated as the problem of describing the corresponding grading on $V[\omega_0 - n\delta]$. Below, we will state a conjecture of what this grading is expected to be.

Fix $(\lambda, k)$. Define the Lie subalgebra $\mathfrak{a} = \mathfrak{a}(\lambda, k)$ of $\widetilde{\mathfrak{g}} \oplus \mathbb{C}$ generated by the Cartan subalgebra $\widetilde{\mathfrak{h}}$ and

(1) $a_{ml}, l \in \mathbb{Z} \setminus 0$, if a singular hyperplane $E_{m,N}$ contains $(\lambda, k)$;

(2) $e_{\alpha + m\delta}, e_{-\alpha - m\delta}$ for each singular hyperplane $H_{\alpha,m,N}$ containing $(\lambda, k)$.

Note that this Lie algebra inherits a polarization from $\widetilde{\mathfrak{g}} \oplus \mathbb{C}$. Also note that it is finite dimensional if and only if $k / \in \mathbb{Q}$.

The above grading on $\text{gr} K_0(\text{Rep} \Gamma_n) \subset V[\omega_0 - n\delta]$ can conjecturally be described in terms of the decomposition of the representation $V$ into $\mathfrak{a}$-isotypic components, in a way described below.

We will say that $\mu \in \mathfrak{h}^*$ is a dominant integral weight for $\mathfrak{a}$ if $(\mu, \beta) \in \mathbb{Z}_+$ for any positive root $\beta$ of $\mathfrak{a}$. Denote the set of such weights by $P^\mathfrak{a}_\mu$. For $\mu \in P^\mathfrak{a}_\mu$, let $L_\mu$ be the corresponding irreducible integrable $\mathfrak{a}$-module. It is clear that

$$V|_\mu \cong \bigoplus_{\mu \in P^\mathfrak{a}_\mu} L_\mu \otimes \text{Hom}_\mathfrak{a}(L_\mu, V).$$

Let $\mathfrak{a}' \subset \mathfrak{a}$ be the subalgebra generated by the elements $e_{\alpha + m\delta}, e_{-\alpha - m\delta}$ for $(\lambda, k) \in H_{\alpha,m,N}$. Note that it is a Kac–Moody algebra (finite dimensional or affine).
6.2. The main conjecture (the case of irrational $k$). Suppose first that $k$ is irrational. In this case, the Lie algebra $\mathfrak{g}$ is a finite dimensional Levi subalgebra of $\tilde{\mathfrak{g}}$. Namely, $\mathfrak{g}$ is generated by $\mathfrak{h}$ and $e_{\alpha + m_i \delta} = e_{\alpha}^t m_i \alpha$ for $\alpha$ running through some root subsystem $R'$ of the root system $R$ of $\tilde{\mathfrak{g}}$.

Our main conjecture in the case of irrational $k$ is the following.

**Conjecture 6.1.** For either of the filtrations $F_\ast$ and $F_\ast\ast$, there exists an isomorphism of vector spaces

$$\text{gr}_i K_0(\text{Rep} \Gamma_n) \cong \bigoplus_{\mu \in P^+_n; \mu^2 = -2i} L_\mu[\omega_0 - n\delta] \otimes \text{Hom}_\mu(L_\mu, V),$$

(5)

**Remark 6.2.** Note that if $\text{Hom}_\mu(L_\mu, V) \neq 0$ then $\mu = \omega_0 - j\delta + \beta$ where $\beta \in Q_\mathfrak{g}$ and $\beta^2 / 2 \leq j$, so $\mu^2 = \beta^2 - 2j$ is a nonpositive even integer. Thus, the right hand side of (5) vanishes if $i < 0$. Also, if $L_\mu[\omega_0 - n\delta] \neq 0$ then $\mu = \omega_0 - n\delta + \sum_{i=0}^p \alpha_i$, $\alpha_i \geq 0$, so $\mu^2 \geq -2n$. Thus, the right hand side of (5) vanishes if $i > n$. Therefore, Conjecture 6.1 is valid upon taking the direct sum over $i$.

