INFINITE FAMILIES OF EXACT PERIODIC SOLUTIONS TO THE NAVIER–STOKES EQUATIONS

OLEG I. BOGOYAVLENSKIJ

Abstract. A complete classification of all periodic solutions to the 3-dimensional Navier–Stokes equations with pairwise non-interacting Fourier modes is obtained. The corresponding sets of the wave vectors \( k \in \mathbb{Z}^3 \) necessarily belong either to the straight lines, the planes, the circumferences or the spheres. The constructed exact periodic solutions are smooth and exist for all values of the time variable \( t > 0 \).


Key words and phrases. Navier–Stokes equations, periodic solutions, Fourier modes, Beltrami equation.

1. Infinite-dimensional linear spaces of exact solutions

I. The Navier–Stokes equations (NSE) for incompressible fluid have the form

\[
\frac{\partial V}{\partial t} + (V \cdot \nabla)V = -\frac{1}{\rho} \nabla p + \nu \Delta V, \quad \text{div} V = 0.
\]

Here \( V(t, x) \) is the fluid velocity vector field and \( p(t, x) \) is the pressure, \( x = (x_1, x_2, x_3) \). We assume that the density \( \rho \) and the kinematic viscosity \( \nu \) are constant. The theory of the NSE can be found in the Leray paper \([L]\) and in the monographs \([CF]\), \([T]\). The classical exact solutions to the NSE are presented in \([Bat]\). The known Landau \([Lan]\) and Squire \([S]\) exact solution describes a steady and axially symmetric laminar jet; the solution has singularity at \( r = 0 \), infinite total kinetic energy and has the exact form \( V(x, y, z) = r^{-1}V(\theta) \) where \( r, \theta, \phi \) are the spherical coordinates.

In the paper \([B1]\), we introduced exact solutions to the Navier–Stokes equations and the viscous magnetohydrodynamics equations (MHD) in the form of the time-dependent Beltrami flows. For these solutions, the fluid velocity and the pressure have the form \([B1]\)

\[
V(t, x) = e^{-\alpha^2 \nu t} \int S^2 \left( (\sin(\alpha k \cdot x)T(k) + \cos(\alpha k \cdot x)k \times T(k) \right) d\sigma,
\]

\[
p(t, x) = C_1 - \rho V^2(t, x)/2.
\]

Received June 27, 2002.

©2003 Independent University of Moscow

263
Here the integral is taken with respect to an arbitrary measure $d\sigma$ on the 2-dimensional unit sphere $S^2$: $k \cdot k = 1$ and $T(k)$ is an arbitrary smooth vector field tangent to the unit sphere, $T(k) \cdot k = 0$ and $\alpha \neq 0$ is an arbitrary parameter. Formula (1.2) gives also the general stationary solutions to the Euler equations, (1.1) for $\nu = 0$, which satisfy also the Beltrami equation $\text{curl} \ V = \alpha V$. The corresponding to (1.2) magnetic field $B(t, x)$ is

$$B(t, x) = C_2 \exp(-\alpha^2(\eta - \nu)t) V(t, x),$$

where $\eta$ is a diffusivity coefficient (or magnetic viscosity) and $C_2$ is an arbitrary constant. For $C_2 = 0$, we get exact solutions to the NSE only.

For the special vector fields $T(k)$ and the Euclidean measure $d\sigma$, solutions (1.2)–(1.3) have the soliton-like properties and are called viscons $B_2$, $B_3$.

If the measure $d\sigma$ has the form $d\sigma = \delta(k_1) + \cdots + \delta(k_P)$, the formulae (1.2) give the exact solutions $V(t, x) = e^{-\alpha^2\nu t} \sum_{i=1}^{P} (\sin(\alpha k_i \cdot x) T(k_i) + \cos(\alpha k_i \cdot x) k_i \times T(k_i))$.

Since $V(t, x)$ is real, the relations

$$V(-k(t)) = V_k(t)$$

hold. The wave vectors $k \in \mathbb{Z}^3$ in (1.6), (1.7) are arbitrary vectors with integral coordinates $k_1, k_2, k_3 \in \mathbb{Z}$.

