INDICES OF 1-FORMS ON AN ISOLATED COMPLETE
INTERSECTION SINGULARITY

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To Vladimir Igorevich Arnold with admiration

Abstract. There are some generalizations of the classical Eisenbud–Levine–Khimshiashvili formula for the index of a singular point of an analytic vector field on $\mathbb{R}^n$ to vector fields on singular varieties. We offer an alternative approach based on the study of indices of 1-forms instead of vector fields. When the variety under consideration is a real isolated complete intersection singularity (icis), we define an index of a (real) 1-form on it. In the complex setting we define an index of a holomorphic 1-form on a complex icis and express it as the dimension of a certain algebra. In the real setting, for an icis $V = f^{-1}(0)$, $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$, $f$ is real, we define a complex analytic family of quadratic forms parameterized by the points $\epsilon$ of the image $(\mathbb{C}^k, 0)$ of the map $f$, which become real for real $\epsilon$ and in this case their signatures defer from the “real” index by $\chi(V_\epsilon) - 1$, where $\chi(V_\epsilon)$ is the Euler characteristic of the corresponding smoothing $V_\epsilon = f^{-1}(\epsilon) \cap B_\delta$ of the icis $V$.


Key words and phrases. Singular varieties, 1-forms, singular points, indices.

Introduction (1-forms versus vector fields)

There are a number of papers devoted to definition and calculation of the index of an analytic vector field on a real analytic variety with an isolated singular point ([ASV], [EGZ], [GMM1], [GMM2]). In [GSV], the index of a holomorphic vector field on an isolated complete intersection singularity (icis) was defined. This notion was invented in the hope that it might be used to calculate the index of a vector field on a real icis.

We offer a different approach. Instead of considering vector fields on a variety, we consider 1-forms. We define the index of a real 1-form on a germ of a real analytic variety with an isolated singular point and of a holomorphic 1-form on a (complex) icis. (For short, we sometimes refer to these two notions as the real and complex indices, respectively.) To a vector field on a real variety $(V, 0) \subset (\mathbb{R}^n, 0)$ (or on a complex analytic variety $(V, 0) \subset (\mathbb{C}^n, 0)$), we can associate a 1-form
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on it (dependent on the choice of coordinates in $(\mathbb{R}^n, 0)$ or in $(\mathbb{C}^n, 0)$). If the vector field has an algebraically isolated singular point on $(V, 0)$ (in the complex setting, algebraic isolatedness is simply isolatedness), then, under a generic choice of coordinates, the corresponding 1-form has an algebraically isolated singular point as well. The converse does not hold. Moreover, the index of a vector field on a real analytic variety with an isolated singular point coincides with that of the corresponding 1-form. The notion of the index of a holomorphic 1-form on an ics is somewhat more natural than that of a holomorphic vector field: in some sense, it is “more complex analytic” (its definition does not use the complex conjugation) and “more geometric” (only objects of the same tensor type are used). Moreover, the index of an isolated singular point of a holomorphic 1-form on a complex ics can be described as the dimension of an appropriate algebra. Finally, the real index of a 1-form on a real ics (with an algebraically isolated singular point) plus-minus the Euler characteristic of a (real) smoothing of the ics can be expressed in terms of the signature of a (nondegenerate) quadratic form on a space the dimension of which is equal to the (complex) index of the corresponding complexification.

The idea to consider indices of 1-forms instead of indices of vector fields (in some situations) was first formulated by Arnold [A].

1. Indices of real 1-forms on singular varieties

A manifold with isolated singularities is a topological space $M$ which has the structure of a smooth (say, $C^\infty$-) manifold outside a discrete set $S$ (the set of singular points of $M$). A diffeomorphism between two such manifolds is a homeomorphism which sends the set of singular points of one of them onto the set of singular points of the other and is a diffeomorphism outside of these sets. We say that $M$ has a cone-like singularity at a (singular) point $P \in S$ if there exists a neighbourhood of the point $P$ diffeomorphic to the cone $CW_P = (I \times W_P)/\{0\} \times W_P$ ($I = [0, 1]$) over a smooth manifold $W_P$ ($W_P$ is called the link of the point $P$). In what follows, we assume all manifolds to have only cone-like singularities. In [EGZ], we discussed the notion of a vector field on a manifold with isolated singularities and the notion of the index of its singular point. Here we adapt the corresponding definitions to the case of 1-forms. A (smooth or continuous) 1-form on a manifold $M$ with isolated singularities is a (smooth or continuous) 1-form on the set $M \setminus S$ of regular points of $M$. The set of singular points $S_\omega$ of a 1-form $\omega$ on a (singular) manifold $M$ is the union of the set of usual singular points of $\omega$ on $M \setminus S$ (i.e., the points at which $\omega$ tends to zero) and of the set $S$ of singular points of $M$ itself.

For an isolated usual singular point $P$ of a 1-form $\omega$, its index $\text{ind}_P \omega$ is defined as the degree of the map $\omega/\|\omega\|: \partial B \to S^{n-1}$ of the boundary of a small ball $B$ centred at the point $P$ in a coordinate neighbourhood of $P$ to the unit sphere in the dual space; $n = \dim M$. If the manifold $M$ is closed (i.e., it is a compact manifold without boundary), has no singularities ($S = \emptyset$), and the 1-form $\omega$ on $M$ has only
isolated singularities, then
\[ \sum_{P \in S} \text{ind}_P \omega = \chi(M) \] (1)

(\(\chi(M)\) is the Euler characteristic of \(M\)).

Let \((M, P)\) be a cone-like singularity (i.e., a germ of a manifold with such a singular point), and let \(\omega\) be a 1-form defined on an open neighbourhood \(U\) of the point \(P\). Suppose that \(\omega\) has no singular points on \(U \setminus \{P\}\). Let \(V\) be a closed cone-like neighbourhood of \(P\) in \(U\) (\(V \cong CW_p\), \(V \subset U\)). On the cone \(CW_p = (I \times W_p) / \{(0) \times W_p\} (I = [0, 1])\), the natural 1-form \(dt\) is defined (\(t\) is the coordinate on \(I\)). Let \(\omega_{\text{rad}}\) be the corresponding 1-form on \(V\), and let \(\tilde{\omega}\) be a smooth 1-form on \(U\) which coincides with \(\omega\) near the boundary \(\partial U\) of the neighbourhood \(U\) and with \(\omega_{\text{rad}}\) on \(V\) and has only isolated singular points.

