NOTES ON THE QUANTUM TETRAHEDRON

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Abstract. This is a set of notes describing several aspects of the space of paths on ADE Dynkin diagrams, with a particular attention paid to the graph $E_6$. Many results originally due to A. Ocneanu are described here in a very elementary way (manipulation of square or rectangular matrices). We recall the concept of essential matrices (intertwiners) for a graph and describe their module properties with respect to right and left actions of fusion algebras. In the case of the graph $E_6$, essential matrices build up a right module with respect to its own fusion algebra, but a left module with respect to the fusion algebra of $A_{11}$. We present two original results: 1) Our first contribution is to show how to recover the Ocneanu graph of quantum symmetries of the Dynkin diagram $E_6$ from the natural multiplication defined in the tensor square of its fusion algebra (the tensor product should be taken over a particular subalgebra); this is the Cayley graph for the two generators of the twelve-dimensional algebra $E_6 \otimes A_3 E_6$ (here $A_3$ and $E_6$ refer to the commutative fusion algebras of the corresponding graphs). 2) To every point of the graph of quantum symmetries one can associate a particular matrix describing the “torus structure” of the chosen Dynkin diagram; following Ocneanu, one obtains in this way, in the case of $E_6$, twelve such matrices of dimension $11 \times 11$, one of them is a modular invariant and encodes the partition function of which corresponding conformal field theory. Our own next contribution is to provide a simple algorithm for the determination of these matrices.

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1. Introduction

1.1. Summary. One purpose of the present paper is to present a simple construction for the “Ocneanu graph” describing the quantum symmetries of the Dynkin diagram $E_6$. Another purpose of our paper is to give a simple method allowing
one to determine, for each point of this Ocneanu diagram, a particular “toric matrix”. One of these matrices, associated with the origin of the graph, is a modular invariant. These toric matrices were first obtained by A. Ocneanu (they were never made available in printed form). Our techniques bring some simplification to the calculations and should improve the understanding of the inter-related structures appearing in this subject. We choose to follow the example of $E_6$ because it exhibits quite generic features$^1$.

We do not intend to give here a detailed account of the constructions of Ocneanu, but the present paper may provide a simple introduction to this theory, since it does not require any particular knowledge of operator algebra or conformal field theory. The mathematical background needed here usually does not involve more than multiplication of rectangular matrices. Therefore, apart from the two above-mentioned results, another purpose of our paper is to show how one can recover several important results of this theory, bypassing many of the steps described in reference $^{[23]}$.

1.2. Historical comments. The classification of conformal field theories of SU(2) type was performed by A. Cappelli, C. Itzykson and J. B. Zuber in $^{[7]}$. They established a correspondence between these conformal field theories and the ADE Dynkin diagrams used in the classification of simple Lie algebras. This correspondence was found, at the same time, by V. Pasquier in $^{[27]}$, within the context of lattice gauge models. This new ADE classification has then been discussed in several papers (see in particular $^{[35]}$). Later, a generalization of lattice models to the case of SU($n$) was studied in $^{[28]}$ and the study of conformal theories of SU(3) type was performed by P. Di Francesco and J. B. Zuber in $^{[15]}$ (see also $^{[36]}$ and reference therein); the last authors related the SU(3) classification problem to new types of graphs generalizing the ADE Dynkin diagrams. Several years ago, in order to study Von Neumann algebras, a theory of “paragroups” was invented by A. Ocneanu $^{[21]}$. Roughly speaking, these paragroups characterize embeddings of operator algebras. The combinatorial data provided by Dynkin diagrams (and corresponding affine Dynkin diagrams) provides a simple example of this general framework. Many details have been worked out by A. Ocneanu himself, who gave several talks on the subject (for instance $^{[22]}$), but this work was not made available in written form, with the exception of a recent set of notes $^{[23]}$. Later, A. Ocneanu discovered how to recover the ADE classification of modular invariant partition functions from his theory of “Quantum Symmetries” on graphs and, more recently $^{[24]}$, how to generalize his method to conformal theories of type SU(3) and SU(4), therefore establishing a direct relationship with the results of $^{[15]}$.

1.3. Structure of the paper. In the first part, we show how to construct a particular finite-dimensional commutative algebra (technically an hypergroup) from the combinatorial data provided by the graph $E_6$; this algebra can be realized in terms of $6 \times 6$ commuting matrices that will be called “graph fusion matrices” (we shall recover, in the corresponding subsection, several results that are more or less

$^1$Note added in proof: the calculations and results, for all ADE Dynkin diagrams, can be found in our subsequent paper $^{[12]}$. 

42 R. COQUEREAUX
well known, and can be found in the book [13]). These matrices can be used to study paths on the $E_6$ graph. In the second part, we use the concept of essential paths (due to Ocneanu) to define what we call “essential matrices” ($E_6$); these are rectangular $11 \times 6$ matrices that generate a bimodule with respect to the fusion algebra of the $E_6$ graph (from one side) and with respect to the fusion algebra of the $A_{11}$ graph (from the other side). In the third part, we build the commutative algebra $E_6 \otimes_{A_3} E_6$, where $E_6$ refers to the fusion algebra of the $E_6$ graph and $A_3$ refers to a particular subalgebra (isomorphic to the fusion algebra of the $A_3$ graph); the tensor product is taken over $A_3$ so that this algebra has dimension $6 \times 6/3 = 12$. Its multiplicative structure is described by a graph with 12 points describing the quantum symmetries of the $E_6$ graph; this graph was originally obtained by A. Ocneanu after diagonalization of the convolution product in the bigebra of endomorphisms of essential paths (a kind of generalized finite-dimensional Racah–Wigner bigebra of dimension 2512). Our approach based on the study of the finite-dimensional commutative algebra $E_6 \otimes_{A_3} E_6$ allows one to obtain the Ocneanu graph directly, therefore bypassing the rather complicated study of the Racah–Wigner bigebra (one of the two multiplications of the latter involves generalized 6j symbols containing 24-th roots of unity). We also give an interpretation for the square matrices (respectively of size $(11, 11)$ or $(6, 6)$) obtained when calculating products of essential matrices $E_6^a E_6^b$ or $E_6^a \widetilde{E}_6^b$; the symbol $\sim$ stands for “transpose” (we sometimes use the symbol $T$). Actually, tables $E \times E \rightarrow A$ and $E \times E \rightarrow S$ given in Sections 3.3.2 and 4.5 describe two different sets of vertices ((3.4) or ((9)), out of which one can nicely encode all the elements of the two different adapted base for the above bigebra (sometimes called the “double triangle algebra”).

In the fourth part, we define “reduced essential matrices” by removing from the essential matrices of $E_6$ the columns associated to the supplement of its $A_3$ subalgebra, and use them to construct twelve “toric matrices” $11 \times 11$ (one for each point of the Ocneanu graph). One of these matrices is a modular invariant, in the sense that it commutes with the generators $S$ and $T$ of $\text{SL}(2, \mathbb{Z})$, in the 11-dimensional representation of Hurwitz-Verlinde. This particular matrix is associated with the unit of the $E_6 \otimes_{A_3} E_6$ algebra and defines a modular invariant sesquilinear form which is nothing but the partition function of Cappelli, Itzykson, and Zuber. The other toric matrices (associated with the other points of the Ocneanu graph) are also very interesting (see footnote 3), but are not invariant under $\text{SL}(2, \mathbb{Z})$. In the last section (Comments), we gather miscellaneous comments about the relation between our approach and the one based on the study of the generalized Ocneanu–Racah–Wigner bigebra. We conclude with several open questions concerning an interpretation in terms of non-semi-simple (but finite-dimensional) quantum groups.

1.4. Remarks. The reader is probably aware of the fact that the seven points of the affine graph $E_6^{(1)}$ are in one-to-one correspondence (McKay correspondence [20]) with the irreducible representations of the binary tetrahedral group (the two-fold covering of the tetrahedral group), and that the corresponding fusion algebra specified by this affine Dynkin diagram encodes the structure of the Grothendieck ring of representations of this finite group (of order 24). When the graph $E_6$ is replaced by the graph $E_6^{(1)}$, many constructions described in this paper can be interpreted in
terms of conventional finite group theory. The binary tetrahedral group is a rather classical (well-known) object, even if its treatment based on the structure of the affine $E_6^{(1)}$ graph is not so well known. The interested reader can refer to [9] for a “non-standard” discussion of the properties of this finite group, along these lines. Removing one node from this affine graph (hence getting $E_6$ itself) leads to entirely new results and to what constitutes the subject of the present article. It would be certainly useful to carry out the analysis in parallel, for both affine and non-affine Dynkin diagrams, but this would dangerously increase the size of this paper... We shall nevertheless make several remarks about the group case situation, all along the text, that should help the reader to perform fruitful analogies and develop some intuition. The above remark partly justifies our title for the present paper.

Actually, an Ocneanu graph encoding quantum symmetries usually involves “connections” between a pair of diagrams (for instance, in the case $E_6 \otimes A_3 E_6$, it involves twice the graph $E_6$ itself). These two graphs should have the same Coxeter number: we have distinct theories for the pairs $A_{11}-A_{11}$, $A_{11}-E_6$, $D_5-D_5$, $A_{11}-D_5$, $D_5-E_6$ and $E_6-E_6$. Here we only describe part of the results relative to the $A_{11}-A_{11}$, $A_{11}-E_6$ and $E_6-E_6$ situations (and especially the last one). The reader may want to know why we restrict our study to these cases and do not present a full description of the situation in all ADE cases. One reason is somehow pedagogical: we believe that it is useful to grasp the main ideas by studying a particular case that exhibits generic features. Another reason is size: the corresponding calculations, or even the presentation of results, can be rather long. The last reason is anteriority: we believe that many results concerning the ADE (or affine ADE), have been fully worked out by A. Ocneanu himself (certainly using other techniques) and will—maybe—appear some day. Our modest contribution should allow the dedicated reader to recover many results in a simple way.

Let us mention that the study of the $E_8$ case is very similar to the $E_6$ case: for $E_8$, the Ocneanu graph possesses $32 = 8 \times 8/2$ points, one for each element of $E_8 \otimes A_2 E_8$. The study of $A_N$ is also similar but is a bit too “simple”, since several interesting constructions just coincide in that case. Diagrams $E_7$ and $D_{odd}$ are special, because their fusion algebra is not a positive hypergroup (this was first noticed long ago, using another terminology, by [27]) but only a module over an hypergroup. The case of $D_{even}$ is also special because the algebra associated with its Ocneanu graph is not commutative. In order to study them, the techniques that we introduce for $E_6$ have to be slightly modified (see [12]).

One interesting direction of research is to generalize the simple algorithms developed here so as to recover and generalize the results relative to conformal field theories with chiral algebra $SU(3)$, $SU(4)$, ... (see [15], [24], [32] and the lectures of J. B. Zuber at Bariloche [36]). Details concerning $SU(n)$ generalizations, following the methods explained in the present paper, should appear in [34].

2See the footnote in Section 1.1.

3After completion of the present work, we received a new preprint by V. B. Petkova and J. B. Zuber [31], giving a physical interpretation of these other toric matrices in terms of partition functions associated with twisted boundary conditions (defect lines) in boundary conformal field theories.
2. The graph $E_6$ and its fusion algebra

2.1. The graph. The labelling of the vertices $\sigma_a$ of the graph (Fig. 1) follows the convention $(0, 1, 2, 5, 4, 3)$. The reader should distinguish this labelling from the order itself that we have chosen to enumerate the vertices (i.e., for instance, the fourth vertex in the list is called $\sigma_5$). To each vertex $\sigma_a$ we associate a basis (column) vector $V_a$ in a six-dimensional vector space; for instance $V_0 = (1, 0, 0, 0, 0, 0)^T$, $V_5 = (0, 0, 1, 0, 1, 0)^T$, ...

![Figure 1. The graph of $E_6$](image)

The adjacency matrix of this graph (we use the above order for labelling the vertices) is:

$$G = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}.$$  

2.2. Norm of the graph and Perron–Frobenius eigenvector. The norm of this graph is, by definition, the largest eigenvalue of its adjacency matrix. It is

$$\beta = \frac{\sqrt{3} + 1}{\sqrt{2}}.$$  

For a given Dynkin diagram, it is convenient to set $\beta = \tilde{q} + 1/\tilde{q}$, with $\tilde{q} = e^{\hat{h}}$ and $\hat{h} = i\pi/N$. Then $\beta = 2\cos(\pi/N)$ and $\tilde{q}$ is a root of unity. In the present case (graph $E_6$), $N = 12$, indeed

$$2\cos(\pi/12) = \frac{\sqrt{3} + 1}{\sqrt{2}}.$$  

Notice that $N = 12$ is the dual Coxeter number of $E_6$ (notice that we do not need to use any knowledge coming from the theory of Lie algebras). In all cases (other Dynkin diagrams), $\beta$ is equal to the “$q$” number $[2]_q$ (where $[n]_q \doteq \frac{q^n - q^{-n}}{q - q^{-1}}$).

Warning: we set $h = 2\hat{h}$ and $q = \tilde{q}^2$, so that $q = e^{2\pi i/N}$ whereas $\tilde{q} = e^{3\pi i/N}$. In the case of affine Dynkin diagrams (for instance $E_6^{(1)}$), $\beta$ is always equal to 2.

The Perron–Frobenius eigenvector $D$ is, by definition, the corresponding normalized eigenvector (the normalization consists in setting $D_0 = 1$). One finds

$$D = \left\{1, \frac{\sqrt{3} + 1}{\sqrt{2}}, 1 + \sqrt{3}, \frac{\sqrt{3} + 1}{\sqrt{2}}; 1, \sqrt{2}\right\}$$
It is nice to write it in terms of \( q \)-numbers (with \( N = 12 \)); one finds\(^4\)
\[
D = \{[1], [2], [3], [2], [1]; [3]/[2]\}
\]
The component of \( D \) associated with the origin \( \sigma_0 \) of the graph is minimal.