In fact, the analysis of Remark 6.2 shows that in the extremal case $i = 0$ (i.e., $\mu^2 = 0$), if $\phi: L_\mu \to V$ is an $\mathfrak{a}$-homomorphism and $v_\mu$ a highest weight vector of $L_\mu$, then $\phi(v_\mu)$ is necessarily an extremal vector of $V$. The space of such vectors for each weight $\mu$ is 1-dimensional. Moreover, any such nonzero vector gives rise to a nonzero homomorphism $\phi$. Indeed, for any simple root $\gamma$ of $\mathfrak{a}$ we have $(\mu + \gamma)^2 = \mu^2 + 2(\mu, \gamma) + \gamma^2 = 2(\mu, \gamma) + 2 > 0$, which implies that $\mu + \gamma$ is not a weight of $V$, and hence $\phi(v_\mu)$ is a highest weight vector for $\mathfrak{a}$.

Thus, setting $i = 0$, we obtain from Conjecture 6.1 the following conjecture on the number of finite dimensional representations.

**Conjecture 6.3.** The number of isomorphism classes of finite dimensional irreducible representations of $H_{c,k}(\Gamma_n)$ is equal to

$$\sum_{\mu \in P^+_n; \mu^2 = 0} \dim L_\mu[\omega_0 - n\delta].$$

Indeed, the above discussion implies that for $\mu = \omega_0 - \frac{\beta^2}{2}\delta + \beta$ such that $\mu \in P^+_n$, one has $\dim \text{Hom}_\mu(L_\mu, V) = 1.$

**Example 6.4.** Assume that $\mathfrak{a}' = (\mathfrak{sl}_2)^\ell$. In this case, the point $(\lambda, k)$ lies on the intersection of the hyperplanes $(\lambda, \alpha_i) + km_i + N_i = 0$, $i = 1, \ldots, \ell$, where $\alpha_i$ are pairwise orthogonal roots of $\mathfrak{g}$, and $m_i$ are integers. Let $\tilde{\alpha}^i = \alpha^i + m_i \delta$. We will pick the signs in the hyperplane equations so that $\tilde{\alpha}_i$ are positive roots; in particular, $m_i \geq 0$. Then, $\mu - \omega_0 + n\delta$ should be a nonnegative integer linear combination of $\tilde{\alpha}^i$:

$$\mu - \omega_0 + n\delta = \sum_{i=1}^\ell a_i \tilde{\alpha}^i, \quad a_i \in \mathbb{Z}_+. $$

The condition $\mu^2 = 0$ reads

$$\sum_{i=1} a_i (a_i + m_i) = n.$$  

(6)
Thus, Conjecture 6.3 predicts that the number of finite dimensional representations in this case equals the number of integer solutions \((a_1, \ldots, a_l)\) of equation (6) such that \(a_i \geq 0\).

If \(k\) is a formal variable and \((\lambda, k)\) (where \(\lambda = \lambda(k)\)) is on the above hyperplanes (but not on any other singular hyperplanes), then this statement is in fact true, and can be deduced from the papers [M] and [G]. Let us sketch a proof.

The paper [M] (and by another method, [G]) constructs an irreducible representation of \(H_{c, k}(\Gamma_n)\) for each solution of equation (6) as above. Indeed, fix such a solution \(a = (a_1, \ldots, a_l)\). Let \(\pi_i\) be the representation of the symmetric group \(S_{a_i(a_i+m_i)}\) whose Young diagram is a rectangle with \(a_i\) columns and \(a_i+m_i\) rows if \(a_i^+ + N_0\delta\) is a positive root, and with \(a_i+m_i\) columns and \(a_i\) rows otherwise. Let \(Y_i\) be the representations of \(\Gamma\) corresponding to dimension vectors \(\pm(a_i^+ + N_0\delta)\) (whichever is a positive root), and let

\[
U_a = \text{Ind} \prod_i S_{a_i(a_i+m_i)} \otimes (\pi_i \otimes Y_i^{\otimes a_i(a_i+m_i)}).
\]