**Definition.** The **index** \(\text{ind}_P \omega\) of the 1-form \(\omega\) at the point \(P\) is equal to
\[ 1 + \sum_{Q \in S \setminus \{P\}} \text{ind}_Q \tilde{\omega} \]
(the sum is over all singular points \(Q\) of the 1-form \(\tilde{\omega}\) except \(P\) itself).

Generally speaking, for a cone-like singularity at a point \(P \in S\), the link \(W_p\) and, thus, the cone structure of a neighbourhood are not well-defined (cones over different manifolds may be **locally** diffeomorphic). However, it is not difficult to show that the index \(\text{ind}_P \omega\) does not depend on the choice of the cone structure on a neighbourhood and of the 1-form \(\tilde{\omega}\).

**Proposition 1.** For a 1-form \(\omega\) with isolated singular points on a closed manifold \(M\) with isolated singularities, relation (1) holds.

**Definition.** A singular point \(P\) of a manifold \(M\) (locally diffeomorphic to the cone \(CW_p\) over a manifold \(W_p\)) is **smoothable** if \(W_p\) is the boundary of a smooth compact manifold \(\tilde{V}_p\).

In what follows, we shall call \(\tilde{V}_p\) a smoothing of \((V, P)\). The class of smoothable singularities includes, in particular, the class of (real) isolated complete intersection singularities. For such a singularity, there is a distinguished cone-like structure on its neighbourhood.

Let \((M, P)\) be a smoothable singularity (i.e., a germ of a manifold with a smoothable singular point), and let \(\omega\) be a 1-form on \((M, P)\) with an isolated singular point at \(P\). Suppose that \(V = CW_p\) is a closed cone-like neighbourhood of the point \(P\); \(\omega\) is supposed to have no singular points on \(V \setminus \{P\}\). Let the link \(W_p\) of the point \(P\) be the boundary of a compact manifold \(\tilde{V}_p\). Identifying \(\partial \tilde{V}_p = W_p\) with \(W_p \times \{1/2\}\) and smoothing turn the union \(\tilde{V}_p \cup_{W_p} (W_p \times [1/2, 1])\) of \(\tilde{V}_p\) and \(W_p \times [1/2, 1] \subset CW_p\) glued together along the common boundary into a smooth manifold (with the boundary \(W_p \times \{1\}\)). The restriction of the 1-form \(\omega\) to \(W_p \times [1/2, 1] \subset CW_p = V\) can be extended to a smooth 1-form \(\tilde{\omega}\) on \(\tilde{V}_p \cup_{W_p} (W_p \times [1/2, 1])\) with isolated singular points.
Proposition 2. The index \( \text{ind}_P \omega \) of the 1-form \( \omega \) at the point \( P \) is equal to
\[
\sum_{Q \in S_0} \text{ind}_Q \tilde{\omega} - \chi(\tilde{V}_P) + 1
\]
(the sum is over all singular points of \( \tilde{\omega} \) on \( \tilde{V}_P \)).

Let \( (V, 0) \subset (\mathbb{R}^n, 0) \) be a real \((n - k)\)-dimensional (from the topological point of view as well) variety with an isolated singularity at the origin. Let \( X \) be an analytic vector field on \((V, 0)\), that is the restriction of an analytic vector field \( \sum X_i \frac{\partial}{\partial x_i} \) (which we also denote by \( X \)) defined on a neighbourhood of the origin in \( \mathbb{R}^n \) and tangent to the variety \( V \) (outside the origin). Suppose that the origin is an isolated singular point of the vector field \( X \) on \( V \), i.e., \( X \) has no zeros on \( V \) outside the origin (in its neighbourhood). In this situation, the index \( \text{ind}_0 X \) of the vector field \( X \) at the origin is defined (see [EGZ]). Let \( \omega \) be the 1-form \( \sum X_i \, dx_i \). The 1-form \( \omega \) on \( V \) has an isolated singular point at the origin as well. Moreover, \( \text{ind}_0 \omega = \text{ind}_0 X \). Thus, in this case, the problem of calculating the index of a vector field can be reduced to the problem of calculating the index of a 1-form. The converse is not true: for a 1-form \( \omega = \sum A_i \, dx_i \), the vector field \( \sum A_i \frac{\partial}{\partial x_i} \), generally speaking, is not tangent to the variety \( V \).

Remark. The described 1-form \( \omega \) corresponding to a vector field \( X \) on \( V \) depends on the choice of the coordinates on \((\mathbb{R}^n, 0)\).

We can hope to get an algebraic formula for the index of a vector field or of a 1-form on a singular variety \((V, 0) \subset (\mathbb{R}^n, 0)\) with an isolated singular point at the origin in the spirit of the Eisenbud–Levine–Khimshiashvili one ([EL], [Kh]) only if the origin is an algebraically isolated singular point of the vector field or of the 1-form. This means that the complexification of the vector field (or of the 1-form) on the complexification \((V_\mathbb{C}, 0) \subset (\mathbb{C}^n, 0)\) of the variety \((V, 0)\) has no zeros on \( V_\mathbb{C} \) outside the origin (in its neighbourhood; we suppose that \( V \) has an algebraically isolated singular point at the origin itself, i.e., \( V_\mathbb{C} \) has an isolated singular point at the origin). Suppose that the vector field \( X = \sum X_i \frac{\partial}{\partial x_i} \) has an algebraically isolated singular point at the origin. In this case, the 1-form \( \omega = \sum X_i \, dx_i \) on \( V_\mathbb{C} \) may have a nonisolated singular point at the origin. For instance, this takes place for \( V = \{(x, y, z) \in \mathbb{R}^3; \, x^2 + y^2 - z^2 = 0\} \), \( X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z} \). In such a case, there is little hope of obtaining an algebraic formula for the index \( \text{ind}_0 \omega \) of the 1-form \( \omega \). The following statement helps to avoid this difficulty.

Lemma 1. Let \( X = \sum X_i \frac{\partial}{\partial x_i} \) be a vector field tangent to \( V \) which has an algebraically isolated (on \( V \)) singular point at the origin. Then there is an analytic (in fact, generic) change of coordinates on \((\mathbb{R}^n, 0)\), after which the origin becomes an algebraically isolated singular point of the associated 1-form \( \omega = \sum X_i \, dx_i \) on \( V \) as well.