In the case of \( E_6^{(1)} \), the seven entries of \( D \) are the (usual) integers 1, 2, 3, 2, 1, 2, 1; these numbers coincide with the dimensions of the seven irreducible representations (“irreps”) of the binary tetrahedral group. For this reason, the six entries of \( D \), in the \( E_6 \) case, should be thought of as quantum dimensions for the irreps of a quantum analogue of this finite group.

Returning to the \( E_8 \) case, we notice that the eigenvalues of the adjacency matrix \( G \) read \( 2 \cos \frac{2\pi}{N} \), with \( N = 12 \) and \( m = 1, 4, 5, 7, 8, 11 \). The value obtained with \( m = 1 \) gives the norm of the graph. These integers are also the Coxeter exponents of the Lie group \( E_8 \).

2.3. Hypergroup structure. In the classical case (i.e., the group case), irreducible representations can be tensorially multiplied and decomposed into a sum of irreps; by considering sums and difference, they actually generate a commutative ring (or a commutative algebra): the Grothendieck ring of virtual characters. By irreps; by considering sums and difference, they actually generate a commutative algebra: the Grothendieck ring of virtual characters. By analogy, we build a finite-dimensional associative and commutative algebra with the above combinatorial data. This was first done, to our knowledge, by V. Pasquier [27] who also noticed that this construction is not always possible: it works for all affine Dynkin diagrams (in which case one recovers the multiplication of characters of the binary groups of Platonic bodies) and for Dynkin diagrams of type \( A_n, D_{2n}, E_6 \) and \( E_8 \). However, for \( E_7 \) and \( D_{odd} \), it does not work in the same way (lack of positivity of structure constants in the \( E_7 \) case).

We restrict ourselves to the \( E_6 \) case and build this algebra as follows (we shall call it the fusion algebra associated with the graph \( E_6 \)):

- The algebra is linearly generated by the six elements \( \sigma_0, \ldots, \sigma_5 \).
- \( \sigma_0 \) is the unit.
- \( \sigma_1 \) is the algebraic generator: multiplication by \( \sigma_1 \) is given by the adjacency matrix (this is just an eigenvalue equation for \( G \)),
\[
G.(\sigma_0, \sigma_1, \ldots, \sigma_5) = \sigma_1(\sigma_0, \sigma_1, \ldots, \sigma_5).
\]
More explicitly: \( \sigma_1 \sigma_0 = \sigma_1, \sigma_1 \sigma_1 = \sigma_0 + \sigma_2, \sigma_1 \sigma_2 = \sigma_1 + \sigma_3 + \sigma_5, \sigma_1 \sigma_5 = \sigma_2 + \sigma_4, \sigma_1 \sigma_4 = \sigma_5, \sigma_1 \sigma_3 = \sigma_2 \). The reader will notice that this is nothing but a quantum analogue of multiplication of spins (in the case of SU(2), the corresponding graph is the infinite graph \( A_{\infty} = \tau_0, \tau_1, \ldots \) where the point \( \tau_{2j} \) refers to the irrep of spin \( j \) (of dimension \( 2j + 1 \)), and where \( \tau_1 \tau_q = \tau_{q-1} + \tau_{q+1} \) (composition of an arbitrary spin with a spin \( 1/2 \)).
- Once multiplication by the generator \( \sigma_1 \) is known, one may multiply arbitrary \( \sigma_\hat{q} \)'s by imposing associativity and commutativity of the algebra. For instance:
\[
\sigma_2 \sigma_2 = (\sigma_1 \sigma_1 - \sigma_0) \sigma_2 = \sigma_1 \sigma_1 \sigma_2 - \sigma_2 = \sigma_1 (\sigma_1 + \sigma_3 + \sigma_5) - \sigma_2 = \sigma_0 + \sigma_2 + \sigma_2 + \sigma_4 - \sigma_2 = \sigma_0 + 2\sigma_2 + \sigma_4.
\]

\(^4\)We have suppressed the sub-index \( \hat{q} \) from the brackets.
This fusion algebra (or graph algebra) is a particular example of what is called a commutative positive integral hypergroup (see general definitions in the collection of papers [8]); prototype of commutative hypergroups are the class hypergroup and the representation hypergroup of a group (which is also the Grothendieck ring of its virtual characters).

2.4. The $E_6 \times E_6 \rightarrow E_6$ multiplication table. In this way, one can construct the following multiplication table (we write $a$ rather than $\sigma_a$), that we call the fusion table for $E_6$:

\[
\begin{array}{cccccc}
E_6 & 0 & 3 & 4 & 1 & 2 & 5 \\
0 & 0 & 4 & 3 & 1 & 2 & 5 \\
3 & 3 & 0 & 4 & 2 & 15 & 2 \\
4 & 4 & 3 & 0 & 5 & 2 & 1 \\
1 & 1 & 2 & 5 & 0 & 135 & 24 \\
2 & 2 & 15 & 2 & 135 & 0224 & 135 \\
5 & 5 & 2 & 1 & 24 & 135 & 02 \\
\end{array}
\]

Notice that all entries are positive integers; this is not trivial (and fails to be true for the graphs of type $E_7$ or $D_{\text{odd}}$). The structure constants of the fusion algebra are the integers $C_{abc}$ that appear in the previous multiplication table ($\sigma_a \sigma_b = C_{abc} \sigma_c$); we have, for instance, $\sigma_2 \sigma_2 = \sigma_0 + 2\sigma_2 + \sigma_4$, therefore $C_{220} = 1$, $C_{222} = 2$, $C_{224} = 1$, and the other $C_{22c}$ are equal to zero.

Notice also that we have chosen the order $\{034125\}$ to display this multiplication table; the reason is that it shows clearly that $\{034\}$ generate a subalgebra with particular properties; we shall come back to this later.

From the above table, we can check that

\[
\begin{align*}
\sigma_0 &= 1, \\
\sigma_1 &= \sigma_1, \\
\sigma_2 &= \sigma_1 \sigma_1 - \sigma_0, \\
\sigma_3 &= -\sigma_1 (\sigma_4 - \sigma_1 \sigma_1 + 2\sigma_0), \\
\sigma_4 &= \sigma_1 \sigma_1 \sigma_1 \sigma_1 - 4\sigma_1 \sigma_1 + 2\sigma_0, \\
\sigma_5 &= \sigma_1 \sigma_4,
\end{align*}
\]

2.5. Graph fusion matrices and paths on the graph $E_6$. What actually turn out to be useful are the integral matrices $N_a$; in the present case ($E_6$ case) these are symmetric matrices:

\[
(N_a)_{bc} = C_{abc}.
\]

Because of the algebraic relations satisfied by the generators $\sigma_a$, the simplest way to obtain the six $6 \times 6$ matrices $N_a$ is to set:

\[
\begin{align*}
N_0 &= 1d_6 \quad \text{(the identity matrix)}, \\
N_1 &= G, \\
N_2 &= G.G - N_0, \\
N_3 &= -G.(N_4 - G.G + 2N_0), \\
N_4 &= G.G.G.G - 4G.G + 2N_0, \\
N_5 &= G.N_4,
\end{align*}
\]

\footnote{Warning: indices do not refer to row and line numbers but to the labels (0, 1, 2, 5, 4, 3) of vertices.}
Again one should notice that \(N_0, N_3\) and \(N_4\) form a subalgebra (graph \(A_3\)):
\[
N_3 N_3 = N_0 + N_4,
\]
\[
N_4 N_3 = N_3,
\]
\[
N_4 N_4 = N_0.
\]

Using the ordered basis \((012543)\), we have
\[
N_0 = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
N_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
N_2 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
N_3 = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
N_4 = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
N_5 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

The ring generated by the matrices \(N_a\) provides a faithful matrix realization of the fusion algebra. In particular, the Dynkin diagram of \(E_6\), considered as the graph of multiplication by \(\sigma_1\), is also the graph of multiplication by the matrix \(N_1\) (indeed, \(N_1 \cdot N_1 = N_0 + N_2\), etc).

Warning: In this paper, the notation \(E_6\) will denote the Dynkin diagram \(E_6\), its fusion algebra, also called the graph algebra (the commutative algebra generated by the \(\sigma_a\)), or the explicit matrix algebra generated by the \(N_a\) matrices. The context should be clear enough to avoid ambiguities.

Since all these \(N_a\) matrices commute with one another, they can be simultaneously diagonalized. If we were working with the graph \(E_6^{(1)}\) rather than with \(E_6\), i.e., in the finite group case (the binary tetrahedral group), the simultaneous diagonalization \(S^{-1} N_a S\) of the \(N_a\) matrices would be done thanks to a matrix \(S\), which is nothing but the character table (this is precisely the method used to recover a character table from the structure constants of the Grothendieck ring, when the multiplication of the group itself is not known). In the present case (graph of \(E_6\)), the \(6 \times 6\) matrix \(S\) is a kind of “noncommutative" character table (or Fourier transform); this matrix is clearly an interesting object, but we shall not make an explicit use of it in the sequel. The matrix \(S\) associated with the graph \(A_{11}\) will appear in Section 3.3.4 and play an important role later on (Verlinde representation of the modular group).

A last comment about the fusion algebra of the graph \(E_6\): it is isomorphic to the algebra \(\mathbb{C}[X]/P[X]\) of complex polynomials modulo \(P[X]\), where \(P[X] = (X^2 - 1)(X^4 - 4X^2 + 1)\) is the characteristic polynomial of the matrix \(G\); this property (already mentioned in [13]) is a direct consequence of the Cayley–Hamilton theorem.

2.6. The \(A_3\) subalgebra. We already noticed that the algebra generated by the vertices \(\sigma_0, \sigma_3\) and \(\sigma_4\) is a subalgebra of the fusion algebra of the \(E_6\) graph. This
algebra is actually isomorphic to the fusion algebra of the Dynkin graph $A_3$. This is almost obvious: consider the following figure (Fig. 2).

\begin{center}
\[ b \quad s \quad v \]
\end{center}

**Figure 2.** The graph of $A_3$

The corresponding fusion algebra is therefore defined by the relations $bs = s$, $s^2 = b + v$, $sv = s$. This implies $vv = (ss - b)v = ss - v = b$. The corresponding multiplication table is the same as the one obtained by restriction of the $E_6$ table to the vertices 0, 3, 4, under the identification $b \rightarrow \sigma_0$, $s \rightarrow \sigma_3$, $v \rightarrow \sigma_4$.

The fusion subalgebra $A_3$ of the fusion algebra of the graph $E_6$ has also the following remarkable property: call $\mathcal{P}$ the vector space linearly generated by $\{\sigma_1, \sigma_2, \sigma_5\}$. From the table of multiplication, we see that

$E_6 = A_3 \oplus \mathcal{P}$,

$A_3.A_3 = A_3$,

$A_3.\mathcal{P} = \mathcal{P}.A_3 = \mathcal{P}$.

Such properties have been already described in various places (see [2] or [14]). The situation is similar to what happens for homogeneous spaces and reductive pairs of Lie algebras, but in the present case we are in an associative (and commutative) algebra. A better analogy comes immediately to mind when we compare the representations of SO(3) and of SU(2): All representations of the former are representations of the latter, the set of irreps of SO(3) is closed under tensor products, and the coupling of an integer spin with a half (odd)-integer spin can be decomposed on half-integer spins. Since $SO(3) = SU(2)/\mathbb{Z}_2$, we could say, by analogy, that the ‘quantum space’ dual to the fusion algebra of the graph $A_3$ is a quotient of the ‘quantum space’ dual to the fusion algebra of the graph $E_6$.

### 2.7. Paths on the $E_6$ graph.

An elementary path on a graph is a path, in the usual sense, starting at some vertex and ending at some other (or at the same) vertex. Its length is counted by the number of edges entering the definition of the path. Of course, paths can backtrack. A (general) path is, by definition, a linear combination of elementary paths. The vector space of paths of length $n$ originating from $\sigma_i$ and ending on $\sigma_j$ will be called $\text{Path}_{ij}^n$.

The fusion graph of the group SU(2) is an infinite half-line $A_\infty$, with the vertices labelled by the representations $\sigma_0 = [1]$, $\sigma_1 = [2]$, $\ldots$, $\sigma_{2s} = [2s + 1]$, $\ldots$, where the integers 1, 2, 3, $\ldots$, $d_s = 2s + 1$ are the dimensions of the irreducible representations of spin $s$. Tensor multiplication by the (two-dimensional) fundamental representation is indeed such that $[d_s] \otimes [2] = [d_s - 1] \oplus [d_s + 1]$. An elementary path of length $n$ starting at the origin (the trivial representation) and ending on some $d_s$ can be put into one-to-one correspondence with a projector that projects the representation $[2]^n$ on the irreducible representation $[d_s]$. The same comment can be made for discrete subgroups of SU(2), for instance any binary polyhedral group, in which case the graph of fusion is given by the affine Dynkin diagrams.
In the case of ‘genuine’ Dynkin diagrams (not affine), we do not have a group theoretical interpretation, but the situation is similar.

Call \( P_n = G.P_{n-1} \), where \( G = N_1 \) is, as usual, the adjacency matrix of the graph, and where \( P_0 = (1, 0, 0, 0, 0, 0)^T \) (in the case of \( E_6 \)). Clearly, the various components of the vector \( P_n \) give the number of paths of length \( n \), starting at the origin \( (\sigma_0) \) and ending on the vertex corresponding to the chosen component.

The following picture, a kind of truncated Pascal triangle, can therefore be generated very simply by considering successive powers of the matrix \( N_1 \) acting on the (transpose) of the vector \( (1, 0, 0, 0, 0, 0)^T \).

\[
\begin{array}{cccccc}
\sigma_0 & \sigma_1 & \sigma_2 & \sigma_3 & \sigma_4 & n \\
* & 1 & 1 & 2 & 3 & 4 \\
1 & 2 & 1 & 1 & 5 & 6 \\
2 & 4 & 1 & 5 & 15 & 20 \\
6 & 4 & 5 & 15 & 56 & 8 \\
21 & 20 & 20 & 20 & 20 & 20 \\
\end{array}
\]

Warning: in this picture, we have chosen the vertex order 012354 rather than 012543 for aesthetic reasons; for instance, one finds \( P_7 = (0, 21, 15, 20, 0, 15) \) but it is displayed as \( (0, 21, 0, 15, 20, 0) \). We have therefore \( 21 + 15 + 20 = 56 \) paths of length 7 starting at the leftmost vertex on the graph \( E_6 \), 21 of them end on the vertex \( \sigma_1 \), 15 end on \( \sigma_3 \), and 20 end on \( \sigma_5 \).

