SINGULAR DIFFERENTIAL, INTEGRAL AND DISCRETE EQUATIONS: THE SEMIPOSITONE CASE

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Abstract. Fixed point methods play a major role in the paper. In particular, we use lower type inequalities together with Krasnoselskii’s fixed point theorem in a cone to deduce the existence of positive solutions for a general class of problems. Moreover, the results and technique are applicable also to positone problems.


Key words and phrases. Singular, differential, integral, discrete, semipositone.

1. Introduction

This paper presents existence results for differential, integral and discrete equations. In particular, our nonlinear term \( f(\cdot, y) \) may be singular at \( y = 0 \), and \( f \) may take on negative values. Problems of this type are referred to as semipositone problems in the literature. Almost all papers in the literature [6], [8]–[12] are devoted to the study of positone problems (i.e., when \( f \) takes nonnegative values), and only recently [3, 4, 7, 11] have a number of papers appeared which discuss the semipositone nonsingular problem. However no paper to date has discussed the semipositone singular problem. This paper attempts to fill this gap in the literature.

Existence in this paper will be established using Krasnoselskii’s fixed point theorem in a cone, which we state here for the convenience of the reader.

Theorem 1.1. Let \( E = (E, \| \cdot \|) \) be a Banach space and let \( K \subset E \) be a cone in \( E \). Assume \( \Omega_1 \) and \( \Omega_2 \) are open subsets of \( E \) with \( 0 \in \Omega_1 \) and \( \Omega_1 \subset \Omega_2 \) and let \( A: K \cap (\Omega_2 \setminus \Omega_1) \to K \) be continuous and completely continuous. In addition suppose either

\[
\| Au \| \leq \| u \| \text{ for } u \in K \cap \partial \Omega_1 \quad \text{and} \quad \| Au \| \geq \| u \| \text{ for } u \in K \cap \partial \Omega_2
\]

or

\[
\| Au \| \geq \| u \| \text{ for } u \in K \cap \partial \Omega_1 \quad \text{and} \quad \| Au \| \leq \| u \| \text{ for } u \in K \cap \partial \Omega_2
\]

hold. Then \( A \) has a fixed point in \( K \cap (\Omega_2 \setminus \Omega_1) \).

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2. Differential equations

In this section we first discuss the singular conjugate boundary value problem

\[
\begin{align*}
(-1)^{n-p}y^{(n)}(t) &= \mu f(t, y(t)), \quad 0 < t < 1, \\
y^{(i)}(0) &= 0, \quad 0 \leq i \leq p - 1, \\
y^{(i)}(1) &= 0, \quad 0 \leq i \leq n - p - 1, 
\end{align*}
\] (2.1)

where \(n \geq 2, 1 \leq p \leq n - 1\), and \(\mu > 0\) are constants. Here our nonlinearity \(f\) may be singular at \(y = 0\).

Before we prove our main result we first recall two well known results from the literature [1, 6].

Lemma 2.1. Suppose \(y \in C^{n-1}[0, 1] \cap C^n(0, 1)\) satisfies

\[
\begin{align*}
(-1)^{n-p} y^{(n)}(t) &\geq 0 \quad \text{for } t \in (0, 1), \\
y^{(i)}(0) &= 0, \quad 0 \leq i \leq p - 1, \\
y^{(i)}(1) &= 0, \quad 0 \leq i \leq n - p - 1.
\end{align*}
\]

Then

\[y(t) \geq t^p (1 - t)^{n-p} |y_0| \quad \text{for } t \in [0, 1];\]

here \(|y_0| = \sup_{t \in [0, 1]} |y(t)|\).

Lemma 2.2. The boundary value problem

\[
\begin{align*}
(-1)^{n-p} y^{(n)}(t) &= 1 \quad \text{for } t \in (0, 1), \\
y^{(i)}(0) &= 0, \quad 0 \leq i \leq p - 1, \\
y^{(i)}(1) &= 0, \quad 0 \leq i \leq n - p - 1,
\end{align*}
\]

has a solution \(w\) with

\[w(t) \leq \frac{1}{n!} t^p (1 - t)^{n-p} \quad \text{for } t \in [0, 1].\]

The above lemmas together with Theorem 1.1 establish our main result.

Theorem 2.3. Suppose the following conditions are satisfied:

\(f : [0, 1] \times (0, \infty) \rightarrow \mathbb{R}\) is continuous and there exists a constant \(M > 0\) with \(f(t, u) + M \geq 0\) for \(t, u \in [0, 1] \times (0, \infty)\). \hfill (2.2)

\(f(t, u) + M \leq g(u) + h(u)\) on \([0, 1] \times (0, \infty)\) with \(g > 0\) continuous and nonincreasing on \((0, \infty)\), \(h \geq 0\) continuous on \((0, \infty)\) and \(h/g\) nondecreasing on \((0, \infty)\). \hfill (2.3)

\[\exists K_0 > 0 \text{ with } g(ab) \leq K_0 g(a)g(b) \quad \forall a > 0, b > 0.\] \hfill (2.4)

\[\int_0^1 g(s^p(1 - s)^{n-p}) \, ds < \infty.\] \hfill (2.5)

\[\exists \mu > 0 \text{ with } \frac{\mu M}{n!} \geq \frac{r}{g\left(1 - \frac{\mu M}{n!}\right)} \left[1 + \frac{h(r)}{g(r)}\right] \geq a_0;\] \hfill (2.6)

here \(a_0 = \mu K_0 \sup_{t \in [0, 1]} \int_0^1 (-1)^{n-p} G(t, s) g(s^p(1 - s)^{n-p}) \, ds.\)
There exists \( a \in (0, 1/2) \) (choose and fix it) and a continuous, nonincreasing function \( g_1 : (0, \infty) \to (0, \infty) \), and a continuous function \( h_1 : (0, \infty) \to [0, \infty) \) with \( h_1/g_1 \) nondecreasing on \((0, \infty)\) and with \( f(t, u) + M \geq g_1(u) + h_1(u) \) for \((t, u) \in [a, 1 - a] \times (0, \infty)\),

and

\[
3R > r \text{ with } R g_1(\epsilon A_0 R) = \mu \int_a^1 (-1)^{n-p} G(\sigma, s) \, ds; \tag{2.7}
\]

here \( \epsilon > 0 \) is any constant (choose and fix it) so that \( 1 - \frac{\mu M}{\epsilon A_0 R} \geq \epsilon \) (note \( \epsilon \) exists since \( R > r > \frac{\mu M}{\epsilon A_0 R} \)).

