MULTIPlicITIES IN THE BRANCHING RULES AND
THE COMPLEXITY OF HOMOGENEOUS SPACES

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Abstract. Let $H$ be an algebraic subgroup of a connected algebraic
group $G$ defined over an algebraically closed field $k$ of characteristic
zero. For a dominant weight $\lambda$ of $G$, let $V_\lambda$ be a simple $G$-module
with highest weight $\lambda$, $d_\lambda = \dim V_\lambda$, and denote by $k[G/H]_{(\lambda)}$ the isotypic
$\lambda$-component in $k[G/H]$. For $G/H$ quasi-affine, we show that the ra-
tio $\dim k[G/H]_{(\lambda)}/d_\lambda$ grows no faster than a polynomial in $\|\lambda\|$ whose
degree is the complexity of the homogeneous space $G/H$. If $H$ is re-
ductive and connected, this yields an estimate of branching coefficients
for the pair $(G, H)$ in terms of the complexity of $G/B_H$, where $B_H$ is
a Borel subgroup of $H$. We classify all affine homogeneous spaces $G/H$
such that $G$ is simple and the complexity of $G/B_H$ is at most 1. Some
explicit descriptions of branching rules are also given.

Key words and phrases. Complexity of a homogeneous space, branching
rule, Grosshans subgroup, algebra of covariants.

Introduction

Let $G$ be a connected reductive algebraic group defined over an algebraically
closed field $k$ of characteristic zero. Consider an algebraic subgroup $H \subset G$ and the
corresponding homogeneous space $G/H$. The algebra of regular functions $k[G/H]$ has a natural structure of a (locally-finite) $G$-module. Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Denote by $U$ the unipotent radical of $B$. Let $X(T)$ denote the character group of $T$ and $X(T)_+ \subset X(T)$ denote the monoid of dominant
weights relative to $B$. The group $X(T)$ is regarded as a lattice in the real vector
space $X(T) = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. For any $\lambda \in X(T)_+$, let $V_\lambda$ be a simple $G$-module
with highest weight $\lambda$ and $k[G/H]_{(\lambda)}$ be the corresponding isotypic component of $k[G/H]$. The set of all $\lambda$'s such that $k[G/H]_{(\lambda)} \neq \{0\}$ is a monoid, denoted $\Gamma(G/H)$.
It is easily seen that each $k[G/H]_{(\lambda)}$ is finite-dimensional. Indeed, the surjective $G$-equivariant mapping $G \to G/H$ induces an injection $k[G/H]_{(\lambda)} \hookrightarrow k[G]_{(\lambda)}$ for

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each $\lambda \in \mathcal{X}(T)_+$, and it is well known that $\dim k[\mathcal{G}]_{(\lambda)} = d^2_{\lambda}$, where $d_{\lambda} = \dim V_{\lambda}$. Define the rank of the monoid $\Gamma(G/H)$, denoted $\text{rk}\Gamma(G/H)$, as the rank of the (free) Abelian group generated by $\Gamma(G/H)$ in $\mathcal{X}(T)$.

Recall that, for any irreducible $G$-variety $X$, the complexity $c_G(X)$ of $X$ with respect to $G$ is the minimal codimension of $B$-orbits in $X$. If there is no danger of confusing $G$ with another reductive group, we write merely $c(X)$ for this complexity. For instance, $c(G/H)$ always stands for $c_G(G/H)$. If $G/H$ is quasi-affine, then the rank of $G/H$, denoted $r_G(G/H) = r(G/H)$, can be defined as $\text{rk}\Gamma(G/H)$. We refer to [14] for general facts about the complexity and rank of homogeneous spaces.

An algebraic subgroup $H \subset G$ is called a Grosshans subgroup if $G/H$ is quasi-affine and $k[G/H]$ is finitely generated. See [7, Ch. I] for a number of various characterizations of Grosshans subgroups. For instance, any reductive or 1-dimensional subgroup is Grosshans.

For any Grosshans subgroup $H \subset G$, the ratio

$$\frac{\dim k[G/H]_{(\lambda)}}{d_{\lambda} \cdot n_G}$$

is bounded for any fixed $\lambda \in \Gamma(G/H)$ as $n \to \infty$, see [14, Th. 2.4.16]. It is also shown there that if $\lambda$ lies in the relative interior of the cone generated by $\Gamma(G/H)$ in $\mathcal{X}_k(T)$, then the exponent $c(G/H)$ cannot be reduced. This shows that if $H$ is Grosshans, then the properties of the mapping

$$\lambda \mapsto \dim k[G/H]_{(\lambda)}, \quad \lambda \in \Gamma(G/H),$$

reflect the geometric properties of $G/H$.

In this paper, we give a uniform estimate of the ratio $k[G/H]_{(\lambda)}/d_{\lambda}$. Let $\lambda \mapsto \|\lambda\|$ be any Euclidean norm function on $\mathcal{X}_k(T)$. Our main result is the following.

**Theorem 1.** For any Grosshans subgroup $H \subset G$ there is a constant $A = A(G, H)$ such that

$$\frac{\dim k[G/H]_{(\lambda)}}{d_{\lambda}} \leq A \cdot (\|\lambda\| + 1)^{c(G/H)}$$

for all $\lambda \in \mathcal{X}(T)_+$. Furthermore, the exponent $c(G/H)$ is the least possible for such an estimate.

This theorem is deduced from a general result about multigraded algebras over arbitrary fields, see Theorem 3.

Theorem 1 can be used to obtain estimates of multiplicities arising in the branching rules. Let $H$ be a reductive subgroup of $G$ and $\text{res}_H V_{\lambda}$ the $H$-module obtained from $V_{\lambda}$ by restriction. Inside $H$, fix a Borel subgroup $B_H$ and a maximal torus $T_H \subset B_H$. Let $U_H$ be the unipotent radical of $B_H$. For $H$ connected, denote by $W_{\mu}$ a simple $H$-module with highest weight $\mu \in \mathcal{X}(T_H)_+$. Then one obtains a decomposition

$$\text{res}_H V_{\lambda} = \bigoplus_{\mu \in \mathcal{X}(T_H)_+} m_{\lambda}(\mu) \cdot W_{\mu},$$

and the branching rule is a device for computing the multiplicities $m_{\lambda}(\mu)$. We set

$$\Gamma(G, H) := \{(\lambda, \mu) : m_{\lambda}(\mu) \neq 0 \} \subset \mathcal{X}(T)_+ \times \mathcal{X}(T_H)_+.$$

M. Brion pointed out that Theorem 1 yields estimate (i) in the following theorem.
Theorem 2. Let \( H \) be a connected reductive subgroup of \( G \).

(i) There is a constant \( D = D(G, H) \) such that

\[
m_\lambda(\mu) \leq D \cdot (\|\lambda\| + 1)^{c(G/B_H)}
\]

for all \( \lambda \in X(T)_+ \) and \( \mu \in X(T_H)_+ \). The exponent \( c(G/B_H) \) is the least possible for such a uniform estimate.

(ii) \( \Gamma(G, H) \) is a finitely generated monoid.

Remark. Notice that the monoids \( \Gamma(G, H) \) and \( \Gamma(G/H) \) are completely different, unless \( H = \{1\} \). More precisely, we have \( \Gamma(G/H) = \Gamma(G, H) \cap (X(T) \times \{0\}) \).