As is known, the interaction between the Fourier modes is absent for any linear equation with constant coefficients and is rather complex for the non-linear ones. For example, the Fourier transform of the Korteweg–de Vries equation

$$\frac{\partial u}{\partial t} = 6u \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}; \quad u(t, x) = \sum_n u_n(t) \exp(\imath nx)$$

has the form

$$\frac{du_n}{dt} = \imath n^3 u_n + 3 \imath n \sum_{k+m=n} u_k u_m.$$

Hence any two $k$- and $m$-modes interact if $k + m \neq 0$. The same is true for any 1-dimensional non-linear partial differential equation.
In contrast with the 1-dimensional non-linear equations, we show that for the 3-dimensional Navier–Stokes equations any two Fourier modes with collinear wave vectors \( \mathbf{k} \) and \( \mathbf{m} \) do not interact and that the Fourier modes with arbitrary wave vectors \( \mathbf{k} \) and \( \mathbf{m} \) do not interact if their Fourier components \( V_k(t), V_m(t) \) are proportional: \( V_k(t) = \lambda(t) V_m(t) \). For the 2-dimensional NSE, the \( \mathbf{k} \)- and \( \mathbf{m} \)-modes do not interact if and only if either \( k^2 = m^2 \) or \( k = \lambda m \).

We obtain a complete classification of all possible periodic solutions to the 3D NSE with pairwise non-interacting Fourier modes and prove that there are only four different cases. The wave vectors \( \mathbf{k} \in \mathbb{Z}^3 \) for these solutions belong to one of the following four families of sets \( S \):

1. the straight lines \( \mathbf{k} = \lambda \mathbf{n} \),
2. the planes \( \mathbf{k} \cdot \mathbf{e} = 0 \),
3. the circumferences \( \mathbf{k} \cdot \mathbf{e} = 0, \mathbf{k} \cdot \mathbf{k} = N \),
4. the spheres \( \mathbf{k} \cdot \mathbf{k} = N \).

Here the admissible integers \( N \neq 4^n(8k + 7) \) where \( a \geq 0 \) and \( k \geq 0 \) are arbitrary integers. The corresponding periodic solutions to the Navier–Stokes equations are given by the exact formulae, are smooth for all \( t > 0 \) and tend to zero as \( t \to \infty \). For these solutions, there is no transfer of energy through the spectrum.

The periodic solutions of the families (1) and (2) belong to the class of orthogonal rotations of the transversal flows and the unidirectional flows [Bat]. However, since the generic rotations do not preserve the periodicity condition (1.5), the exact form of the periodic solutions (1) and (2) is new. The solutions of class (3) belong to the class of orthogonal rotations of the general solutions [B4] for the NSE and viscous MHD that depend on time and two spatial variables. Since the rotations do not preserve the periodicity, the different linear spaces of solutions (3) have different dimensions and are non-isomorphic to each other. The solutions of class (4) belong to the general class of Beltrami flows (1.2), (1.4) [B1], [B2] and depend on all four variables \( t, x_1, x_2, x_3 \). The dimensions of the linear spaces of solutions (4) can be arbitrarily large and is equal to the number \( r_3(N) \) of representations of the integer \( N \) as sums of three integral squares. These solutions for \( \nu = 0 \) and \( N > 1 \) form an infinite family of the stationary periodic generalizations of the known Arnold–Beltrami–Childress-flows (ABC-flows) [A] that correspond to \( N = 1 \).

2. Non-interacting Fourier modes

For the Fourier series (1.6), the incompressibility equation \( \text{div} \ V(t, \mathbf{x}) = 0 \) takes the form \( \sum_n \exp(i \mathbf{n} \cdot \mathbf{x}) \mathbf{n} \cdot V_n(t) = 0 \) that yields that vectors \( V_n(t) \) are orthogonal to the wave vectors \( \mathbf{n} \):

\[
\mathbf{n} \cdot V_n = 0. \tag{2.1}
\]

As is known the NSE (1.1) have the equivalent form:

\[
\frac{\partial \text{curl} V}{\partial t} + \text{curl}(\text{curl} V \times V) = \nu \Delta \text{curl} V. \tag{2.2}
\]
For the Fourier series (1.6), equation (2.2) yields the system of equations
\[ n \times \left( \frac{dV_n}{dt} + i \sum_{k+m=n} (k \times V_k) \times V_m + \nu n^2 V_n \right) = 0. \] (2.3)

Therefore the vector in the brackets is equal to \( F_n(t)n \) where \( F_n(t) \) is some function. The vectors \( dV_n/dt \) and \( V_n \) are orthogonal to \( n \) in view of the equations (2.1). Hence equations (2.3) imply the dynamical system for the Fourier components
\[ \frac{dV_n}{dt} = -\nu n^2 V_n - i \sum_{k+m=n} (k \times V_k) \times V_m + \frac{i}{n^2} \left( \sum_{k+m=n} n \cdot ((k \times V_k) \times V_m) \right) n. \] (2.4)

The dynamical system (2.4) is equivalent to the Navier–Stokes equations (1.1) for the periodic solutions.