Next let

\[
A_0 = \left\{ \begin{array}{ll}
da^p(1-a)^{n-p} & \text{if } n \leq 2p, \\
(1-a)^{n-p} & \text{if } n > 2p,
\end{array} \right.
\]

\( G(t, s) \) is the Green function (see [1, 7] for an explicit representation) for

\[
\begin{align*}
y^{(n)}(t) &= 0 & \text{on } (0, 1), \\
y^{(i)}(0) &= 0, & 0 \leq i \leq p - 1, \\
y^{(i)}(1) &= 0, & 0 \leq i \leq n - p - 1,
\end{align*}
\]

and \( 0 \leq \sigma \leq 1 \) is such that

\[
\int_a^1 (-1)^{n-p} G(\sigma, s) \, ds = \sup_{t \in [0,1]} \int_a^1 (-1)^{n-p} G(t, s) \, ds.
\]

Then (2.1) has a solution \( y \in C^{n-1}[0, 1] \cap C^n(0, 1) \) with \( y(t) > 0 \) for \( t \in (0, 1) \).

**Proof.** To show (2.1) has a nonnegative solution we will look at the boundary value problem

\[
\begin{align*}
(-1)^{n-p} y^{(n)}(t) &= \mu f^*(-t, y(t) - \phi(t)), \quad 0 < t < 1, \\
y^{(i)}(0) &= 0, \quad 0 \leq i \leq p - 1, \\
y^{(i)}(1) &= 0, \quad 0 \leq i \leq n - p - 1,
\end{align*}
\]

where \( \phi(t) = \mu M w(t) \) (\( w \) is as in Lemma 2.2) and

\( f^*(t, v) = f(t, v) + M, \quad v > 0 \).

We will show, using Theorem 1.1, that there exists a solution \( y_1 \) to (2.9) with \( y_1(t) > \phi(t) \) for \( t \in (0, 1) \) (note \( \phi(t) > 0 \) for \( t \in (0, 1) \)). If this is true then clearly \( u(t) = y_1(t) - \phi(t) \) is a nonnegative solution (positive on \( (0, 1) \)) of (2.1). As a result we will concentrate our study on (2.9). Let \( E = (C[0, 1], \frac{\cdot}{\cdot}_0) \) and

\[
K = \left\{ u \in C[0, 1] : u(t) \geq t^p (1 - t)^{n-p} |u|_0 \text{ for } t \in [0, 1] \right\}.
\]

Clearly \( K \) is a cone of \( E \). Let

\[
\Omega_1 = \left\{ u \in C[0, 1] : |u|_0 < r \right\} \text{ and } \Omega_2 = \left\{ u \in C[0, 1] : |u|_0 < R \right\}.
\]

Next let \( A : K \cap (\Omega_2 \setminus \Omega_1) \to C[0, 1] \) be defined by

\[
Ay(t) = \mu \int_0^1 (-1)^{n-p} G(t, s) f^*(s, y(s) - \phi(s)) \, ds.
\]
First we show $A$ is well defined. To see this notice that if $y \in K \cap (\Omega_2 \setminus \Omega_1)$ then

$$r \leq |y|_0 \leq R$$

and so $y(t) \geq t^p(1-t)^{n-p}|y|_0 \geq t^p(1-t)^{n-p}$. Also notice for $t \in (0, 1)$ that Lemma 2.1 and Lemma 2.2 imply

$$y(t) - \phi(t) = y(t) - \mu_M w(t) \geq y(t) - \frac{\mu M}{n!} t^p(1-t)^{n-p}$$

$$\geq y(t) \left[ 1 - \frac{\mu M}{n! r} \right] \geq t^p(1-t)^{n-p} \left[ 1 - \frac{\mu M}{n! r} \right],$$

so for $t \in (0, 1)$ we have

$$f^*(t, y(t) - \phi(t)) = f(t, y(t) - \phi(t)) + M \leq g(y(t) - \phi(t)) + h(y(t) - \phi(t))$$

$$= g(y(t) - \phi(t)) \left\{ 1 + \frac{h(y(t) - \phi(t))}{g(y(t) - \phi(t))} \right\}$$

$$\leq g \left( t^p(1-t)^{n-p} \left[ 1 - \frac{\mu M}{n! r} \right] \right) \left\{ 1 + \frac{h(y(t))}{g(y(t))} \right\}$$

$$\leq g \left( t^p(1-t)^{n-p} \left[ 1 - \frac{\mu M}{n! r} \right] \right) \left\{ 1 + \frac{h(R)}{g(R)} \right\}.$$
so $A: K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ is continuous. Also for $y \in K \cap (\bar{\Omega}_2 \setminus \Omega_1)$ we have
\[
|Ay|_0 \leq \left\{ 1 + \frac{h(R)}{g(R)} \right\} \sup_{t \in [0,1]} \int_0^1 (-1)^{n-p}G(t, s) g \left( s^p(1-s)^{n-p} \right) \left[ 1 - \frac{\mu M}{n! r} \right] \, ds
\]
and for $t, t' \in [0, 1]$ we have
\[
|Ay(t) - Ay(t')| \\
\leq \left\{ 1 + \frac{h(R)}{g(R)} \right\} \int_0^1 (-1)^{n-p} G(t, s) - G(t', s) g \left( s^p(1-s)^{n-p} \right) \left[ 1 - \frac{\mu M}{n! r} \right] \, ds.
\]
Now the Arzela–Ascoli Theorem guarantees that $A: K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$ is compact.