Proof. Consider the subset \( \Xi \) of the space \( J^1(V_\mathbb{C} \setminus \{0\}; \mathbb{C}^n) \) of 1-jets of maps from \( V_\mathbb{C} \setminus \{0\} \) to \( \mathbb{C}^n \) which consists of the jets \((F(x), dF(x)) \mid x \in V_\mathbb{C} \setminus \{0\}, \, F(x) \in \mathbb{C}^n, \, dF(x) \colon T_x V_\mathbb{C} \to T_{F(x)} \mathbb{C}^n = \mathbb{C}^n \) such that \( dF(x)(X(x)) \) are orthogonal to \( \text{Im} \, dF(x) \) in the sense of the quadratic form \( \sum_{i=1}^n z_i^2 \) on \( \mathbb{C}^n \). The set \( \Xi \) is a submanifold of
The 1-forms $V$ are more similar to covectors. The variety $V$ at isolated points. Now, the fact that, after the change of coordinates, the origin becomes an algebraically isolated singular point of the 1-form corresponding to the vector field under consideration follows from the curve selection lemma. 

2. THE INDEX OF A HOLOMORPHIC 1-FORM ON AN ICIS

Let $f = (f_1, \ldots, f_k): (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$ be an analytic map which defines an $(n-k)$-dimensional icis $V = f^{-1}(0) \subset (\mathbb{C}^n, 0)$ ($f_i: (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$).

For a germ of a holomorphic vector field $X$ which is tangent to $V$ and has an isolated singular point on $V$ at the origin, the index is defined in [GSV], [SS]. We recall the definition of this index. Let $U$ be a neighbourhood of the origin in $\mathbb{C}^n$ where all the functions $f_i$ $(i = 1, \ldots, k)$ and the vector field $X$ are defined. Take a sufficiently small sphere $S_d \subset U$ around the origin which intersects $V$ transversally. Let $K = V \cap S_d$ be the link of the icis $(V, 0)$. We define grad $f_i$ by

$$\text{grad } f_i = \left(\frac{\partial f_i}{\partial x_1}, \ldots, \frac{\partial f_i}{\partial x_n}\right).$$

We have the map

$$(X, \text{grad } f_1, \ldots, \text{grad } f_k): K \to W_{k+1}(\mathbb{C}^n),$$

where $W_{k+1}(\mathbb{C}^n)$ is the Stiefel manifold of $(k+1)$-frames in $\mathbb{C}^n$. It is well known that $W_{k+1}(\mathbb{C}^n)$ is $(2(n-k) - 2)$-connected and

$$H_{2(n-k) - 1}(W_{k+1}(\mathbb{C}^n)) \cong \pi_{2(n-k) - 1}(W_{k+1}(\mathbb{C}^n)) \cong \mathbb{Z}$$

(see, e.g., [H]). On the other hand, $K$ is a smooth manifold of dimension $2(n-k) - 1$, and it has natural orientation as the boundary of the complex manifold $V \setminus \{0\}$. Therefore, the map from $K$ to $W_{k+1}(\mathbb{C}^n)$ has a degree. The index of the vector field $X$ at the origin is defined to be the degree of this map.

**Remark.** Note that this definition uses complex conjugation, and the components of the above map are of different tensor nature: $X$ is a vector field, whereas grad $f_i$ are more similar to covectors.

Now, we adapt the definition of the index of a holomorphic vector field to the case of a holomorphic 1-form on $V$. Let $\omega = \sum A_i \, dx_i$ ($A_i = A_i(x)$) be a germ of a holomorphic 1-form on $(\mathbb{C}^n, 0)$ which has (at most) an isolated singular point at the origin as a 1-form on $V$ (thus it does not vanish on the tangent space $T_P V$ to the variety $V$ at all points $P$ from a punctured neighbourhood of the origin in $V$). The 1-forms $\omega$, $df_1$, $\ldots$, $df_k$ are linearly independent for all $P \in K$. Thus we have the map

$$(\omega, df_1, \ldots, df_k): K \to W_{k+1}(\mathbb{C}^n).$$

The intersection points of the image of the jet extension $j^1 F$ vanishes on $T_{F(x)} F(\mathbb{C}^n, 0))$. The strong transversality theorem implies that, under a generic change of coordinates, the image of $V \setminus \{0\}$ intersects $\Sigma$ (transversally) at isolated points. Now, the fact that, after the change of coordinates, the origin becomes an algebraically isolated singular point of the 1-form corresponding to the vector field under consideration follows from the curve selection lemma. 


Definition. We define the index \( \text{ind}_{C, 0} \omega \) of the 1-form \( \omega \) at 0 to be the degree of the map
\[
(\omega, df_1, \ldots, df_k): K \to W_{k+1}(\mathbb{C}^n)
\]
(here \( W_{k+1}(\mathbb{C}^n) \) is the manifold of \((k + 1)\)-frames in the dual \( \mathbb{C}^n \)).

Remark. The Stiefel manifold \( W_{k+1}(\mathbb{C}^n) \) of \((k + 1)\)-frames in \( \mathbb{C}^n \) is homotopy equivalent to the (Stiefel) manifold \( \tilde{W}_{k+1}(\mathbb{C}^n) \) of orthonormal (with respect to the Hermitian scalar product \( \sum x_i \bar{y}_j \)) \((k + 1)\)-frames in \( \mathbb{C}^n \). The homotopy equivalence is defined by the Gram–Schmidt process. Thus, in the definition, we can replace \( W_{k+1}(\mathbb{C}^n) \) by \( \tilde{W}_{k+1}(\mathbb{C}^n) \) and the map
\[
(\omega, df_1, \ldots, df_k): K \to W_{k+1}(\mathbb{C}^n)
\]
by the corresponding map
\[
(\omega, df_1, \ldots, df_k)^\sim: K \to \tilde{W}_{k+1}(\mathbb{C}^n).
\]
However, \( \tilde{W}_{k+1}(\mathbb{C}^n) \) is not a complex analytic manifold, and thus the map
\[
(\omega, df_1, \ldots, df_k)^\sim: U \setminus \{0\} \to \tilde{W}_{k+1}(\mathbb{C}^n)
\]
of a punctured neighbourhood \( U \setminus \{0\} \) of the origin in \( U \) is not complex analytic (in contrast with the map \( (\omega, df_1, \ldots, df_k): U \setminus \{0\} \to W_{k+1}(\mathbb{C}^n) \)). We prefer to give a “more complex analytic” definition, using sometimes the map to \( \tilde{W}_{k+1}(\mathbb{C}^n) \) for calculations.