In what follows, we study the conditions when all non-zero modes do not interact pairwise that means that for any two \( k \)- and \( m \)-modes we have
\[ Z_{km} = (k \times V_k) \times V_m + (m \times V_m) \times V_k = \lambda_{km}(t)(k + m). \] (2.5)

It is evident that any \( k \)-mode and \((-k)\)-mode do not interact. Indeed, using the Jacobi identity we find for the interaction term in (2.4)
\[ Z_{k,-k} = (k \times V_k) \times V_{-k} + (-k \times V_{-k}) \times V_k = -(V_k \times V_{-k}) \times k = 0 \]

because both vectors \( V_k \) and \( V_{-k} \) are orthogonal to the vector \( k \).

**Lemma 1.** The \( k \)- and \( m \)-modes do not interact if and only if one of the following three conditions is met:

1. The wave vectors \( k \) and \( m \) are parallel,
2. The vectors \( V_k \) and \( V_m \) have the form \( V_k = C_ke, \ V_m = C_me \),
3. The vectors \( V_k \) and \( V_m \) have the form \( V_k = C_k(ae + bk \times e), \ V_m = C_m(ae + bm \times e) \),

where \( e \) is a unit vector orthogonal to the vectors \( k \) and \( m \), \( a, b, C_k \) and \( C_m \) are arbitrary constants, and vectors \( k \) and \( m \) have equal norms, \( k^2 = m^2 \).

**Proof.** Since vector \( V_k \) is orthogonal to \( k \) and \( V_m \) is orthogonal to \( m \), they have the form
\[ V_k = a_1e + b_1k \times e, \ V_m = a_2e + b_2m \times e, \] (2.8)

where \( e \cdot k = e \cdot m = 0, \ e \cdot e = 1 \). The interaction of the \( k \) - and \( m \)-modes in the NSE equations (2.4) is described by the term
\[ Z_{km} = (k \times V_k) \times V_m + (m \times V_m) \times V_k \]
\[ = -a_1a_2(k + m) - b_1b_2(m^2k + k^2m) + (a_1b_2 - a_2b_1)(e \cdot (k \times m))e. \] (2.9)

It is evident that formula (2.9) has form (2.5) if vectors \( k \) and \( m \) are parallel. If they are not parallel then \( e \cdot (k \times m) \neq 0 \) and formulae (2.9), (2.5) imply \( a_1b_2 - a_2b_1 = 0 \). Hence if \( b_1 = 0 \) then \( b_2 = 0 \) and formula (2.9) has form (2.5) for \( \lambda_{km} = -a_1a_2 \).
and formulae (2.8) take form (2.6). If \( b_1 \neq 0 \) then \( b_2 \neq 0 \) and hence the formulae (2.9), (2.5) yield \( k^2 = m^2 \). Since \( a_1 b_2 - a_2 b_1 = 0 \) the formulae (2.8) take the form (2.7).

**Corollary 1.** If \( k \)- and \( m \)-modes did not interact and the wave vectors \( k \) and \( m \) were not parallel then

\[
(V_k \times V_m) \cdot (k + m) = 0.
\]  

**Proof.** Lemma 1 implies that vectors \( V_k \) and \( V_m \) have form (2.7). Hence we find

\[
V_k \times V_m = C_k C_m (ab(m - k) + b^2 (e \cdot (k \times m))e).
\]

Hence \((V_k \times V_m) \cdot (k + m) = C_k C_m ab(m^2 - k^2) = 0\) because \( b = 0 \) for the case (2.6) and \( m^2 = k^2 \) for the case (2.7).

**Lemma 2.** If a \( k \)-mode with vector \( V_k \) and a \( m \)-mode with vector \( V_m \) did not interact and \( k^2 = m^2 = N \) then the same modes with vectors

\[
\tilde{V}_k = \alpha V_k - i\beta k \times V_k, \quad \tilde{V}_m = \alpha V_m - i\beta m \times V_m
\]

did not interact either. Here \( \alpha \) and \( \beta \) are arbitrary reals.

