AN ANALYTIC SEPARATION OF SERIES OF REPRESENTATIONS FOR SL(2; \mathbb{R})

SIMON GINDIKIN

To Yuri Manin with warmest regards

ABSTRACT. For the group SL(2; \mathbb{R}), holomorphic wave fronts of the projections on different series of representations are contained in some disjoint cones. These cones are convex for holomorphic and antiholomorphic series, which corresponds to the well-known fact that these projections can be extended holomorphically to some Stein tubes in SL(2; \mathbb{C}) [GG]. For the continuous series, the cone is not convex, and the projections are boundary values of 1-dimensional \( \overline{\partial} \)-cohomology in a non-Stein tube.


Key words and phrases. Integral geometry, horospherical transform, series of representations, \( \overline{\partial} \)-cohomology, holomorphic wave front.

A central place in the harmonic analysis on pseudo Riemannian symmetric spaces, including real semisimple Lie groups, is occupied by the existence of several series of representations. In the Plancherel formula, decomposition into representation series is the crucial step, since decomposition into irreducible representations inside the series reduces to the commutative Mellin transform for the corresponding Cartan subgroups. In my opinion, the analytic nature of the decomposition into series needs to be clarified. Of course, from the algebraic point of view, different series correspond to different classes of Cartan subgroups. From the point of view of functional analysis, different series correspond to different types of spectra, but the difference of discrete series in spectra is not so instructive.

I believe that it is natural to separate series by means of complex analysis. In a sense, the decomposition of \( L^2 \) over a symmetric space \( X \) into components corresponding to different series is a multidimensional version of the decomposition of \( L^2 \) over the line into Hardy spaces of holomorphic functions in the upper and lower half-planes. Specifically, the holomorphic wave fronts of functions from different series lie in some disjoint cones. Such cones are convex only for holomorphic discrete series. In these cases, the functions from the series holomorphically extend to some tubes in the complexified symmetric space \( X_C \) with wedge \( X \). This phenomenon was discovered in our joint paper with Gelfand [GG], and it was investigated in
full generality by Olshansky, Stanton, Hilgert, Ólafsson, et al. (see [HO]). For other series, these cones are nonconvex, and the corresponding functions can be characterized as boundary values of higher $\partial$-cohomology in non-Stein tubes with wedge $X$. The flat version of such decompositions was considered in [G1], [G2]. We emphasize that relating discrete series to complex analysis is quite customary, but the approach under consideration assumes strong connections with complex analysis of all series, including maximally continuous ones. Moreover, the method is hopefully strong enough to separate multiplicities when the decomposition in the Plancherel formula is not multiplicity-free. The separation of multiplicities by intrinsic tools of representations theory looks very problematic. We cannot yet describe a general picture. Since the results are new and nontrivial even for $\text{SL}(2; \mathbb{R})$, we decided that it makes sense to investigate this simplest case. The basic tool in this paper is the integral geometry on $\text{SL}(2; \mathbb{R})$ developed in [G3]. Let us recall that the main idea of this construction is the replacement of real horospheres in the usual horospherical transform by complex horospheres without real points and consideration of Cauchy type integrals along real horospheres.

The universal separation of series in the language of complex analysis is a manifestation of the fundamental multidimensional phenomenon of the extension of real functions in the complex domain not only as holomorphic functions, but also as higher cohomology. Of course, the concept of hyperfunctions is the most important example, but it is not the only one.

1. **Real and complex geometries associated with $\text{SL}(2; \mathbb{R})$**

All details concerning the considerations of this and the following sections can be found in [G3]. We consider the group $G = \text{SL}(2; \mathbb{R})$ as the hyperboloid of signature $(2, 2)$ in $\mathbb{R}^4$ determined by

$$\Box(x) = x_1x_4 - x_2x_3 = 1, \quad x = \begin{pmatrix} x_1 \\ x_3 \\ x_2 \\ x_4 \end{pmatrix} \in \mathbb{R}^4.$$  \hfill (1)

Accordingly, we consider the complex group $G_{\mathbb{C}} = \text{SL}(2; \mathbb{C})$ as the complex hyperboloid $\Box(z) = 1, \quad z = x + iy \in \mathbb{C}$.