We now show
\[
|Ay|_0 \leq |y|_0 \quad \text{for } y \in K \cap \partial \Omega_1. \quad (2.10)
\]
To see this let $y \in K \cap \partial \Omega_1$. Then $|y|_0 = r$ and $y(t) \geq t^p(1-t)^{n-p} r$ for $t \in [0, 1]$. Now for $t \in (0, 1)$ we have (as above)
\[
y(t) - \phi(t) \geq y(t) - \frac{\mu M}{n!} t^p(1-t)^{n-p} \geq t^p(1-t)^{n-p} \left[ 1 - \frac{\mu M}{n! r} \right],
\]
so for $t \in [0, 1]$ we have
\[
Ay(t) \leq \mu \int_0^1 (-1)^{n-p} G(t, s) \{ g(y(s) - \phi(s)) + h(y(s) - \phi(s)) \} \, ds
\]
\[
= \mu \int_0^1 (-1)^{n-p} G(t, s) g(y(s) - \phi(s)) \left\{ 1 + \frac{h(y(s) - \phi(s))}{g(y(s) - \phi(s))} \right\} \, ds
\]
\[
\leq \mu \left\{ 1 + \frac{h(r)}{g(r)} \right\} \int_0^1 (-1)^{n-p} G(t, s) g \left( s^p(1-s)^{n-p} \right) \left[ 1 - \frac{\mu M}{n! r} \right] \, ds
\]
\[
\leq \mu K_0 g \left( r \left[ 1 - \frac{\mu M}{n! r} \right] \right) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \int_0^1 (-1)^{n-p} G(t, s) g(s^p(1-s)^{n-p}) \, ds.
\]
This together with (2.6) yields
\[
|Ay|_0 \leq a_0 g \left( r \left[ 1 - \frac{\mu M}{n! r} \right] \right) \left\{ 1 + \frac{h(r)}{g(r)} \right\} \leq r = |y|_0,
\]
so (2.10) is satisfied. Next we show
\[
|Ay|_0 \geq |y|_0 \quad \text{for } y \in K \cap \partial \Omega_2. \quad (2.11)
\]
To see this let $y \in K \cap \partial \Omega_2$ so $|y|_0 = R$ and $y(t) \geq t^p(1-t)^{n-p} R$ for $t \in [0, 1]$. Also for $t \in (0, 1)$ we have
\[
y(t) - \phi(t) = y(t) - \mu M w(t) \geq y(t) - \frac{\mu M}{n!} t^p(1-t)^{n-p}
\]
\[
\geq y(t) \left[ 1 - \frac{\mu M}{n! R} \right] \geq c y(t)
\]
\[
\geq c t^p(1-t)^{n-p} R.
\]
As a result
\[
y(t) - \phi(t) \geq c A_0 R \quad \text{for } t \in [a, 1-a].
\]
Now with \( \sigma \) as in the statement of Theorem 2.3, we have
\[
Ay(\sigma) \geq \mu \int_{0}^{1} (-1)^{n-p}G(\sigma, s)\left\{ g_1(y(s) - \phi(s)) + h_1(y(s) - \phi(s)) \right\} ds
\]
\[
= \mu \int_{0}^{1} (-1)^{n-p}G(\sigma, s)g_1(y(s) - \phi(s))\left\{ 1 + \frac{h_1(y(s) - \phi(s))}{g_1(y(s) - \phi(s))} \right\} ds
\]
\[
\geq \mu g_1(R) \int_{a}^{1-a} (-1)^{n-p}G(\sigma, s)\left\{ 1 + \frac{h_1(\epsilon A_0 R)}{g_1(\epsilon A_0 R)} \right\} ds.
\]
This together with (2.6) yields
\[
Ay(\sigma) \geq \mu g_1(R)\left\{ 1 + \frac{h_1(\epsilon A_0 R)}{g_1(\epsilon A_0 R)} \right\} \int_{a}^{1-a} (-1)^{n-p}G(\sigma, s) ds
\]
\[
\geq R = |y|_0.
\]
Thus |Ay|_0 \geq |y|_0, so (2.11) holds.

Now Theorem 1.1 implies \( A \) has a fixed point \( y_1 \in K \cap (\bar{\Omega_2} \setminus \Omega_1) \), i.e., \( r \leq |y_1|_0 \leq R \) and \( y_1(t) \geq t^p(1-t)^{n-p}r \) for \( t \in [0, 1] \). To finish the proof we need to show \( y_1(t) > \phi(t) \) for \( t \in (0, 1) \). This is immediate since Lemma 2.2 with the fact that \( r > \mu M/n! \) implies for \( t \in (0, 1) \) that
\[
y_1(t) \geq t^p(1-t)^{n-p}r > \frac{\mu M}{n!}t^p(1-t)^{n-p} = \mu M \omega(t) = \phi(t).
\]

\[
□
\]

Remark 2.1. From the proof it is easily seen that (2.4) can be removed provided we adjust assumption (2.7); we leave the details to the reader.

Example. Consider the boundary value problem
\[
\begin{align*}
g'' + \mu(y^{-\alpha} + y^\beta - 1) &= 0, & 0 < t < 1, \\
y(0) = y(1) &= 0, & 0 < \alpha < 1 < \beta,
\end{align*}
\]
with \( \mu \in (0, \mu_0) \) is such that
\[
\left( \frac{4}{1 - \alpha} \right)^{1/\alpha} \mu_0^{1/\alpha} + \frac{\mu_0}{2} \leq 1.
\]

Then (2.12) has a solution \( y \) with \( y(t) > 0 \) for \( t \in (0, 1) \).