The proof of the previous theorem, as well as “explicit” results described below, rely on the interpretation of the multiplicities \( m_\lambda(\mu) \) as dimensions of homogeneous components in the multigraded algebra \( k[G]_{U \times U_H} \), see details in Section 3. For practical applications of Theorem 2, it is important to know the exponent \( c(G/B_H) \) and the rank of \( \Gamma(G, H) \). Obviously, one may assume that \( H \) contains no infinite normal subgroups of \( G \). We prove that in this case

\[
\text{rk} \, \Gamma(G, H) = \text{rk} \, G + \text{rk} \, H, \quad c(G/B_H) = \dim U - \dim B_H,
\]

see Theorem 4. Combining the last formula and Theorem 2, we recover the result of [1] for Levi subgroups (of parabolic subgroups), see Theorem 5.

By a result of E. B. Vinberg and B. Kimelfeld [15, Cor. 1], the following properties of the pair \((G, H)\) are equivalent:

(a) for all \( \lambda \in X(T)_+ \) the \( H \)-module \( \text{res}_H V_\lambda \) is multiplicity free, i.e.,

\[
\dim \text{Hom}_H(W, \text{res}_H V_\lambda) \leq 1
\]

for each simple \( H \)-module \( W \);

(b) \( c(G/B_H) = 0 \).

Note that, for \( H \) connected, (a) means \( m_\lambda(\mu) \leq 1 \) for all \((\lambda, \mu)\). Thus, the implication ‘(a)⇒(b)’ is a consequence of Theorem 2(i). All connected reductive subgroups of simple algebraic groups satisfying (a) were described by M. Krämer [10].

In Section 4, we recall Krämer’s list, which corresponds to the case \( c(G/B_H) = 0 \), and obtain the list of all pairs \((G, H)\) such that \( c(G/B_H) = 1 \), under the same constraints on \( G \) and \( H \). Then we explicitly describe the branching rules for the pairs \((G, H)\) in Krämer’s list and for those pairs with \( c(G/B_H) = 1 \), where \( H \) is semisimple. The word “explicitly” means that we present the minimal generating system of \( \Gamma(G, H) \) and explain how to determine the value \( m_\lambda(\mu) \) for any \((\lambda, \mu) \in \Gamma(G, H) \).

The second part is trivial for the case of complexity 0, since all multiplicities are equal to 1. In case of complexity 1, Theorem 2(i) only says that the growth of multiplicities is linear. However, if \( G \) is semisimple and simply connected, then results of [14] allow us to get precise information about all multiplicities in \( \Gamma(G, H) \), see Theorem 6. It should be noted that, for \( c(G/B_H) = 0 \), there is a classical way to describe \( \Gamma(G, H) \) in terms of certain interlacing conditions, see e.g. [2, Ch. 5], [17, Ch. 18], [6, Ch. 8]. In this way, one obtains a collection of inequalities describing the cone generated by the monoid \( \Gamma(G, H) \), while Theorems 7 and 8 list the generators of \( \Gamma(G, H) \) thereby providing the “dual” presentation of the monoid in
question. Our approach to branching rules is also justified by the fact that this yields a uniform presentation for the cases \( c(G/B_H) = 0 \) or 1.

1. Multigraded algebras

Let \( \Gamma \) be a commutative semigroup with neutral element, i.e., a monoid. The operation in \( \Gamma \) is written additively and the neutral element is denoted by 0. We assume that \( \Gamma \) can be imbedded into some \( \mathbb{Z}^r \). In particular, the notation \((-\Gamma)\) makes sense. Let \( \mathbb{Z} \cdot \Gamma = \{ \gamma_1 - \gamma_2 : \gamma_1, \gamma_2 \in \Gamma \} \) be the group generated by \( \Gamma \) and \( \text{rk} \Gamma \) be the rank of \( \mathbb{Z} \cdot \Gamma \).

Let \( k \) be any field, and let \( R \) be a \( \Gamma \)-graded commutative \( k \)-algebra with identity 1.

Let \( k \) be any field, and let \( R \) be a \( \Gamma \)-graded commutative \( k \)-algebra with identity 1. Thus,

\[
R = \bigoplus_{\gamma \in \Gamma} R_{(\gamma)} \quad \text{(a vector space direct sum)}, \quad R_{(\gamma)} \cdot R_{(\delta)} \subset R_{(\gamma+\delta)}, \quad k = k \cdot 1 \subset R_{(0)}.
\]

The spaces \( R_{(\gamma)} \) are called homogeneous components of \( R \). Elements \( x \in R_{(\gamma)} \) are said to be homogeneous of degree \( \gamma \), denoted \( \gamma = \deg x \).

Lemma 1. Let \( R \) be a \( \Gamma \)-graded algebra without zero divisors, with \( \dim_k R_{(0)} < \infty \). Then \( R_{(0)} \) is a field and \( \dim_k R_{(\gamma)} = \dim_k R_{(0)} \) for any \( \gamma \in \Gamma \cap (-\Gamma) \). Moreover, a nonzero homogeneous element of degree \( \beta \in \Gamma \) is invertible in \( R \) if and only if \( \beta \in \Gamma \cap (-\Gamma) \).

Proof. Since \( R_{(0)} \) is a finite-dimensional domain, it is a field. Take any \( x \in R_{(\gamma)} \setminus \{0\} \), where \( \gamma \in \Gamma \cap (-\Gamma) \). Since \( R \) is a domain, the multiplication mappings

\[
R_{(-\gamma)} \xrightarrow{x} R_{(0)} \xrightarrow{x} R_{(\gamma)}
\]

are injective. Therefore \( \dim R_{(-\gamma)} \leq \dim R_{(0)} \leq \dim R_{(\gamma)} \). By symmetry between \( \gamma \) and \( -\gamma \), we conclude that all dimensions in question are equal. This also proves the “if” part of the last assertion. The “only if” part is obvious. \( \square \)

An ideal \( a \subset R \) is called homogeneous if \( a = \bigoplus_{\gamma \in \Gamma} a_{(\gamma)} \), where \( a_{(\gamma)} = a \cap R_{(\gamma)} \).

For a homogeneous ideal \( a \), let

\[
\Gamma(a) = \{ \gamma \in \Gamma : a_{(\gamma)} \neq R_{(\gamma)} \}.
\]

If \( p \) is a homogeneous prime ideal, then \( \Gamma(p) \) is a monoid and \( R/p \) has a natural \( \Gamma(p) \)-grading.

If \( R \) is a finitely generated prime ideal, then \( \dim R \) stands for the Krull dimension of \( R \), i.e., the transcendence degree of \( R \) over \( k \). Whenever a \( \Gamma \)-graded algebra \( R \) is a domain, we found it convenient to assume that \( R_{(\gamma)} \neq \{0\} \) for all \( \gamma \in \Gamma \). (For, otherwise one may replace \( \Gamma \) with a smaller monoid.) Thus, for a finitely generated \( \Gamma \)-graded domain \( R \), we set

\[
d(R) = \dim R - \text{rk} \Gamma.
\]

Remark. For \( k \) algebraically closed, \( d(R) \) has a transparent geometric meaning. A \( \Gamma \)-grading in \( R \) gives rise to an effective action of an \( \text{rk} \Gamma \)-dimensional torus \( T \) on \( \text{Spec} R \). Then \( d(R) \) is nothing but the complexity \( c_T(\text{Spec} R) \).
Lemma 2. Let $R$ be a finitely generated $\Gamma$-graded algebra without zero divisors. Then $d(R) \geq 0$. Furthermore, one has $d(R/p) \leq d(R)$ for any homogeneous prime ideal $p \subset R$.