Example. Take \( n = 2, k = 1 \), and \( f_1(x, y) = x^2 + y^3 \). Consider the 1-form
\[
\omega = 3y^2 \, dx - 2x \, dy.
\]
The form \( \omega \) on \( V = \{f_1 = 0\} \) has an isolated zero at the origin. It is easy to verify that the degree of the map
\[
(\omega, df_1)^\sim: K \to \tilde{W}_2(\mathbb{C}^2) \cong U(2)
\]
is 6. Therefore, \( \text{ind}_{C, 0} \omega = 6 \) (see also Proposition 3). This 1-form corresponds (in the above sense) to the vector field
\[
X = 3y^2 \frac{\partial}{\partial x} - 2x \frac{\partial}{\partial y}.
\]
This vector field is (by chance) tangent to the hypersurface \( V = \{f_1 = 0\} \) and has an isolated singular point at the origin on it, but its index is 0, since the map
\[
(X, \text{grad} f_1)^\sim: K \to \tilde{W}_2(\mathbb{C}^2) \cong U(2)
\]
maps \( K \) to \( SU(2) \), which is simply connected.

Let \( B_\delta \) be the ball of radius \( \delta \) around the origin in \( \mathbb{C}^n \) with boundary \( S_\delta \). Suppose that the functions \( f_1, \ldots, f_k \) and the 1-form \( \omega \) are defined in a neighbourhood of \( B_\delta \) and take \( \varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) \in \mathbb{C}^k \) small enough for the level set \( V_\varepsilon := f_1^{-1}(\varepsilon) \cap B_\delta \) to be transversal to the sphere \( S_\delta \) and such that this level set \( V_\varepsilon \) is nonsingular.
Definition. Let $M$ be a complex $m$-dimensional manifold, and let $\eta$ be a holomorphic 1-form on $M$. A zero $P \in M$ of the 1-form $\eta$ is called nondegenerate if in local coordinates $y_1, \ldots, y_m$ on $M$ around $P$ in which the 1-form $\eta$ is written as $\eta = C_1 dy_1 + \cdots + C_m dy_m$, the Hessian of $\eta$ at the point $P$, i.e., the determinant of the matrix

$$\left( \frac{\partial C_i}{\partial y_j}(P) \right)_{i,j=1,\ldots,m},$$

is nonzero.

There exists a perturbation $\tilde{\omega}$ of the 1-form $\omega$ which has only nondegenerate zeros on $V_\varepsilon$. (In fact, a generic perturbation of $\omega$, in particular, a perturbation of the form $\tilde{\omega} = \omega - \lambda \eta$ for a generic 1-form $\eta$ on $\mathbb{C}^n$ (where $\lambda \neq 0$ is small enough), possesses this property.)

Proposition 3. The index $\text{ind}_{C,0} \omega$ of the 1-form $\omega$ on the icis $V$ at the origin is equal to the number of zeros of $\omega$ on $V_\varepsilon$, counted with multiplicities. It is also equal to the number of zeros of $\tilde{\omega}$ on $V_\varepsilon$ for a small perturbation $\tilde{\omega}$ of the 1-form $\omega$ with only nondegenerate zeros on $V_\varepsilon$.

Proof. Let $P_1, \ldots, P_\nu$ be the zeros of the form $\omega$ on $V_\varepsilon$. In local coordinates $y_1, \ldots, y_k, y_{k+1}, \ldots, y_n$ centred at the point $P_i$ such that $V_\varepsilon = \{y_1 = \cdots = y_k = 0\}$ (we can take $y_i = f_i - \varepsilon_i$ for $1 \leq i \leq k$), let

$$\omega|_{V_\varepsilon} = C_{k+1} dy_{k+1} + \cdots + C_n dy_n.$$

Take a small open ball $B_i$ centred at the point $P_i$. The degree of the map

$$(\omega, df_1, \ldots, df_k): \partial B_i \cap V_\varepsilon \to W_{k+1}(\mathbb{C}^n)$$

is equal to the degree of the map $S^{2(n-k)-1} \to S^{2(n-k)-1}$ given by

$$x \mapsto (C_{k+1}, \ldots, C_n)/\| (C_{k+1}, \ldots, C_n)\|;$$

thus it is equal to the multiplicity $\mu_i$ of the zero $P_i$ (see, e.g., [AGV]). Now, consider the manifold

$$M := V_\varepsilon \setminus \bigcup_i B_i.$$ 

Since the 1-form $\omega$ has no zeros on $M$, the degree of the mapping

$$(\omega, df_1, \ldots, df_k): \partial M \to W_{k+1}(\mathbb{C}^n)$$

is equal to zero. This implies

$$\text{ind}_{C,0} \omega = \sum_{i=1}^\nu \mu_i.$$ 

If $\tilde{\omega}$ is a perturbation of the 1-form $\omega$ with only nondegenerate zeros on $V_\varepsilon$, then it has precisely $\mu_i$ nondegenerate zeros on $B_i \cap V_\varepsilon$. \qed
3. An algebraic formula for the index

Our aim in this section is to derive the following algebraic formula for the complex index.

**Theorem 1.** Let $\mathcal{O}_{\mathbb{C}^n,0}$ be the algebra of germs of holomorphic functions at the origin in $\mathbb{C}^n$, and let $I \subset \mathcal{O}_{\mathbb{C}^n,0}$ be the ideal generated by $f_1, \ldots, f_k$ and the $(k+1) \times (k+1)$ minors of the matrix

$$
\left(\begin{array}{ccc}
\frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\cdots & \cdots & \cdots \\
\frac{\partial f_k}{\partial x_1} & \cdots & \frac{\partial f_k}{\partial x_n} \\
A_1 & \cdots & A_n
\end{array}\right).
$$

Then

$$\text{ind}_{\mathbb{C},0} \omega = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C},0}/I.$$

**Remark.** A similar result for the case when the 1-form $\omega$ is the differential $df_{k+1}$ of a function $f_{k+1}$ was first proven by Lê Dũng Tráng [Le] and Greuel [G, Lemma 1.9].