To see this we will apply Theorem 2.3 with (here \( R > 1 \) will be chosen later; in fact here we choose \( R > 1 \)) so that \( \epsilon = 1/2 \) works, i.e., we choose \( R \) so that \( 1 - \mu/2R \geq 1/2 \)

\[
n = 2, \ p = 1, \ g(y) = g_1(y) = y^{-\alpha}, \ h(y) = h_1(y) = y^\beta, \ M = 1, \ \epsilon = \frac{1}{2} \text{ and } a = \frac{1}{4}.
\]

Clearly (2.2), (2.3), (2.4) (with \( K_0 = 1 \)), (2.5) (since \( 0 < \alpha < 1 \)) and (2.7) hold. Next notice that
\[
\mu K_0 \sup_{t \in [0,1]} \int_{0}^{1} (-1)^{n-p}G(t, s)s^{-\alpha}(1-s)^{-\alpha} ds \leq \frac{2\mu}{1 - \alpha}
\]
so (2.6) is true with \( r = 1 \) since
\[
\frac{\mu M}{n!} = \frac{\mu}{2} < \frac{\mu_0}{2} \leq 1 = r.
\]
and

\[
\mu K_0 \sup_{t \in [0, 1]} \int_0^1 (-1)^{n-r} G(t, s) s^{-\alpha} (1 - s)^{-\alpha} \, ds \leq \frac{2\mu}{1 - \alpha} \leq \frac{2\mu_0}{1 - \alpha} \leq \frac{(1 - \mu_0/2)^{\alpha}}{2} = \frac{r}{g(r [1 - \mu M]) \{1 + \frac{h(r)}{g(r)}\}}
\]

from (2.13). Finally notice (2.8) is satisfied for \( R \) large since

\[
R g_1(\epsilon A_0 R) g_1(\epsilon A_0 R) + g_1(\epsilon A_0 R) h_1(\epsilon A_0 R) = \frac{(\epsilon A_0)^{-\alpha} R^{1+\alpha}}{(\epsilon A_0)^{-\alpha} + (\epsilon A_0)^{\beta} R^{\alpha+\beta}} \to 0
\]

as \( R \to \infty \), since \( \beta > 1 \). Thus all the conditions of Theorem 2.3 are satisfied, so existence is guaranteed.

Next we consider the \((n, p)\) boundary value problem

\[
\begin{aligned}
&y^{(n)}(t) + \mu f(t, y(t)) = 0, \quad 0 < t < 1, \\
y^{(i)}(0) = 0, \quad 0 \leq i \leq n - 2, \\
y^{(p)}(1) = 0
\end{aligned}
\]

where \( n \geq 2, 1 \leq p \leq n - 1 \) is fixed and \( \mu > 0 \) is a constant. The following two lemmas can be found in the literature [1, 6].

**Lemma 2.4.** Suppose \( y \in C^{n-1}[0, 1] \cap C^n(0, 1) \) satisfies

\[
\begin{aligned}
&y^{(n)}(t) \leq 0 \quad \text{for } t \in (0, 1), \\
y^{(i)}(0) = 0, \quad 0 \leq i \leq n - 2, \\
y^{(p)}(1) = 0
\end{aligned}
\]

Then

\[
y(t) \geq t^{n-1}|y_0| \quad \text{for } t \in [0, 1].
\]

**Lemma 2.5.** The boundary value problem

\[
\begin{aligned}
&y^{(n)}(t) + 1 = 0 \quad \text{for } t \in (0, 1), \\
y^{(i)}(0) = 0, \quad 0 \leq i \leq n - 2, \\
y^{(p)}(1) = 0
\end{aligned}
\]

has a solution \( w \) with

\[
w(t) \leq \frac{1}{(n - 1)! (n - p)} t^{n-1} \quad \text{for } t \in [0, 1].
\]

Essentially the same reasoning as in Theorem 2.3 (only obvious adjustments are needed) establishes the following result.
Theorem 2.6. Suppose (2.2)–(2.4) hold. In addition assume the following conditions are satisfied:

\[ \int_0^1 g(s^{n-1}) \, ds < \infty. \]  

(2.15)

\[ \exists r > \frac{\mu M}{(n-1)! (n-p)} \text{ with } \int_0^1 \frac{r}{g(r)} \left[1 - \frac{\mu M}{(n-1)! (n-p)} \right](1 + \frac{h(\epsilon)}{\epsilon a}) \geq b_0; \]  

(2.16)

where \( b_0 = \mu K_0 \sup_{t \in [0,1]} \int_0^1 G_1(t, s) g(s^{n-1}) \, ds \).

There exists \( a \in (0, 1/2) \) (choose and fix it) and a continuous, nonincreasing function \( g_1 : (0, \infty) \to (0, \infty) \), and a continuous function \( h_1 : (0, \infty) \to [0, \infty) \) with \( h_1/g_1 \) nondecreasing on \((0, \infty)\) and with \( f(t, u) + M \geq g_1(u) + h_1(u) \text{ for } (t, u) \in [a, 1] \times (0, \infty) \),

and

\[ \exists R > r \text{ with } \frac{R g_1(\epsilon a^{n-1} R)}{g_1(R) g_1(\epsilon a^{n-1} R) + g_1(R) h_1(\epsilon a^{n-1} R)} \leq \mu \int_a^{1-a} G_1(\sigma, s) \, ds; \]  

(2.18)

here \( \epsilon > 0 \) is any constant (choose and fix it) so that \( 1 - \frac{\mu M}{(n-1)! (n-p)} \geq \epsilon, G_1 \) is the Green function (see [1, 7] for an explicit representation) for

\[ \begin{cases}
- y^{(n)} = 0 & \text{on } (0, 1), \\
y^{(i)}(0) = 0, & 0 \leq i \leq n-2, \\
y^{(n-2)}(1) = 0
\end{cases} \]

and \( 0 \leq \sigma \leq 1 \) is such that

\[ \int_a^1 G_1(\sigma, s) \, ds = \sup_{t \in [0,1]} \int_a^1 G_1(t, s) \, ds. \]

Then (2.14) has a solution \( y \in C^{n-1}[0, 1] \cap C^n(0, 1) \) with \( y(t) > 0 \) for \( t \in (0, 1) \).