Notice that the picture stabilizes after a few steps: the whole structure appearing below can be graphically generated from the folded \( E_6 \) graph appearing at steps \( n = 4 \) and 5 (the Bratteli diagram) by reflection and repetition down to infinity. This structure will be understood, at the end of this section, in terms of a tower of algebras (inclusions).
In the group case, SU(2) for example, the vector $P_n$ has infinitely many components (almost all are zero), and the value of the component $P_n(\sigma)$ gives the multiplicity of the representation $[\sigma]$ in the $n$-th tensor power of the fundamental representation. In that case, the semi-simple matrix algebra $TL(n)$, defined as a sum of simple blocks of dimensions $P_n(\sigma)$, is nothing but the centralizer algebra for the group SU(2), also called the “Schur algebra” or the “Temperley–Lieb–Jones algebra” (for the index 1/4), and is a well known quotient of the group algebra of the permutation group $S_n$; the dimension of $TL(n)$, in that case, is given by the Catalan numbers 1, 2, 5, 14, ... When the graph is $E_6$, rather than $A_\infty$, there is no group theoretical interpretation, but the construction of the commutant is similar (it is a particular case of the Jones tower construction), and what we are describing here is the path model for (the analogue of) a centralizer algebra. Still in the case of the graph $E_6$, we see on the previous picture that, for example, the algebra $TL(7)$ is isomorphic to $M(21, \mathbb{C}) \oplus M(15, \mathbb{C}) \oplus M(20, \mathbb{C})$ and is of dimension $21^2 + 15^2 + 20^2$.

### 3. Essential paths and essential matrices

#### 3.1. Essential paths. Let us start with the case of SU(2). The consecutive $n$-th tensor powers of the fundamental representation $[2]$ can be decomposed into irreducible representations ($[2]^2 = [1] + [3]$, $[2]^3 = 2[2] + [4]$, $[2]^4 = 2[1] + 3[3] + [5]$, ...). A given irreducible representation of dimension $d$ appears for the first time in the decomposition of $[2]^{d-1}$ and corresponds to a particular projector in the vector space $(\mathbb{C}^2)^{\otimes d-1}$ which is totally symmetric and therefore projects on the space of symmetric tensors. These symmetric tensors provide a basis of this particular representation space and are, of course, in one-to-one correspondence with symmetric polynomials in two complex variables $u, v$ (representations of given degree). From the point of view of paths, these representations (projectors) correspond to non-backtracking paths of length $d - 1$ starting at the origin (walking to the right on the graph $A_\infty$). However, irreducible representations of dimension $d$ appear not only in the reduction of $[2]^{d-1}$ but also in the reduction of $[2]^f$, when $f = d + 1$, $d + 3$, ... . These representations are equivalent to the symmetric representations already described but they are nevertheless distinct as explicitly given representations; the associated projectors are not symmetric and correspond to paths on $A_\infty$ that can backtrack. The notion of “essential path”, due to A. Ocneanu formalizes and generalizes the above remarks. In the case of SU(2), essential paths from the origin are just non-backtracking-right-moving paths starting from the origin ($\tau_0 = [1]$) of $A_\infty$. There is a one-to-one correspondence between such paths and irreducible symmetric representations. Clearly, “essentiality” is a meaningful property for a path or a projector, but a given explicit irreducible representation, associated with an essential path, may very well be equivalent to another explicit representation which is still irreducible (of course, by definition of equivalence), but which is not associated with an essential path. For instance, the representation $[3]$ that appears in the reduction of $[2]^3$ corresponds to an essential path (starting from the origin), but the three equivalent representations $[3]$ that appear in the reduction of $[2]^4$ do not correspond to such paths. More generally, essential paths of length $n$ starting...
at a given irreducible representation of dimension \([a]\) (not necessarily the identity) correspond to projectors appearing in the decomposition of \([a] \otimes [n+1]\) into irreducible summands (here \([a]\) and \([n+1]\) are explicitly realized in terms of symmetric representations\(^6\)).

When we move from the case of SU(2) to the case of finite subgroups of SU(2), in particular the binary polyhedral groups whose representation theory is described by the affine Dynkin diagrams \(E_6^{(1)}, E_7^{(1)}\) and \(E_8^{(1)}\), the notion of essential paths can be obtained very simply by declaring that a path on the corresponding diagram is essential if it describes an irreducible representation that appears in the branching of a symmetric representation of SU(2) with respect to the chosen finite subgroup. A novel feature of these essential paths is that they can backtrack (in general, such paths will be linear combinations of elementary paths). Essential paths for the finite subgroups of SU(2) can be of arbitrary length, since symmetric representations of SU(2) can be of arbitrary degree (horizontal Young diagrams with an arbitrary number of boxes).

In more general situations (like the \(E_6\) case which is the example that we are following in this paper), we need a general definition that encompasses all the previous concepts and provides a meaningful generalization. This definition was given by A. Ocneanu (several seminars in 1995) and published in [23], and is as follows.

Take a graph \(\Gamma\) described by an adjacency matrix \(G\). In this paper, edges of the graph are not oriented (as for the ADE); we may replace every unoriented edge by a pair of edges with the same endpoints but carrying two opposite orientations. The notion of essential paths makes sense for more general graphs — for example those encoding the fusion by the two inequivalent fundamental representations of SU(3) and its subgroups — but we shall not be concerned with them in the present paper.

Call \(\beta\) the norm of the graph \(\Gamma\) (the biggest eigenvalue of its adjacency matrix) and \(D_i\) the components of the (normalized) Perron–Frobenius eigenvector. Call \(\sigma_i\) the vertices of \(\Gamma\) and, if \(\sigma_j\) is a neighbour of \(\sigma_i\), call \(\xi_{ij}\) the oriented edge from \(\sigma_i\) to \(\sigma_j\). If \(\Gamma\) is unoriented (the case for ADE and affine ADE diagrams), each edge should be considered as carrying both orientations.

An elementary path can be written either as a finite sequence of consecutive (i.e., neighbours on the graph) vertices, \([\sigma_{a_1}, \sigma_{a_2}, \sigma_{a_3}, \ldots]\), or, better, as a sequence \((\xi(1)\xi(2)\ldots)\) of consecutive edges, with \(\xi(1) = \xi_{a_1a_2} = \sigma_{a_1}\sigma_{a_2}\), \(\xi(2) = \xi_{a_2a_3} = \sigma_{a_2}\sigma_{a_3}\), etc. Vertices are considered as paths of length 0.

The length of the (possibly backtracking) path \((\xi(1)\xi(2)\ldots\xi(p))\) is \(p\). We call \(r(\xi_{ij}) = \sigma_j\), the range of \(\xi_{ij}\) and \(s(\xi_{ij}) = \sigma_i\), the source of \(\xi_{ij}\).

For all edges \(\xi(n+1) = \xi_{ij}\) that appear in an elementary path, we set \(\xi(n+1)^{-1} = \xi_{ji}\).

For every integer \(n > 0\), the annihilation operator \(C_n\), acting on elementary paths of length \(p\), is defined as follows: if \(p \leq n\), \(C_n\) vanishes, whereas if \(p \geq n+1\),

\(^6\)The author acknowledges interesting comments by A. Garcia and R. Trinchero about this topic.
then
\[ C_n(\xi(1)\xi(2)\ldots\xi(n)\xi(n+1)\ldots) = \sqrt{\frac{D_r(\xi(n))}{D_s(\xi(n))}} \delta_{\xi(n),\xi(n+1)} \cdot (\xi(1)\xi(2)\ldots\hat{\xi}(n)\hat{\xi}(n+1)\ldots). \]

Here, the symbol “hat” (like in \( \hat{\xi} \)) denotes omission. The result is therefore either 0 or a path of length \( p - 2 \). Intuitively, \( C_n \) chops the round trip that possibly appears at positions \( n \) and \( n + 1 \).

Acting on an elementary path of length \( p \), the creating operators \( C_n^\dagger \) are defined as follows: if \( n > p + 1 \), \( C_n^\dagger \) vanishes, and if \( n \leq p + 1 \), then, setting \( j = r(\xi(n-1)), \)
\[ C_n^\dagger(\xi(1)\ldots\xi(n-1)\ldots) = \sum_{d(j,k)=1} \sqrt{\frac{D_k}{D_j}} (\xi(1)\ldots\xi(n-1)\xi_k\xi_j\ldots). \]

The above sum is taken over the neighbours \( \sigma_k \) of \( \sigma_j \) on the graph. Intuitively, this operator adds one (or several) small round trip(s) at position \( n \). The result is therefore either 0 or a linear combination of paths of length \( p + 2 \).

For instance, on paths of length zero (i.e., vertices),
\[ C_1^\dagger(\sigma_j) = \sum_{d(j,k)=1} \sqrt{\frac{D_k}{D_j}} \xi_k\xi_j = \sum_{d(j,k)=1} \sqrt{\frac{D_k}{D_j}} [\sigma_j\sigma_k\sigma_j]. \]

We already mentioned the fact that the Temperley–Lieb–Jones algebra \( \text{TL}(n) \) could be constructed as endomorphism algebra of the vector space of paths of length \( n \) (path model). The Jones’ projectors \( e_k \) are defined (as endomorphisms of \( \text{Path}^n \)) by
\[ e_k = \frac{1}{\beta} C_k^\dagger C_k. \]

The reader can indeed check that all Jones–Temperley–Lieb relations between the \( e_i \) are satisfied. We remind the reader that \( \text{TL}(n) \) is usually defined as the \( \mathbb{C}^* \) algebra generated by \( \{1, e_1, e_2, \ldots, e_{n-1}\} \) with the relations
\[ e_i e_{i \pm 1} e_i = \tau e_i, \]
\[ e_i e_j = e_j e_i \quad \text{whenever} \ |i - j| \geq 2, \]
\[ e_i^2 = e_i, \]
with
\[ \tau = 1/\beta^2. \]

We can now define what are “essential paths” for a general graph: a path is called essential if it belongs to the intersection of the kernels of all the Jones projectors \( e_i \)'s (or if it belongs to the intersection of the kernels of all the annihilators \( C_i \)'s). The dedicated reader will show that this definition indeed generalizes the naive definition given previously in the case of graphs associated with \( SU(2) \) and its subgroups.

The following difference of non-essential paths of length 4 starting at \( \sigma_0 \) and ending at \( \sigma_2 \) is an essential path of length 4 on \( E_6 \):
\[ \sqrt{2} \left[ \xi_{01} \xi_{12} \xi_{23} \xi_{32} \right] - \sqrt{\frac{3}{2}} \left[ \xi_{01} \xi_{12} \xi_{25} \xi_{52} \right] = \sqrt{2} [0, 1, 2, 3, 2] - \sqrt{\frac{3}{2}} [0, 1, 2, 5, 2]. \]
Here the brackets denote the \( q \)-numbers: \([2] = \frac{\sqrt{2}}{\sqrt{3} - 1}\) and \([3] = \frac{2}{\sqrt{3} - 1}\).

The Wenzl projector \( p_n \) is, by definition, the projector that takes arbitrary paths of length \( n \) and projects them on the vector subspace of essential paths (call \( \text{EssPath}^n \) this vector subspace). The original definition of these projectors did not use the path model, but this equivalent definition will be enough for our purpose.

In the case of \( SU(2) \), those elements of Jones’ algebra corresponding to projectors of \([2]^n \) on symmetric irreducible representations are Wenzl projectors, and each Wenzl projector of that type is associated with an essential path from the origin. In the case of \( SU(2) \), there is only one symmetric representation in any dimension: the space of essential paths of length \( n \) starting at the origin of the graph (identity representation) is one-dimensional, and the map projecting the whole space of paths \( \text{Path}^n \) on this one-dimensional space is the Wenzl projector \( p_n \). It is of rank one.

In the case of finite subgroups of \( SU(2) \) or in the “quantum” cases corresponding to the graphs ADE, this is not so: the space \( \text{EssPath}^n \) is generally not of dimension 1; for instance, the 4-dimensional irreducible representation of \( SU(2) \) (which appears for the first time in \([2]^3 \), and it can be associated with a single essential path of length 3 starting at the identity representation) can be decomposed into two irreps \([2'] + [2'']\) of its binary tetrahedral subgroup; the Wenzl projector is of rank 2.

### 3.2. Dimension of \( \text{EssPath} \). Essential matrices.

The only motivation for the previous discussion was to put what follows in its proper context. Indeed, we shall not need, in this paper, to manipulate explicitly with essential paths themselves (the interested reader can do it by using the previous general definitions). What we want to do here is only to give a simple method to count them; the method will be explicitly illustrated for the \( E_6 \) graph.

The main observation is that the dimension of the space of essential paths of length \( n + 1 \) starting at \( a \) and ending at \( b \) is given by

\[
\dim \text{EssPath}^{n+1}_{a,b} = \dim(H_{n+1}) - \dim \text{EssPath}^{n-1}_{a,b},
\]

where \( H_{n+1} \) is the space of linear combinations of paths of length \( n + 1 \) which are essential on their first \( n \) segments. This result was obtained by A. Ocneanu (see also the lectures by J. B. Zuber [36]). The dimensions of spaces of essential paths can be encoded by a set of rectangular matrices that we shall call “essential matrices”, but they are also called “intertwiners” in other contexts, like conformal field theory or statistical mechanics lattice models.