Proof. Set $r = \text{rk } \Gamma$. There exist homogeneous elements $x_1, \ldots, x_r \in R$ such that $\deg x_1, \ldots, \deg x_r$ are linearly independent over $\mathbb{Z}$. This clearly implies that $x_1, \ldots, x_r$ are algebraically independent over $k$, whence the first inequality. One can find $y_1, \ldots, y_d \in R$ such that $x_1, \ldots, x_r, y_1, \ldots, y_d$ are algebraically independent and $r + d = \dim R$. Notice that $d = d(R)$. Let $\bar{y}_i$ (resp. $\bar{x}_j$) be the image of $y_i$ (resp. $x_j$) in $R/p$. Then each element of $R/p$ is algebraic over $k(\bar{x}_1, \ldots, \bar{x}_r, \bar{y}_1, \ldots, \bar{y}_d)$. Now, assume $x_i/p$ for $i \leq s$ and $x_i \in p$ for $i > s$. Since 0 occurs at least $r - s$ times in the sequence $(\bar{x}_1, \ldots, \bar{x}_r, \bar{y}_1, \ldots, \bar{y}_d)$, we obtain

$$\dim R/p \leq d + s = \dim R + s - \text{rk } \Gamma.$$

On the other hand, $\deg x_1, \ldots, \deg x_s \in \Gamma(p)$, and therefore $s \leq \text{rk } \Gamma(p)$. This proves the second inequality. $\square$

If $\Gamma$ is as above and $V = \bigoplus_{\gamma \in \Gamma} V(\gamma)$ is an arbitrary direct sum of finite-dimensional $k$-vector spaces, we will use the notation

$$m(V, \gamma) = \dim_k V(\gamma), \quad \gamma \in \Gamma.$$

It is sometimes convenient to regard $m(V, \gamma)$ as a function on all of $\mathbb{Z}^r$ by putting $m(V, \gamma) = 0$ for $\gamma \notin \Gamma$. This notation will be frequently used in the following situation.

Suppose $R$ is finitely generated and $\dim_k R(0) < \infty$. Then all the vector spaces $R(\gamma)$ are finite-dimensional. Indeed, let $\{x_i\}_{i=1,2,\ldots}$ be any infinite sequence in $R(\gamma)$. Since $R$ is a Noetherian algebra, there exists a number $m$ such that $x_i \in R \cdot x_1 + \cdots + R \cdot x_m$ for all $i$. Since all $x_i$ have the same degree, we see that $x_i \in R(0) \cdot x_1 + \cdots + R(0) \cdot x_m$ for all $i$.

Lemma 3. Let $R$ be as in Lemma 2. For any homogeneous ideal $a \subset R$, its radical $\sqrt{a}$ and the associated prime ideals are homogeneous as well. Let $p_1, \ldots, p_s$ be the minimal prime ideals of $a$. Then

$$m(R/\sqrt{a}, \gamma) \leq \sum_{j=1}^s m(R/p_j, \gamma).$$

Furthermore, there exist $\gamma_1, \ldots, \gamma_d \in \Gamma$ such that

$$m(R/a, \gamma) \leq \sum_{j=1}^d m(R/\sqrt{a}, \gamma - \gamma_j).$$

Proof. It is a standard fact that the associated primes of $a$ are homogeneous, see [5, 3.5]. Since

$$\sqrt{a} = \bigcap_{j=1}^s p_j,$$
it follows that $\sqrt{a}$ is homogeneous as well. Moreover, the canonical homomorphism
\[ R/\sqrt{a} \to \bigoplus_{j=1}^{s} R/p_j \]
is an imbedding which induces an injection on homogeneous components. This gives the first inequality. To prove the second one, let $r = \sqrt{a}$. Since $R$ is a Noetherian algebra, $a$ contains $r^p$ for some $p \geq 1$. Thus, it suffices to prove that
\[ m(R/r^p, \gamma) \leq \sum_{j=1}^{d_p} m(R/r, \gamma - \gamma_j^{(p)}), \quad (1) \]
where $d_p$ and $\gamma_j^{(p)}$ depend only on $r$. For each $p \geq 1$, the exact sequence
\[ \{0\} \to r^{p-1}/r^p \to R/r^p \to R/r^{p-1} \to \{0\} \]
shows that
\[ m(R/r^p, \gamma) = m(R/r^{p-1}, \gamma) + m(r^{p-1}/r^p, \gamma). \quad (2) \]
Let $\{u_1, \ldots, u_N\}$ be a homogeneous basis for the ideal $r^{p-1}$ and let $\beta_i = \deg u_i$, $i = 1, \ldots, N$. Given $\bar{x} \in (R/r)_{(\sigma)}$, let $x$ denote a representative of $\bar{x}$ in $R_{(\sigma)}$. It is easily seen that the linear map
\[ (R/r_{(\gamma - \beta_1)} \oplus \cdots \oplus (R/r_{(\gamma - \beta_N)}) \to r^{p-1}/r^p, \quad (\bar{x}_1, \ldots, \bar{x}_N) \mapsto \sum_{i=1}^{N} x_i u_i \quad (\text{mod } r^p), \]
is well defined and surjective. Hence
\[ m(r^{p-1}/r^p, \gamma) \leq \sum_{i=1}^{N} m(R/r, \gamma - \beta_i), \]
and (1) follows from (2) by induction. \qed

2. Growth of multiplicities in multigraded algebras

In this section, $k$ is a field and $\Gamma$ is a (commutative) monoid that can be imbedded into some $\mathbb{Z}^r$. We study the multiplicity function $\gamma \mapsto m(R, \gamma)$ for a $\Gamma$-graded $k$-algebra $R$ with $\dim_k R_{(0)} < \infty$. Associated with the imbedding $\Gamma \hookrightarrow \mathbb{Z}^r \subset \mathbb{R}^r$, one obtains the Euclidean norm function
\[ \gamma = (t_1, \ldots, t_r) \mapsto ||\gamma|| = \sqrt{t_1^2 + \cdots + t_r^2}, \quad \gamma \in \Gamma. \]
Without loss of generality, we may assume that $\mathbb{Z} \cdot \Gamma = \mathbb{Z}^r$, i.e., $r = \text{rk} \Gamma$. For the proof of the following theorem, we borrow some technique from [8] (cf. Theorem 4.1 therein).

**Theorem 3.** Let $R$ be a finitely generated integral $\Gamma$-graded $k$-algebra such that $\dim_k R_{(0)} < \infty$. Then there exists a positive constant $A = A(R)$ such that
\[ m(R, \gamma) \leq A \cdot (||\gamma|| + 1)^{d(R)} \]
for all $\gamma \in \Gamma$. 

follows after a change of the constant. Obviously, $n \gamma \in \Gamma \cap (-\Gamma)$, see Lemma 1. Let $\{s_1, \ldots, s_N\}$ be a minimal system of homogeneous generators of $R$ and $\alpha_i = \deg s_i$ ($i = 1, \ldots, N$). Assume that the numbering of the generators is chosen so that $\alpha_i \notin \Gamma \cap (-\Gamma)$ for $i = 1, \ldots, M$ and $\alpha_i \in \Gamma \cap (-\Gamma)$ for $i = M + 1, \ldots, N$. In other words, the generator $s_i$ is invertible in $R$ if and only if $i \geq M + 1$. Among all minimal systems $\{s_1, \ldots, s_N\}$ choose one with the smallest possible $M$. This number depends only on $R$ and is sometimes denoted by $M_R$.

The proof is by induction on $\dim R$ and, for $dim R$ fixed, by induction on $M = M_R$.

If $\dim R = 0$, then $R = R_{(0)}$ and $d(R) = 0$. Thus, the assertion is obvious. If $M = 0$, then $m(R, \gamma) = 1$ for all $\gamma \in \Gamma$, so that the desired inequality is also clear. In what follows, we assume that $\dim R > 0$ (or, equivalently, $r > 0$) and $M > 0$. The subalgebra $R_i = k[s_1, \ldots, s_i, \ldots s_N] \subset R$ inherits a grading. The corresponding monoid $\Gamma_i \subset \Gamma$ is generated by $\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_N$. The induction argument splits into three cases.