3. Integral equations

In this section we present a new result for the semipositone Fredholm integral equation

\[ y(t) = \mu \int_0^1 k(t, s) f(s, y(s)) \, ds \quad \text{for } t \in [0, 1]; \]  

(3.1)

here \( \mu > 0 \) is a constant.

Theorem 3.1. Let \( 1 \leq p \leq \infty \) be a constant and \( q \) be such that \( 1/p + 1/q = 1 \). Suppose the following conditions are satisfied:

There exists \( a \in C[0, 1] \) and \( t^* \in [0, 1] \) with \( a(t) > 0 \) for a.e. \( t \in [0, 1] \) and \( a(t^*) > 0 \), there exists \( \kappa \in L^p[0, 1] \) with \( \kappa(t) \geq 0 \)

a.e. \( t \in [0, 1] \) and \( \|\kappa\|_p = \left( \int_0^1 \kappa^p(s) \, ds \right)^{1/p} > 0 \) such that

\[ a(t) \kappa(s) \leq k(t, s) \text{ for all } t \in [0, 1], \text{ a.e. } s \in [0, 1]. \]  

(3.2)
\[ k_1(s) = k(t, s) \leq \kappa(s) \quad \text{for all } t \in [0, 1], \text{ a.e. } s \in [0, 1]. \] (3.3)

The map \( t \mapsto k_t \) is continuous from \([0, 1]\) to \( L^p[0, 1] \).

\[ f: [0, 1] \times (0, \infty) \to \mathbb{R} \text{ is Carathéodory } [12] \text{ and there exists a constant } M > 0 \text{ with } f(t, u) + M \geq 0 \text{ for } (t, u) \in [0, 1] \times (0, \infty). \] (3.4)

\[ f(t, u) + M \leq g(u) + h(u) \text{ on } [0, 1] \times (0, \infty) \text{ with } g > 0 \text{ continuous and nonincreasing on } (0, \infty), h \geq 0 \text{ continuous on } (0, \infty) \text{ and } h/g \text{ nondecreasing on } (0, \infty). \] (3.5)

\[ \exists C > 0 \text{ with } \int_0^1 k(t, s) ds \leq Ca(t) \text{ for } t \in [0, 1]. \] (3.6)

\[ \exists K_0 > 0 \text{ with } g(ab) \leq K_0 g(a)g(b) \forall a > 0, b > 0. \] (3.7)

\[ \int_0^1 |g(a(s))|^q ds < \infty. \] (3.8)

\[ \exists r > \mu MC \text{ with } \frac{r}{g(r[1 - \frac{\mu MC}{r}]) \{1 + \frac{\mu(r)}{g(r)}\}} \geq c_0; \] (3.9)

\[ \text{here } c_0 = \mu K_0 \sup_{t \in [0,1]} \int_0^1 k(t, s) ds. \]

\[ \exists a \text{ continuous, nonincreasing function } g_1: (0, \infty) \to (0, \infty) \text{ and a continuous function } h_1: (0, \infty) \to [0, \infty) \text{ with } h_1/g_1 \text{ nondecreasing on } (0, \infty) \text{ and with } f(t, u) + M \geq g_1(u) + h_1(u) \text{ for } (t, u) \in [0, 1] \times (0, \infty), \] (3.10)

and

\[ \exists R > r \text{ with } \frac{R}{g_1(R)} \leq \mu \int_0^1 k(t^*, s) \left\{1 + \frac{h_1(\epsilon Ra(s))}{g_1(\epsilon Ra(s))}\right\} ds; \] (3.11)

\[ \text{here } \epsilon > 0 \text{ is any constant (choose and fix it) so that } 1 - \mu MC/R \geq \epsilon. \text{ Then (3.1) has a nonnegative solution } y \in C[0, 1] \text{ with } y(t) > 0 \text{ for a.e. } t \in [0, 1] \text{ (in fact } y(t) > 0 \text{ at those } t \text{'s where } a(t) > 0). \]

**Proof.** To show (3.1) has a nonnegative solution we will look at

\[ y(t) = \mu \int_0^1 k(t, s) f^*(s, y(s) - \phi(s)) ds \] (3.12)

where \( \phi(t) = \mu M \int_0^1 k(t, s) ds \) and

\[ f^*(t, v) = f(t, v) + M, \quad v > 0. \]