Once an (arbitrary) ordering of the vertices of the graph has been chosen — so that we know how to associate a positive integer (say \( a \)) to any chosen vertex — it is easy to show that the number of essential paths of length \( n \) starting at some vertex \( a \) and ending on the vertex \( b \) is given by \( b \)-th component (for the chosen vertex ordering) of the row vector \( E_n(a) \) defined as follows:

- \( E_n(0) \) is the (line) vector caracterizing the chosen initial vertex,
- \( E_n(1) = E_n(0).G \),
- \( E_n(n) = E_n(n-1).G - E_n(n-2) \).

This is a kind of moderated Pascal rule: the number of essential paths (with fixed origin) of length \( n \) reaching a particular vertex is obtained from the sum of number
of paths of length $n - 1$ reaching the neighbouring points (as in the Pascal rule) by subtracting the number of paths of length $n - 2$ reaching the chosen vertex.

In the case of $E_6$, we order7 the six vertices $\sigma_0$, $\sigma_1$, $\sigma_2$, $\sigma_3$, $\sigma_4$, $\sigma_5$ as before, so that $E_6(0) = (1, 0, 0, 0, 0, 0)$, $E_6(1) = (0, 1, 0, 0, 0, 0)$, $E_6(2) = (0, 0, 1, 0, 0, 0)$, $E_6(3) = (0, 0, 0, 1, 0, 0)$, $E_6(4) = (0, 0, 0, 0, 1, 0)$, $E_6(5) = (0, 0, 0, 0, 0, 1)$.

Starting from $E_a(0)$, one obtains in this way six rectangular matrices $E_a$ with infinitely many rows (labelled by $n$) and six columns (labelled by $b$). The reader can check that all $E_a(n)$ are positive integers provided $0 \leq n \leq 10$, but this ceases to be true as soon as $n > 10$: $E_a(11) = (0, 0, 0, 0, 0, 0)$, $E_a(12) = (0, 0, 0, 0, -1, 0)$, ....

We shall call by “essential matrices” the six rectangular $11 \times 6$ matrices obtained by keeping only the first 11 rows of the $E_a(n)$’s, and these finite dimensional rectangular matrices will be still denoted by $E_a$. The particular matrix $E_0$, interpreted as an intertwiner, was obtained long ago (see for instance [27], [30]); it is indeed easy to check that it intertwines the adjacency matrices $G^{E_6}$ and $G^{A_{11}}$ of the Dynkin diagrams $E_6$ and $A_{11}$:

$$E_0 G^{E_6} = G^{A_{11}} E_0 .$$

For all ADE graphs, the number of rows of essential matrices is always given by the (dual) Coxeter number of the graph minus one. In the case of $E_6$ this number is indeed $12 - 1 = 11$. The components of the six rectangular matrices $E_a$ are denoted by $E_a[n, b]$.

Once $E_0$ is known, one can obtain all other essential matrices, thanks to the simple relation

$$E_a = E_0 N_a .$$

Rather than gathering all the information concerning essential paths on the diagram $E_6$ into a set of six rectangular matrices $11 \times 6$ (our essential matrices $E_a$’s), one can also define a set of eleven square matrices $6 \times 6$ sometimes called “fused adjacency matrices”

$$F_n^{E_a}[n, b] = E_a^{E_6}[n, b] .$$

In terms of the $F$ matrices, the definition given for the $E$ matrices reads simply $F_0 = 1$, $F_1 = G$, and

$$F_n F_1 = F_{n-1} + F_{n+1} .$$

The important observation is that these last matrices $F$ (relative to the Dynkin diagram $E_6$) build up a representation of the fusion algebra of the Dynkin diagram $A_{11}$; this is clear, since the previous relation is the usual SU(2) recurrence relation. We shall come back to the relations between $E_6$ and $A_{11}$ in a later section.

3.2.1. Essential matrices and essential paths for $E_6$. The six essential matrices of $E_6$ are determined by straightforward calculations using the previous recurrence relations. The reader can also easily obtain explicit expression for the corresponding eleven fused adjacency matrices $F_n$. Here are the results for the rectangular

7 Warning: again, the $E_0$-indices used to label matrices always refer to the “name” of the chosen vertices and not to the integer that labels corresponding rows or columns.
matrices $E_a$.

$$E_0 = \begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . \\
\end{pmatrix}, \quad E_1 = \begin{pmatrix}
1 & . & . & . & . & . & . & . & . & . & . \\
. & 1 & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . \\
\end{pmatrix}, \quad E_2 = \begin{pmatrix}
. & 1 & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & 1 \\
\end{pmatrix}$$

$$E_3 = \begin{pmatrix}
. & 1 & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & 1 \\
\end{pmatrix}, \quad E_4 = \begin{pmatrix}
. & 1 & . & . & . & . & . & . & . & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & 1 \\
\end{pmatrix}, \quad E_5 = \begin{pmatrix}
. & . & 1 & . & . & . & . & . & . & . & . \\
. & . & . & 1 & . & . & . & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . \\
. & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & . & . & . & 1 & . & . \\
. & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & . & . & . & . & . & . & . & 1 \\
\end{pmatrix}$$

It is clear that nonzero entries of the matrix $E_a$ also graphically encode the structure of essential paths starting from $a$. For instance, we can “read”, from the $E_0$ matrix, the following graph (Fig. 3) giving all the essential paths leaving $σ_0$; this particular figure appears explicitly in [23]. In this picture, for aesthetic reasons, the order of vertices was chosen as $σ_0, σ_1, σ_2, σ_3, σ_5, σ_4$, whereas the vertex order chosen for essential matrices was $σ_0, σ_1, σ_2, σ_5, σ_4, σ_3$.

### 3.2.2 Essential paths and matrices for $A_{11}$

The space of paths and space of essential paths can be defined for arbitrary ADE Dynkin diagrams (extended or not); it is therefore natural to denote by $E_{a}^{X}$ the essential matrices relative to the choice of the graph $X$. Our previous results (called $E_a$ in the case of the graph $E_6$) should therefore be denoted by $E_{a}^{E_6}$.

We shall not only need the essential matrices for the graph $E_6$, but also those for the graph $A_{11}$. The technique is exactly the same: first we call $τ_0, τ_1, \ldots, τ_{10}$, from left to right, the vertices of $A_{11}$ and order them in a natural way; then we build the adjacency matrix for $A_{11}$ that we may call again $G$ (but this should be understood now as $G^{A_{11}}$, since we want to refer to this particular Dynkin diagram). We then build the associated graph fusion algebra and its matrix representation: we obtain in this way 11 square matrices $N^{A_{11}}_a$ of size $11 \times 11$. $N^{A_{11}}_0$ is the unit matrix, $N^{A_{11}}_1 = G^{A_{11}}$ is the generator, and the other fusion matrices are determined by the graph $A_{11}$; for $m + 1 \leq 10$, we observe that $N^{A_{11}}_{m+1} = N^{A_{11}}_{m} \cdot N^{A_{11}}_{m} - N^{A_{11}}_{m-1}$; so that multiplication by $N^{A_{11}}_1$ therefore describes the usual coupling to spin $1/2$, but we have also $N^{A_{11}}_9 = N^{A_{11}}_1 \cdot N^{A_{11}}_{10}$. More generally, it is easy to prove (recurrence) the...
Figure 3. Essential paths for $E_6$ starting from $\sigma_0$

following formula (that can be interpreted in terms of couplings to higher spins):

$$N^{A_{11}}_m N^{A_{11}}_n = \sum_{p \in A_{11}} (N^{A_{11}}_p)_{n,m} N^{A_{11}}_p.$$ 

This last relation shows in particular that the structure constants $C_{nmp}$ of the associative algebra $A_{11}$ can be identified with the matrix elements $(N^{A_{11}}_p)_{n,m}$.

One of the immediate consequences of the above is that, for $A_{11}$, and actually for all $A_N$ Dynkin diagrams, there will be no difference between graph fusion matrices, fused adjacency matrices and essential matrices, since essential matrices in general (for any ADE graph) are precisely defined by the recurrence formula that characterizes the fusion matrices of the $A_n$. In other words:

$$N^{A_{11}}_m[i,j] = E^{A_{11}}_m[i,j] = F^{A_{11}}_m[i,j].$$

In the case of $A_{11}$, we obtain therefore eleven square $11 \times 11$ matrices of essential paths; this is to be contrasted with the case of $E_6$, where the six graph fusion matrices $N^{E_6}_a(b_1, b_2)$ are square ($6 \times 6$) but where the six essential matrices $E^{E_6}_a(n, b)$ are rectangular ($11 \times 6$).
It is easy to show that the dimension of the space of essential paths of length $n$, for $A_N$ Dynkin diagrams, $n \leq 10$, is $(N - n)(n + 1)$. In the case of $A_{11}$, this dimension $d_n$ is

$$
\begin{pmatrix}
 n: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{pmatrix}.
$$

Notice that

$$\sum_n d_n = 536, \quad \sum_n (d_n)^2 = 8294.$$

From the above data, the reader may easily write down the multiplication table for the fusion algebra and the eleven square matrices $(11 \times 11)$ associated with $A_{11}$; we shall not write them explicitly.

### 3.3. Relations between $E_6$ and $A_{11}$

The main relation defining the matrices $E_a$ for the $E_6$ diagram can actually be understood without having to use the notion of essential paths: written $E_a(n - 1)G = E_a(n - 2) + E_a(n)$, this relation describes the usual way of coupling irreducible representations of $A_{11}$ (a truncated version of $SU(2)$), and this observation leads to the (already defined) notion of “fused adjacency matrices”, namely the eleven square matrices $F^E_0[a, b] \doteq E^E_n[a, b]$. The known equality $E^E_n = E^E_0 \cdot N^E_n$, when written in terms of $F$ matrices, leads to the following relation between the six fusion graph matrices $N^E_n$ and the eleven fused adjacency matrices $F^E_n$:

$$F^E_n = \sum_{e \in E_6} (F^E_n)_{e0} N^E_e.$$

Explicitly (we drop the $E_6$ label): $F_0 = N_0$, $F_1 = N_1$, $F_2 = N_2$, $F_3 = N_3 + N_5$, $F_4 = N_2 + N_4$, $F_5 = N_1 + N_5$, $F_6 = N_0 + N_2$, $F_7 = N_1 + N_3$, $F_8 = N_2$, $F_9 = N_5$, $F_{10} = N_4$.

We prefer to use essential matrices $E_a$’s, but it is clear that the following discussion could also be carried out in terms of the $F_n$ matrices.

### 3.3.1. $A_{11}$ labellings of the $E_6$ graph

The six essential matrices $E_a$, or, equivalently, the eleven matrices $F_n$, or the six graphs of essential paths, encode a good deal of information. Let us consider for instance $E_5$; we see that, going from top to bottom, and following essential paths starting from $\sigma_0$, after 0-step, we are at $\sigma_0$, after 1 step, we reach $\sigma_1$, after 2 steps, we reach $\sigma_2$, after 3 steps, we reach either $\sigma_3$ or $\sigma_5$, etc. Let us take another example, $E_2$; we see that, starting from the vertex $\sigma_2$, and following essential paths, after 6 steps (line $7 = 6 + 1$), we may reach the vertex $\sigma_0$ (column 1), the vertex $\sigma_2$ (column 3) or the vertex $\sigma_4$; notice that there are 3 different ways to reach $\sigma_2$ (three linearly independent paths of this type).

All these results just restate the fact that $E^{(a)}_n = E_a[n, b]$ is the number of linearly independent essential paths of length $n$ starting from $a$ and reaching $b$.

The index $n$ — the length — can be thought of as a label for a vertex of $A_{11}$, but it can be also understood as a (horizontal) Young diagram with $0 \leq n \leq 10$ boxes.

A particularly instructive way of illustrating these results is to give, for each vertex chosen as initial point (marked by a star on the picture), a graph of $E_6$ with the length of all possible essential paths indicated under the diagram. In this way, we get immediately:
We may look differently at the above correspondence(s) established between $A_{11}$ and $E_6$ by displaying the results as Figure 6: every graph of the previous kind (choice of the origin for the space of essential paths on $E_6$) gives rise to a new graph, where the bottom line refers to $\sigma_a$ (the points of $E_6$), the top line to $\tau_j$ (the points of $A_{11}$), and there is a connecting line between the two, whenever $\tau_j$ (actually the index $j$) appears below the vertex $\sigma_a$ in the figure describing essential paths from a chosen vertex (Figure 4, 5, ...). For instance, if we choose the origin at $\sigma_0$ (essential paths from 0), we see that 6 (denoting $\tau_6$) should be linked both to $\sigma_0$ (the leftmost point) and $\sigma_2$ (the middle point). The graph gotten in this way is always disconnected; this is the case when we consider essential paths starting from $\sigma_0$, since we obtain a graph with two connected components $\Gamma_1$ and $\Gamma_2$, namely Figure 6.

Although we do not need this information here, it can be seen (cf. the last section) that the eleven points 0, ..., 10 of $A_{11}$ correspond to representations of a finite dimensional quantum group (a quotient of SU(2)$_q$ when $q^{12} = \tilde{q}^{24} = 1$).

The two graphs $\Gamma_1$ and $\Gamma_2$ can be also used to describe the conformal embedding of the affine loop groups LSU(2)$_{10} \subset$ LSO(5)$_1$, but we shall not elaborate this ([6]). Let us mention nevertheless that the basic $b$, vector $v$, and spinor $s$ representations of LSO(5) obey the (Ising) fusion rules of the graph $A_3$; we know that $(b, s, v)$ have
the same fusion rule as the $A_3$ fusion subalgebra of $E_6$ generated by $(\sigma_0, \sigma_3, \sigma_4)$.

For this reason we can also write $b \simeq \sigma_0 \rightarrow \tau_0 + \tau_6$, $s \simeq \sigma_3 \rightarrow \tau_3 + \tau_7$, and $v \simeq \sigma_4 \rightarrow \tau_4 + \tau_{10}$. As it is well known, the same $A_3$ graph labels the three blocks of the modular invariant partition function relative to conformal field theory (the $E_6$ case), a result that we shall recover later when we deal with the toric matrices associated with the $E_6$ Dynkin diagram.