Consider the bilinear form $x \cdot y$ corresponding to the quadratic form $\Box x$. On $G_{\mathbb{C}}$, we have

$$\Box x - \Box y = 1, \quad x \cdot y = 0.$$  \hfill (2)

The group $G \times G$ acts on $G$ by left and right translations, and this group is locally isomorphic to $\text{SO}(2, 2)$. In our considerations, three domains in $G_{\mathbb{C}} \setminus G$ play a central role. These are the two connected components $G_{\pm}$ of the set of those $z = x + iy \in G_{\mathbb{C}}$ where $\Box y > 0$, and the domain $G_0$ where $\Box y < 0$, $\Box x > 0$. All these domains have $G$ as their wedges (Shilov boundaries), and the domains $G_{\pm}$ are Stein. They are all invariant relative to $G \times G$ (but, of course, not homogeneous). Using our bilinear form, we identify dual spaces with the spaces $\mathbb{R}^4_{\xi}$ and $\mathbb{C}^4_{\zeta}$, where $\zeta = \xi + i\eta$, and consider the cones

$$\Xi = \{ \Box \xi = 0 \}, \quad \Xi_{\mathbb{C}} = \{ \Box \zeta = 0 \}.$$  \hfill (3)
We use the term horospheres for the sections $L(\zeta, p)$ of the hyperboloid $G_\mathbb{C}$ by the isotropic hyperplanes 
$$\zeta \cdot z = p, \quad \zeta \in \Xi_\mathbb{C}.$$ 
We are interested only in horospheres having no real points (not intersecting $G$). Among them are the horospheres with real $\zeta$ and complex $p$, that is, 
$$\xi \cdot z = p, \quad p \in \mathbb{C} \setminus \mathbb{R}. \quad (4)$$
Let $\hat{\Xi}_0$ be the set of all such horospheres. Then the set 
$$\square \xi = \square \eta > p^2 \quad (5)$$
has two connected components $\hat{\Xi}_\pm$. We choose the signs in the subscripts in such a way that 
$$L(\zeta, p) \cap G_{\mp} = \emptyset \quad (6)$$
if $L(\zeta, p) \in \hat{\Xi}_\pm$. Moreover, the union of horospheres from $\hat{\Xi}_\pm$ coincides with the complement $G^\pm$ to the closure of $G_{\mp}$. The pairs $(\zeta, p)$ give a homogeneous parametrization of the horospheres. If $p \neq 0$, then we can always put $p = 1$ and parameterize the corresponding horospheres by the points of $\Xi_\mathbb{C}$. By $\Xi_\pm$ and $\Xi_0$, we denote the domains in $\Xi_\mathbb{C}$ corresponding to the families of horospheres $\hat{\Xi}_\pm$ and $\hat{\Xi}_0$, respectively. Outside these classes, there is a set of horospheres without real points, which depend on fewer parameters and are not involved in the results of this paper [G3].

The horospheres corresponding to the boundary points of $\hat{\Xi}_\pm$, which are 
$$\zeta \cdot z = 1, \quad z = \xi + i\eta, \quad \square \xi = \square \eta = 1, \quad (7)$$
have the unique intersection point $x = \xi$ with $G$. The boundary of $\hat{\Xi}_0$ contains horospheres with real $(\zeta = \xi, p)$ which intersect $G$ by real horospheres.

Along with horospheres, we consider horocycles, that is, isotropic lines on the hyperboloid $G_\mathbb{C}$. They have the form 
$$z = a + t\zeta, \quad \square \zeta = 0, \quad a \cdot \zeta = 0. \quad (8)$$
Let us denote such a horocycle as $l(\zeta, a)$; $t$ is a parameter along the horocycle, and $a$ is defined up to a multiple of $\zeta$. The union of horocycles with fixed $\zeta$ coincides with the horosphere $L(\zeta, 0)$. The horospheres $L(\zeta, p)$ with $p \neq 0$ are fibered into the horocycles $l(\lambda, a)$, where $\lambda \cdot \zeta = 0, \lambda \neq c\zeta$, and $\zeta \cdot a = p$. For a dual parameterization of horocycles, we use their special representation as an intersection of horospheres: 
$$l(\lambda, \nu; p) = L(\lambda, 0) \cap L(\nu, p), \quad \lambda \cdot \nu = 0, \quad p \neq 0. \quad (9)$$

This is the representation of the horocycle $l(\lambda, a)$, $\nu \cdot a = p$. In (9), the vector $\nu \in \Xi$ modulo $\lambda$ can take two values $\nu', \nu''$ up to multiplicative factors. Thus, all horospheres containing the horocycle $l(\nu; p)$ have either the form $L(c_1 \lambda + c_2 \nu')$, or the form $L(c_1 \lambda + c_2 \nu'')$.