Greuel informed us that arguments used in that paper can be applied to prove Theorem 1 as well.

The proof relies on the basic fact stated below.

Let $F : (\mathbb{C}^n \times \mathbb{C}^M, 0) \to (\mathbb{C}^k \times \mathbb{C}^M, 0)$ be a versal deformation of $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^k, 0)$, and let $N := n + M$, $K := k + M$. By the same symbol $F$ we denote a representative $U \to \mathbb{C}^K$ of this deformation defined in a small open neighbourhood $U \subset \mathbb{C}^N$ of the origin. Let $A_i = 0$ for $i = n + 1, \ldots, N$, and let $\eta = \sum_{i=1}^N B_i dx_i$ be a 1-form such that, for a regular value $s \in F(U)$ of $F$ and for $\lambda \in W$, $\lambda \neq 0$, where $W \subset \mathbb{C}$ is a suitable small open neighbourhood of the origin, the form $\omega - \lambda \eta$ has only isolated zeros on $F^{-1}(s)$.

Consider the matrix

$$
\Phi = \left(\begin{array}{ccc}
\frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_N} \\
\cdots & \cdots & \cdots \\
\frac{\partial F_k}{\partial x_1} & \cdots & \frac{\partial F_k}{\partial x_N} \\
A_1 - \lambda B_1 & \cdots & A_N - \lambda B_N
\end{array}\right).
$$

Consider the vector space $\mathbb{C}^{N+1}$ with coordinates $x_1, \ldots, x_N, \lambda$. By $\mathcal{C}(U \times W)$ we denote the ideal in $\mathcal{O}_{\mathbb{C}^{N+1}}(U \times W)$ generated by the $(K+1) \times (K+1)$ minors of the matrix $\Phi$. Let $\mathcal{C} \subset \mathcal{O}_{\mathbb{C}^{N+1}}$ denote the corresponding ideal sheaf, and let $C \subset U \times W$ be the analytic space defined by $\mathcal{C}$. Then $C$ consists of the points $x \in U \times W$ such that $F$ is not a submersion at $x$ or $\omega - \lambda \eta$ does not have an isolated zero at $x \in F^{-1}(F(x))$. Let $\mathcal{O}_C := \mathcal{O}_{\mathbb{C}^{N+1}}/\mathcal{C}$ be the structure sheaf of $C$.

Let $\tilde{F} : U \times C \to \mathbb{C}^{K+1}$ be the mapping defined by $(x, \lambda) \mapsto (F(x), \lambda)$. We define $\Sigma := \tilde{F}(C)$ and $\pi := \tilde{F}|_C : C \to \Sigma$. Then $\Sigma$ is an analytic space endowed with the structure sheaf $\mathcal{O}_\Sigma := \mathcal{O}_{\mathbb{C}^{N+1}}/\mathcal{F}_0(\pi_*(\mathcal{O}_C))$ (restricted to $\pi(C)$), where $\mathcal{F}_0$ denotes the 0-th Fitting ideal (cf. [L, 4.E]). The mapping $\pi : C \to \Sigma$ is a finite mapping.

Theorem 1 is implied by the following theorem.

**Theorem 2.** For every $x \in C$, $\mathcal{O}_{C,x}$ is a flat $\mathcal{O}_{\Sigma,x}$-module.
Proposition. The proof follows the same lines as [L, (4.4) and (4.8)].

Let \( x \in C \) and \( s = \pi(x) \). The minors of the matrix \( \Phi \) vanish at a point \( x \in U \times W \) if and only if the rank of the matrix \( \Phi(x) \) is not maximal. The set of \( N \times (K + 1) \) matrices with complex entries with rank \( < K + 1 \) is an affine algebraic variety of codimension \( N - K \) in the set of all \( N \times (K + 1) \) matrices with complex entries. Hence \( C \) has dimension \( K + 1 \). Therefore, \( \text{depth}(\mathcal{C}_x; \mathcal{O}_{\Sigma^{K+1},s}) = N - K \). By [BR, Corollary (2.7)], this implies that the homological dimension \( \text{hd} \mathcal{C}_x \) of \( \mathcal{C}_x \) is equal to \( N - K \).

By the formula of Auslander–Buchsbaum (see, e.g., [Ma, p. 114]), we have
\[
\text{depth} \mathcal{O}_{C,x} = N + 1 - (N - K) = K + 1.
\]
Since \( \text{dim} \mathcal{O}_{C,x} = K + 1 \), it follows that \( \mathcal{O}_{C,x} \) is a Cohen–Macaulay ring.

Since \( \bar{F}|_C : C \to \mathbb{C}^{K+1} \) is a finite mapping, \( \mathcal{O}_{C,x} \) is a finite \( \mathcal{O}_{\Sigma^{K+1},s} \)-module. Therefore, \( \mathcal{O}_{C,x} \) is a Cohen–Macaulay module over \( \mathcal{O}_{\Sigma^{K+1},s} \) and
\[
\text{depth}_{\mathcal{O}_{\Sigma^{K+1},s}} \mathcal{O}_{C,x} = \text{dim} \mathcal{O}_{C,x} = K + 1.
\]
Since \( \mathcal{O}_{\Sigma^{K+1},s} \) is a \( (K + 1) \)-dimensional regular ring, it follows from the Auslander–Buchsbaum formula that the homological dimension of \( \mathcal{O}_{C,x} \) as an \( \mathcal{O}_{\Sigma^{K+1},s} \)-module is 1. This means that there is an exact sequence of \( \mathcal{O}_{\Sigma^{K+1},s} \)-modules
\[
0 \to \mathcal{O}_{\Sigma^{K+1},s}^{q} \overset{\alpha}{\to} \mathcal{O}_{\Sigma^{K+1},s}^{p} \to \mathcal{O}_{C,x} \to 0.
\]
Moreover, \( q \) must be equal to \( p \) and the 0-th Fitting ideal of \( \mathcal{O}_{C,x} \) viewed as an \( \mathcal{O}_{\Sigma^{K+1},s} \)-module is generated by the determinant of \( \alpha \). Hence \( \Sigma \) is a hypersurface and \( \mathcal{O}_{\Sigma,s} \) is a Cohen–Macaulay ring, too.