One may understand the above results in terms of an analog of group theoretical elementary induction-restriction of representations: if, instead of the graphs $A_{11}$ and $E_6$, we were considering a finite group $G$ together with a finite subgroup $H$, then, to each irreducible representation $\sigma$ of $H$ on a vector space $V$ we could associate a vector bundle $G \times_\sigma V$ over the quotient space $G/H$, and the spaces $\Gamma(G \times_\sigma V)$ of corresponding sections could be decomposed in terms of irreducible representations of $G$ (induction). Therefore, to each irreducible representation $\sigma$ of $H$, one can associate a (finite) list of numbers, namely the dimensions of the irreducible representations of $G$ appearing in this decomposition. This is precisely the classical analog of what is described by Figure 4. Notice that when the chosen representation $\sigma$ of $H$ is the trivial representation (the analog of the $\sigma_0$ vertex, the marked endpoint of our Dynkin diagram), the space of sections is nothing but the space of functions over the homogeneous space $G/H$. The dedicated reader can easily work out this example when $G = \tilde{I}$ is the binary icosahedral group and $H = \tilde{T}$, is a binary tetrahedral subgroup; the corresponding induction-restriction theory is then described by the affine Dynkin diagram $E_6^{(1)}$ (replacing Figure 4) which is then “decorated” with the dimensions\(^8\) of representations of $\tilde{I}$.

Another interpretation for the $A_{11}$ labelling of the $E_6$ graph (and generalizations) can be obtained in the theory of induction-restriction of sectors, applied to conformal field theory, see \cite{4}.

### 3.3.2. The $E_6 \times \tilde{E}_6 \rightarrow A_{11}$ table.

The previous information can be also gathered in the following table which can be directly read from the essential matrices (for instance, the fourth column of $E_1$ (referring to vertex $\sigma_5$ of the graph $E_6$) has entries 0, 0, 1, 0, 2, 0, 1, 0, 1, 0, 1 referring to the vertices $\tau_2, \tau_4, \tau_6, \tau_8, \tau_{10}$ of the graph $A_{11}$).

<table>
<thead>
<tr>
<th>$E_a \times \tilde{E}_b$</th>
<th>0</th>
<th>3</th>
<th>4</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>06</td>
<td>37</td>
<td>4,10</td>
<td>157</td>
<td>2468</td>
<td>359</td>
</tr>
<tr>
<td>3</td>
<td>37</td>
<td>046,10</td>
<td>37</td>
<td>2468</td>
<td>1357,29</td>
<td>2468</td>
</tr>
<tr>
<td>4</td>
<td>4,10</td>
<td>37</td>
<td>06</td>
<td>359</td>
<td>2468</td>
<td>157</td>
</tr>
<tr>
<td>1</td>
<td>157</td>
<td>2468</td>
<td>359</td>
<td>024628</td>
<td>1325,729</td>
<td>24268,10</td>
</tr>
<tr>
<td>2</td>
<td>2468</td>
<td>135279</td>
<td>2468</td>
<td>1325,729</td>
<td>02243638,10</td>
<td>1325,729</td>
</tr>
<tr>
<td>5</td>
<td>359</td>
<td>2468</td>
<td>157</td>
<td>24268,10</td>
<td>1325,729</td>
<td>024628</td>
</tr>
</tbody>
</table>

Multiplying a matrix $E_a$ (of dimension $(11, 6)$) by the transpose of a matrix $E_b$ (of dimension $(6, 11)$) gives a matrix $(11, 11)$, and it can be checked that the above table gives the decomposition of a product $E_a^T \cdot E_b^T$ of essential matrices of $E_6$ in \footnote{This pedagogical calculation has been worked out in \cite{9}, after completion of the present article.}
terms of matrices of $A_{11}$ (remember that for $A_{11}$, and in general for $A_N$ graphs, we have the equality between fused adjacency matrices, graph fusion matrices and essential matrices: $F_{n}^{A_{11}} = E_{n}^{A_{11}} = N_{n}^{A_{11}}$). For instance,

$$
E_{1}^{E_{6}} \tilde{E}_{1}^{E_{6}} = E_{2}^{A_{11}} + 2E_{4}^{A_{11}} + E_{6}^{A_{11}} + E_{8}^{A_{11}} + E_{10}^{A_{11}}
$$

$$= A_{2}^{A_{11}} + 2A_{4}^{A_{11}} + A_{6}^{A_{11}} + A_{8}^{A_{11}} + A_{10}^{A_{11}}.
$$

The general proof uses the fact that the (symmetric) $F_{E_{6}}$ matrices constitute a representation of the $A_{11}$ algebra and that the structure constants of this algebra are given by matrix elements of the fusion graph matrices themselves; it goes as follows: $(E_{E_{6}}a_{E_{6}} \tilde{E}_{E_{6}}b_{E_{6}})_{mn} = \sum_{c_{E_{6}} \in E_{6}} (E_{E_{6}}a_{E_{6}})_{mc} (\tilde{E}_{E_{6}}b_{E_{6}})_{cn} = \sum_{c_{E_{6}} \in E_{6}} (F_{E_{6}}m_{E_{6}})_{ac} (F_{E_{6}}n_{E_{6}})_{cb} = \sum_{p_{E_{11}} \in A_{11}} C_{mnp}(F_{E_{6}})_{ap} = \sum_{p_{E_{11}} \in A_{11}} (E_{E_{6}}a_{E_{6}})_{pb} (N_{E_{6}}p_{E_{6}})_{m,n}.$

Therefore $E_{E_{6}}a_{E_{6}} \tilde{E}_{E_{6}}b_{E_{6}} = \sum_{p_{E_{11}} \in A_{11}} (E_{E_{6}}a_{E_{6}})_{pb} (N_{E_{6}}p_{E_{6}})_{m,n}$. The same relation could be also written in a more symmetric way as

$$E_{E_{6}}a_{E_{6}} \tilde{E}_{E_{6}}b_{E_{6}} = \sum_{p_{E_{11}} \in A_{11}} (F_{E_{6}})_{ab} F_{A_{11}}^{p_{E_{6}}}.$$

3.3.3. Invariants and para-invariants. Following the terminology due to A. Ocneanu, a para-invariant of degree $n$, relative to the vertex $x$, and denoted $E_{x}^{(n)}$, is an essential path of length $n$ starting at $x$ and coming back at $x$. When $x = 0$, we get an invariant in the usual sense. Notice that the index $n$ can be thought of as a particular kind of Young diagram made of a single horizontal row with $n$ boxes.

Here we call by $I_{n}$ the total number of para-invariants of degree $n$.

An invariant of degree $n$ is an essential path of length $n$; starting at $\sigma_0$ (origin) and coming back to $\sigma_0$ (extremity). Their number is $E_{0,0}^{(n)}$.

$$
\begin{pmatrix}
  n: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  E_{0,0}^{(n)}: & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  E_{1,1}^{(n)}: & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\
  E_{2,2}^{(n)}: & 1 & 0 & 2 & 0 & 3 & 0 & 3 & 0 & 2 & 0 & 1 \\
  E_{3,3}^{(n)}: & 1 & 0 & 1 & 0 & 1 & 0 & 2 & 0 & 1 & 0 & 0 \\
  E_{4,4}^{(n)}: & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
  E_{5,5}^{(n)}: & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
  E_{6,6}^{(n)}: & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
  I^{(n)}: & 6 & 0 & 4 & 0 & 6 & 0 & 10 & 0 & 4 & 0 & 2 
\end{pmatrix}
$$

Here we call by $I_{n}$ the total number of para-invariants of degree $n$.

These invariants (in the case of $E_{6}$ we have only one, in degree 6) are quantum analogs of the famous Klein invariants for polyhedra. Let us explain this: take a classical polyhedron, put its vertices on the sphere, make a stereographical projection, build a polynomial that vanishes precisely at the location of the projected
vertices (or centers of faces, or mid-edges): you get a polynomial which, by construction, is invariant under the symmetry group of the polyhedron (at least projectively), since group elements only permute the roots. This is the historical method—see in particular the famous little book \[19\]. In the case of the tetrahedron, for instance, you get the three polynomials (in homogeneous coordinates):

\[V = u^4 + 2i\sqrt{3}u^2v^2 + v^4, \quad E = uv(u^4 - v^4)\]  and  \[F = u^4 - 2i\sqrt{3}u^2v^2 + v^4.\]

Actually \(V\) and \(F\) are only projectively invariant, but \(X = \frac{108}{1} E, \quad Y = -VF = \frac{1}{2}(u^8 + v^8 + 14u^4v^4)\) and \(Z = V^3 - iX^2 = (u^{12} + v^{12}) - 33(u^8v^4 + u^4v^8)\) are (absolute) invariants, of degrees 6, 8, 12. Together with the relation \(X^4 + Y^3 + Z^2 = 0\), they generate the whole set of invariants. Alternatively you can build the \(p\)-th power of the fundamental representation (it is 2-dimensional) of the symmetry group of the chosen binary polyhedral group, and choose \(p\) so that there exists one essential path of length \(p\) starting at the origin of the graph of tensorisation by the fundamental representation (therefore one of the affine ADE graphs) that returns to the origin. Therefore you get a symmetric tensor (since the path is essential), hence a homogeneous polynomial of degree \(p\); moreover, this polynomial is invariant, since the path goes back to the origin (the identity representation). By calculating explicitly the projectors corresponding to the (unique) essential path of \([2^6, 2^8\) and \([2^{12}\) on the affine \(E_6^{(1)}\) graph, one can recover the polynomials \(X, Y, Z\). The reader may refer to the set of notes \([9]\) where this (tedious) calculation can be found.

Returning to the quantum tetrahedron case (the \(E_6\) Dynkin diagram), we do not have a polyhedron to start with... Nevertheless, we still have essential paths, so the above notion of invariants (defined as essential paths starting to the origin and returning at the origin) makes sense. Notice that it would be nice to be able to exhibit a polynomial with non-commuting variables \(u\) and \(v\) manifesting some invariance with respect to an appropriate quantum group action. This was not obtained, so far.

### 3.3.4. Diagonalization of the fusion algebra of \(A_{11}\).

The eleven \(11 \times 11\) square matrices \(N_{n}^{A_{11}}\) that we just introduced commute with one another, therefore, they can be simultaneously diagonalized: one can find a matrix \(S\) which is such that all the \(S^{-1}.N_{n}^{A_{11}}.S\) are diagonal. This matrix, which is itself \(11 \times 11\), can be considered as the “non-commutative character table” of \(A_{11}\) (see the remark made at the end of Section 2.5); it will be explicitly given in Section 5.1.

### 3.4. The algebra \(A\) of endomorphisms of essential paths.

The dimension of the vector space of essential paths of length \(n\) (with arbitrary origin and extremity) is \(d_n = \sum a,b E_{a,b}^{(n)}\): we take the sum of all matrix elements of the row \(n + 1\) of each matrix \(E_{a,}\) (since length 0 corresponds to the first row of the essential matrices), then we sum over \(a\).

\[
\begin{pmatrix}
  n: & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
  d_n: & 6 & 10 & 14 & 18 & 20 & 20 & 18 & 14 & 10 & 6
\end{pmatrix}
\]

One may interpret \(n\) as a length, or as a particular vertex \(\tau_n\) of the \(A_{11}\) graph. An essential path \(\xi\) of length \(n\) from \(a\) to \(b\) can be denoted by \(\xi_{a,b}^{n}\) and pictured as
We introduce one such vertex $\xi_{n, a, b}$ whenever $n$ appears, in the previous $E_a \times \tilde{\text{E}}_b$ table, at the intersection of rows and columns $a$ and $b$ (in case of multiplicity, one has to introduce different labels $\xi_{n, a, b}$, $\zeta_{n, a, b}$, ...) Therefore, we have 6 vertices of type $\xi^0$, 10 vertices of type $\xi^1$, 14 of type $\xi^2$, etc., and the dimension of $\text{EssPath}$ is

$$\sum_n d(n) = 156.$$ 

The space of essential paths is graded by the length; $\text{EssPath} = \bigoplus E_{n} \text{EssPath}^{n}$. Let us consider $A^{n} = \text{End} \text{EssPath}^{n} \simeq \text{EssPath}^{n} \otimes \text{EssPath}^{n}$, the algebra of endomorphisms of this particular vector subspace. Notice that $A^{n}$ is isomorphic to an algebra of square matrices of dimension $d_{n}^{2}$, where $d_{n}= \text{dim} \text{EssPath}^{n}$. Let us also consider the graded algebra $A = \bigoplus E_{n} \text{End} \text{EssPath}^{n}$ and call it “the algebra of endomorphisms of essential paths”. $A$ is a direct sum of matrix algebras. Its dimension is $\sum_{n} d_{n}^{2}$:

$$\dim(A) = 6^2 + 10^2 + 14^2 + 18^2 + 20^2 + 20^2 + 20^2 + 18^2 + 14^2 + 10^2 + 6^2 = 2512.$$ 

A first basis for the subalgebra $A^{n}$ is given by the tensor products $\xi \otimes \eta$, where $\xi$ and $\eta$ run in a basis of $\text{EssPath}^{n}$. Such tensor products can be described by the following picture that looks like a diffusion graph in particle physics (dually, this is a double triangle).

The dimension of $A$ can be recovered by a simple exercise in combinatorics: the counting of all possible labelled diffusion graphs.