Then by [M], [G], \(U_a\) extends to an \(H_{c, k}(\Gamma_n)\) module, and \(U_a\) is not isomorphic to \(U_{a'}\) (even as a \(\Gamma_n\)-module) if \(a \neq a'\). Moreover, we claim that any finite dimensional irreducible \(H_{c, k}(\Gamma_n)\) module is of this form. Indeed, suppose \(U\) is a finite dimensional irreducible \(H_{c, k}(\Gamma_n)\) module (defined over \(\mathbb{C}[k]\)), and \(\bar{U} = U/kU\) be the corresponding module over \(H_{c,0}(\Gamma_n) = \mathbb{C}S_n \rtimes H_{c,0}(\Gamma)^{\otimes n}\). Since the roots \(\alpha_i^\prime\) are orthogonal, the category of finite dimensional modules over the algebra \(\mathbb{C}S_n \rtimes H_{c,0}(\Gamma)^{\otimes n}\) is semisimple, so we have \(\bar{U} = \bigoplus_j U_j\), where \(U_j\) are simple modules. By [M], this means that each of the representations \(\bar{U}_j\) should have a deformation \(U_j\), and \(U = \bigoplus_j U_j\) (as \(\text{Ext}^1(\bar{U}_j, U_j) = 0\), so \(J = 1\) and by [M] \(U = U_a\) for some \(a\).

Note that if the roots \(\alpha^i\) are not orthogonal (i.e., \(a'\) contains components of rank > 1) then this proof fails (and the conjecture, in general, predicts more representations than constructed in [M], [G]). Presumably, the missing representations are constructed by \(k\)-deforming reducible but not decomposable representations of \(H_{c,0}(\Gamma_n)\).

Example 6.5. Suppose \(n = 1\). Let \(a'^\prime = a' \cap g\). Since the algebra \(H_{c, k}(\Gamma_1) = H_c(\Gamma)\) does not depend on \(k\), it follows from the above that \(\dim F_0 = \dim F_0'\) is the semisimple rank of \(a'^\prime\). Let us show that this agrees with the prediction of Conjecture 6.3 (so Conjectures 6.1 and 6.3 are true in this case). Indeed, if \(L_{\mu} [\omega_0 - \delta] \neq 0\), we have \(\mu = \omega_0 - \delta + \beta\), where \(\beta\) is a root of \(a'^\prime\). Also, for any positive root \(\gamma\) of \(a'^\prime\) we must have \((\mu, \gamma) = (\beta, \gamma) \geq 0\), which implies that \(\beta\) is the maximal root of \(a'^\prime\). So dim \(L_{\mu} [\omega_0 - \delta]\) is the semisimple rank of \(a'^\prime\), as desired.

6.3. The case of rational \(k\)

6.3.1. Integer \(k\). In the case when \(k\) is rational, the situation becomes more complicated. First consider the situation when \(k\) is an integer. In this case, the situation is similar to the case of irrational \(k\), except that the Kac-Moody Lie algebra \(a'\) is affine, rather than finite dimensional. More precisely, \(a' = \tilde{g}'\), where \(g' \subset g\) is the semisimple subalgebra generated by the root elements corresponding to roots
\( \alpha \) with \((\lambda, \alpha) \in \mathbb{Z}\). Thus, Conjecture 6.1 predicts that
\[
\text{gr}_s K_0(\text{Rep}\, \Gamma_n) \cong F^{\otimes s}[i - n] \otimes F^{\otimes r + 1 - s}[i],
\]
where \( s \) is the rank of \( g' \), and the numbers in the square brackets mean the degrees.