**Proof.** A direct calculation gives for the interaction term

\[
\tilde{Z}_{km} = (k \times \tilde{V}_k) \times \tilde{V}_m + (m \times \tilde{V}_m) \times \tilde{V}_k
\]

\[
= (\alpha^2 - \beta^2 N)(k \times V_k) \times V_m + (m \times V_m) \times V_k = (\alpha^2 - \beta^2 N)C_{km}(k + m).
\]

Hence the interaction is absent. It is evident that transforms (2) with real \( \alpha \) and \( \beta \) preserve the conditions (1.7).

The transform (2) gives the linearly dependent vectors \( \tilde{V}_k \) and \( V_k \) only if \( k \times V_k = \lambda V_k \). Cross-multiplying this equality with vector \( k \) we get \( k \times (k \times V_k) = -k^2 V_k = \lambda k \times V_k = \lambda^2 V_k \). Hence \( \lambda = \pm i|k| \). Thus only vectors \( V_k \) satisfying the equations

\[
k \times V_k = \pm i|k| V_k
\]  

(2.12)

span 1-dimensional invariant subspaces for the transforms (2). For such complex vectors \( V_k \) equations (2.12) yield \( \tilde{V}_k \cdot V_k = 0 \).

A complex conjugation of (2.12) and the conditions (1.7) give the equations

\[
-k \times V_{-k} = \pm i|k| V_{-k}
\]  

that have the same form as (2.12). The vectors

\[
V_{k \pm} = A_k \pm \frac{i}{|k|} k \times A_k
\]  

(2.13)

represent all solutions to the equation (2.12), where \( A_k \) is a real vector orthogonal to \( k \): \( A_k \cdot k = 0 \), \( A_{-k} = A_k \).

Any four modes with vectors \( k \), \( -k \), \( m \) and \( -m \) which satisfy equations (2.12) and \( k^2 = m^2 = N \) do not interact pairwise because

\[
(k \times V_k) \times V_m + (m \times V_m) \times V_k = \pm i\sqrt{N}(V_k \times V_m + V_m \times V_k) = 0.
\]

Let \( S \) be a set of modes that contains along with a \( k \)-mode also the \((-k)\)-mode.

**Theorem 1.** The \( k \)-modes of the set \( S \) do not interact pairwise if and only if one of the following four conditions is met:
(1) All wave vectors \( k \in S \) are parallel;
(2) All wave vectors \( k \) lie in one plane \( L \) and the corresponding vectors \( \mathbf{V}_k \) are orthogonal to \( L \);
(3) The vectors \( k \) belong to a circumference \( \mathbf{k} \cdot \mathbf{e} = 0, \mathbf{k} \cdot \mathbf{k} = N \) and vectors \( \mathbf{V}_k \) have the form
\[
\mathbf{V}_k = C_k(\alpha \mathbf{e} + i\beta \mathbf{k} \times \mathbf{e}),
\]
where \( \mathbf{e} \) is a unit vector and \( \alpha, \beta \) are arbitrary reals, \( C_k = \overline{C_k} \);
(4) The vectors \( k \) belong to a sphere \( \mathbf{k} \cdot \mathbf{k} = N \) and vectors \( \mathbf{V}_k \) satisfy the equations
\[
\mathbf{k} \times \mathbf{V}_k = \pm i\sqrt{N} \mathbf{V}_k,
\]
with the same sign for all vectors \( k \in S \).

**Proof.** (1) Lemma 1 implies that all \( k \)-modes with parallel wave vectors \( k \) do not interact.

(2–3) Let all wave vectors \( k \) of the set \( S \) lie in a plane \( L \) and let \( \mathbf{e} \) be the orthogonal unit vector, \( \mathbf{e} \cdot \mathbf{k} = 0, \mathbf{e} \cdot \mathbf{e} = 1 \). Let \( k \) and \( m \) be some non-parallel wave vectors in \( S \). Lemma 1 implies that vectors \( \mathbf{V}_k \) and \( \mathbf{V}_m \) have form (2.7). Any other wave vector \( p \in S \) is non-parallel either to \( k \) or to \( m \). Hence by Lemma 1 \( \mathbf{V}_p = C_p(\alpha \mathbf{e} + \beta \mathbf{p} \times \mathbf{e}) \).