4. The algebra $E_6 \otimes_{A_3} E_6$

4.1. Definition. In the present section, we introduce and discuss the properties of the algebra

$$S = E_6 \otimes_{A_3} E_6.$$ 

It will be called “the algebra of quantum symmetries”. Let us first explain this definition. We start from the fusion algebra of $E_6$; this is a commutative algebra, and, in particular, a vector space. We may consider its tensor square $E_6 \otimes E_6$, which is a vector space of dimension $6^2$ and can be endowed with a natural multiplication:

$$(a_1 \otimes b_1)(a_2 \otimes b_2) \doteq (a_1a_2 \otimes b_1b_2).$$

Here the tensor product $\otimes$ is the usual tensor product. In order to construct the algebra $S$, we do not take the tensor product...
as above, over the complex numbers, but over the subalgebra $A_3$; this means that, given $a$ and $b$ in the fusion algebra $E_6$, we identify $ax \otimes b \equiv a \otimes xb$ not only when $x$ is a complex number, but also when $x$ is an element of the subalgebra $A_3$ generated by $\sigma_0$, $\sigma_4$, $\sigma_3$. This tensor product will be denoted by $\otimes$ rather than $\otimes$. We remember that $A_3$ is a very particular subalgebra of the (commutative) algebra $E_6$, on which it acts non-trivially by multiplication. The dimension of the algebra $S$ just constructed is therefore not equal to $6^2$ but to $6^2/3 = 12$.

4.2. A linear basis for $S$. The following elements build up a set $(L \cup R \cup A \cup C)$ of 12 linearly independent generators for $S$. Here and below, we write $a$ rather than $\sigma_a$.

$L = \{1 \otimes 0, 2 \otimes 0, 5 \otimes 0\},$

$R = \{0 \otimes 1, 0 \otimes 2 = 3 \otimes 1, 0 \otimes 5 = 4 \otimes 1\},$

$A = \{0 \otimes 0, 3 \otimes 0 = 0 \otimes 3, 4 \otimes 0 = 0 \otimes 4\},$

$C = \{1 \otimes 1, 2 \otimes 1 = 1 \otimes 2, 5 \otimes 1 = 1 \otimes 5\}.$

For reasons that will be explained later, we have split this basis, made of twelve elements, into four subsets: $L, R, A, C$.

Showing that these 12 elements are linearly independent is straigtforward, and since $S$ is 12-dimensional, we have a basis. What are not totally obvious are the above-mentioned equalities; let us prove them. The calculations use (of course) the multiplication table for the fusion algebra of the $E_6$ graph, and the fact that $0, 3, 4$ can “jump” over the tensor product sign $\otimes$. 

$3 \otimes 1 = 0 \otimes 3.1 = 0 \otimes 2,$

$4 \otimes 1 = 0 \otimes 4.1 = 0 \otimes 5,$

$3 \otimes 0 = 0 \otimes 3.0 = 0 \otimes 3,$

$4 \otimes 0 = 0 \otimes 4.0 = 0 \otimes 4,$

$2 \otimes 1 = 1.3 \otimes 1 = 1 \otimes 3.1 = 1 \otimes 2,$

$5 \otimes 1 = 1.4 \otimes 1 = 1 \otimes 4.1 = 1 \otimes 5.$

The reader can prove, in the same way, many other identities, like for instance

$2 \otimes 2 = 5.3 \otimes 2 = 5 \otimes 3.2 = 5 \otimes (1 + 5) = 5 \otimes 1 + 5 \otimes 5,$

$5 \otimes 5 = 1.4 \otimes 5 = 1 \otimes 4.5 = 1 \otimes 1.$

4.3. Structure and multiplication table of $S$. Using the previous technique, one can build a multiplication table $12 \times 12$. However, it is enough to observe the following.

The subalgebra $E_6 \otimes 0$ of $S$ linearly generated by $L \cup R \cup A$, i.e., by the six elements $a \otimes 0$ (where $a$ runs in the set $\{0, 3, 4, 1, 2, 5\}$), is obviously isomorphic to the fusion algebra of the graph $E_6$ itself; it is called the “chiral left subalgebra”. We can make a similar remark for the “chiral right subalgebra” $0 \otimes E_6$ linearly generated by $R \cup A$. Since 1 is the (algebraic) generator of the fusion algebra, we see that
1 \otimes 0 and 0 \otimes 1 separately generate (algebraically) the left and right subalgebras $E_6 \otimes 0$ and $0 \otimes E_6$. The “ambichiral subalgebra” is the intersection of the left and right chiral subalgebras; it is linearly generated by the elements of the set $A$ and is isomorphic to the algebra generated by $0, 3, 4$, i.e., with the graph fusion algebra of $A_3$. Notice that the vector space linearly generated by the elements of $C$ (standing for “Complement”) is not a subalgebra; a final observation is that the three basis vectors of $C$ can be obtained by multiplying elements of $L$ and elements of $R$. Indeed:

1 \otimes 1 = (0 \otimes 1)(1 \otimes 0),
2 \otimes 1 = (3.1 \otimes 1) = (3 \otimes 1)(1 \otimes 0),
5 \otimes 1 = (4.1 \otimes 1) = (4 \otimes 1)(1 \otimes 0).

The conclusion is that $1 \otimes 0$ and $0 \otimes 1$ algebraically generate the algebra $S$. These two elements are called the left and right generators; it is therefore sufficient to know the multiplication of arbitrary elements by these two generators to reconstruct the whole multiplication table of $S$.

4.4. Ocneanu graph of quantum symmetries. The multiplication of arbitrary elements by the two (left and right) generators can be best summarized by the corresponding Cayley graph.

Multiplication by $1 \otimes 0$ is given by continuous lines, and multiplication by $0 \otimes 1$ is given by dotted lines. For instance, we read from this graph the equalities

$(2 \otimes 1)(0 \otimes 1) = 1 \otimes 1 + 2 \otimes 0 + 5 \otimes 1,$
$(5 \otimes 1)(1 \otimes 0) = 2 \otimes 1 + 0 \otimes 5.$
Let us prove, for instance, the first equality. The left-hand side is also equal to $(2.0 \otimes 1.1) = 2 \otimes (0 + 2) = 2 \otimes 0 + 2 \otimes 2$, but $2 \otimes 2 = 5 \otimes 1 + 1 \otimes 1$, as shown previously; hence the result.

This graph was obtained by A. Ocneanu as a graph encoding the quantum symmetries of $E_6$, defined in a totally different way (irreducible connections on a graph—we shall come back to this original definition in one of the appendices), and interpreted as a Cayley graph describing multiplication by two particular generators. One of our observations, in the present paper, is to notice that this algebra of quantum symmetries is isomorphic to the associative algebra $E_6 \otimes A_3 E_6$.

In the case of $AN$, the Ocneanu graph is obtained by setting $S = A_N \otimes A_N A_N$ (there are $N$ points). In the case of $E_8$, it is obtained by setting $S = E_8 \otimes A_2 E_8$ (there are $8 \times 8/2 = 32$ points). The cases of $E_7$ and $D_{odd}$ are special, since the fusion table of those Dynkin diagrams cannot be constructed (they do not define an hypergroup with positive structure constants); this does not mean, of course, that one cannot consider their quantum symmetries, but the technique that we are explaining here should be adapted. The case of $D_{even}$ is also special, because the two vertices that constitute the “fork” of the graph do not behave like the others and give rise to an algebra of quantum symmetries which is not commutative, contrarily to the other examples encountered so far (the number of points of the Ocneanu graph, for $D_{2n}$ is $(2n - 2) \times (2n - 2)/(n - 1) + 2^2 = 4n$ (the algebra itself being isomorphic to $\mathbb{C}^{4n-4} \oplus M(2, \mathbb{C})$). The results themselves can be found in the paper [23], the details and proofs should appear in the work of A. Ocneanu, when available, or, following the techniques adapted from what is explained here, in [12] and in part of the thesis [34].

4.5. The $E_6 \times E_6 \rightarrow S$ table. To each element of $S = E_6 \otimes A_3 E_6$ (for instance, $\sigma_5 \otimes \sigma_1$), we may associate one representative in $E_6 \otimes E_6$ (for instance, $\sigma_5 \otimes \sigma_1$). Then, we may apply the multiplication map $a \otimes b \rightarrow ab$ to get one element in $E_6$ (here it is $\sigma_5 \sigma_1 = \sigma_3 + \sigma_1$). The result is obviously independent on the choice of the representative, since both $ax \otimes b$ and $a \otimes xb$, with $x \in A_3$, have the same image $AXB$ in $E_6$. Now we can represent the obtained element of $E_6$ ($\sigma_5 \sigma_1$ in our example) by the corresponding graph fusion matrix (namely $N_5 N_1$ in our example). The result is a $6 \times 6$ matrix with rows and columns labelled by the Dynkin diagram of $E_6$. In our example, using the (ordered) basis $012543$, we have the correspondence

$$\sigma_5 \otimes \sigma_1 \rightarrow S_{51} \equiv \begin{pmatrix} . & 1 & 1 & . & . & . \\ . & 1 & 2 & . & 1 & . \\ 1 & 3 & 1 & . & . & . \\ . & 2 & 1 & 1 & . & . \\ 1 & 1 & . & 1 & . & . \\ . & 1 & 1 & 1 & . & . \end{pmatrix}$$

By this construction we obtain twelve $(6, 6)$ matrices $S_{ab}$, each one being associated with a particular basis element $a \otimes b$ of $S$. We shall define $d_{a \otimes b} = \sum_{c,d} (S_{ab})_{cd}$, for instance $d_{5 \otimes 1} = 20$.

The following table summarizes the results: the element $a \otimes b$ of $S$ (denoted $ab$ in the table, to save space) appears at the intersection of $\tilde{E}_c \times E_d$ (with possible
multiplicity \( m \)) whenever the entry \((c, d)\) (labelling the \( E_6 \) graph) of the matrix \( S_{ab} \) is equal to \( m \). For instance, the entry 51 appears in the table with multiplicity 3 at the intersection of row 2 and column 2, since the matrix element of \( S_{11} \) relative to the row and column indexed by \((\sigma_2, \sigma_2)\) is equal to 3.

One may use these results to define a kind of generalized fusion law \( \circledast : E_6 \times E_6 \to S \) (hence the name of the section). In this way, one obtains the following particularly useful equality:

\[
\sigma_0 \circledast \sigma_0 = 0 \circledast 0 + 1 \circledast 1.
\]

Here comes the table:\footnote{This table is actually symmetric, but for reasons of space, and also to make it easier to read, we have removed the lower entries.}

<table>
<thead>
<tr>
<th>( E_6 \times E_6 )</th>
<th>0</th>
<th>3</th>
<th>4</th>
<th>1</th>
<th>2</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>00,11</td>
<td>30,21</td>
<td>40,51</td>
<td>10,01,21</td>
<td>20,11,31,51</td>
<td>50,21,41</td>
</tr>
<tr>
<td>3</td>
<td>00,40,11,51</td>
<td>30,21</td>
<td>20,11,31,51</td>
<td>10,50,01,21,41</td>
<td>20,11,31,51</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>00,11</td>
<td>50,21,41</td>
<td>20,11,31,51</td>
<td>10,01,21</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>00,20,112,31,51</td>
<td>10,30,50,01,21,41</td>
<td>20,40,11,31,51</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>00,20,40,113,312,513</td>
<td>10,30,50,01,21,41</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>00,20,112,31,51</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The reader who does not want to use the previous simple matrix manipulations to generate the whole table may proceed as follows (this is equivalent): first we remember that \( S \) is both a left and right \( E_6 \) module and that \( \sigma_0 \) is the unit of \( E_6 \). Therefore we have

\[
\sigma_a \circledast \sigma_b = \sigma_a(\sigma_0 \circledast \sigma_0)\sigma_b = \sigma_a(0 \circledast 0 + 1 \circledast 1)\sigma_b.
\]

Let us compute for instance \( \sigma_1 \circledast \sigma_2 \) and write only the subscripts to save space.

\[
1 \circledast 2 = 1(0 \circledast 0 + 1 \circledast 1)2 = (1.0) \circledast (0.2) + (1.1) \circledast (1.2)
\]

\[
= 1 \circledast 2 + (0 + 2) \circledast (1 + 3 + 5)
\]

\[
= 1 \circledast 2 + 0 \circledast 1 + 0 \circledast 3 + 0 \circledast 5 + 2 \circledast 1 + 2 \circledast 3 + 2 \circledast 5
\]

\[
= 2 \circledast 1 + 0 \circledast 1 + 3 \circledast 0 + 4 \circledast 1 + 2 \circledast 1 + 1 \circledast 0 + 5 \circledast 0 + 2 \circledast 1.
\]

At the last line, we used the fact that, in \( S \), \( 1 \circledast 2 = 2 \circledast 1, 0 \circledast 3 = 3 \circledast 0, 0 \circledast 5 = 4 \circledast 1, 2 \circledast 3 = 2.3 \circledast 0 = 1 \circledast 0 + 5 \circledast 0 \), and that \( 2 \circledast 5 = 2 \circledast 4.1 = 2 \circledast 1 = 2 \circledast 1 \).

Notice that this table looks very much like the table that was called \( E_6 \times E_6 \to \tilde{A}_{11} \) (or \( E_6 \times E_6 \to A_{11} \)) in the previous section, but now lengths of essential paths are replaced by the \( x = a \circledast b \) labels of the algebra \( S \). We could also encode this structure in terms of six matrices of dimension \((12, 6)\) (exactly as we encoded all data concerning essential paths in terms of six matrices of dimension \((11, 6)\)), now the rows would be labelled by the twelve basis elements of \( S \) rather than by the essential paths of \( A_{11} \); in other words, the twelve matrices \( S_{x=a \circledast b} \) replace the eleven matrices \( F^E_n \).