**Theorem 6.6.** Conjectures 6.1 and 6.3 hold for \( k = 0 \).

**Proof.** This follows from the fact that \( H_{c,0}(\Gamma_n) = CS_n \times H_c(\Gamma)^{\otimes n} \), so the understanding the filtrations \( F_* \) and \( F_* \) reduces to the case \( n = 1 \). \( \square \)

We expect that the situation is the same for any \( k \in \mathbb{Z} \), because of the existence of translation functor \( k \to k + 1 \) defined in [EGGO]. This functor is an equivalence of categories outside of aspherical hyperplanes, and expected to always be a derived equivalence.

### 6.3.2. Non-integer \( k \in \mathbb{Q} \)

Now consider the case of rational non-integer \( k \), with denominator \( m > 1 \). Then we have \( a = \mathfrak{h} + \mathfrak{a}' + \mathcal{H}_m \), where \( \mathcal{H}_m \) is the Heisenberg algebra generated by \( a_{mi} \). In this case, the one-index filtrations and gradings we have considered above can actually be refined to two-index ones. More specifically, it is easy to see that the possible annihilator varieties are the varieties \( Y_{p,j} \subset V_n \) of points with some \( p \) (\( V \)-valued) coordinates equal to zero and \( j \) collections of \( m \) other coordinates whose images in \( V/\Gamma \) coincide (where \( p, j \in \mathbb{Z}_+ \) and \( p + jm \leq n \)).\(^3\)

Clearly, \( Y_{p,j} \) contains \( Y_{p+1,j}, Y_{p,j+1} \) and \( Y_{p+m,j-1} \) and moreover \( Y_{p,j}^0 := Y_{p,j} \setminus (Y_{p+1,j} \cup Y_{p,j+1} \cup Y_{p+m,j-1}) \) are locally closed smooth subvarieties which define a finite stratification of \( V \). So we can define \( F_{i,j} K_0(\text{Rep}\, \Gamma_n) \) to be spanned by the classes of the modules \( M \) with \( AV(M) \subset Y_{n-i-jm,j} \), and set
\[
\text{gr}^F_{i,j} K_0 = F_{i,j} K_0 / (F_{i-m,j+1} K_0 + F_{i-1,j} K_0 + F_{i,j-1} K_0).
\]
A similar definition is made for the filtration \( F \). Namely, we define \( X_{p,j} \subset \mathbb{C}^n \) to be the varieties of points with some \( p \) coordinates equal to zero and \( j \) collections of \( m \) of coordinates whose images in \( \mathbb{C}/\Gamma \) coincide, and define \( F_{i,j} K_0(\text{Rep}\, \Gamma_n) \) to be spanned by classes of modules with support (as a \( \mathbb{C}[x_1, \ldots, x_n] \)-module) contained in \( X_{n-i-jm,j} \). Then we set
\[
\text{gr}^F_{i,j} K_0 = F_{i,j} K_0 / (F_{i-m,j+1} K_0 + F_{i-1,j} K_0 + F_{i,j-1} K_0).
\]

Note that for either of the filtrations,
\[
\text{gr}_s K_0 = \bigoplus_{i,j \geq 0 : i + j = s} \text{gr}_{i,j} K_0
\]

Now we will formulate a conjectural description of the space \( \text{gr}_{i,j} K_0 \). To this end, define the element
\[
\partial_{m} = \frac{1}{m} \sum_{l=1}^{\infty} a_{-ml} a_{ml}.
\]

Obviously, \( \partial_{m} \) has zero weight, and its eigenvalues are in \( \mathbb{Z}_+ \) on every module \( L_{\mu} \).

We denote by \( L_{\mu}[\lambda, j] \) the subspace of \( V \) of weight \( \lambda \) and eigenvalue \( j \) of \( \partial_{m} \).