The tube $G_0$ is a union of some horocycles with real $\zeta = \xi$, and its closure $\bar{G}_0$ coincides with the union of all such horocycles. Accordingly, $G_0$ is the union of horospheres $L(\xi, 0)$ with real $\xi \in \Xi$. The union of all horocycles $l(\lambda, a) \subset G_0$
with \( \lambda \in \Xi \) has two connected components \( L^\pm(\lambda) \), half-horospheres. Using the parameters in (9), we can define them as

\[
L^\pm(\lambda) = \{ z \in L(\lambda, 0); \pm \Im(\nu \cdot z) > 0 \}.
\]

Only the choice of sign in \( L^\pm \) depends on \( \nu \). The tube \( G_0 \) is the disjoint union of semi-horospheres \( L^\pm(\lambda) \), \( \lambda \in \Xi \).

2. Integral geometry on \( G = \text{SL}(2; \mathbb{R}) \)

The integral geometry on the hyperboloid \( G \) is based on the following decomposition of the \( \delta \)-function on \( G \) into planar waves. For a function \( f \) on \( G \), let us consider the differential form

\[
\kappa f[x] = \frac{f(x)}{[\zeta \cdot (u - x - \delta)]^2} \mu(u, du) \wedge [u + x, \zeta, d\zeta, d\zeta], \quad \Box x = \Box u = 1
\]

with a fixed \( x \in G \), where

\[
\mu(u, du) = d(\Box u) \wedge du
\]

is an invariant 3-form on \( G \) and \( [a_1, a_2, a_3, a_4] \) denotes the determinant of the matrix with columns \( a_j \) where the matrix coefficient-forms are multiplied exteriorly.

The difference between this situation and the classical case of decomposition into planar waves in \( \mathbb{R}^n \) is that the former is overdetermined, because it involves a 5-form on a 6-dimensional manifold. The crucial point is that the form \( \kappa f[x] \) is closed and, if \( f \in C^\infty_0(G) \), then the integration of the restrictions of this form to appropriate cycles over the support of \( f \) gives various reconstructions of \( f(x) \) (decompositions of the \( \delta \)-function on \( G \) into planar waves). By cycles here we mean relative cycles, i.e., such that the projections of their boundaries on \( G \) do not intersect the support of \( f \). Here we consider only functions with compact supports, and the formulas can be extended to various functional spaces by using the invariance. In our constructions it is important that we can take a complex \( \zeta \) in (9) although the hyperboloid \( G \) under consideration is real. Such a possibility is crucial for the construction of a horospherical transform on \( \text{SL}(2; \mathbb{R}) \). Following [G3], in the plane \( x \cdot \lambda = 0 \), we take any cycle \( \Lambda(x) \) intersecting each ray from 0 in one point (e.g., a sphere), and in \( \mathbb{C}^4_\zeta \) we take the cycle

\[
\gamma(x) = \{ \zeta = \sqrt{\Box \lambda} x + \lambda, \ \lambda \in \Lambda(x) \},
\]

where the root is either positive (if it is real) or of the form \( ia \) with \( a < 0 \) (if it is imaginary). We have \( \Box \zeta = 0 \) on these cycles. Let us integrate \( \kappa f[x] \) along the cycle

\[
\Gamma^{(1)} = (\text{supp}(f) \times \gamma(x)).
\]

The form \( \kappa f[x] \) is regular on this cycle, because, as can be checked directly, the horospheres \( L(\zeta, \zeta \cdot x + i\varepsilon) \), where \( \varepsilon > 0 \) and \( x \) is fixed, have no real points: they are in \( \tilde{\Xi}_0 \) if \( \Box \lambda > 0 \) and in \( \tilde{\Xi}_\pm \) if \( \Box \lambda < 0 \). As the result,

\[
\frac{1}{4(\pi i)^3} \int_{\Gamma^{(1)}} \kappa f[x] = f(x).
\]
This inversion formula was proved in [G3] by using a homotopy between the cycle \( \Gamma^{(1)} \) and the real cycle corresponding to the inversion formula for the projective Radon transform in \( \mathbb{R}P^3 \).