Since \( \pi : C \to \Sigma \) is a finite mapping, \( \mathcal{O}_{C,x} \) is a finite \( \mathcal{O}_{\Sigma,s} \)-module. But a finitely generated \( \mathcal{O}_{\Sigma,s} \)-module is flat if and only if it is free (see, e.g., [Ma, Proposition 3.G]). Therefore, it suffices to show that \( \mathcal{O}_{C,x} \) is a free \( \mathcal{O}_{\Sigma,s} \)-module. By the Auslander–Buchsbaum formula,
\[
\text{hd} \mathcal{O}_{\Sigma,s} \mathcal{O}_{C,x} + \text{depth} \mathcal{O}_{\Sigma,s} \mathcal{O}_{C,x} = \text{depth} \mathcal{O}_{\Sigma,s}.
\]
Since we have \( \text{depth} \mathcal{O}_{\Sigma,s} \mathcal{O}_{C,x} = \text{depth} \mathcal{O}_{\Sigma,s} = \text{dim} \mathcal{O}_{\Sigma,s} = K + 1 \), it follows that \( \text{hd} \mathcal{O}_{\Sigma,s} \mathcal{O}_{C,x} = 0 \). But this means that \( \mathcal{O}_{C,x} \) is a free \( \mathcal{O}_{\Sigma,s} \)-module. \( \square \)

Proof of Theorem 1. Consider again the mapping \( \pi : C \to \Sigma \) and let \( s \in \Sigma \). For each \( x \in C(s) := \pi^{-1}(s) \), \( \mathbb{C} \otimes_{\mathcal{O}_{\Sigma,s}} \mathcal{O}_{C,x} \) is a finite-dimensional vector space over \( \mathbb{C} \). We denote its dimension by \( \nu(x) \) and define
\[
\nu(s) = \sum_{x \in C(s)} \nu(x).
\]
By Theorem 2 and [D, §5, Theorem 1], \( \nu(s) \) is a locally constant function of \( s \). Now, for a point \( s = (s', \lambda) \) where \( s' \) is a regular value of \( F \), \( \nu(s) = \text{ind}_{\mathbb{C},0} \omega \) by Proposition 3. On the other hand, \( \nu(0) = \text{dim}_{\mathbb{C}} \mathcal{O}_{\Sigma,0}/I \). \( \square \)
Now, let us discuss the index of a real analytic 1-form $\omega$ on a real closed interval $(V, 0) = \{f_1 = \cdots = f_k = 0\} \subset \mathbb{R}^n, 0$ (with an algebraically isolated singular point at the origin). The 1-form $\omega$ on $V$ is the restriction of an analytic 1-form $\sum_{i=1}^{n} A_i \, dx_i$ defined on $\mathbb{R}^n$ in a neighbourhood of the origin (we denote this 1-form by the same symbol $\omega$). We consider the (analytic) functions $f_1, \ldots, f_k$ and the 1-form $\omega$ as being defined in a neighbourhood of the origin in $\mathbb{C}^n$ as well. Let $\delta$ be a positive number small enough for the functions $f_1, \ldots, f_k$ and the 1-form $\omega$ to be defined on the ball $B_{\delta}^c$ of radius $\delta$ centred at the origin in $\mathbb{C}^n$ and for the variety $V_{\mathbb{C}} = \{f_1 = \cdots = f_k = 0\} \subset (\mathbb{C}^n, 0)$ to intersect the sphere $S_{\mathbb{R}}^n$ of any positive radius $\delta' < \delta$ centred at the origin transversally. For a sufficiently small $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k)$ ($|\varepsilon| \ll \delta$), we set $V_{\varepsilon, \mathbb{C}} = \{f = \varepsilon\} \cap B_\delta \subset \mathbb{C}^n$ and, if $\varepsilon$ is real, $V_\varepsilon = V_{\varepsilon, \mathbb{C}} \cap \mathbb{R}^n$. Let $\Sigma \subset (\mathbb{C}^k, 0)$ be the bifurcation diagram of the map $f$ (the set of $\varepsilon \in (\mathbb{C}^k, 0)$ critical for $f$), and let $\Sigma_\mathbb{R} = \Sigma \cap (\mathbb{R}^k, 0)$.

By Proposition 2, for each real $\varepsilon$ outside the bifurcation diagram (i.e., for an $\varepsilon \in \mathbb{R}^k \setminus \Sigma_\mathbb{R}$), the (real) index $\text{ind}_0 \omega$ differs from the sum of indices of zeros of the 1-form $\omega$ on the real smooth manifold $V_\varepsilon$ by $(\chi(V_\varepsilon) - 1)$. By Proposition 3, the complex index $\text{ind}_{\mathbb{C}, 0} \omega$ counts the number of zeros of the same 1-form on the complex manifold $V_{\varepsilon, \mathbb{C}}$. The Euler characteristics $\chi(V_\varepsilon)$ of the real level manifolds $V_\varepsilon$ are different for different $\varepsilon \in \mathbb{R}^k \setminus \Sigma_\mathbb{R}$ (at least for even $n - k$). It is constant on each component of the complement to the bifurcation diagram $\Sigma_\mathbb{R}$. Thus we cannot expect that the (real) index $\text{ind}_0 \omega$ of the 1-form $\omega$ is the signature of a nondegenerate quadratic form on a vector space of dimension $\text{ind}_{\mathbb{C}, 0} \omega = (\dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0}) I$.

Such a signature may be equal to $\text{ind}_0 \omega + (\chi(V_\varepsilon) - 1)$ and, thus, must take different values for different components of the complement to the bifurcation diagram. We want to obtain a somewhat finer

Theorem 3. There exists a family $Q_\varepsilon$ of quadratic forms on the space $\mathcal{C}^L$ of dimension $L = \dim_{\mathbb{C}} \mathcal{O}_{\mathbb{C}^n, 0} / I$ (i.e., a family of symmetric $L \times L$ matrices) defined for $\varepsilon \in \mathbb{C}^k$ from a neighbourhood of the origin and analytically dependent on $\varepsilon$ such that:

1. the quadratic form $Q_\varepsilon$ is nondegenerate for $\varepsilon$ from the complement to the bifurcation diagram $\Sigma$;
2. for a real $\varepsilon$, the quadratic form (i.e., the matrix) $Q_\varepsilon$ is real and, for a real $\varepsilon$ outside the bifurcation diagram (i.e., for $\varepsilon \in \mathbb{R}^k \setminus \Sigma_\mathbb{R}$), its signature is equal to

$$\sum_{\varepsilon \in \Sigma_\mathbb{R}} \text{ind}_\varepsilon \omega = \text{ind}_0 \omega + (\chi(V_\varepsilon) - 1).$$