**Case 1.** $r \ker \Gamma_i = r - 1$ for some $i \in \{1, \ldots, N\}$.

Fix $i$ with this property. Since $-\alpha_i \notin \Gamma$, we actually have $i \leq M$. It is clear that $s_i$ is transcendental over $R_i$. Therefore $\dim R_i = \dim R - 1$ and $d(R_i) = d(R)$. The homomorphism

$$\pi_i: R_i[T] \rightarrow R, \quad T \mapsto s_i,$$

is surjective (because the image contains all the generators $s_1, \ldots, s_N$) and injective (because $s_i$ is transcendental over $R_i$). The assumption $r \ker \Gamma_i = r \ker \Gamma - 1$ implies that any $\gamma \in \Gamma$ has a unique presentation of the form $\gamma = n\alpha_i + \beta$, where $\beta \in \Gamma$, and $n = n(\gamma) \in \mathbb{Z}_{\geq 0}$. Furthermore, the isomorphism $\pi_i$ shows that $R_i(\gamma) = s_i^n(R_i)_{(\beta)}$.

Therefore, applying the induction hypothesis to $R_i$, we obtain

$$m(R, \gamma) = m(R_i, \beta) \leq A \cdot (\|\beta\| + 1)^d(R_i) = A \cdot (\|\beta\| + 1)^d(R)
\quad = A \cdot (\|\gamma - n(\gamma)\alpha_i\| + 1)^d(R) \leq A \cdot (\|\gamma\| + n(\gamma)\|\alpha_i\| + 1)^d(R).$$

Obviously, $n(\gamma) \leq \|\gamma\|/\|\alpha_i\|^k$, where $\alpha_i^k$ is the projection of $\alpha_i$ to the normal direction of the hyperplane generated by $\Gamma_i$ in $\mathbb{R}^r$. Therefore the required inequality follows after a change of the constant.

**Case 2.** $r \ker \Gamma_j = r$ for all $j \in \{1, \ldots, N\}$ and $\dim R_i = \dim R$ for at least one $i \in \{1, \ldots, M\}$.

Fix $i$ with the latter property. Since $s_i$ is algebraic over $R_i$, there exist a homogeneous element $p \in R_i$ and $d \geq 1$ such that

$$ps_i^d \in V, \quad \text{where } V = \sum_{t=0}^{d-1} s_i^t R_i.$$

Since $R = s_i^d R + V$, the multiplication by $s_i^d$ defines a surjective homogeneous linear map

$$R/R p \rightarrow s_i^d R / s_i^d R \cap V.$$
which shifts the grading by \(d \cdot \alpha_i\). Therefore

\[
m(R, \gamma) = m(V, \gamma) + m(R/V, \gamma) = m(V, \gamma) + m(s_i^d R/s_i^d R \cap V, \gamma)
\]

\[
\leq \sum_{t=1}^{d-1} m(s_i^t R_i, \gamma) + m(R/Rp, \gamma) = \sum_{t=1}^{d-1} m(R_i, \gamma - t\alpha_i) + m(R/Rp, \gamma).
\]

By the assumption of Case 2, we have \(\dim R = \dim R_i\) and \(M_{R_i} < M_R\). It follows that

\[m(R_i, \gamma) \leq A_1 \cdot (\|\gamma\| + 1)^{d(R)}\]

for some \(A_2 > 0\). To finish the proof in Case 2, it suffices to demonstrate that

\[m(R/Rp, \gamma) \leq A_3 \cdot (\|\gamma\| + 1)^{d(R)}\]

for some \(A_3 > 0\). If \(p\) is invertible, this is obvious. Otherwise, Lemma 3 shows that it suffices to prove the similar inequality for \(m(R/p, \gamma)\), where \(p\) is an associated prime ideal of the principal ideal \(Rp\). Here the induction hypothesis applies, since \(\dim R/p < \dim R\). Together with the second inequality in Lemma 2, this completes the proof in Case 2.

**Case 3.** \(\text{rk} \Gamma_j = r\) for all \(j \in \{1, \ldots, N\}\) and \(\dim R_i < \dim R\) for all \(i \in \{1, \ldots, M\}\).

If \(M = 1\), then \(\text{rk} \Gamma_1 = r - 1\), i.e., we are in Case 1 and not in Case 3. Hence \(M > 1\).

We first prove that the image of \(s_j\) in \(R/s_i R\) is not a zero divisor for all \(i \leq M\), \(j \leq M\), and \(i \neq j\). Assume that this is not the case. Then there exist \(a, b \in R, a \notin s_i R\), such that \(s_j a = s_j b\). Write \(a = a_0 s_i^d + a_1 s_i^{d-1} + \cdots + a_d\), where \(a_0, a_1, \ldots, a_d \in R_i\). We have \(a_d \neq 0\), hence

\[s_i \left(b - \sum_{t=0}^{d-1} a_t s_i^{d-1-t}\right) = s_j a_d \in R_i \setminus \{0\}.
\]

Since \(b\) can also be written as a polynomial in \(s_i\) with coefficients in \(R_i\), we see that \(s_i\) is algebraic over \(R_i\), which contradicts the assumption of Case 3.

Let \(p\) be an associated prime ideal of \(s_i R\). We have just proved that \(s_j \notin p\) if \(j \neq i\). This implies that \(\alpha_j \in \Gamma(p)\) and therefore \(\Gamma_i \subseteq \Gamma(p)\). It follows that \(\text{rk } \Gamma(p) = r\), since we are in Case 3.

Because \(s_i\) is not invertible for \(i \leq M\), \(\dim R/p = \dim R - 1\) for every minimal prime ideal of \(s_i R\). Since \(\text{rk } \Gamma(p) = \text{rk } \Gamma\), we have \(d(R/p) = d(R) - 1\). Applying the induction hypothesis to \(R/p\) yields

\[m(R/p, \gamma) \leq A \cdot (\|\gamma\| + 1)^{d(R)-1}.
\]

By Lemma 3, we get a similar estimate for \(R/s_i R\) after a change of the constant:

\[m(R/s_i R, \gamma) \leq A_1 \cdot (\|\gamma\| + 1)^{d(R)-1}.
\]

We may assume that the constant \(A_1\) in this estimate is the same for all \(s_i (i \leq M)\).

Invoking the exact sequence

\[\{0\} \to R \xrightarrow{s_i} R \to R/s_i R \to \{0\}\]
arising from the multiplication by \(s_i\), we get
\[
m(R, \gamma) = m(R, \gamma - \alpha_i) + m(R/s_i R, \gamma), \quad \gamma \in \Gamma.
\]
For any \(\gamma \in \Gamma\), take a presentation
\[
\gamma = \beta + n_1 \alpha_1 + \cdots + n_M \alpha_M,
\]
where \(n_i \in \mathbb{Z}_{\geq 0}\) and \(\beta \in \Gamma \cap (-\Gamma)\). Using repeatedly (4) with \(s_M, \ldots, s_1\), we obtain
\[
m(R, \gamma) = m(R, \beta) + \sum_{j=1}^{M} \sum_{t=1}^{n_j} m(R/s_j R, n_1 \alpha_1 + \cdots + n_{j-1} \alpha_{j-1} + t \alpha_j).
\]
Recall that \(m(R, \beta) = 1\). Therefore
\[
m(R, \gamma) \leq 1 + A_1 \cdot \sum_{j=1}^{M} \sum_{t=1}^{n_j} \left(\|n_1 \alpha_1 + \cdots + n_{j-1} \alpha_{j-1} + t \alpha_j\| + 1\right)^{d(R)-1}
\]
by (3). Clearly,
\[
\|n_1 \alpha_1 + \cdots + n_{j-1} \alpha_{j-1} + t \alpha_j\| \leq \max\{\|\alpha_1\|, \ldots, \|\alpha_M\|\} \cdot (n_1 + \cdots + n_M)
\]
for all \(j\) and \(t\).