This formula gives the inversion of the horospherical transform on \( G \). If the horosphere \( L(\zeta, p) \) has no real points, we define the (complex) horospherical transform

\[
\hat{f}(\zeta, p) = \int_G \frac{f(u)}{\zeta \cdot u - p} \mu(du), \quad \text{Im} \, p \neq 0.
\]

Integration with respect to \( u \) in the inversion formula (12) transforms this formula into

\[
f(x) = \frac{1}{4(\pi i)^3} \int_{\Lambda(x)} \mathcal{L} \hat{f}(\sqrt{\Box} \lambda x + \lambda, \sqrt{\Box} \lambda + i0) \det \lambda, \, d\lambda, \, d\lambda',
\]

where

\[
\mathcal{L} = \frac{1}{p} \frac{\partial}{\partial p} + \frac{\partial^2}{\partial p^2}.
\]

In (14), we integrate over the set of horospheres passing through the point \( x \) and take the boundary values of the horospherical transform corresponding to the shifts of these horospheres to horospheres without real points. The cycles \( \gamma(x) \) were picked so that such shifts exist. In the formula \([\lambda, d\lambda, d\lambda']\) we assume \( \lambda \) to be represented by some coordinates in the plane \( x \cdot \xi = 0 \) (the integrand does not depend on the choice of coordinates) and take the determinant of order 3.

Thus, we have the horospherical transform (13) \( f \mapsto \hat{f} \) and its inverse (14) \( f = S\hat{f} \). It is natural to decompose the horospherical transform \( \hat{f} \) into the parts \( \hat{f}_\pm, \hat{f}_0 \), corresponding to \( \Xi_\pm, \Xi_0 \). It is shown in [G3] that \( \hat{f}_\pm, \hat{f}_0 \) decompose into representations of holomorphic, antiholomorphic, and continuous series, respectively. Let us decompose the cycle \( \gamma(x) \) into parts \( \gamma_\pm(x), \gamma_0(x) \) on the boundaries of \( \Xi_\pm, \Xi_0 \). They correspond to the division of the cycle \( \Lambda(x) \) into parts \( \Lambda_\pm(x), \Lambda_0(x) \); on \( \Lambda_\pm(x) \), we have \( \Box \lambda < 0 \), and on \( \Lambda_0(x) \), we have \( \Box \lambda > 0 \). Integrating in (14) over the parts \( \Lambda_\pm(x), \Lambda_0(x) \), we obtain the projections \( f_\pm, f_0 \) of \( f \) on the corresponding holomorphic, antiholomorphic, and continuous series; they are expressed through \( \hat{f}_\pm \) and \( \hat{f}_0 \), respectively. The projection operators \( S_\pm, S_0 \) are the compositions of the horospherical transforms \( f \mapsto f_\pm, f_0 \) and the inverse horospherical transform (14) where the integration is only along the corresponding part \( \Lambda_\pm(x), \Lambda_0(x) \). Our aim is to investigate the analytic nature of \( f_\pm, f_0 \) by using the integral representations of these operators following from (13), (14).

3. Holomorphic wave fronts of projections on series

The description of holomorphic wave fronts of projections on series is a direct corollary of the inversion formula (14). Recall that, for a distribution \( F \) on a manifold \( G \), a covector \( a \in T^*_a(G) \setminus 0 \) is not contained in the holomorphic wave front of \( F \) at the point \( x \), if \( F \) can be locally (in a neighborhood of \( x \)) represented as a sum of boundary values of holomorphic functions in some tubes inside of the half-space \( \{ \Im(a \cdot z) < 0 \} \). For our purposes is sufficient to recall that, as directly follows from the definition, if \( F \) locally holomorphically extends in the half-space \( \{ \Im(a \cdot z) > 0 \} \), then the holomorphic front cannot be larger than the ray generated
by $a$. A more general fact is that if $F$ locally holomorphicaly extends in a tube with a convex cone $V$, then its holomorphic wave front is contained in the dual cone $V^*$. To calculate the wave fronts of $f_\pm, f_0$, let us investigate the decomposition (14) in a neighborhood of a point $x \in G$. In this neighborhood we have the superposition of horospherical waves $\mathcal{L} \hat{f}(\zeta, \zeta \cdot x)$, $\zeta \in \gamma(x)$; the cycles depend on $x$, but locally, this dependence is not essential, and we can consider

$$\int_{\gamma(x)} \mathcal{L} \hat{f}(\zeta, \zeta \cdot w)[\lambda, d\lambda, d\lambda],$$

(15)

where $w$ is in a neighborhood of $x$.