Proof. It is convenient to define the family $Q_\varepsilon$ for a larger space of parameters. Let $F : (\mathbb{C}^n \times \mathbb{C}^M, 0) \rightarrow (\mathbb{C}^k \times \mathbb{C}^M, 0)$ be a real (i.e., invariant with respect to complex conjugation) versal deformation of $f : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}^k, 0), F(x, \varepsilon') = (f_\varepsilon(x), \varepsilon')$, $f_0 = f$. Here $\varepsilon_{k+1}, \ldots, \varepsilon_{k+M}$ are the coordinates on $\mathbb{C}^M, \varepsilon' = (\varepsilon_{k+1}, \ldots, \varepsilon_{k+M}) \in \mathbb{C}^M$. Suppose that $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_{k+M}) \in \mathbb{C}^{k+M} = \mathbb{C}^k \times \mathbb{C}^M, \mathbb{C}^n_\alpha$ is the $n$-dimensional affine space with the coordinates $\alpha_1, \ldots, \alpha_n$ and $\varepsilon = (\bar{\varepsilon}, \alpha)$ ($\varepsilon \in \mathbb{C}^{k+M+n} = \mathbb{C}^{k+M} \times \mathbb{C}^n_\alpha$). We use the same letter $F$ to denote the trivial
extension \((\mathbb{C}_x \times \mathbb{C}_x' \times \mathbb{C}_x^0, 0) \rightarrow (\mathbb{C}_x^k \times \mathbb{C}_x^M \times \mathbb{C}_x^0, 0) = (\mathbb{C}^{k+M+n}, 0)\) of the chosen versal deformation: \(F(x, \varepsilon, \alpha) = (f_\varepsilon(x), \varepsilon, \alpha)\). Let \(\Omega\) be the 1-form on \((\mathbb{C}_x^k \times \mathbb{C}_x^M \times \mathbb{C}_x^0, 0)\) defined by

\[
\Omega = \omega - \sum_{i=1}^{n} \alpha_i \, dx_i = \sum_{i=1}^{n} (A_i - \alpha_i) \, dx_i.
\]

Again, we use the same symbols \(\Sigma\) and \(\Sigma_n\) as above to denote the complex and the real bifurcation sets of the map \(F\) (the real one being the intersection of the complex one with the real space \(\mathbb{R}^{k+M+n}\)) and define \(V_{c, \Sigma} := F^{-1}(\Sigma) \cap B_3 \subset \mathbb{C}^{n+M+n}\) and, if \(\varepsilon\) is real, \(V_{\varepsilon, \Sigma} := V_{c, \Sigma} \cap \mathbb{R}^{n+M+n}\) (in fact, \(V_{c, \Sigma}\) is a subvariety of the \(n\)-dimensional affine space \(\mathbb{C}_x^n \times \{(\varepsilon, \alpha)\}\)).

For a point \(P \in \mathbb{C}_x^k \times \mathbb{C}_x^M \times \mathbb{C}_x^0\), let

\[
\Delta = \Delta(P) := \left| \begin{array}{ccc}
\frac{\partial f_{\varepsilon 1}}{\partial x_1} & \cdots & \frac{\partial f_{\varepsilon 1}}{\partial x_k} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_{\varepsilon k}}{\partial x_1} & \cdots & \frac{\partial f_{\varepsilon k}}{\partial x_k}
\end{array} \right|,
\]

where \(f_{\varepsilon i}\) \((i = 1, \ldots, k)\) are the components of the map \(f_{\varepsilon}: f_{\varepsilon} = (f_{\varepsilon 1}, \ldots, f_{\varepsilon k})\).

If \(P \in V_{c, \Sigma}\) and \(\Delta(P) \neq 0\), the variables \(x_{k+1}, \ldots, x_n\) are local coordinates on the \((n-k)\)-dimensional variety \(V_{c, \Sigma}\) in a neighbourhood of the point \(P\). For a generic \(\varepsilon \in (\mathbb{C}^{k+M+n}, 0) \setminus \Sigma\), we have \(\Delta(P) \neq 0\) for all zeros \(P\) of the restriction of the 1-form \(\Omega\) to the level manifold \(V_{c, \Sigma}\). Moreover, for a generic \(\varepsilon \in (\mathbb{C}^{k+M+n}, 0) \setminus \Sigma\) which does not possess this property (such \(\varepsilon\)'s form a subset of codimension 1), all zeros \(P\) of the restriction of the 1-form \(\Omega\) to the level manifold \(V_{c, \Sigma}\) are simple (i.e., nondegenerate).

For a local system of coordinates \(y_1, \ldots, y_m\) on a manifold \(M\), the Hessian \(h = h_\omega\) of the 1-form \(\omega\) equal to \(A_1 \, dy_1 + \cdots + A_m \, dy_m\) in these coordinates is defined as the determinant

\[
\left| \frac{\partial A_i}{\partial y_j} \right|_{i,j=1,\ldots,m}.
\]

(The Hessian is a function on the manifold \(M\); it depends on the choice of the coordinates.)

For \(i = k + 1, \ldots, n\), let

\[
m_i := \left| \begin{array}{ccc}
\frac{\partial f_{\varepsilon 1}}{\partial x_1} & \cdots & \frac{\partial f_{\varepsilon 1}}{\partial x_k} & \frac{\partial f_{\varepsilon 1}}{\partial x_i} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{\partial f_{\varepsilon k}}{\partial x_1} & \cdots & \frac{\partial f_{\varepsilon k}}{\partial x_k} & \frac{\partial f_{\varepsilon k}}{\partial x_i} \\
A_1 - \alpha_1 & \cdots & A_k - \alpha_k & A_i - \alpha_i
\end{array} \right|.
\]