\(^3\)For example, for \( m = 2 \) and \( n = 3 \), \( Y_{1,1} \subset V_3 \) is the set of points \((v, \gamma v, 0), (v, 0, \gamma v), \text{and} (0, v, \gamma v)\), where \( \gamma \in \Gamma \).
Conjecture 6.7. For either of the filtrations $F_\ast$ and $F_\ast$, there exists an isomorphism of vector spaces

$$\text{gr}_{i,j} K_0(\text{Rep } \Gamma_n) \cong \bigoplus_{\mu \in \left(\mathbb{Z}^+, \mu_2 = -2i\right)} L_\mu[\omega_0 - n\delta, j] \otimes \text{Hom}_n(L_\mu, V). \quad (7)$$

Note that this subsumes Conjecture 6.1 (by putting $m = \infty$).

For finite dimensional representations, $j$ must be zero, so we have

Conjecture 6.8. The number of isomorphism classes of finite dimensional irreducible representations of $H_{c,k}(\Gamma_n)$ is equal to

$$\sum_{\mu \in \left(\mathbb{Z}^+, \mu_2 = 0\right)} \dim L_\mu[\omega_0 - n\delta, 0].$$

Example 6.9. Let $\Gamma = 1$. In this case, $\mathfrak{a} = \mathcal{H}_m$, and the Fock space $V = \mathcal{F}$ factors as $\mathcal{F} = \mathcal{F}_m \otimes \mathcal{F}'_m$, where $\mathcal{F}_m$ is the Fock space for $\mathcal{H}_m$, and $\mathcal{F}'_m$ is the Fock space of the Heisenberg algebra $\mathcal{H}'_m$ generated by $a_s$ with $s$ not divisible by $m$. Also we have $\mu = \omega_0 - i\delta$ for some $i \in \mathbb{Z}_+$. Thus Conjecture 6.7 predicts that

$$\text{gr}_{i,j} K_0(\text{Rep } \Gamma_n) \cong L_{\omega_0 - i\delta}[\omega_0 - n\delta, j] \otimes \text{Hom}_n(L_{\omega_0 - i\delta}, \mathcal{F}).$$

For the first factor to be nonzero, we need $i = n - jm$, so the prediction is that $\text{gr}_{i,j}$ vanishes unless $i = n - jm$, in which case it simplifies to

$$\text{gr}_{n-jm,j} K_0(\text{Rep } \Gamma_n) \cong \text{gr}_{n-j(m-1)} K_0(\text{Rep } \Gamma_n) \cong \mathcal{F}_m[-jm] \otimes \mathcal{F}'_m[jm - n].$$

In particular, there are no finite dimensional representations, since one never has $i, j = 0$ (as $i + jm = n$).

For filtration $F$, this prediction is in fact a theorem. Namely, first of all, it is clear from [BE] that $p = n - i - jm$ must be zero, since the Weyl algebra has no finite dimensional representations. Also, one has the following much stronger result, proved recently by S. Wilcox.

Assume that $k < 0$ (the case $k > 0$ is similar). Let $\mathcal{C}_j$ be the category of modules over the rational Cherednik algebra $H_{1,k}(S_n)$ from category $O$ which are supported on $X_j := X_{0,j}$ modulo those supported on $X_{j+1}$ (for $0 \leq j \leq n/m$). Let $H^j_q(N)$ be the finite dimensional Hecke algebra corresponding to the symmetric group $S_N$ and parameter $q$.

Theorem 6.10 (S. Wilcox, [W]). The category $\mathcal{C}_j$ is equivalent to

$$\text{Rep } S_j \boxtimes \text{Rep } H^j_q(n - jm),$$

where $q = e^{2\pi i k}$.