On the other hand, let \(C\) be the convex cone in \(\mathbb{R}^r\) generated by \(\Gamma\), and let \(C^*\) be the dual cone. Note that \(C^*\) does not contain straight lines. Thus, \(C^*\) is the convex hull of a finite set \(\Phi\), the elements of which determine the extreme rays of \(C^*\). Now, let \(\lambda\) be any linear function from the relative interior of \(C^*\), so that if \(\lambda(\alpha) = 0\) for some \(\alpha \in C\) then \(\phi(\alpha) = 0\) for all \(\phi \in \Phi\). Assuming \(\lambda(\alpha_j) = 0\) for some \(j \in \{1, \ldots, M\}\), we obtain \(-\alpha_j \in C\) by Farkas’ Lemma (see e.g. [18, Prop. 1.9]). However, this implies \(-\alpha_j \in \Gamma\), contradictory to our assumption. We see that there exists a linear function \(\lambda\) such that \(\lambda|_{\Gamma \cap (-\Gamma)} = 0\) and \(\lambda(\alpha_j) > 0\) for all \(j \in \{1, \ldots, M\}\). It follows that
\[
\lambda(\gamma) = n_1 \lambda(\alpha_1) + \cdots + n_M \lambda(\alpha_M) \geq \min_{j \in \{1, \ldots, M\}} \lambda(\alpha_j) \cdot (n_1 + \cdots + n_M),
\]
hence
\[
n_1 + \cdots + n_M \leq D \cdot \|\gamma\|
\]
for some \(D > 0\). Therefore
\[
m(R, \gamma) \leq A_2 \cdot (n_1 + \cdots + n_M + 1)^{d(R)} \leq A_3 \cdot (\|\gamma\| + 1)^{d(R)}
\]
with some \(A_2, A_3 > 0\).

3. THE ALGEBRA OF COVARIANTS AND BRANCHING RULES

For the rest of the paper, \(k\) is algebraically closed and \(\text{char } k = 0\). Keep the notation of the introduction. We first deduce Theorem 1 from Theorem 3.

Proof of Theorem 1. Since \(H\) is a Grosshans subgroup, \(k[G/H]\) is finitely generated. Let \(R := k[G/H]^U\) be the algebra of covariants on \(G/H\). Then \(R\) is finitely generated by the Hadžiev–Grosshans theorem, see e.g. [7, Thm. 9.4]. In general, the isotypic decomposition
\[
k[G/H] = \bigoplus_{\lambda \in \Gamma(G/H)} k[G/H]_{(\lambda)}
\]
is not a grading, but letting $R(\lambda) = R \cap k[G/H]_{(\lambda)}$ one obtains a $\Gamma(G/H)$-grading in $R$. Clearly, $\dim k[G/H]_{(\lambda)} = d_\lambda \cdot m(R, \lambda)$ and $k[G/H]_{(0)} = R_{(0)} = k$. Since $G/H$ is quasi-affine, the rank of the monoid $\Gamma(G/H)$ is equal to $r(G/H)$ (see [14, 1.3]). Thus, it follows from [14, 1.2.9] that
\[ c(G/H) + \text{rk } \Gamma(G/H) = \dim R. \]
Therefore $c(G/H) = d(R)$ and, by Theorem 3,
\[ \frac{\dim k[G/H]_{(\lambda)}}{d_\lambda} = m(R, \lambda) \leq A \cdot (\| \lambda \| + 1)^c(G/H) \]
for all $\gamma \in \Gamma(G/H)$. The second assertion follows from the corresponding “non-uniform” result mentioned in the introduction. Namely, if
\[ \frac{\dim k[G/H]_{(\lambda)}}{d_\lambda} \leq A \cdot (\| \lambda \| + 1)^k \]
for some $k$ and all $\lambda \in \Gamma(G/H)$, then $\dim k[G/H]_{n\lambda/d_\lambda} n^k$ is bounded as $n \to \infty$ for each $\lambda \in \Gamma(G/H)$. Hence $k \geq c(G/H)$. \hfill $\square$

For any $V_\lambda$, $\lambda \in \mathcal{X}(T)_+$, the highest weight of the dual $G$-module is denoted by $\lambda'$.

**Proof of Theorem 2.** (i) Recall that $H$ is a connected reductive subgroup of $G$ and we have fixed a Borel subgroup $B_H \subset B$ and a maximal torus $T_H \subset B_H$. Whenever it is convenient hereafter, we may assume that $T_H \subset T$ and $B_H \subset B$. We are going to apply Theorem 1 to the transitive action of $G \times H$ on $G$ which is defined by $(g, h) \cdot \tilde{g} = ggh^{-1}$. This means that we identify $G$ with the homogeneous space $(G \times H) / \Delta_H$, where $\Delta_H = \{(h, h^{-1}); h \in H\}$. By definition, the $G \times H$-complexity of $G$ is the minimal codimension of $B \times B_H$-orbits in $G$, where $B$ (resp. $B_H$) acts by left (resp. right) translations. Clearly, this is the same as the minimal codimension of $B$-orbits in $G/B_H$ or $B_H$-orbits in $B \setminus G$. In particular,
\[ c(G \times H / \Delta_H) = c_{G \times H}(G) = c_{G}(G/B_H) = c(G/B_H). \tag{5} \]

To understand the structure of the corresponding isotypic components, consider first the larger group $\tilde{G} = G \times H$ acting on $G$ by left and right translations. As is well known, the $G$-module $k[G]$ has the decomposition
\[ k[G] = \bigoplus_{\lambda \in \mathcal{X}(T)_+} (V_\lambda \otimes V_{\lambda'}) \]
(see e.g. [9, II.3.1, Satz 3]). In other words, for a $\tilde{G}$-isotypic component $k[G]_{(\lambda, \nu)}$ one has $k[G]_{(\lambda, \nu)} \neq \{0\}$ if and only if $\nu = \lambda'$, and each such isotypic component is a simple $\tilde{G}$-module. Restricting the $\tilde{G}$-action to $G \times H$, we obtain
\[ k[G] = \bigoplus_{\lambda \in \mathcal{X}(T)_+} V_\lambda \otimes (\text{res}_H V_{\lambda'}) = \bigoplus_{\lambda \in \mathcal{X}(T)_+} V_\lambda \otimes \left( \bigoplus_{\mu} m_{\lambda'}(\mu) W_\mu \right) = \bigoplus_{\lambda, \mu} m_{\lambda'}(\mu) (V_\lambda \otimes W_\mu), \]
where $\mu$ ranges over $\mathcal{X}(T_H)_+$. Thus, the $G \times H$-isotypic components of $k[G]$ are given by
\[ k[G]_{(\lambda, \mu)} = m_{\lambda'}(\mu) (V_\lambda \otimes W_\mu), \quad (\lambda, \mu) \in \mathcal{X}(T)_+ \times \mathcal{X}(T_H)_+. \tag{6} \]
Making use of Theorem 1 and (5), we then conclude that
\[ m_{\chi}(\mu) \leq A(\|\lambda\| + \|\mu\| + 1)^{c(G/B_H)}. \]

Let \( g: \chi(T) \to \chi(T_H) \) be the restriction homomorphism. Then \( \|g(\eta)\| \leq C \cdot \|\eta\| \) for all \( \eta \in \chi(T) \) and some \( C > 0 \). Next, if \( m_{\chi}(\mu) \neq 0 \) then \( \mu = g(\eta) \), where \( \eta \) is a weight of \( V_\chi \), whence \( \|\mu\| \leq C \cdot \|\chi\| \). There is no loss of generality in assuming that the norm function \( \lambda \mapsto \|\lambda\| \) is invariant under the Weyl group. Thus, \( \|\lambda\| = \|\lambda\| \) and changing the constant \( A \), we finally obtain
\[ m_{\chi}(\mu) \leq D \cdot (\|\lambda\| + 1)^{c(G/B_H)}. \]

(ii) Equation (6) shows that \( \Gamma(G, H) \) is a monoid, which is isomorphic to \( \Gamma(G \times H/\Delta_H) \). The latter monoid is finitely generated, since \( G \times H/\Delta_H \) is affine.