If \( b = 0 \) then all vectors \( \mathbf{V}_p = C_p \mathbf{e} \) and the norms of vectors \( p \in S \) can be arbitrary, that proves the case 2). If \( b \neq 0 \) then Lemma 1 gives \( k^2 = m^2 = p^2 = N \). Hence all vectors of the set \( S \) lie on the circumference \( \mathbf{k} \cdot \mathbf{e} = 0, \mathbf{k} \cdot \mathbf{k} = N \). The formulae (2.7) for the non-interacting modes with vectors \( k, -k, m, -m \) are compatible with the conditions (1.7) only if \( b/a \) is pure imaginary. Hence equations (2.7) are reduced to (2.14) with arbitrary reals \( \alpha, \beta \), that proves the case 3).

(4) Let the set \( S \) contains some three linearly independent wave vectors \( k, m, p \).

By Lemma 1 vectors \( \mathbf{V}_k \) and \( \mathbf{V}_m \) have form (2.7) where \( \mathbf{k} \cdot \mathbf{e} = 0, \mathbf{m} \cdot \mathbf{e} = 0 \). If \( b \neq 0 \) then Lemma 1 gives \( k^2 = m^2 \). If \( b = 0 \) then \( \mathbf{V}_k = C_k \mathbf{e} \) and \( \mathbf{V}_m = C_m \mathbf{e} \). Hence the linear independence of the vectors \( k, m, p \) implies \( \mathbf{V}_k \cdot \mathbf{p} \neq 0 \) and \( \mathbf{V}_m \cdot \mathbf{p} \neq 0 \) and Lemma 1 for the pairs \( p, k \) and \( p, m \) yields \( p^2 = k^2 \) and \( p^2 = m^2 \). Thus for \( b \neq 0 \) and for \( b = 0 \) we have \( k^2 = m^2 \). Hence we get \( k^2 = m^2 = p^2 = N \). Since any vector \( q \in S \) is linearly independent with some two of the vectors \( k, m, p \), we get \( q^2 = N \) for all vectors \( q \in S \).

Hence Lemma 2 is applicable to the set \( S \) and gives the pairwise non-interacting \( k \)-modes with the new vectors \( \tilde{V}_k = \alpha \mathbf{V}_k + i\beta \mathbf{k} \times \mathbf{V}_k \).

Let us prove by contradiction that all vectors \( \mathbf{V}_k \) satisfy the equations (2.15) with the same sign. If for some vector \( \mathbf{V}_k \) the equation (2.15) did not hold then we consider two wave vectors \( m, p \in S \) that form a linearly independent triple \( k, m, p \). Since the modes \( k, -k, m, -m \) do not interact, Lemma 1 gives \( \mathbf{V}_k = C_k(\alpha \mathbf{e} + i\beta \mathbf{k} \times \mathbf{e}) \), \( \mathbf{V}_m = C_m(\alpha \mathbf{e} + i\beta \mathbf{m} \times \mathbf{e}) \) where \( \mathbf{k} \cdot \mathbf{e} = 0, \mathbf{m} \cdot \mathbf{e} = 0 \) and \( \alpha, \beta \) are some reals. For these vectors equations (2.15) are equivalent to the equalities \( \alpha = \mp i\beta \sqrt{N} \). Hence if vector \( \mathbf{V}_k \) did not satisfy equations (2.15) we have \( (\alpha^2 - \beta^2 N) \neq 0 \). Applying transform (2):
\[
\tilde{V}_q = \alpha \mathbf{V}_q - i\beta \mathbf{q} \times \mathbf{V}_q,
\]
we obtain due to Lemma 2 the non-interacting \( k, -k, m \) and \(-m\)-modes with the vectors

\[
\vec{V}_k = C_k \lambda e, \quad \vec{V}_{-k} = \overline{C_k} \lambda e, \quad \vec{V}_m = C_m \lambda e, \quad \vec{V}_{-m} = \overline{C_m} \lambda e,
\]