If $\zeta \in \Xi_\pm$, then the function $\hat{f}(\zeta, \zeta \cdot z)$ is holomorphic in the domain on $G_C$ defined by conditions

$$G^\pm(\zeta) = \{z \in G_C: |\zeta \cdot z|^2 < \square \xi, \xi = \Re \zeta\}.$$  

(16)

We call such domains horospherical cylinders, they are unions of horospheres without real points from $\Xi_\pm$ with given $\zeta$. We have

$$G^\pm(\zeta) = G^\pm(\lambda \zeta), \quad |\lambda| = 1.$$  

(17)

Of course, this definition makes sense only for $\zeta$ with $\square \xi > 0$. We can suppose in the definition of horospherical cylinders that $\square \xi = 1$ and, hence, $G_{\pm}(\zeta) = \{|\zeta \cdot w| < 1\}$; for such $\zeta$, only one of the cylinders $G^\pm(\zeta)$ is nonempty. Their boundaries intersect $G$ on the circles $\{x = a_\xi + b_\eta, a^2 + b^2 = 1\}$, $\zeta = \xi + \eta$. The horospherical cylinders are Stein, and $G^\pm(\zeta)$ give coverings of $G^\pm$, respectively. For real $\zeta = \xi \in \Xi$, the functions $\hat{f}(\xi, \xi \cdot z)$ are holomorphic in the half-spaces $\{z \in G_C: \Im(\xi \cdot x) > 0\}$. Of course, all these holomorphy properties are preserved under the action of the differential operator $\mathcal{L}$.

Let us localize the picture in a neighborhood of $x \in G$ on the tangent plane $x \cdot u = 1$. We take $u = x + v$, where $x \cdot v = 0$, and use $v$ as coordinates. We also identify $T^*_x(G)$ with the plane $x \cdot \lambda = 0$. Recall that precisely these $\lambda$ parameterize the points of the cycle $\gamma(x)$. The holomorphy domains described above locally, on the tangent space, correspond to the half-spaces $\Im(\lambda \cdot v) > 0$ for all $\zeta \in \gamma(x)$. This means that the holomorphic wave front of $\mathcal{L} \hat{f}(\zeta, \zeta \cdot w)$, $\zeta = \sqrt{\square} \lambda x + \lambda x$, reduces to $c\lambda, c > 0$, which immediately gives estimates for the wave fronts of the results of integration along $\gamma_\pm(x)$, $\gamma_0(x)$. Namely, in the cotangent plane $x \cdot \lambda = 0$, consider three cones: the two connected components $V_\pm(x)$ of the set $\Box \lambda < 0$ and the set $V_0(x)$ where $\Box \lambda > 0$. We choose the subscripts $\pm$ such that

$$L(\sqrt{\square} \lambda x + \lambda, \sqrt{\square} \lambda + i0) \in \Xi_\pm$$

for $\lambda \in V_\pm(x)$.

We obtain the following assertion.

**Proposition 1.** The holomorphic wave fronts of the projections on the series $f_\pm, f_0$ of $f \in C_0^\infty$ are contained in the cones $V_\pm(x), V_0(x)$, respectively.

This proposition demonstrates the fundamental difference of the projections on series from the point of view of local complex analysis; we will see that this difference extends to a global complex analytic picture.
4. Cohomological extensions of projections on discrete series

Since the cones $V^\pm(x)$ are convex, it is natural to expect that $f^\pm$ admit holomorphic extensions to some Stein tubes. Moreover, it was proved in [GG] that they indeed admit holomorphic extensions to $G^\pm$, but this result was obtained as a consequence of representations of $S^\pm$ through characters and Plancherel coefficients for discrete series and the holomorphic extensions of explicit formulas for these characters. Here we obtain the holomorphic extensions directly from the inversion formulas.

In the flat case, holomorphicity in tube domains is connected with decomposition into planar waves [G2]. Its horospherical analogue (12) does not give an extension directly, but it admits another complex analytical interpretation, as cohomological extension of higher dimension. Let us consider the integrand in (15)

$$
\sigma[\hat{f}](w | \zeta, d\zeta) = cL\hat{f}(\zeta, \zeta \cdot w)[w, \zeta, d\zeta, d\zeta],
$$

as a (closed) 2-form on $\zeta \in \Xi^\pm$ holomorphically depending on the parameters $w \in G^\pm(\zeta)$. Following the concept of Čech cohomology for the smoothly parameterized Stein covering $G^\pm(\zeta)$ [G1], [G2], we relate with such closed forms $\phi(w | \zeta, d\zeta)$ with parameters to some 2-dimensional analytic cohomology class on $G^\pm$. It follows from [BEG] that this cohomology is isomorphic to the Dolbeault cohomology. A morphism in the Dolbeault cohomology can be constructed very explicitly. Let $Z(w)$ be a smooth function on $G^\pm$ such that $w \in G^\pm(Z(w))$. Then the $(0, 2)$-part of the form $\phi(w | \zeta, d\zeta)|_{\zeta=Z(w)}$ is $\bar{\partial}$-closed and determines the Dolbeault cohomology class. Apparently, such functions exist, since the sets $F(w) = \{\zeta; w \in G^\pm(\zeta)\}$, $w \in G^\pm$, are contractible.