This completes the proof of Theorem 2.

The following simple formula for the complexity \( c(G/B_H) \) is a useful complement to Theorem 2. Note first that if \( F \subset H \) is a (reductive) normal subgroup of \( G \), then passing from \( (G, H) \) to \( (G/F, H/F) \) does not change the complexity under consideration. Indeed, without loss of generality, assume \( F \) connected and, making a finite covering, reduce to the case \( G = F \times G^*, H = F \times H^* \), where \( H^* \subset G^* \). Then, in the obvious notation, \( G/B_H \cong G^*/B_{H^*} \times F/B_F \) and hence \( c_G(G/B_H) = c_{G^*}(G^*/B_{H^*}) + c_F(F/B_F) = c_{G^*}(G^*/B_{H^*}). \)

**Theorem 4.** Let \( H \) be a connected reductive subgroup of \( G \). Suppose \( H \) does not contain infinite normal subgroups of \( G \). Then

(i) \( c(G/B_H) = \dim U - \dim B_H \);
(ii) \( \text{rk} \Gamma(G, H) = \text{rk} G + \text{rk} H \).

**Proof.** By (5), the complexity in question is equal to the complexity of the affine homogeneous space \( (G \times H)/\Delta_H \). Therefore it can be computed via the isotropy representation of the subgroup \( \Delta_H \subset G \times H \), i.e., the representation of \( \Delta_H \) on the tangent space of \( (G \times H)/\Delta_H \) at the point \( \{\Delta_H\} \), see [14, 2.2]. As \( \Delta_H \simeq H \), this isotropy representation is nothing but \( \text{res}_H g \), where \( g = \text{Lie} G \) is the adjoint \( G \)-module. In order to compute the complexity and rank of \( (G \times H)/\Delta_H \), one may exploit a generic stabilizer for the isotropy representation. Let \( S \subset H \) be such stabilizer. Since \( g \) is an orthogonal \( H \)-module, \( S \) is reductive by a result of D. Luna. By [14, 2.2], the following holds:

\[ r(G \times H/\Delta_H) + 2c(G \times H/\Delta_H) = \dim(G \times H/\Delta_H) - \dim \Delta_H + \dim S, \]
\[ r(G \times H/\Delta_H) = \text{rk}(G \times H) - \text{rk} S. \]

It follows from (6) and the definition of rank that \( r(G \times H/\Delta_H) = \text{rk}(G, H) \). We shall prove that \( S \) is finite. Let \( m \) be an \( H \)-stable complement to \( \mathfrak{h} = \text{Lie} H \) in \( g \). Then \( \text{res}_H g = \mathfrak{h} \oplus m \). A generic stabilizer for the \( H \)-module \( \mathfrak{h} \) is \( T_H \). Therefore \( S \) is a generic stabilizer for the \( T_H \)-module \( \text{res}_{T_H} m \). By the hypothesis, \( \mathfrak{h} \) contains no ideals of \( g \). Hence \( m \) is a locally faithful \( H \)-module. In particular, \( \text{res}_{T_H} m \) is a locally faithful \( T_H \)-module. Therefore \( S \) is finite. As \( \dim S = \text{rk} S = 0 \), we conclude
by (7) and (8) that
\[
c(G \times H/\Delta_H) = \frac{1}{2} (\dim G - \dim H - \text{rk} G - \text{rk} H) = \dim U - \dim B_H
\]
and \(\text{rk} \Gamma(G, H) = \text{rk} G + \text{rk} H\).

Combining Theorems 2 and 4, we recover the following result of [1].

**Theorem 5.** Let \(P\) be a parabolic subgroup of \(G\) and \(L\) a Levi subgroup of \(P\). Let \(H \subset G\) be a connected subgroup such that \([L, L] \subset H \subset L\). Denote by \(Z\) the connected centre of \(G\). Set \(l = \text{rk} [G, G], d = \dim G/P\), and \(s = \text{rk} G - \text{rk} HZ\).

Assume that \(P\) contains no simple factor of \(G\). Then there is a positive constant \(D = D(G, H)\) such that
\[
m_\lambda(\mu) \leq D \cdot (\|\lambda\| + 1)^{d-l+s}
\]
for all dominant weights \(\lambda\) of \(G\) and \(\mu\) of \(H\). \(\square\)

4. **Applications**

In [10, Prop. 3], M. Krämer found all pairs \((G, H)\) such that \(G\) is simple and \(m_\lambda(\mu) = 1\) for all \((\lambda, \mu) \in \Gamma(G, H)\). By [15], this condition is equivalent to the fact that \(c(G/B_H) = 0\). Up to a local isomorphism, Krämer’s list contains two series: \((\text{SL}_{n+1}, \text{GL}_n), n \geq 1\), and \((\text{SO}_n, \text{SO}_{n-1}), n \geq 3, n \neq 4\), and one sporadic case: \((\text{SO}_8, \text{Spin}_7)\). Actually, Krämer worked in the category of compact connected Lie groups, but this makes no essential difference. Without loss of generality, one may assume that \(G\) is simply connected. It is then easily seen that the above sporadic case becomes an item of the orthogonal series. More precisely, let \(G = \text{Spin}_8\) and let \(H_1, H_2 \subset G\) be the two copies of \(\text{Spin}_7\) defined, respectively, by the spin representation of the latter group and by the standard imbedding \(\text{SO}_7 \hookrightarrow \text{SO}_8\). One can show that there is an outer automorphism \(\phi: G \to G\) such that \(\phi(H_1) = H_2\). Thus, we essentially have only two series. But, for the sake of future exposition, it is convenient (a) to split the orthogonal series into two series, according to the parity of \(n\), and (b) to write \(\text{SL}_n \cdot k^*\) in place of \(\text{GL}_n\). Here ‘·’ stands for an almost direct product, which means that one has a direct sum for the respective Lie algebras. We collect this information in Table 1.

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>(G)</td>
<td>(\text{SL}_{n+1} (n \geq 1))</td>
<td>(\text{Spin}_{2n+1} (n \geq 2))</td>
<td>(\text{Spin}_{2n} (n \geq 3))</td>
</tr>
<tr>
<td>(H)</td>
<td>(\text{SL}_n \cdot k^*)</td>
<td>(\text{Spin}_{2n})</td>
<td>(\text{Spin}_{2n-1})</td>
</tr>
<tr>
<td>(\text{rk} \Gamma(G, H))</td>
<td>(2n)</td>
<td>(2n)</td>
<td>(2n - 1)</td>
</tr>
</tbody>
</table>

All pairs \((G, H)\) such that \(G\) is simple and simply connected, \(H\) is connected reductive, and \(c(G/B_H) = 1\) are gathered in Table 2. This classification can be quickly obtained in the following way. If \(c(G/B_H) = 1\), then \(c(G/H) \leq 1\). The list of reductive subgroups \(H\) with the last property is known (see [11, Tabelle 1] for
Theorem 6. Suppose $G$ is simply connected and semisimple, and $c(G/B_H) = 1$. Then there exists a unique weight $(\lambda_0, \mu_0) \in \Gamma(G, H)$ such that

(i) $m_{\lambda_0}(\mu_0) = 2$;

(ii) For any $(\lambda, \mu) \in \Gamma(G, H)$, let $n = n(\lambda, \mu) \in \mathbb{Z}_{\geq 0}$ be the greatest number such that $(\lambda, \mu) - n(\lambda_0, \mu_0) \in \Gamma(G, H)$. Then $m_{\lambda}(\mu) = n + 1$. □

The weight $(\lambda_0, \mu_0)$ is called remarkable.