(2.17)

where \( \lambda = \alpha^2 - \beta^2 N \neq 0 \). The transform \( 2.16 \) has the inverse \( \vec{V}_q = \lambda^{-1}(\alpha \vec{V}_q + i\beta q \times \vec{V}_q) \). Hence vector \( \vec{V}_p = \alpha \vec{V}_p - i\beta p \times \vec{V}_p \neq 0 \). The formulae \( 2.17 \) imply that the four vectors

\[
\vec{V}_k \times \vec{V}_p, \quad \vec{V}_{-k} \times \vec{V}_p, \quad \vec{V}_m \times \vec{V}_p, \quad \vec{V}_{-m} \times \vec{V}_p
\]

are proportional to the vector \( \vec{U} = e \times \vec{V}_p \). Since the wave vectors \( k, m, p \) are linearly independent and the equations \( e \cdot k = 0, e \cdot m = 0, \vec{V}_p \cdot p = 0 \) hold, we have \( e \times \vec{V}_p = \vec{U} \neq 0 \). Applying equation \( 2.10 \) to the four pairs of non-interacting modes \( (k, p), (-k, p), (m, p), (-m, p) \) we obtain the equations

\[
\vec{U} \cdot (k + p) = 0, \quad \vec{U} \cdot (-k + p) = 0, \quad \vec{U} \cdot (m + p) = 0, \quad \vec{U} \cdot (-m + p) = 0.
\]

Hence the nonzero vector \( \vec{U} \) is orthogonal to the three linearly independent vectors \( k, m, p \), a contradiction. Hence any vector \( \vec{V}_k \) satisfies one of the equations \( 2.15 \).

Suppose there are two vectors \( k \in S \) and \( m \in S \) for which the signs in the equations \( 2.15 \) are different. Let \( p \in S \) be any vector linearly independent with \( k \) and \( m \). With no loss of generality, let the signs in the equations \( 2.15 \) for the \( k, m, p \)-modes be +, −, +. Then we have for the interaction terms

\[
(k \times \vec{V}_k) \times \vec{V}_m + (m \times \vec{V}_m) \times \vec{V}_k = 2i\sqrt{N}V_k \times \vec{V}_m,
\]

\[
(p \times \vec{V}_p) \times \vec{V}_m + (m \times \vec{V}_m) \times \vec{V}_p = 2i\sqrt{N}V_p \times \vec{V}_m.
\]

(2.18)

Since the interactions between the \( k \)- and \( m \)-modes and between the \( p \)- and \( m \)-modes are absent we find from the equations \( 2.5 \) and \( 2.18 \):

\[
\vec{V}_k \times \vec{V}_m = \lambda_{km}(t)(k + m), \quad \vec{V}_p \times \vec{V}_m = \lambda_{pm}(t)(p + m).
\]

(2.19)

Corollary 1 gives

\[
(\vec{V}_k \times \vec{V}_m) \cdot (k + m) = 0, \quad (\vec{V}_p \times \vec{V}_m) \cdot (p + m) = 0.
\]

(2.20)

The equations \( 2.19 \) and \( 2.20 \) imply that \( \vec{V}_k \times \vec{V}_m = 0 \) and \( \vec{V}_p \times \vec{V}_m = 0 \). Hence the vectors \( \vec{V}_k, \vec{V}_m, \vec{V}_p \) are parallel to a vector \( \vec{U}_1 \). Hence the vector \( \vec{U}_1 \) is orthogonal to the three linearly independent vectors \( k, m, p \), a contradiction. Hence equations \( 2.15 \) for all vectors \( q \in S \) have the same sign. \( \square \)

Remark 1. The admissible integers \( N = k_1^2 + k_2^2 + k_3^2 \) in the formula \( 2.15 \) are \( N \neq 4^a(8k + 7) \) where \( a \geq 0 \) and \( k \geq 0 \) are arbitrary integers, because by Legendre’s theorem [Gr] only these \( N \)’s are sums of three integral squares.

3. Exact periodic solutions to the NSE

To each of the four sub-cases of Theorem 1 there correspond the linear subspaces of exact periodic solutions.
Corollary 2. For the Navier–Stokes equations (1.1), there exist only four families of the periodic solutions with pairwise non-interacting modes:

1. The convergent series defined for any integral vector \( n \):
   \[
   V_n(t, x) = \sum_{k=1}^{\infty} e^{-k^2 N \nu t} (A_{kn} \cos(ks \cdot x) + B_{kn} \sin(ks \cdot x)),
   \tag{3.1}
   \]
   where \( N = n \cdot n \) and vectors \( A_{kn} \) and \( B_{kn} \) are orthogonal to the vector \( n \).