The domains $G^\pm$ are tubes with edge $G$. It is possible to define the boundary values of the 2-dimensional cohomology $H^2(G^\pm, \mathcal{O})$ on $G$. In the language of forms with parameters, we take the boundary values of $\phi$ on the boundaries of the horospherical cylinders $G^\pm(\zeta)$ (which all include $G$) and define

$$
b^\pm \phi(x) = \int_{\gamma^\pm(x)} \phi(x | \zeta, d\zeta).
$$

For a general $\phi$, we can define the boundary values as hyperfunctions on $G$, but in the case under consideration ($\phi = \sigma[\hat{f}]$), we can use the classical boundary values.

The projections $f^\pm$ then coincide with the boundary values of $\sigma[\hat{f}]$:

$$
f^\pm = b^\pm \sigma[\hat{f}].
$$

Thus, we have obtained the following result:

**Proposition 2.** Formula (19) gives the canonical extensions of $f^\pm$ as elements of $H^{(2)}(G^\pm, \mathcal{O})$. 
5. Holomorphic extensions of projections on discrete series

It is natural to expect that \( H^{(2)}(G^\pm) \sim H(G^\pm) \). To obtain the holomorphic extensions of \( f_\pm \) to \( G^\pm \), we use yet another trick. Namely, we replace the cycle \( \Gamma^{(1)} \) in (12) by the other cycle
\[
\Gamma^{(2)} = \bigcup_{u \in \text{supp } f} \gamma(u).
\]
If the choice of \( \zeta \) in \( \Gamma^{(1)} \) was determined by \( x \), then in \( G^{(2)} \), it is determined by \( u \). Note that two similar types of cycles appear in the Cauchy–Fantappie formula \( [GKh] \). First of all, the form \( \kappa f[x] \) is regular on this cycle, since the horospheres \( L(\zeta, \zeta \cdot u + i0), \) where \( u \in G \), and \( \zeta \in \gamma(u) \), have no real points. For the same reason, we can build a homotopy of \( \Gamma^{(1)} \) to \( \Gamma^{(2)} \) through cycles on which \( \kappa f \) is regular (by moving \( x \) to \( u \) along some paths and using the cycles \( \gamma \)). In such a way, we can reconstruct \( f \) using \( \Gamma^{(2)} \) instead of \( \Gamma^{(1)} \). Let us decompose the cycle \( \Gamma^{(2)} \) into the parts \( \Gamma^{(2)}_0, \Gamma^{(2)}_\pm \) corresponding to the parts \( \gamma_0(u), \gamma_{\mp}(u) \) (we change the signs in the subscripts). Consider the integrals along these parts. We see that the sets \( \Gamma^{(1)}_0 \) and \( \Gamma^{(2)}_0 \) coincide, and hence they both give \( f_0 \). Correspondingly, integration along \( \Gamma^{(2)}_\pm \) give the projections \( f_\pm \) on the discrete series:
\[
f_\pm(x) = c \int_{\Gamma^{(2)}_\pm} \kappa[f], \quad c = \frac{1}{4(\pi i)^3}. \quad (20)
\]
We can remove \( -i0 \) and note that for each \( u \in G \), the integrand is holomorphic for \( z \in G^\pm \), since the horospheres \( L(\zeta, \zeta \cdot u) \), where \( u \in G \) and \( \zeta \in \gamma_{\mp}(u) \), lie in the closure of \( G^{\mp} \) and do not intersect \( G^\pm \) (they intersect \( G \) only in the point \( u \)). It directly gives the holomorphic extensions of \( f_\pm \) to \( G^\pm \).

Integrating \( \kappa[f] \) over \( \zeta \in \gamma_{\mp}(u) \) for a fixed \( u \), we obtain the Szegő reproducing kernel:
\[
S_{\pm}(z, u) = c \int_{\Lambda_{\pm}(u)} \frac{[z + u, u + i\lambda, d\lambda, d\lambda]}{(u + i\lambda)(u - z)}^{3}, \quad u \in G, \ z \in G_{\pm}, \quad (21)
\]
where \( \Lambda_{\pm}(u) \) are the sheets of the two-sheeted hyperboloid \( \{\lambda: \ u \cdot \lambda = 0, \ \Box \lambda = -1\} \). The kernels \( S_{\pm} \) have convolutor form on the group \( G \).