We can also read the previous table in terms of essential matrices and graph fusion matrices. Multiplying the transpose of a matrix \( E_6 \) (of dimension \((6, 11)\)) by a matrix \( E_6 \) (of dimension \((11, 6)\)) gives a matrix \((6, 6), \) and it can indeed be
checked that the above table gives the decomposition of a product \( \tilde{E}_a \tilde{E}_b \) in terms of (product of) graph fusion matrices \( N_{\tilde{E}_a} \) for the graph \( E_a \). For instance,
\[
\tilde{E}_1 E_0 \tilde{E}_0 = N_{E_2} N_{E_0} + N_{E_4} N_{E_0} + N_{E_4} N_{E_1} N_{E_0} + N_{E_4} N_{E_2} N_{E_0} + 2 N_{E_5} N_{E_0}.
\]
For this reason, the above table \( E_a \times E_b \to S \) may also be called “the \( \tilde{E}_a \times \tilde{E}_b \) table”. More generally, these relations read:
\[
\tilde{E}_a \tilde{E}_b = \sum_{x \in \text{Oc}(E_a)} (S_x)_{ab} S_x.
\]
Here the sum is over all twelve vertices \( x \) of the Ocneanu graph of \( E_a \) (compare with Section 3.4). Once the \( E_a \times E_b \to S \) table is known (and we have explained how to get it), it is a simple matter to check all these equations for \( \tilde{E}_a \tilde{E}_b \). Admittedly, this is not an enlightening method. The simplest and most direct proof uses the fact that \( \tilde{E}_a \tilde{E}_b = E_0^{E_a} N_{E_a} \) and that (only one equation to check)
\[
\tilde{E}_0 \tilde{E}_0 = S_{0\otimes 0} + S_{1\otimes 1} = N_{E_0} + N_{E_0} + N_{E_2}.
\]
The conclusion follows from the fact that \( S_{a\otimes b} = N_{E_a} N_{E_b} \). A more formal proof (which would not use the above explicit result for \( \tilde{E}_0 \tilde{E}_0 \)) cannot be as straightforward as its analog (for \( E_a E_b \tilde{E}_0 \)) presented in Section 3.4; the difficulty, now, is to relate the eleven matrices \( F_{E_a}^{E_b} \) to the twelve matrices \( S_x \) (this can be a posteriori done in terms of explicit Fourier-like matrices \( 11 \times 12 \)). In any case, the proof using the explicit calculation of \( \tilde{E}_0 \tilde{E}_0 \) is easy.

Here again it is handy to describe the nonzero entries of the table by a new kind of vertices. This should be compared with those introduced in Section 3.4. The former table \( (E_6 \times E_6 \to A_{11}) \) gives all possible vertices of the type displayed in Section 3.4, where they are associated with nonzero matrix elements \( (F_{E}^{E})_{ab} \); the entries of the latter table \( (E_6 \times E_6 \to S) \) are associated with the nonzero matrix elements \( (S_x)_{ab} \), and gives all possible vertices of the following type:

\[
\begin{array}{c}
&d\\
\downarrow & a \otimes b \\
&c \\
I
\end{array}
\]

We introduce one such vertex, called \( I_{a \otimes b} \), whenever \( a \otimes b \) appears in the previous \( E_c \times E_d \)-table, at the intersection of row \( c \) and column \( d \). We have therefore 6 vertices of type \( I^{0\otimes 0} \), 8 vertices of type \( I^{1\otimes 0} \), etc. The integer \( d_{a \otimes b} \) is also the number of vertices of type \( I_{a \otimes b} \); we find
\[
\begin{pmatrix}
a \otimes b & 0 & 0 & 3 & 0 & 0 & 1 & 1 & 2 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 2 & 1 & 5 & 1 & 5 & 1
\end{pmatrix}.
\]

The sum of the squares of these numbers is \( \sum_{a \otimes b \in S} d_{a \otimes b}^2 = 2512 \).

We notice immediately that this sum is also equal to the sum \( \sum_{n \in A_{11}} d_n^2 = 2512 \) found previously in the section devoted to the study of essential paths.
4.6. The algebra $\mathcal{A}$ of endomorphisms of essential paths (again). The fact that $\sum_{a, b \in S} d_{a \otimes b}^2 = \sum_{n \in A_{11}} d_n^2 (= 2512)$ suggests immediately that the algebra $\mathcal{A} = \bigoplus_n \text{EssPath}^n \otimes \text{EssPath}^n$ carries two algebra structures. For the first structure, it is a direct sum of eleven blocks (square matrices) of dimensions $d_n$; for the second structure, it is a direct sum of twelve blocks (square matrices) of dimensions $d_{a \otimes b}$.

In other words, rather than decomposing $\mathcal{A}$ into the tensor products $\xi \otimes \eta$, as we did in Section 3.4, we decompose it into tensor products $I_1 \otimes I_2$, where $I_1$ and $I_2$ refer to the vertices appearing in the $E_6 \times E_6 \mapsto S$ table. The two decompositions of the bi-algebra $\mathcal{A}$ as two distinct sums of blocks correspond to a diagonalization of the two algebra structures.

The first algebra structure is the composition of endomorphisms of essential paths, and it is directly given by the very definition of $\mathcal{A}$; its block decomposition is labelled by the points of $A_{11}$. The second algebra structure (call it $\star$) comes from the fact that essential paths (on which the elements of $\mathcal{A}$ act) are endowed with a partial multiplication, namely concatenation of paths, and one can use this to define, by duality, the new multiplication on $\mathcal{A}$.

One technical difficulty is that the concatenation of two essential paths is not necessarily essential, so that one has to reproject the result of concatenation to obtain an essential path. We shall not describe this construction explicitly and refer to [23].

The block decomposition of $\mathcal{A}$ with respect to the second algebra structure is labelled by the basis elements $J = a \otimes b$ of $S$: $\mathcal{A} = \bigoplus_J \mathcal{A}_J = \bigoplus_J H_J \otimes H_J$. The index $J$ labelling the different blocks is therefore also associated with minimal central projectors for the product $\star$ (for instance, the central projector associated with the block of dimension $28^2$ is a direct sum of twelve matrices, its restriction to this chosen block is the identity matrix, and all other blocks are zero). This is (probably) how the Ocneanu graph of quantum symmetries was first defined and obtained. From a pictorial point of view, the dimension of the vector space $H_{a \otimes b}$ is $d_{a \otimes b}$ and given, as we know, by the number of all vertices of type $I_{a \otimes b}^d$; the dimension of the block $A_{a \otimes b} = H_{a \otimes b} \otimes H_{a \otimes b}$ is therefore given by the counting of all the possible labelled dual diffusion graphs with fixed internal line labelled by $a \otimes b$. Elements of $A_{a \otimes b}$ can be depicted by the following figure that looks like a dual diffusion graph of particle physics.

5. Modular structure of the $E_6$ graph

5.1. Verlinde representation. Consider the following $11 \times 11$ matrices $S$ and $T$ (here $N = 12$)

$$S[m, n] = (-2I/(\sqrt{2\sqrt{N}}))\sin[\pi m n/N],$$

$$T[m, n] = \exp[i\pi (m^2/(2N) + 1/4)]\delta_{m,n}.$$
We may check that $S^4 = 1$ and that $(ST)^3 = 1$; we have also $S^2 = -1$. The above matrices $S$ and $T$ are therefore representatives for the generators $\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \}$ and $\{ \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \}$ of the modular group. This is an eleven dimensional representation of the group $\text{SL}(2, \mathbb{Z})$.

The above $11 \times 11$ matrix $S$ is nothing but the “noncommutative character table” of the graph $A_{11}$; one may indeed check that the eleven matrices $S^{-1}, N_n^{A_{11}}, S$ are diagonal (like in 3.3.4, the $N_n^{A_{11}}$ denote the fusion matrices for the graph $A_{11}$).

As for the diagonal $T$ matrix, its eigenvalues can be directly obtained from the central element that defines the ribbon structure of the quantum group $U_q(\text{SL}_2)$. One may notice that eigenvalues of $T$ corresponding to irreducible representations of $A_{11}$ associated by induction (see Figure 4) with the three extremal points of the graph $E_6$ (the $A_3$ subalgebra) are equal. This is not so for the other points of the $E_6$ graphs.

One may also notice that in the above representation of the modular group, $T^{48} = 1$ (and $T^s \neq 1$ for smaller powers of $T$); one can actually prove [17] that the representation factors over the finite quotient $\text{SL}(2, \mathbb{Z}/48\mathbb{Z})$.

5.2. Reduced essential matrices. The reduced essential matrices, in the case of $E_6$, are obtained (definition) from the essential matrices by putting to zero all the matrix elements of the columns associated with the vertices $\{ \sigma_1, \sigma_2, \sigma_5 \}$. Because of the order chosen for basis elements, this corresponds to the columns 2, 3, 4 of our essential matrices. In general, the reduced essential matrices $E_a$ should be obtained by keeping only the matrix elements of the columns relative to the subalgebra defining the ambichiral part of the Ocneanu graph, and putting all others to zero. In our example, this subalgebra is $A_3$ and spanned by $\sigma_0, \sigma_4, \sigma_3$.

We shall see later why it is useful to introduce these objects.

\[
E'_0 = \begin{pmatrix} 1 & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}, \quad E'_1 = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}, \quad E'_2 = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}
\]

\[
E'_3 = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}, \quad E'_4 = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}, \quad E'_5 = \begin{pmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & 1 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{pmatrix}.
\]
5.3. Torus structure of the $E_6$ graph. A set of matrices describing what is called “the torus structure of a Dynkin diagram” was mentioned by A. Ocneanu in various talks since 1995 (for instance [22]), but — to our knowledge — this has not been made available in written form. We shall therefore neither comment about the original definition of these matrices nor relate Ocneanu’s construction to ours... but it is clear that the Ocneanu matrices describing the “torus structure” of the Dynkin diagrams and our “toric matrices” are the same objects. We shall introduce them directly, in terms of our essential matrices and reduced essential matrices.

To every point $a \otimes b$ of the Ocneanu graph of quantum symmetries (i.e., the twelve points corresponding to linear generators of $S = E_6 \otimes A_3 E_6$, in the case of the Dynkin diagram $E_6$), we associate a matrix $11 \times 11$ (more generally, a square matrix $(N - 1) \times (N - 1)$ if $N$ is the dual Coxeter number of the chosen Dynkin diagram) defined by $W_{ab} = E_{a} \tilde{E}_{b}$ ($= E_{a}^r \tilde{E}_{b}^r$). Starting from the point $a \otimes b$ of the Ocneanu graph, the number $(W_{ab})_{ij}$ counts the number of independent ways to reach the origin $0 \otimes 0$ after having performed essential paths of length $i$ (resp. $j$) on the left and right chiral subgraphs.

$$W_{60} = \begin{pmatrix} 1 & . & . & . & . & 1 \\ . & . & . & . & . & 1 \\ . & . & . & 1 & . & 1 \\ . & 1 & . & . & . & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad W_{11} = \begin{pmatrix} . & 1 & . & . & . & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

$$W_{30} = \begin{pmatrix} . & . & . & . & 1 & 1 \\ . & . & . & . & 1 & 1 \\ . & . & . & 1 & . & 1 \\ . & . & . & 1 & . & 1 \\ . & 1 & . & . & . & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad W_{21} = \begin{pmatrix} . & . & . & . & 1 & 1 \\ . & . & . & . & 1 & 1 \\ . & . & . & 1 & . & 1 \\ . & . & . & 1 & . & 1 \\ . & 1 & . & . & . & 1 \\ . & 1 & 1 & 2 & 2 & 2 \\ . & 1 & 1 & 2 & 2 & 2 \\ . & 1 & 1 & 2 & 2 & 2 \\ . & 1 & 1 & 2 & 2 & 2 \\ . & 1 & 1 & 2 & 2 & 2 \\ . & 1 & 1 & 2 & 2 & 2 \\ . & 1 & 1 & 2 & 2 & 2 \\ . & 1 & 1 & 2 & 2 & 2 \end{pmatrix}$$

$$W_{40} = \begin{pmatrix} . & . & . & . & 1 & 1 \\ . & . & . & . & 1 & 1 \\ . & . & . & 1 & . & 1 \\ . & . & . & 1 & . & 1 \\ . & 1 & . & . & . & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \quad W_{51} = \begin{pmatrix} . & . & . & . & 1 & 1 \\ . & . & . & . & 1 & 1 \\ . & . & . & 1 & . & 1 \\ . & . & . & 1 & . & 1 \\ . & 1 & . & . & . & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \\ . & 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$
5.4. The modular invariant. The toric matrix associated with the origin $0 \otimes 0$ of the Ocneanu graph of $E_6$ is

\[
W_{00} = \begin{pmatrix}
1 & \ldots & 1 & \ldots \\
\vdots & \ddots & \vdots & \ddots \\
\vdots & \ddots & 1 & \ddots \\
\vdots & \ddots & \vdots & \ddots \\
1 & \ldots & 1 & \ldots \\
\end{pmatrix}
\]
It can be checked that it commutes with $S$ and $T$, and therefore with the whole $SL_2(\mathbb{Z})$ group:

$$SW_{00} = W_{00}S, \quad TW_{00} = W_{00}T.$$ Notice that $W_{00}$ is normalized ($W_{00}[1, 1] = 1$) and that all the entries of this matrix are positive integers. The following $SL(2, \mathbb{Z})$-invariant sesquilinear form on $\mathbb{C}^{11}$,

$$Z = \sum_{i,j=1}^{11} W_{00}[i, j] \chi_i \bar{\chi}_j = |\chi_1 + \chi_7|^2 + |\chi_4 + \chi_8|^2 + |\chi_5 + \chi_{11}|^2,$$

gives a solution of the Cappelli–Itzykson–Zuber problem and can therefore be interpreted as the modular invariant partition function of a quantum field model.

This partition function was of course obtained long ago, but it is interesting to notice that it can be recovered as the toric matrix associated with the origin of the Ocneanu graph of $E_6$.

As stated in [23], each entry of $W_{00}[i, j]$ can be considered as the dimension of a representation of the algebra of quantum symmetries (for instance, in the case of $E_6$, we have twelve nonzero components and all the components are equal to 1, this reflects the fact that $S$ is a twelve-dimensional abelian algebra). Notice finally that the Coxeter numbers of the graph $(1, 4, 5, 7, 8, 11)$ appear on the diagonal of this matrix.

The above reconstruction of the partition function also works for all ADE (i.e., chiral algebra of $\hat{SU}(2)$ type) statistical models and can probably be generalized to more general situations ($\hat{SU}(3)$, ...). The interpretation of the other toric matrices $W_{ab}$ in terms of conformal field theory models still requires some work. 