We will show, using Theorem 1.1, that there exists a solution \( y_1 \) to (3.13) with \( y_1(t) \geq \phi(t) \) for \( t \in [0, 1] \) and \( y_1(t) > \phi(t) \) for those \( t \)’s where \( a(t) > 0 \). If this is true then clearly \( u(t) = y_1(t) - \phi(t) \) is a nonnegative solution (positive a.e. on \([0, 1]\)) of (3.1). Let \( E = (C[0, 1], | \cdot |_0) \) and

\[ K = \{u \in C[0, 1]: u(t) \geq a(t)|u|_0 \text{ for } t \in [0, 1]\}. \]
Also let
\[ \Omega_1 = \{ u \in C[0, 1] : |u|_0 < r \} \quad \text{and} \quad \Omega_2 = \{ u \in C[0, 1] : |u|_0 < R \} \]
and let \( A: K \cap (\Omega_2 \setminus \Omega_1) \to C[0, 1] \) be defined by
\[
Ay(t) = \mu \int_0^1 k(t, s) f^*(s, y(s) - \phi(s)) \, ds.
\]
First notice \( A \) is well defined. To see this note if \( y \in K \cap (\Omega_2 \setminus \Omega_1) \) then \( r \leq |y|_0 \leq R \) and so \( y(t) \geq a(t)|y|_0 \geq a(t)r \) for \( t \in [0, 1] \). Let \( Q = \{ t \in [0, 1] : a(t) > 0 \} \). Now for \( t \in Q \) we have
\[
y(t) - \phi(t) = y(t) - \mu M \int_0^1 k(t, s) ds \geq y(t) - \mu M a(t)
\]
\[
\geq y(t) \left[ 1 - \frac{\mu MC}{r} \right] \geq a(t)r \left[ 1 - \frac{\mu MC}{r} \right],
\]
so for \( t \in Q \) we have
\[
f^*(t, y(t) - \phi(t)) \leq g(y(t) - \phi(t)) \left\{ 1 + \frac{h(y(t) - \phi(t))}{g(y(t) - \phi(t))} \right\}
\]
\[
\leq g \left( a(t)r \left[ 1 - \frac{\mu MC}{r} \right] \right) \left\{ 1 + \frac{h(R)}{g(R)} \right\}.
\]
This with (3.2), (3.3), (3.4), and (3.9) guarantees that \( A: K \cap (\Omega_2 \setminus \Omega_1) \to C[0, 1] \) is well defined. Next we show \( A: K \cap (\Omega_2 \setminus \Omega_1) \to K \). If \( y \in K \cap (\Omega_2 \setminus \Omega_1) \) and \( t \in [0, 1] \) then (3.2) implies
\[
Ay(t) \geq \mu a(t) \int_0^1 \kappa(s) f^*(s, y(s) - \phi(s)) \, ds. \tag{3.14}
\]
On the other hand (3.3) implies
\[
|Ay|_0 \leq \mu \int_0^1 \kappa(s) f^*(s, y(s) - \phi(s)) \, ds,
\]
and this together with (3.14) yields
\[
Ay(t) \geq a(t)|Ay|_0 \quad \text{for } t \in [0, 1].
\]
Consequently \( Ay \in K \) so \( A: K \cap (\Omega_2 \setminus \Omega_1) \to K \). It is easy to check (a slight modification of the argument in Theorem 2.1) that \( A: K \cap (\Omega_2 \setminus \Omega_1) \to K \) is continuous and compact (see also [12]).

A slight modification of the argument in Theorem 2.1 establishes

\[
|Ay|_0 \leq |y|_0 \quad \text{for } y \in K \cap \partial \Omega_1. \tag{3.15}
\]

Next we show
\[
|Ay|_0 \geq |y|_0 \quad \text{for } y \in K \cap \partial \Omega_2. \tag{3.16}
\]
To see this let \( y \in K \cap \partial \Omega_2 \) so \(|y_0| = R\) and \( y(t) \geq a(t)R \) for \( t \in [0, 1] \). For \( t \in \{ t \in [0, 1] : a(t) > 0 \} \) we have
\[
y(t) - \phi(t) \geq y(t) - \mu MC a(t) \geq y(t) \left[ 1 - \frac{\mu MC}{R} \right] \\
\geq \epsilon y(t) \geq \epsilon a(t) R.
\]
Now with \( t^* \) as in (3.2), we have
\[
A y(t^*) = \mu \int_0^1 k(t^*, s) f^*(s, y(s) - \phi(s)) \, ds \\
\geq \mu \int_0^1 k(t^*, s) g_1(y(s) - \phi(s)) \left\{ 1 + \frac{h_1(y(s) - \phi(s))}{g_1(y(s) - \phi(s))} \right\} \, ds \\
\geq \mu g_1(R) \int_0^1 k(t^*, s) \left\{ 1 + \frac{h_1(\epsilon a(s) R)}{g_1(\epsilon a(s) R)} \right\} \, ds,
\]
and this together with (3.12) gives
\[
A y(t^*) \geq R = |y_0|,
\]
so (3.16) holds.

Now Theorem 1.1 implies \( A \) has a fixed point \( y_1 \in K \cap (\Omega_2 \setminus \Omega_1) \), i.e., \( r \leq |y_1| \leq R \) and \( y_1(t) \geq a(t) r \) for \( t \in [0, 1] \). To finish the proof we need to show \( y_1(t) > \phi(t) \) for \( t \in Q = \{ s \in [0, 1] : a(s) > 0 \} \). This is immediate since for \( t \in Q \) we have
\[
y_1(t) \geq a(t)r > a(t)\mu MC \geq \mu M \int_0^1 k(t, s) \, ds = \phi(t).
\]

**Remark 3.1.** From the proof it is easily seen that (3.8) can be removed provided we adjust assumption (3.10).

**Remark 3.2.** It is also possible to obtain another existence result for (3.1) if we use the ideas here with those in [4, Theorem 2.3]; we leave the details to the reader.

### 4. Discrete equations

In this section we discuss the singular discrete \((n, p)\) boundary value problem
\[
\begin{align*}
\Delta^n y(k - n + 1) + \mu f(k, y(k)) &= 0, & k \in J_{n-1}, \\
\Delta^i y(0) &= 0, & 0 \leq i \leq n - 2, \\
\Delta^p y(T + n - p) &= 0, & 1 \leq p \leq n - 1 \ (p \text{ is fixed}),
\end{align*}
\]
where \( \mu > 0, \ T \in \{ 1, 2, \ldots \}, \ J_{n-1} = \{ n - 1, n, \ldots, T + n - 1 \} \) and \( y: I_n = \{ 0, \ldots, T + n \} \to \mathbb{R} \). Here our nonlinearity \( f \) may be singular at \( y = 0 \).