This Szegő kernel was defined and evaluated in [GC] by using explicit formulas for characters and the Plancherel formula on \( \text{SL}(2; \mathbb{R}) \); the final expression is
\[
S_{\pm}(z, u) = c\left[\left(\text{tr}(zx^{-1}) - 2\right)\left(\text{tr}(zx^{-1}) + 2\right)\right]^{-1/2}.
\]

The Szegő kernel was investigated for other symmetric spaces of Hermitian type, but explicit formulas are known only in a few cases. I believe that it is interesting to obtain explicit formulas for the kernels, but it would be more important to find their integral representations similar to (21). Such representations are similar to classical Bochner’s representation of the Szegő kernel for tube domains, but the difference between them is quite substantial. The following assertion is valid.

**Proposition 3.** The projections \( f_\pm \) on the discrete series admit holomorphic extensions to the tubes \( G_{\pm} \), where they are represented by the Szegő-type integral...
AN ANALYTIC SEPARATION OF SERIES OF REPRESENTATIONS FOR $\text{SL}(2; \mathbb{R})$

Formulas
\[ f_{\pm}(z) = \int_G S_{\pm}(z, u)f(u)\mu(du), \quad z \in G_{\pm} \]  
with kernels (21).

Thus, we have constructed two integral representations of the projections on discrete series; one of them relates the projections to the horospherical transform, and the other implements holomorphic extension.

6. REAL HOROCYCLIC TRANSFORM AND COHOMOLOGICAL EXTENSION OF $f_0$

Since the holomorphic wave front of the projection $f_0$ on the continuous series is contained in the nonconvex cones $V_0(u)$, we cannot hope to extend it holomorphically. Instead, we extend it as a 1-dimensional cohomology in the nonconvex tube $G_0$.

First, the inversion formula (12) (as well as its analogue for the cycle $\Gamma^{(2)}$) gives the following representation of the projection $f_0$:
\[ f_0(x) = \frac{1}{4(\pi i)^3} \int_{G \times S \Xi} \frac{f(u)}{(\xi \cdot (u - x) - i\theta)^3} \mu(u, du) \wedge [x + u, \xi, d\xi, d\xi], \]  
(23)
where $S \Xi$ is the factorization of the cone $\Xi$ (3) relative to the dilatation $\xi \mapsto r\xi$, $r > 0$; we integrate along any cycle intersecting each generating ray of this cone in one point. Integrating this formula with respect to $u$, we obtain
\[ f_0(x) = \frac{1}{2(\pi i)^3} \int_{G \times S \Xi} \mathcal{L} \hat{f}(\xi, \xi \cdot x + i\theta)[x, \xi, d\xi, d\xi]. \]  
(24)
We have used the observation that, since $\xi \cdot \xi = 0$,
\[ [y, \xi, d\xi, d\xi] \]
\[ y \cdot \xi \] is independent of $y$. To see this, is sufficient to replace a row in the determinant by the combination of all rows with coefficients $\xi_i$.

Formula (23) gives also the Szegő-type kernel reconstructing $f_0$:
\[ S_0(x, u) = \frac{1}{4(\pi i)^3} \int_{S \Xi} \frac{[x + u, \xi, d\xi, d\xi]}{(\xi \cdot (u - x) - i\theta)^3}. \]  
Of course,
\[ S_0(x, e) = \delta(x) - S_+(x, e) - S_-(x, e). \]

Note that the integrand in (23) is not even. As usual, we can leave only its even part by replacing $\{\xi \cdot (u - x)\}^{-3}$ with $\pi \delta''(\xi \cdot (u - x))$. Accordingly, we can obtain a local inversion formula from (24) by replacing the complex horospherical transform $\hat{f}$ (13) with the real horospherical transform
\[ Rf(\xi, p) = \int_G f(u)\delta(\xi \cdot u - p)\mu(du). \]

We cannot connect (24) with an extension of $f_0$ as a 2-dimensional cohomology. One of reasons for this is that the set of horospheres from $\tilde{\Xi}_0$ passing through a given point is not contractible. This problem can be corrected by replacing the
horospherical transform by the horocyclic one. Technically, in (23), we make the substitution

$$\xi = \lambda + t\nu, \quad t > 0, \quad \lambda, \nu \in \Xi, \quad x \cdot \lambda = 0, \quad \lambda \cdot \nu = 0, \quad \nu \cdot x \neq 0. \quad (25)$$

The last condition in (25) can be replaced by the condition that \( \lambda \) and \( \mu \) are not proportional, because two elements of \( \Xi \) orthogonal to \( x \in G \) and mutually orthogonal are proportional. Let \( \Delta(x) \) be the set of such pairs \( \xi, \eta \) factorized by the equivalency relation \( (\xi, \eta) \sim (r\xi, s\eta) \), where \( r > 0, s > 0 \). Then, substituting \( \xi = \lambda + t\nu \) from (25) in (23) and integrating over \( 0 < t < \infty \), we obtain

$$f_0(x) = \frac{1}{16\pi} \int_{G \times \Delta(x)} \frac{f(u)\delta(\lambda \cdot u)}{(\nu \cdot (u - x) - i0)^2} \mu(du) \wedge [x, \lambda, \nu, d\nu]. \quad (26)$$

Obviously, (26) is independent of the specific choice of the cycle \( \Delta(x) \) satisfying (25), and \( \nu \) is a function of \( \lambda \).