In more detail, the irreducible module $L_k(\lambda)$ whose lowest weight is the partition $\lambda$ of $n$ is supported on $X_j$ but not $X_{j+1}$ if and only if $\lambda = m\mu + \nu$, where $\mu$ is a partition of $j$ and $\nu$ is an $m$-regular partition (i.e., each part occurs $< m$ times), and in $\text{Rep } S_j \boxtimes \text{Rep } H^j_q(n - jm)$ it is expected to correspond to $\pi_\mu \boxtimes D_\nu$, where $\pi_\mu$ is the irreducible representation of $S_j$ corresponding to $\mu$, and $D_\nu$ is the irreducible Dipper-James module corresponding to $\nu$ [DJ].
Remark 6.11. We note that in the important special case when $a'$ is the winding subalgebra 
\[ a' = g[[t^m, t^{-m}]] \oplus C K, \]
$2 \leq m \leq \infty$ (where for $m = \infty$, $g[[t^m, t^{-m}]] \oplus C K$ should be replaced by $g$), the decomposition of $V_0$ as an $a'$-module is completely described in the paper [F2] (see also [KW]). This decomposition easily yields the decomposition of $V$ as an $a$-module, which appears in our conjectures.

Remark 6.12. After this paper appeared online, Conjecture 6.7 for cyclic groups $\Gamma$ and filtration $F$ under some technical assumptions on the parameters was proved by Shan and Vasserot [SV].

6.4. Motivation for Conjectures 4.1 and 4.7. Now we are ready to explain the motivation behind Conjectures 4.1 and 4.7. Namely, we expect that the inner product $(\cdot, \cdot)$ on finite dimensional modules is obtained by restriction of the Shapovalov form on $V$ to appropriate isotypic components. Note that this was shown in [SV] for cyclic $\Gamma$ and rational $k$, under some restriction on the other parameters [SV, Remark 5.14]. This would imply Conjecture 4.1, since $V$ is a unitary representation, and the Shapovalov form is positive definite. We also expect that the form $(\cdot, \cdot)_q$ is obtained by a similar restriction to isotypic components of the Shapovalov form for the quantum analog of $V$, (i.e., the basic representation $V_q$ of the corresponding quantum affine algebra $U_q(\widehat{g \oplus \mathbb{C}})$). This would imply Conjecture 4.7, since this form degenerates only at roots of unity.

6.5. Relation to quantum connections. Consider the special case when $(\lambda, k)$ is a generic point of a singular hyperplane $H$. In this case, it was conjectured by Bezrukavnikov and Okounkov that the spaces $gr_i K_0(\operatorname{Rep} \Gamma_n \mathbb{C})$ should be eigenspaces of the residue $C_H$ on the hyperplane $H$ of the quantum connection on the equivariant quantum cohomology of the Hilbert scheme of the minimal resolution of the Kleinian singularity $\mathbb{C}^2/\Gamma$ (cf. [MO]). This conjecture agrees with our conjectures (which, in the case of a generic point on a hyperplane, should be provable by using deformation theory). Namely, in the case of $\Gamma = 1$ and the hyperplane $H = E_{m,N}$ (i.e., $k = -N/m$), this is explained in [BE]; the case of general $\Gamma$ is similar, see [MO]. So let us consider the remaining case $H = H_{\alpha,m,N}$. In this case, $a'$ is the $\mathfrak{sl}_2$-subalgebra corresponding to the root $\alpha + m\delta$, and $C_H = C_{\alpha + m\delta}$ is the Casimir of this $\mathfrak{sl}_2$-subalgebra. So the conjecture of Bezrukavnikov and Okounkov says that the decomposition into $gr_i$ is the decomposition into eigenspaces of the Casimir $C_{\alpha + m\delta}$. But this decomposition is clearly the same as the decomposition into isotypic components for $a$, which implies that the conjectures agree.

More generally, suppose $(\lambda, k)$ is a generic point of the intersection of several singular hyperplanes. According to the insight of Bezrukavnikov and Okounkov, local monodromy of the quantum connection near $(\lambda, k)$ should preserve the space $gr_i$ (after an appropriate conjugation). One also expects that in the neighborhood of $(\lambda, k)$ this connection is equivalent to its “purely singular part”, i.e., the connection obtained from the full quantum connection by deleting all the regular terms of the Taylor expansion. This agrees with our conjectures, since the residues of the quantum connection on all the singular hyperplanes passing through $(\lambda, k)$ lie in
(a completion of) $U(a)$ and hence preserve the decomposition of $V$ into $a$-isotypic components.