The next two lemmas may be found in [2, 5].

**Lemma 4.1.** Suppose \( y: I_n \to \mathbb{R} \) satisfies
\[
\begin{align*}
\Delta^n y(k - n + 1) &\leq 0, & k \in J_{n-1}, \\
\Delta^i y(0) &= 0, & 0 \leq i \leq n - 2, \\
\Delta^p y(T + n - p) &= 0, & 1 \leq p \leq n - 1.
\end{align*}
\]
Then
\[ y(k) \geq \frac{k^{(n-1)}}{(T + n)(n-1)} |y|_0 \quad \text{for} \quad k \in I_n; \]
here \( |y|_0 = \sup_{j \in I_n} |y(j)| \).

**Lemma 4.2.** The boundary value problem
\[
\begin{cases}
\Delta^n y(k - n + 1) + 1 = 0, & k \in J_{n-1}, \\
\Delta^i y(0) = 0, & 0 \leq i \leq n - 2, \\
\Delta^p y(T + n - p) = 0, & 1 \leq p \leq n - 1,
\end{cases}
\]
has a solution \( w \) with
\[ w(k) \leq \frac{(T + 1)}{(n-p)(n-1)!} k^{(n-1)} \quad \text{for} \quad k \in J_{n-1}; \]
here
\[ w(k) = \sum_{j=0}^{T+n-1} G(k, j) = \sum_{j=0}^{T} G_1(k, j) = \frac{k^{(n-1)}}{(n-1)!} \left[ \frac{(T+1)}{(n-p)} - \frac{k-n+1}{n} \right] \]
for \( k \in I_n \), where \( G(k, j) = G_1(k, j-n+1) \) and \( G_1 \) is the Green function for (see [2] for an explicit representation)
\[
\begin{cases}
\Delta^n y(k) = 0, & k \in I_0 = \{0, 1, \ldots, T\}, \\
\Delta^i y(0) = 0, & 0 \leq i \leq n - 2, \\
\Delta^p y(T + n - p) = 0, & 1 \leq p \leq n - 1.
\end{cases}
\]

The above lemmas together with Theorem 1.1 establish our main result in this section.

**Theorem 4.3.** Suppose the following conditions are satisfied:
\[ f : J_{n-1} \times (0, \infty) \to \mathbb{R} \] is continuous and there exists a constant \( M > 0 \) with \( f(i, u) + M \geq 0 \) for \((i, u) \in J_{n-1} \times (0, \infty)\).
\[ f(i, u) + M \leq g(u) + h(u) \quad \text{on} \quad J_{n-1} \times (0, \infty) \] with \( g > 0 \) continuous and nonincreasing on \((0, \infty)\), \( h \geq 0 \) continuous on \((0, \infty)\) and \( h/g \) nondecreasing on \((0, \infty)\).
\[ \exists r > \mu M(T + n)^{(n-1)}(T + 1) \]
\[ (n-1)!(n-p) \]
\[ g \left( \frac{(n-1)^{(n-1)}}{(T+n)^{(n-1)}} \left( 1 - \frac{\mu M(T+n)^{(n-1)}(T+1)}{(n-1)!(n-p)r} \right) \right) \left( 1 + \frac{h(r)}{g(r)} \right) \frac{1}{r} \geq d_0; \]
here \( d_0 = \mu \sup_{k \in J_n} \sum_{j=0}^{T+n-1} G(k, j) \).
\[ \exists \] a continuous, nonincreasing function \( g_1 : (0, \infty) \to (0, \infty) \),
and a continuous function \( h_1 : (0, \infty) \to [0, \infty) \) with \( h_1/g_1 \)
nondecreasing on \((0, \infty)\) and with \( f(i, u) + M \geq g_1(u) + h_1(u) \)
for \((i, u) \in J_{n-1} \times (0, \infty)\).
and

\[ R > r \text{ with } \frac{R g_1 \left( \frac{c R(n-1)(n-1)}{T+n} \right)}{g_1(R) g_1 \left( \frac{c R(n-1)(n-1)}{T+n} \right) + g_1(R) h_1 \left( \frac{c R(n-1)(n-1)}{T+n} \right)} \leq \mu e_0; \quad (4.6) \]

here

\[ e_0 = \sum_{j=0}^{T+n-1} G(\sigma, j) \]

and \( \epsilon > 0 \) is any constant (choose and fix it) so that

\[ 1 - \frac{\mu M(T + n)^{(n-1)}(T + 1)}{(n-1)! (n-p)R} \geq \epsilon \]

and \( \sigma \in \mathbb{I}_n \) is such that

\[ \sum_{j=0}^{T+n-1} G(\sigma, j) = \max_{i \in \mathbb{I}_n} \sum_{j=0}^{T+n-1} G(i, j). \]

Then (4.1) has a solution \( y \in C(\mathbb{I}_n) \) with \( y(i) > 0 \) for \( i \in \mathbb{I}_{n-1} \) (here \( C(\mathbb{I}_n) \) denotes the class of maps \( w \) continuous on \( \mathbb{I}_n \) (discrete topology) with norm \( |w|_0 = \sup_{k \in \mathbb{I}_n} |w(k)| \)).