2. The convergent series defined for any two non-parallel integral vectors \( n \) and \( m \):
   \[
   V_{n,m}(t, x) = \sum_{k, \ell = -\infty}^{\infty} e^{-(kn+\ell m)^2 \nu t} (a_{k\ell} \cos((kn+\ell m) \cdot x)
   + b_{k\ell} \sin((kn+\ell m) \cdot x)) n \times m,
   \tag{3.2}
   \]
   where \( k, \ell \) are arbitrary integers.

3. The exact solutions defined for any integer \( N \neq 4^a(8k+7) \):
   \[
   V_{Ne}(t, x) = \alpha e^{-N \nu t} \left( \sum_k (a_k \cos(k \cdot x) - b_k \sin(k \cdot x)) \right) e
   - \beta e^{-N \nu t} \sum_k (a_k \sin(k \cdot x) + b_k \cos(k \cdot x)) k \times e,
   \tag{3.3}
   \]
   where the integral vectors \( k \) and the unit vector \( e \) satisfy the equations
   \[ k \cdot e = 0, \quad k \cdot k = N \tag{3.4} \]
   and \( \alpha, \beta, a_k, b_k \) are arbitrary real numbers, \( a_{-k} = a_k \) and \( b_{-k} = -b_k \).

4. The two families (for the sign + and −) of exact solutions defined for any integer \( N \neq 4^a(8k+7) \):
   \[
   V_{N\pm}(t, x) = \exp(-N \nu t) \sum_k \left( \cos(k \cdot x) A_k \pm \frac{1}{\sqrt{N}} \sin(k \cdot x) k \times A_k \right),
   \tag{3.5}
   \]
   where \( k \cdot k = N \) and vectors \( A_k \) satisfy the conditions \( A_k \cdot k = 0, A_{-k} = A_k \). The linear spaces of the exact solutions (3.5) have dimension \( r_3(N) \) that can be arbitrarily large.

Proof. For any set \( S \) of pairwise non-interacting modes, the Navier–Stokes dynamical system (2.4) takes the form
   \[
   \frac{dV_n}{dt} = -\nu n^2 V_n, \quad V_n(t) = \exp(-n^2 \nu t)V_n(0).
   \]
   Therefore the exact solutions for the sub-cases (1) and (2) of Theorem 1 take the form (3.1) and (3.2). The sub-case (3) for \( C_k = a_k + ib_k \) gives exact solutions
   \[
   V_{Ne}(t, x) = \exp(-N \nu t) \sum_k \exp(ik \cdot x) C_k (\alpha e + i\beta k \times e),
   \]
that have the form (3.3). The condition $N = k_1^2 + k_2^2 + k_3^2 \neq 4^s(8k + 7)$ follows from the Legendre theorem [Gr]. The exact solutions for the sub-case (4) have the form

$$V_{N \pm}(t, x) = \exp(-Nvt) \sum_k \exp(ik \cdot x)V_{k \pm},$$

(3.6)

where constant vectors $V_{k \pm}$ satisfy equations (2.15) and have the form (2.13): $V_{k \pm} = A_k \mp ik \times A_k/\sqrt{N}$ where $A_k \cdot k = 0$, $A_{-k} = A_k$. These formulae imply that solutions (3.5), (3.6) satisfy the Beltrami equation curl $V_{N \pm} = \mp \sqrt{N} V_{N \pm}$. Let $r_3(N)$ be the number of integral solutions to the equation $k_1^2 + k_2^2 + k_3^2 = N$.

As is known [Gr], the number $r_3(N)$ can be arbitrarily large. Each pair of $k$- and $(-k)$-modes defines a 2-dimensional family of the exact solutions (3.5). Hence the linear subspaces $S_{N \pm}$ of exact solutions (3.5), (3.6) have dimension $r_3(N)$.

**Remark 2.** For any three distinct positive integers $k_1$, $k_2$, $k_3$, there are 48 vectors $k = \pm k_1 e_1 \pm k_2 e_2 \pm k_3 e_3$. Hence $r_3(k_1^2 + k_2^2 + k_3^2) \geq 48$. For example, $62 = 1^2 + 5^2 + 6^2 = 2^2 + 3^2 + 7^2$; hence the subspaces $S_{62 \pm}$ have dimension 96.