6. Miscellaneous comments

6.1. Remarks about conformal embeddings. The condition for conformal embeddings of affine Kac–Moody algebras $\mathcal{G}_1 \subset \mathcal{G}_2$ at levels $k_1$ and $k_2$ is the identity of their central charges, where $c$ is given by $c = c_{\text{dim}(\mathcal{G})}$, $\tilde{g}$ being the dual Coxeter number of the corresponding finite dimensional Lie algebra. If one takes $\mathcal{G}_1 = \hat{A}_1$ and $k_2 = 1$, there are only three nontrivial solutions (of course, levels should be integers): $(\hat{A}_1)_{10} \subset (\hat{B}_2)_1 = (\hat{C}_2)_1$, $(\hat{A}_1)_{28} \subset (\hat{G}_2)_1$, and $(\hat{A}_1)_{4} \subset (\hat{A}_2)_{1}$, since the Coxeter numbers (resp. dimensions) of $A_1$, $A_2$, $B_1$, and $G_2$ are given by $2, 3, 3, 4$ (resp. $3, 8, 10, 14$). The first two conformal embeddings are respectively described by the Dynkin diagrams of $E_6$ and $E_5$; the last one gives rise to the $D_4$ invariant. This was observed in the literature long ago [6] and discussed, in terms of inclusions of Von Neumann algebras, by [4], see also [5] and references therein.

6.2. Remarks about generalized quantum recoupling theory.

6.2.1. Wigner and Racah multiplications: the pure $SU(2)$ case. Using all possible spin triples $j_1, j_2, J$ such that $|j_1 - j_2| \leq J \leq |j_1 + j_2|$, we can draw elementary vertices decorated with such allowed triples and then build elementary “diffusion graphs” looking like the following one. Spin values (integer or half-integers) describe, as usual, the irreducible representations of $SU(2)$.

\footnote{See however the last footnote of Section 1.4.}
Since we have infinitely many possible (labelled) vertices at our disposal, we have also infinitely many such diffusion graphs. Since a triangle inequality has to be satisfied at each vertex, the above diffusion graphs could also be called (dually) “double triangles” and pictured as two triangles glued together, sharing this time an horizontal common edge.

Following Ocneanu, we define the vector space $A$ generated by a linear basis indexed by the (infinite) set of diffusion graphs — or double triangles. The vector space $A$ comes therefore equipped with a particular basis, and every element of $A$ is a linear combination, over the complex numbers, of diffusion graphs. One also introduces another class of graphs: the “dual” diffusion graphs (here the internal line is horizontal, not vertical).

The next step is to endow the vector space $A$ with two compatible multiplications. These two multiplications appear in the works of Racah and Wigner and are mentioned in the book [3]. The first multiplication (Wigner) amounts to compose these “spin diffusion graphs” vertically, the other (Racah) amounts to compose them horizontally. The precise definition involves appropriate coefficients, and we refer to the book [3] for explicit formulae. The following identity (a kind of Fourier transform) relates the previous diffusion graphs and their dual:

$$\sum_n \left( \begin{array}{ccc} a & b & n \\ c & d & m \end{array} \right)$$

Although the subject itself is quite old, and besides a few lines in the book just quoted, even in the pure SU(2) case we cannot give any reference providing a precise study of this bigebra structure (that we suggest to call the “Racah–Wigner bigebra”).
6.2.2. From SU(2) to its finite subgroups. Rather than using representations of $G = SU(2)$, we can use the data provided by a finite subgroup $K$ of SU(2) and consider several variants of the above bigebra.

- We may use irreps of $K$ alone.
- We may use simultaneously irreps of $K$ and of $G$. Indeed, an irrep of $G$, restricted to $K$, can be tensorially multiplied by irreps of $K$ and the result can be decomposed into irreps of $K$.
- We may even choose two subgroups $H$ and $K$ of SU(2) (for instance the binary tetrahedral and binary icosahedral groups).

6.2.3. Generalized quantum recoupling theory and Ocneanu bigebra. The next step is to replace $G$ by the fusion algebra of the graph $A_N$ and $K$ by the fusion algebra of another graph with the same norm (the norms have to match, otherwise, one cannot consider meaningful bimodules). The ideas are the same as before, but instead of the usual triangular condition of composition of spins (case of SU(2)), we have more complicated conditions which are direct consequences of the multiplication tables (like those expressing the fusion rules $A_{11} \times A_{11} \mapsto A_{11}$, $E_6 \times E_6 \mapsto E_6$, $A_{11} \times E_6 \mapsto E_6$ that we have considered previously). In the same way as for SU(2), one may consider double triangles, generalized $6j$ symbols (represented by tetrahedra carrying bimodules labels on their six edges, and algebra labels at the four vertices), etc. The construction of the Ocneanu bigebra $A$ is a direct generalization of the Racah–Wigner bigebra. It was introduced before in terms of endomorphisms of essential paths. Its dimension is finite (2512 in the case of $E_6$). The diagonalization of this algebra, for the product $\ast$, is $A = \bigoplus_{J \in S} \text{End}(H_J)$. The Hilbert spaces $H_J$ are labelled by minimal central projections of $(A, \ast)$. Elements of $A$, in particular the particular basis elements described by diffusion graphs (also seen as particular endomorphisms of the space of essential paths or as two triangles with vertices $A_{11}$, $A_{11}$, $E_6$ sharing a common edge of type $A_{11} - A_{11}$), can be decomposed into linear combinations of elements in $H_J \otimes H_J$, i.e., in terms of “dual diffusion graphs”. This decomposition generalizes the equation given in 6.2.1 and involves generalized $6j$ symbols (see [23] for details).

6.2.4. Connections on Ocneanu cells, quantum symmetries. Here we just give the following references. The general notion of connections on a system of four graphs was introduced in [21]. Such “cells” have been explicitly written and used to reconstruct systems of Boltzmann weights in [27]. The general formalism of Ocneanu cells was “translated” and adapted to the situation of statistical mechanics in [33]. In the framework of RSOS models, an explicit description of several cell systems can be found in [30]. The notion of quantum symmetries on such a system is due to A. Ocneanu; it was presented at several meetings since 1995, and it is described in the recent article [23].

6.3. Graph algebras and finite dimensional quantum groups. The reader will have noticed that we did not provide, so far, any quantum group (Hopf algebra) interpretation for the above constructions. The main reason is that such a purely algebraic interpretation, in terms of finite dimensional — but not necessarily semi-simple — Hopf algebras is not known for arbitrary ADE graphs. However,
for $A_N$ Dynkin diagrams, such an interpretation can be found. Here it is: Take the quantized enveloping algebra $\mathcal{U} \cong U_q(SL_2)$ at a primitive $(N + 1)$-th root of unity. Take $N + 1$ odd for the moment (the analysis can be done for even $N + 1$, but this is slightly more involved). Call by $\mathcal{H}$ the quotient of $\mathcal{U}$ by the ideal generated by the relations $K^{N+1} = 1$, $X_{\pm}^{N+1} = 0$ (here $K$ and $X_{\pm}$ denote the usual generators of $\mathcal{U}$). This ideal $\mathcal{H}$ is a Hopf ideal and the quotient is a finite dimensional Hopf algebra of dimension $(N + 1)^3$. As an algebra, it is isomorphic to $M(N + 1) \oplus M(1 \mid N)_0 \oplus M(2 \mid N - 1)_0 \oplus \cdots$, where the first term is the algebra of $(N + 1) \times (N + 1)$ matrices over complex numbers and where $M(p \mid N + 1 - p)_0$ is the even part of the algebra of $(N + 1) \times (N + 1)$ matrices with elements in the Grassmann algebra with two generators $(\begin{matrix}1 \\ 25\end{matrix})^{11}$. This algebra is not semi-simple. For instance, if $q^5 = 1$, $\mathcal{H} = M(5) \oplus M(4 \mid 1)_0 \oplus M(3 \mid 2)_0$. Projective indecomposable modules over this algebra are given by the columns, so (let us continue our example with $q^5 = 1$), we get one irreducible and projective representation of dimension 5 and four inequivalent projective indecomposable representations of dimension 2 · 5 = 10. Irreducible representations are obtained by taking the quotient of each projective representation by its own radical; in this way we get finite-dimensional irreducible (but not projective) representations of dimensions 1, 2, 3, 4. We label each projective indecomposable by the corresponding irreducible, so, besides the particular 5-dimensional representation, that is both irreducible and projective, we have four projective indecomposable labelled $10_1, 10_2, 10_3, 10_4$. The notion of quantum dimension makes sense for this algebra; all the projective indecomposable representations (including the 5-dimensional irrep) have quantum dimension 0. The four irreducible representations of dimensions 1, 2, 3, 4 are of $q$-dimension $[1]_5, [2]_5, [3]_5, [4]_5$. We can tensorially multiply these representations and draw, in particular, the diagram of tensorialization (up to equivalence of representations) by the 2-dimensional irrep. Here is the diagram that we get:

If we now decide to discard the representations of $q$-dimension 0, therefore removing all the projective indecomposable, in particular disregarding as well the special irrep of dimension 5, we just obtain the $A_4$ diagram. The fusion graph of $A_4$ describes the tensor products of irreducible representations of $\mathcal{H}$ which are not

\footnote{More information about $\mathcal{H}$ can be gathered from [10], [11] and references therein.}
of zero $q$-dimension. In particular, the equation $2 \otimes 2 \simeq 1 \oplus 3$ reads, in terms of $q$-dimensions (a quantity that is multiplicative under tensor products): $\beta^2 = 1 + \beta$, and we recover the fact that the norm $\beta$ of $A_4$ is the $q$-integer $[2]_q$, i.e., the golden number. The centralizer algebra, in the tensor powers (truncated as explained above) of the fundamental representation of $H$, is given by the Jones algebra for a particular value of the index $(1/\beta^2)$. Explicitly, this commutant is isomorphic to $M(F_\ast, \mathbb{C}) \oplus M(F_{\ast+1}, \mathbb{C})$ where $F_\ast$ are Fibonacci numbers, since

$$[2]^{2p} \simeq F_{2p-2}[1] + F_{2p-1}[3], \quad [2]^{2p+1} \simeq F_{2p}[2] + F_{2p-1}[4].$$

Its dimension is no longer given by Catalan numbers (like for $SU(2)$) but by the sum of the squares of two consecutive Fibonacci numbers (so again a Fibonacci number). Moreover, they are such that $U(\omega)$ (they are "classical" in the sense that they are neither cyclic nor semi-cyclic).

Irreducible representations of $H$ are particular irreducible representations $\rho$ of $U$ (they are "classical" in the sense that they are neither cyclic nor semi-cyclic). Moreover, they are such that $\omega = \rho(K^{N+1}) = \pm 1$. When $N+1$ is odd, in order to get irreps with $\omega = -1$, one has to replace the condition $K^{N+1} = 1$ by $K^{2(N+1)} = 1$ in the definition of $H$ (we may call $\hat{H}$ this algebra, whose dimension is twice the dimension of $H$). When $N+1$ is even, the analysis is slightly different and there are two cases, depending upon the parity of $(N+1)/2$. In any case, the tensor product of irreducible nonprojective representations of the finite dimensional Hopf algebra $H$ can be expressed in terms of the Dynkin diagram $A_N$ (with $q^{N+1} = 1$ if $N + 1$ is odd, and $q^{2(N+1)} = 1$ if $N + 1$ is even). In the case $q^{12} = q^{24} = 1$ the algebra $\hat{H}$ is $M(1 | 1)_{0} \oplus M(3 | 9)_{0} \oplus \cdots \oplus M(9 | 3)_{0} \oplus M(11 | 1)_{0}$, and $\hat{H} = H \oplus M(0 | 12) \oplus M(2 | 10)_{0} \oplus \cdots \oplus M(10 | 2)_{0} \oplus M(12 | 0)$.

In any case, the conclusion is that the fusion algebra of Dynkin diagrams of $A_N$ type can be given a purely finite dimensional interpretation in terms of finite dimensional (not semi-simple) quantum groups. Such an interpretation is, at the moment, still lacking in the case of $E_6$ or $E_8$, but we believe that it should be possible. An interpretation of the algebraic constructions of the type we considered in this paper can certainly be also formulated in categorical terms (truncated tensor products, braided categories etc.), but we think that it is interesting to be able to use finite dimensional Hopf algebras to describe such situations, even if these Hopf algebras are not semi-simple. Notice that modules appearing in discussions involving conformal embeddings (for instance $LSU(2)_{10} \subset LSpin(5,1)$) are modules for affine algebras and are typically infinite-dimensional.

7. Conclusion

The main results of this paper can be summarized as follows. Take the Dynkin diagram $E_6$, consider its associated fusion algebra and its matrix realization (it is generated by $G$, the adjacency matrix of the diagram); call $E_a$ the essential matrices defined, for each vertex $a$, as the $11 \times 6$ rectangular matrix $E_a(row p) = E_a(row p - 1)G - E_a(row p - 2)$, where $E_a(row 0)$ is the row vector that labels the chosen vertex $a$. We recover the Ocneanu graph of quantum symmetries of this Dynkin diagram as the Cayley graph of multiplication by the two generators of the 12-dimensional algebra $S = E_6 \otimes A_2 E_6$. The twelve toric $11 \times 11$ matrices $W_{a \otimes b}$
associated to the points of the Ocneanu graph can be obtained as $E_a \tilde{E}_b$ where the reduced essential matrices $E_a$ are gotten from the $E_a$’s by keeping only the columns associated with the fusion subalgebra $A_3$. The toric matrix $W_0 \otimes 0$ is the modular invariant of $E_6$. The choice of the $E_6$ example exhibits rather generic features, and it is not too hard to generalize the various constructions to other cases (see the subsequent paper [12]).

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References


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12The present paper was posted to the archives in November 2000 and several articles mentioned in the coming references section were not available at that time; for the convenience of the reader, we have added these papers to our references when the final version of our paper was sent to the publisher.
NOTES ON THE QUANTUM TETRAHEDRON 79


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