6.6. Relation to the work of Gordon-Martino and Shan and to level-rank duality. Let $\Gamma = \mathbb{Z}_\ell$ ($\ell = r + 1$). We will assume for simplicity that $(\lambda, \alpha_i) = 0$ for $i = 1, \ldots, \ell - 1$ (we expect that the arguments below extend to the general case).

6.6.1. Irrational $k$. First consider the case when $k$ is irrational. In this case, a construction due to I. Gordon and M. Martino and independently P. Shan [Sh] gives (under some technical conditions) a (projective) action of $\mathfrak{gl}_\infty$ of level $\ell$ on $\bigoplus_n K_n(\text{Rep} \Gamma_n)^{\mathbb{C}} = \mathcal{F}^{\otimes \ell}$ of categorical origin. Here by $\mathfrak{gl}_\infty$ we mean the Lie algebra of infinite matrices with finitely many nonzero diagonals [K], and the above representation is the $\ell$-th power of the usual Fock module (with highest weight $\omega_0$).

The action of Gordon–Martino and Shan is based on induction and restriction functors for the rational Cherednik algebra $H_{c,k}(\Gamma_n)$ constructed in [BE]; in particular, finite dimensional representations are singular vectors (and one should expect that conversely, any singular vector is a linear combination of finite dimensional representations).

So let us find the singular vectors of this action of $\mathfrak{gl}_\infty$. To this end, we need to decompose $\mathcal{F}^{\otimes \ell}$ into isotypic components for $\mathfrak{gl}_\infty$. For this purpose, consider the product

$$E = \mathcal{F}^{\otimes \ell} \otimes \mathbb{C}[\mathbb{Z}_\ell],$$

where for $a \in \mathbb{Z}_\ell$ the vector $1 \otimes e^a$ is put in degree $-a^2/2$. By the boson-fermion correspondence ([K]),

$$E = (\wedge^{\infty + \bullet} U)^{\otimes \ell},$$

where $U$ is the vector representation of $\mathfrak{gl}_\infty$, and $\wedge^{\infty + \bullet}$ denotes the space of semi-infinite wedges (of all degrees). Thus,

$$E = \wedge^{\infty + \bullet}(U \otimes \mathbb{C}^\ell).$$

By an infinite dimensional analog of the Schur–Weyl duality between $\mathfrak{gl}(U)$ and $\mathfrak{gl}(W)$ inside $\wedge(U \otimes W)$ for finite dimensional spaces $U$ and $W$ (which is a limiting case of level-rank duality, see below), this implies that the centralizer of $\mathfrak{gl}_\infty$ in this representation is the algebra generated by $\mathfrak{gl}_\ell$ acting by degree-preserving transformations. More precisely, we have a decomposition [F1, Theorem 1.6]:

$$E = \bigoplus_\nu F_{\nu^*} \otimes L_\nu,$$

where $\nu$ runs over partitions with at most $\ell$ parts, $L_\nu$ are the corresponding representations of $\mathfrak{gl}_\ell$, and $F_{\nu^*}$ is the irreducible representation of $\mathfrak{gl}_\infty$ corresponding to the dual partition $\nu^*$. Thus, the space of singular vectors is given by the formula

$$E_{\text{sing}} = \bigoplus_\nu L_\nu,$$

and hence

$$F^{\otimes \ell}_{\text{sing}} = \bigoplus_\nu L_\nu[0].$$
Moreover, $L_{\nu}[0]$ sits in degree $-\nu^2/2$. Thus we get that the number of finite dimensional representations equals

$$\sum_{\nu: \nu^2=2n} \dim L_{\nu}[0],$$

as predicted by Conjecture 6.3.