**Remark 3.** Let the unit vector $e$ be one of the coordinate unit orts, for example $e_3$. Then the equations (3.4) imply $k = k_1 e_1 + k_2 e_2$ and $N = k_1^2 + k_2^2$. The solutions (3.3) take the form

$$V_{N^3}(t, x) = \alpha \exp(-Nvt) \left( \sum_k (a_k \cos(k_1 x + k_2 y) - b_k \sin(k_1 x + k_2 y)) \right) e_3$$

$$- \beta \exp(-Nvt) \sum_k (a_k \sin(k_1 x + k_2 y) + b_k \cos(k_1 x + k_2 y)) k \times e_3.$$  

(3.7)

The number of such vectors $k = k_1 e_1 + k_2 e_2$ is equal to the $r_2(N)$, where $r_2(N)$ is the number of integer solutions to the equation $k_1^2 + k_2^2 = N$. Let $N = 2^p m_1 m_2$ where $m_1 = \prod p^r$, $p \equiv 1$ (mod 4) and $m_2 = \prod q^s$, $q \equiv 3$ (mod 4) where $p$ and $q$ are prime divisors of $N$. By Euler’s theorem [Gr] the number $r_2(N)$ is zero if any of $s$ is odd. If all $s$ are even, then Gauss theorem [Gr] states that $r_2(N) = 4d(m_1)$ where $d(m_1)$ is the number of divisors of $m_1$. Therefore the number $r_2(N)$ can be arbitrarily large. We have a free parameter $\beta/\alpha$ and for each pair $k$ and $-k$ two arbitrary parameters $a_k$ and $b_k$ in (3.7). Hence the family of exact solutions (3.7) depends on $r_2(N) + 1$ parameters. For any two distinct positive integers $k_1$, $k_2$ there are 8 integral vectors $\pm k_1 e_1 \pm k_2 e_2$. Hence $r_2(k_1^2 + k_2^2) \geq 8$. For example $65 = 1^2 + 5^2 = 2^2 + 7^2$, hence the subspace $S_{65 \pm}$ of exact solutions (3.7) has dimension $r_2(65) + 1 = 17$.

**Remark 4.** Since the number of integral vectors $k$ satisfying the equations $k \cdot k = N$ is equal to $r_3(N)$, there are $r_3(N)/2$ pairs $(k, -k)$ of such vectors and $r_3(N)(r_3(N) - 2)/8$ distinct sets of vectors $k_1$, $-k_1$, $k_2$, $-k_2$. Evidently there are $L(N) = r_3(N)(r_3(N) - 2)/8 - 3r_2(N)(r_2(N) - 2)/8$ subsets of vectors $k_1$, $-k_1$, $k_2$, $-k_2$ that do not belong to the coordinate planes. For each such a set, we define a subspace of exact solutions (3.3) where the unit vector $e = k_1 \times k_2/[|k_1 \times k_2|]$. Such solutions (3.3) are different from (3.7) and depend on 5 arbitrary parameters $a_1, b_1, a_2, b_2$ and $\beta/\alpha$. 
Remark 5. For the simplest case \( N = 1 \), the solutions (3.5) after the change of time \( d\tau/dt = \exp(-N\nu t) \) turn into the known Arnold–Beltrami–Childress-flows (ABC-flows) [A]:

\[
\begin{align*}
\dot{x}_1 &= A \sin x_3 + C \cos x_2, \\
\dot{x}_2 &= B \sin x_1 + A \cos x_3, \\
\dot{x}_3 &= C \sin x_2 + B \cos x_1.
\end{align*}
\]

The corresponding vectors \( k \) and \( A_k \) in (3.5) are:

\[
\begin{align*}
k_1 &= (1, 0, 0), \\
A_{k_1} &= (0, 0, B), \\
k_2 &= (0, 1, 0), \\
A_{k_2} &= (C, 0, 0), \\
k_3 &= (0, 0, 1), \\
A_{k_3} &= (0, A, 0) \\
\end{align*}
\]

and the minus sign is chosen in (3.5). Hence the periodic solutions (3.5) for an arbitrary \( N \neq 4^a(8k+7) \) form an infinite family of generalizations of the ABC-flows of dimensions \( r_3(N) \) that can be arbitrarily large. The solutions also satisfy the Beltrami equation and the Euler equations (1.1) for \( \nu = 0 \).

Acknowledgements. The author thanks Benno Fuchssteiner and Paulo Ribenboim for useful discussions. The work was supported by a Killam Fellowship of Canada and an Alexander von Humboldt Research Award of Germany.

References


Department of Mathematics, Queen’s University, Kingston, K7L 3N6 Canada and Steklov Mathematical Institute, Moscow, 117966 Russia

E-mail address: bogoyavl@maqst.queensu.ca