**Proof.** To show (4.1) has a nonnegative solution we will look at the boundary value problem

\[
\begin{cases}
\triangle^n y(k - n + 1) + \mu f^*(k, y(k) - \phi(k)) = 0, & k \in J_{n-1}, \\
\triangle^i y(0) = 0, & 0 \leq i \leq n - 2, \\
\triangle^p y(T + n - p) = 0, & 1 \leq p \leq n - 1,
\end{cases}
\]

where \( \phi(i) = \mu M w(i) \) (\( w \) is as in Lemma 4.2) and

\[ f^*(i, v) = f(i, v) + M, \quad v > 0. \]

We will show, using Theorem 1.1, that there exists a solution \( y_1 \) to (4.7) with \( y_1(i) > \phi(i) \) for \( i \in J_{n-1} \). Let \( E = (C(\mathbb{I}_n), \ | \cdot |_0) \) and

\[ K = \left\{ u \in C(\mathbb{I}_n) : u(k) \geq \frac{k^{(n-1)}}{(T+n)^{(n-1)}} |u|_0 \text{ for } k \in \mathbb{I}_n \right\}, \]

and let

\[ \Omega_1 = \{ u \in C(\mathbb{I}_n) : |u|_0 < r \} \quad \text{and} \quad \Omega_2 = \{ u \in C(\mathbb{I}_n) : |u|_0 < R \}. \]

Next let \( A: K \cap (\Omega_2 \setminus \Omega_1) \to C(\mathbb{I}_n) \) be defined by

\[ Ay(k) = \mu \sum_{j=0}^{T+n-1} G(k, j) f^*(j, y(j) - \phi(j)). \]

A standard argument [3], see also the ideas in Theorem 2.1, guarantees that \( A: K \cap (\Omega_2 \setminus \Omega_1) \to K \) is continuous and compact.

We now show

\[ |Ay|_0 \leq |y|_0 \quad \text{for } y \in K \cap \partial \Omega_1. \quad (4.8) \]
To see this let \( y \in K \cap \partial \Omega_1 \), so \(|y_0| = r \) and \( g(k) \geq \frac{k^{(n-1)}}{(T+n)^{(n-1)}} r \) for \( k \in I_n \). Also for \( j \in J_{n-1} \) we have from Lemma 4.2 that
\[
y(j) - \phi(j) \geq y(j) - \phi(j) \geq y(j) \left[ 1 - \frac{\mu M(T+1)(T+n)^{(n-1)}}{(n-1)!(n-p)r} \right]
\[
\geq \frac{j^{(n-1)}}{(T+n)^{(n-1)}} r \left[ 1 - \frac{\mu M(T+1)(T+n)^{(n-1)}}{(n-1)!(n-p)r} \right]
\[
\geq \frac{(n-1)^{(n-1)}}{(T+n)^{(n-1)}} r \left[ 1 - \frac{\mu M(T+1)(T+n)^{(n-1)}}{(n-1)!(n-p)r} \right].
\]

Now for \( k \in I_n \) we have
\[
Ay(k) \leq \mu \sum_{j=n-1}^{T+n-1} G(k, j) g(y(j) - \phi(j)) \left\{ 1 + \frac{h(y(j) - \phi(j))}{g(y(j) - \phi(j))} \right\}
\[
\leq \mu \left\{ 1 + \frac{h(r)}{g(r)} \right\} g \left( \frac{(n-1)^{(n-1)}}{(T+n)^{(n-1)}} \left[ 1 - \frac{\mu M(T+1)(T+n)^{(n-1)}}{(n-1)!(n-p)r} \right] \right)
\times \sup_{k \in I_n} \sum_{j=n-1}^{T+n-1} G(k, j),
\]
and this together with (4.4) yields
\[
|Ay|_0 \leq r = |y|_0.
\]
so (4.8) holds.

Next we show
\[
|Ay|_0 \geq |y|_0 \quad \text{for} \quad y \in K \cap \partial \Omega_2.
\]
(4.9)
To see this let \( y \in K \cap \partial \Omega_2 \) so \(|y_0| = R \) and \( y(k) \geq \frac{k^{(n-1)}}{(T+n)^{(n-1)}} R \) for \( k \in I_n \). Also for \( j \in J_{n-1} \) we have
\[
y(j) - \phi(j) \geq y(j) \left[ 1 - \frac{\mu M(T+1)(T+n)^{(n-1)}}{(n-1)!(n-p)R} \right] \geq \epsilon y(j)
\[
\geq \epsilon \frac{(n-1)^{(n-1)}}{(T+n)^{(n-1)}} R \geq \epsilon \frac{(n-1)^{(n-1)}}{(T+n)^{(n-1)}} R.
\]
Now
\[
Ay(\sigma) \geq \mu \sum_{j=n-1}^{T+n-1} G(\sigma, j) g_1(y(j) - \phi(j)) \left\{ 1 + \frac{h_1(y(j) - \phi(j))}{g_1(y(j) - \phi(j))} \right\}
\[
\geq \mu g_1(R) \left\{ 1 + \frac{h_1(\epsilon (n-1)^{(n-1)})}{g_1(\epsilon (n-1)^{(n-1)})} \right\} \sum_{j=n-1}^{T+n-1} G(\sigma, j)
\[
\geq R = |y|_0,
\]
using (4.6).
Now Theorem 1.1 implies $A$ has a fixed point $y_1 \in K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, i.e., $r \leq |y_1| \leq R$ and $y_1(k) \geq \frac{k^{(n-1)}}{(T+n)(n-1)} r$ for $k \in I_n$. To finish the proof we need to show $y_1(i) > \phi(i)$ for $i \in I_{n-1}$. This is immediate since

\[
y_1(i) \geq \frac{i^{(n-1)}}{(T+n)(n-1)} r > \frac{i^{(n-1)}\mu M(T+1)}{(n-1)! (n-p)} \geq \mu M w(i) = \phi(i).
\]

□

\textbf{Remark 4.1.} It is also possible to combine the ideas here with those in [3] to establish existence results for semipositone singular conjugate discrete problems. We leave the details to the reader.

\textbf{References}


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