Below we give a complete description of branching rules for the items of Table 1 and items 1–5 of Table 2. To this end, it suffices to specify the generators of $\Gamma(G, H)$ and, in the second case, the remarkable weight. Denote by $\{\tilde{\varphi}_i\}$ the fundamental weights of $G$ and by $\{\varphi_i\}$ the fundamental weights of $H$. A generator of the character group of $k^*$ is denoted by $\varepsilon$. For item 2 of Table 2, where $H$ is not simple, the fundamental weight of $SL_2$ is denoted by $\psi$. The sum of the fundamental weights of $G$ (resp. $H$) is denoted by $\tilde{\rho}$ (resp. $\rho$).
Remarks. 1. Fortunately, for the cases we are interested in, the numbering of simple roots in [3] and [16] coincide.

2. For all cases with $c(G \times H/\Delta H) = 0$, there is a well-known description of $\Gamma(G, H)$ in terms of a certain interlacing condition for highest weights, see e.g. [2, Ch. 5], [17, Ch. 18], [6, Ch. 8]. In this way, one obtains a collection of inequalities describing the cone generated by $\Gamma(G, H)$, while Theorem 7 lists the generators of $\Gamma(G, H)$ thereby giving the “dual” presentation of the monoid in question. It should be stressed that the interlacing condition is not needed in the proof. This approach to branching rules is also justified by the fact that we obtain a uniform presentation for the two cases $c(G \times H/\Delta H) = 0$ or 1.

Theorem 7. 1. The generators of $\Gamma(\text{SL}_{n+1}, \text{SL}_n, \mathfrak{k}^*)$ are

$$(\tilde{\varphi}_1, \varphi_1 + \varepsilon), (\tilde{\varphi}_2, \varphi_2 + 2\varepsilon), \ldots, (\tilde{\varphi}_{n-1}, \varphi_{n-1} + (n-1)\varepsilon), (\tilde{\varphi}_n, n\varepsilon), (\tilde{\varphi}_1, -n\varepsilon), (\tilde{\varphi}_2, \varphi_1 + (1-n)\varepsilon), \ldots, (\tilde{\varphi}_{n-1}, \varphi_{n-2} - 2\varepsilon), (\tilde{\varphi}_n, \varphi_{n-1} - \varepsilon).$$

2. The generators of $\Gamma(\text{Spin}_{2n+1}, \text{Spin}_{2n})$ are

$$(\tilde{\varphi}_1, \varphi_1), \ldots, (\tilde{\varphi}_{n-2}, \varphi_{n-2}), (\tilde{\varphi}_{n-1}, \varphi_{n-1} + \varphi_n), (\tilde{\varphi}_n, \varphi_n), (\tilde{\varphi}_1, 0), (\tilde{\varphi}_2, \varphi_1), \ldots, (\tilde{\varphi}_{n-1}, \varphi_{n-2}), (\tilde{\varphi}_n, \varphi_{n-1}).$$

3. The generators of $\Gamma(\text{Spin}_{2n}, \text{Spin}_{2n-1})$ are

$$(\tilde{\varphi}_1, \varphi_1), \quad i = 1, \ldots, n-1; (\tilde{\varphi}_1, \varphi_{i-1}), \quad i = 1, \ldots, n; (\tilde{\varphi}_{n-1} + \tilde{\varphi}_n, \varphi_{n-2}).$$

Proof. A general idea that applies to all cases is the following. Since $\Gamma(G, H)$ is a free monoid of known rank, it suffices to find the required number of generators in $\Gamma(G, H)$. This amounts to verifying that the pairs of dominant weights $(\lambda, \mu)$ listed above are contained in and are indecomposable in $\Gamma(G, H)$. The second is automatic if $\lambda$ is fundamental. Note that the only item in the list for which this is not the case is the last one in part 3.

1. We assume that the group $\mathfrak{k}^*$ is imbedded into $\text{SL}_{n+1}$ as the group of diagonal matrices with entries $(\varepsilon, \ldots, \varepsilon, \varepsilon^{-n})$. It is easy to describe the restriction to $H = \text{SL}_n \cdot \mathfrak{k}^*$ of the tautological representation of $\text{SL}_{n+1}$:

$$\text{res}_H V_{\tilde{\varphi}_1} = (W_{\varphi_1} \otimes \varepsilon) \oplus \varepsilon^{-n}.$$ 

This yields two elements of $\Gamma(G, H)$, $(\tilde{\varphi}_1, \varphi_1 + \varepsilon)$ and $(\tilde{\varphi}_1, -n\varepsilon)$, which are certainly generators. Since $V_{\tilde{\varphi}_1}$ is isomorphic to $\wedge^1 V_{\tilde{\varphi}_1}$, one easily finds that

$$\text{res}_H V_{\tilde{\varphi}_i} = (W_{\varphi_i} \otimes \varepsilon^i) \oplus (W_{\varphi_{i-1}} \otimes \varepsilon^{i-1-n}) \quad (i = 1, \ldots, n).$$

It is assumed here that $W_{\varphi_0}$ and $W_{\varphi_n}$ are trivial 1-dimensional representations of $\text{SL}_n$. These formulas yield the required number of generators, which are just the ones listed in the statement of the theorem.

2. Here we have $\text{res}_H V_{\tilde{\varphi}_1} = W_{\varphi_1} \oplus W_0$. Hence

$$\text{res}_H V_{\tilde{\varphi}_i} = W_{\varphi_i} \oplus W_{\varphi_{i-1}} \quad (i = 1, \ldots, n - 2),$$

$$\text{res}_H V_{\tilde{\varphi}_{n-1}} = W_{\varphi_{n-1} + \varphi_n} \oplus W_{\varphi_{n-2}}, \quad \text{res}_H V_{\tilde{\varphi}_n} = W_{\varphi_{n-1}} \oplus W_{\varphi_n}.$$ 

The last formula gives the (well-known) restriction of the spinor representation, while the previous ones follow from the equalities $\bigwedge^i V_{\tilde{\varphi}_1} = V_{\tilde{\varphi}_i}$ for $i \leq n - 1$, $\bigwedge^i W_{\varphi_1} = W_{\varphi_i}$ for $i \leq n - 2$, and $\bigwedge^{n-1} W_{\varphi_1} = W_{\varphi_{n-1} + \varphi_n}$. 