6.6.2. Rational $k$. Now assume that $k$ is a rational number with denominator $m > 1$. Then the construction of Gordon–Martino and Shan [Sh] yields an action of $\hat{\mathfrak{sl}}_m$ of level $\ell$ on $\bigoplus_n K_0(\text{Rep } \Gamma_n) \subset \mathcal{F}^\otimes \ell$, where $\mathcal{F}$ is the Fock representation of $\mathfrak{gl}_\infty$ of highest weight $\omega_0$ restricted to $\hat{\mathfrak{sl}}_m \subset \mathfrak{gl}_\infty$. Namely, the action is defined by induction and restriction functors of [BE]. We expect that by using the restriction functors $O_{c,k}(\Gamma_n) \to O_{c,k}(\Gamma_{n-mj})$ (restricting to the locus with $j$ $m$-tuples of equal coordinates), and the induction functors in the opposite direction, this action can be upgraded to an action of $\hat{\mathfrak{gl}}_m$.

As before, finite dimensional representations are singular vectors for this action, and one should expect that conversely, any singular vector is a linear combination of finite dimensional representations.

So let us find the singular vectors of this action of $\hat{\mathfrak{gl}}_m$. To this end, we need to decompose $\mathcal{F}^\otimes \ell$ into isotypic components for $\hat{\mathfrak{sl}}_m$. For this purpose, as before, consider the space $E$. The space $U$ considered above now equals $\mathbb{C}^m[t, t^{-1}]$, so the boson-fermion correspondence yields

$$E = \bigwedge^\infty_{+} (\mathbb{C}^m \otimes \mathbb{C}[t, t^{-1}])^\otimes \ell,$$

Thus,

$$E = \bigwedge^\infty_{+} (\mathbb{C}^m \otimes \mathbb{C}[t, t^{-1}] \otimes \mathbb{C}^\ell).$$

By an affine analog of the Schur–Weyl duality, this implies that the centralizer of $\hat{\mathfrak{gl}}_m$ in this representation is the algebra generated by $\hat{\mathfrak{sl}}_\ell$ acting at level $m$.

More precisely, this is an instance of the phenomenon called the level-rank duality (cf. [F1]). Namely, let $P_{+0}^{m, \ell}$ be the set of dominant integral weights for $\hat{\mathfrak{sl}}_m$ at level $\ell$ which are trivial on the center of $\text{SL}_m$. (We don’t take into account the action of $D$ here, so this is a finite set). Then level-rank duality implies that there is a natural bijection $\dagger: P_{+0}^{m, \ell} \simeq P_{+0}^{\ell, m}$, and we have a decomposition

$$E = \bigoplus_{\nu \in P_{+0}^{\ell, m}} L_{\nu} \otimes \mathcal{F} \otimes L_{\nu},$$

where $\hat{\mathfrak{gl}}_m$ acts irreducibly in the product of the first and the second components, and $\hat{\mathfrak{sl}}_\ell$ acts irreducibly on the third component in each summand.

This shows that the space of singular vectors is

$$E_{\text{sing}} = \bigoplus_{\nu \in P_{+0}^{\ell, m}} L_{\nu}.$$
Now, the highest weight vector of $L_\nu$ sits in degree $-\nu^2/2$, so we get that the number of finite dimensional irreducible representations of $H_{c,k}(\Gamma_n)$ is

$$\sum_{\nu \in \mathcal{P}_{c,m} : n - \nu^2/2 \in m\mathbb{Z}_{\geq 0}} \dim L_\nu \left[0, \frac{\nu^2}{2} - n\right],$$

where the first entry in the square brackets is weight under $h$, and the second one is the degree (noting that the degrees of vectors in $L_\nu$ are multiples of $m$). This is exactly the prediction of Conjecture 6.8.

References


Department of Mathematics, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

E-mail address: etingof@math.mit.edu