To derive (26), we have used the following facts. For a fixed \( x \in G \) such that \( \xi \cdot x \neq 0 \), we determine \( \lambda \) from the conditions \( \lambda \cdot x = 0 \) and \( \lambda \cdot \xi = 0 \). Such \( \lambda \) lie on two rays. Then, we determine \( \nu \in \Xi \) from the condition \( \nu \cdot \xi = 0 \). Such \( \nu \) modulo \( \xi \) lie on two lines. Then, we apply the formula

$$\int_0^\infty \frac{t \, dt}{(a + bt - i0)^2} = -1/2(a - i0)^{-1}(b - i0)^{-2}. \quad (27)$$

Here \( a = \lambda \cdot u \) and \( b = \nu \cdot u \). Finally, we replace \( (\lambda \cdot u - i0)^{-1} \) by its even part \( \pi \delta(\lambda \cdot u) \).

Then, we note that, on \( \Delta(x) \), \( d\lambda \) is proportional to \( \nu \) and \( [x + u, \lambda, \nu, d\lambda] = 0 \).

Let us replace the integrand in (26) by its even part, i.e., take \( \{\nu \cdot (u - x)\}^2 \) instead of \( \{\nu \cdot (u - x) - i0\}^2 \), so that we could use the distribution \( \pi^{-2} \) (we take the Cauchy principal value). Formula (26) allows us to reconstruct \( f_0 \) by means of the real horocyclic transform

$$f(\lambda, \nu, p) = \int_G f(x)\delta(\lambda \cdot u)\delta(\nu \cdot u - p)\mu(du), \quad (27)$$

$$\lambda, \quad \nu \in \Xi, \quad \lambda \cdot \nu = 0, \quad \lambda \neq \nu, \quad p \in \mathbb{R}. \quad (27)$$

We integrate \( f \) in (27) along the real horocycle \( x = a + t\lambda, \, t \in \mathbb{R}, \, \nu \cdot a = p \). Then (26) means that

$$f_0(x) = \frac{1}{\pi} \int_{\Delta(x)} \frac{f(\lambda, \nu, p)}{(p - \nu \cdot x)^2} [x, \lambda, \nu, d\nu]. \quad (28)$$

In fact, integrating in (28) is over the set \( H(x) \) of real horocycles passing through \( x \in G \). They are parameterized by the projectivization of the set of \( \xi \in \Xi \) orthogonal to \( x \). The cycle \( \Delta(x) \) covers \( H(x) \) with multiplicity 16.

To obtain a cohomological extension, we extend the integrand in (26) for any fixed \( \lambda \in \xi \) (and \( \nu \)) on \( x \) to the half-horospheres \( L^\pm(\lambda) \). Thus, we have a 1-form on \( \Xi \) with coefficients in \( L^\pm \):

$$\tau[f](z \mid \lambda, \, d\lambda) = \frac{1}{16\pi} f(u)\delta(\lambda \cdot u)(\nu \cdot (u - z))^{-2}, \quad \lambda \in \Xi, \quad z \in L^\pm(\lambda). \quad (29)$$
This form determines a 1-dimensional cohomology class on $G_0$, and (26) realizes $f_0$ as its boundary values. Since $L^\pm(\lambda)$ have codimension 1, there are some technical differences with [G1], [G2]; we will consider them elsewhere.

**Proposition 4.** The projection $f_0$ on the continuous series admits the representation (23), which can be transformed into the reconstructions of $f_0$ through either the horospherical transform $\hat{f}_0$ (24) or the real horocyclic transform $\hat{\gamma}_0$ (28). The last one admits an extension of $f_0$ in $G_0$ as a 1-dimensional analytic cohomology.

**Acknowledgments.** The author thanks the Max-Planck-Institut für Mathematik in Bonn for their hospitality.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, HILL CENTER, RUTGERS UNIVERSITY, 110 FRELINGHUYSEN ROAD, PISCATAWAY, NJ 08854-8019

E-mail address: gindikin@math.rutgers.edu