3. In a sense, the roles of $G$ and $H$ (i.e., $V$ and $W$) are interchanged here. We have
\[
\text{res}_H V_{\tilde{\varphi}_i} = W_{\varphi_i} \oplus W_{\varphi_{i-1}} \quad (i = 1, \ldots, n-2),
\]
\[
\text{res}_H V_{\tilde{\varphi}_{n-1}} = W_{\varphi_{n-1}}, \quad \text{res}_H V_{\tilde{\varphi}_n} = W_{\varphi_{n-1}}.
\]
But these restrictions provide us with $2n - 2$ generators for $\Gamma(G, H)$, so that one generator is still missing. We recover this generator from the equalities $\bigwedge^{n-1} V_{\tilde{\varphi}_1} = V_{\varphi_{n-1} + \tilde{\varphi}_n}$ and $\bigwedge^{n-1} W_{\varphi_1} = W_{\varphi_{n-1}}$, and the restriction:
\[
\text{res}_H V_{\tilde{\varphi}_{n-1} + \tilde{\varphi}_n} = W_{\varphi_{n-1} + \tilde{\varphi}_n} \oplus W_{\varphi_{n-2}}.
\]
The previous formulas show that neither $(\varphi_{n-1}, \varphi_{n-2})$ nor $(\tilde{\varphi}_n, \varphi_{n-2})$ lie in $\Gamma(G, H)$. Thus, $(\tilde{\varphi}_{n-1} + \tilde{\varphi}_n, \varphi_{n-2})$ is indecomposable, and we are done. \hfill \Box

**Theorem 8.** 1. The generators of $\Gamma(\text{SL}_{n+1}, \text{SL}_n)$ are:

$$(\tilde{\varphi}_1, \varphi_1), \ldots, (\tilde{\varphi}_{n-1}, \varphi_{n-1}), (\tilde{\varphi}_n, 0), (\tilde{\varphi}_1, 0), (\tilde{\varphi}_2, \varphi_1), \ldots, (\tilde{\varphi}_n, \varphi_{n-1}).$$

The remarkable weight is $(\tilde{\rho}, \rho)$.

2. The generators of $\Gamma(\text{SP}_6, \text{Sp}_4 \times \text{SL}_2)$ are:

$$(\tilde{\varphi}_1, \varphi_1), (\tilde{\varphi}_2, \varphi_1), (\tilde{\varphi}_2, \varphi_2), (\tilde{\varphi}_3, \varphi_1), (\tilde{\varphi}_3, \varphi_2), (\tilde{\varphi}_1 + \tilde{\varphi}_3, \varphi_2).$$

The remarkable weight is $(\tilde{\rho}, \rho)$.

3. The generators of $\Gamma(\text{Sp}_{10}, G_2)$ are:

$$(\varphi_1, \varphi_1), (\varphi_2, \varphi_1), (\varphi_2, \varphi_2), (\varphi_3, \varphi_1), (\varphi_3, 0), (\varphi_1 + \varphi_3, \varphi_2), (\varphi_1 + \varphi_2, \varphi_2).$$

The remarkable weight is $(\tilde{\rho}, \rho)$.

4. The generators of $\Gamma(G_2, \text{SL}_3)$ are:

$$(\varphi_1, \varphi_1), (\varphi_2, \varphi_1), (\varphi_2, \varphi_2), (\varphi_1, 0), (\varphi_2, \varphi_1 + \varphi_2).$$

The remarkable weight is $(\tilde{\rho}, \rho)$.

5. The generators of $\Gamma(\text{SL}_3, \text{SO}_3)$ are:

$$(\varphi_1, 2\varphi_1), (\varphi_2, 2\varphi_1), (\varphi_1 + \varphi_2, 2\varphi_1), (2\varphi_1, 0), (2\varphi_2, 0).$$

The remarkable weight is $(2\tilde{\rho}, 4\rho)$.

**Proof.** The description of the monoid $\Gamma(G, H)$ comes up as a by-product of describing the algebraic structure of $k[G]_{U \times U}$. Since $c(G \times H/\Delta_H) = 1$ here, it follows from [13, Theorem 1.6] that this algebra is a complete intersection. Actually, we shall prove that it is a polynomial algebra in case 1 and is a hypersurface in cases 2–5.

1. This easily follows from Theorem 7(1). Indeed, although the group $H$ has become smaller, the subgroup $U_H$ has remained the same. Therefore we have the same algebra $k[G]_{U \times U}$, with the same free generators. The only thing to be changed is the weight of generators: one has just to erase all the $\varepsilon$’s. Thus, we have $2n$ algebraically independent generators whose multidegrees are given above. Let $x_1, \ldots, x_n$ (resp. $y_1, \ldots, y_n$) be the functions in $k[G]_{U \times U}$ whose multidegrees form the first (resp. second) half in the list of weights. Then $\prod_{i=1}^n x_i$, $\prod_{i=1}^n y_i$ have the multidegree $(\tilde{\rho}, \rho)$, which means that $\dim k[G]_{(\tilde{\rho}, \rho)} \geq 2$, i.e., $m_{\tilde{\rho}}(\rho) \geq 2$. An
easy verification proves that actually $m_{\tilde{\rho}}(\rho) = 2$ and that this is the first occurrence of a homogeneous component whose dimension is $> 1$. Using Theorem 6, we conclude that $(\tilde{\rho}, \rho)$ is the remarkable weight.

2–4. By [14], the Krull dimension of $k[G]^{U \times U_H}$ is equal to $\text{rk}(G \times H/\Delta_H) + c(G \times H/\Delta_H) = \text{rk} G + \text{rk} H + 1$. We are going to prove that $k[G]^{U \times U_H}$ is a hypersurface. The arguments for all three cases are similar. First, we deduce from the explicit description of $\text{res}_H V_\lambda$ for small $\lambda$’s that the weights given in the theorem lie in $\Gamma(G, H)$ and are indecomposable there, and that the corresponding multiplicities are equal to 1. Hence the corresponding homogeneous elements of $k[G]^{U \times U_H}$ are determined up to a constant multiple and form a part of minimal generating system. This means that we have already detected $\text{rk} G + \text{rk} H + 2$ generators in each case. These generators are said to be the initial ones. Second, we are to verify that the set of initial generators is complete. To prove this, we argue as follows. It turns out that in each case there are exactly three monomials in initial generators of multidegree $(\tilde{\rho}, \rho)$. Since $k[G]^{U \times U_H}$ is factorial, we conclude that $\dim k[G]^{U \times U_H} > 1$. On the other hand, it is easily seen that the multidegree $(\tilde{\rho}, \rho)$ is the minimal one where the multiplicity jumps up. It then follows from Theorem 6 that $\dim k[G]^{U \times U_H} = 2$, $(\tilde{\rho}, \rho)$ is the remarkable weight, and there exists a “three monomial” relation connecting the initial generators. Next, a straightforward computation with weights shows that any proper subset of the initial generators is algebraically independent. Therefore the “three monomial” relation is the unique basic relation connecting them. Assume that $k[G]^{U \times U_H}$ has an extra generator, say $f$. By Krull dimension reason, $f$ must occur in another basic relation. It is shown in [13] that the basic relations between generators (if any) have the same multidegree and this multidegree is nothing but the remarkable weight. In particular, the multidegree of $f$, say $(\lambda, \lambda)$, lies under $(\tilde{\rho}, \rho)$, i.e., $\tilde{\rho} - \lambda \in \mathcal{X}(T)_+$ and $\rho - \lambda \in \mathcal{X}(T_H)_+$. However, in each case it is easily seen that there are no new generators of $k[G]^{U \times U_H}$ whose multidegree lies under $(\tilde{\rho}, \rho)$. This contradiction completes the proof for cases 2–4.

5. The argument is essentially the same as in the previous part. The only difference is that multiplicity jumps up for the multidegree $(2\tilde{\rho}, 4\rho)$.

Remarks. 1. It can be shown in a similar fashion that $k[G]^{U \times U_H}$ is also a hypersurface for items 6–8 in Table 2.

2. Since $k[G]^{U \times U_H}$ is a hypersurface and the multidegrees of the generators and the relation are known, one immediately writes down a formula for the Poincaré series of $k[G]^{U \times U_H}$ as a rational function. Such a formula in the case $(G_2, \text{SL}_3)$ was found earlier in [4]. Yet another approach to the branching rule for $(\text{Spin}_7, G_2)$ is presented in [12].

References


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