For a point \(P \in V_{c, \Sigma}\) with \(\Delta(P) \neq 0\), let \(h(P)\) be the Hessian of the 1-form \(\Omega|_{V_{c, \Sigma}}\) in the (local) coordinates \(x_{k+1}, \ldots, x_n\).
Proposition 4. The Hessian $h$ of the 1-form $\Omega|_{V_C,\kappa}$ in the coordinates $x_{k+1}, \ldots, x_n$ is given by the formula

$$h = \frac{1}{\Delta_{2^{(n-k)}}} \begin{vmatrix} \Delta & \frac{\partial \Delta}{\partial x_1} & \cdots & \frac{\partial \Delta}{\partial x_n} \\ 0 & \frac{\partial f_{\varepsilon 1}}{\partial x_1} & \cdots & \frac{\partial f_{\varepsilon 1}}{\partial x_n} \\ 0 & \frac{\partial f_{\varepsilon 2}}{\partial x_1} & \cdots & \frac{\partial f_{\varepsilon 2}}{\partial x_n} \\ \vdots & \ddots & \ddots & \vdots \\ m_{k+1} & \frac{\partial m_{k+1}}{\partial x_1} & \cdots & \frac{\partial m_{k+1}}{\partial x_n} \\ m_n & \frac{\partial m_n}{\partial x_1} & \cdots & \frac{\partial m_n}{\partial x_n} \end{vmatrix}. $$

Lemma 2. In the coordinates $x_{k+1}, \ldots, x_n$ the 1-form $\Omega|_{V_C,\kappa}$ can be expressed as

$$\Omega|_{V_C,\kappa} = \frac{m_{k+1}}{\Delta} dx_{k+1} + \cdots + \frac{m_n}{\Delta} dx_n.$$

Proof. We have

$$0 = df_{\varepsilon 1} = \frac{\partial f_{\varepsilon 1}}{\partial x_1} dx_1 + \cdots + \frac{\partial f_{\varepsilon 1}}{\partial x_n} dx_n,$$

$$0 = df_{\varepsilon k} = \frac{\partial f_{\varepsilon k}}{\partial x_1} dx_1 + \cdots + \frac{\partial f_{\varepsilon k}}{\partial x_n} dx_n.$$

By Cramer’s rule

$$dx_1 = (-1)^k \frac{1}{\Delta} \left( \frac{\partial (f_{\varepsilon 1}, \ldots, f_{\varepsilon k})}{\partial (x_2, \ldots, x_k, x_{k+1})} dx_{k+1} + \cdots + \frac{\partial (f_{\varepsilon 1}, \ldots, f_{\varepsilon k})}{\partial (x_2, \ldots, x_k, x_n)} dx_n \right),$$

$$dx_k = -\frac{1}{\Delta} \left( \frac{\partial (f_{\varepsilon 1}, \ldots, f_{\varepsilon k})}{\partial (x_1, \ldots, x_{k-1}, x_{k+1})} dx_{k+1} + \cdots + \frac{\partial (f_{\varepsilon 1}, \ldots, f_{\varepsilon k})}{\partial (x_1, \ldots, x_{k-1}, x_n)} dx_n \right).$$

Substitution in the form $\Omega$ yields

$$\Omega|_{V_C,\kappa} = \frac{m_{k+1}}{\Delta} dx_{k+1} + \cdots + \frac{m_n}{\Delta} dx_n. \quad \square$$

By some abuse of notation, we shall denote the partial derivatives in the coordinates $x_{k+1}, \ldots, x_n$ of functions on the manifold $V_{C,\kappa}$ by $\frac{d}{dx_j} (j = k+1, \ldots, n)$ (in order to distinguish them from the partial derivatives of the corresponding functions on $\mathbb{C}^n$). Thus the Hessian $h$ is the determinant of the matrix

$$\left( \frac{d}{dx_j} \left( \frac{m_i}{\Delta} \right) \right)_{i,j=k+1,\ldots,n}.$$

For $1 \leq \ell \leq k$ and $k+1 \leq j \leq n$, we have

$$\frac{dx_\ell}{dx_j} = (-1)^{k+\ell+1} \frac{1}{\Delta} \frac{\partial (f_{\varepsilon 1}, \ldots, f_{\varepsilon k})}{\partial (x_1, \ldots, x_{\ell-1}, x_{\ell+1}, \ldots, x_k, x_j)}. $$
Therefore we get
\[
\frac{d}{dx_j} \left( \frac{m_i}{\Delta} \right) = \frac{\partial}{\partial x_j} \left( \frac{m_i}{\Delta} \right) + \frac{\partial}{\partial x_1} \left( \frac{m_i}{\Delta} \right) \frac{dx_1}{dx_j} + \cdots + \frac{\partial}{\partial x_k} \left( \frac{m_i}{\Delta} \right) \frac{dx_k}{dx_j}
\]
\[
= \frac{1}{\Delta^3} \left( \Delta \frac{\partial m_i}{\partial x_j} \Delta - m_i \frac{\partial \Delta}{\partial x_j} \Delta + (-1)^{k} \frac{\partial m_i}{\partial x_1} \frac{\partial (f_{e^1}, \ldots, f_{e^k})}{\partial (x_2, \ldots, x_k, x_j)}
\right.
\]
\[
\left. - (-1)^{k} m_i \frac{\partial \Delta}{\partial x_k} \frac{\partial (f_{e^1}, \ldots, f_{e^k})}{\partial (x_1, \ldots, x_{k-1}, x_j)} + m_i \frac{\partial \Delta}{\partial x_k} \frac{\partial (f_{e^1}, \ldots, f_{e^k})}{\partial (x_1, \ldots, x_{k-1}, x_j)} \right)
\]
\[
= \frac{1}{\Delta^3} \begin{vmatrix}
\Delta & \frac{\partial \Delta}{\partial x_1} & \cdots & \frac{\partial \Delta}{\partial x_k} \\
0 & \frac{\partial f_{e^1}}{\partial x_1} & \cdots & \frac{\partial f_{e^k}}{\partial x_k} \\
m_i & \frac{\partial m_i}{\partial x_1} & \cdots & \frac{\partial m_i}{\partial x_k}
\end{vmatrix}
\]

Now, Proposition 4 is implemented by the following two lemmas.

**Lemma 3.** Let $A$ be an invertible $l \times l$ matrix, $B$ be an $l \times m$ matrix, $C$ be an $m \times l$ matrix, and $D$ be an $m \times m$ matrix. Then
\[
\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \cdot \det (D - CA^{-1}B).
\]

**Proof.** Let $E_m$ denote the $m \times m$ identity matrix. Then we have
\[
\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det A \cdot \det \begin{pmatrix} A^{-1} & 0 \\ 0 & E_m \end{pmatrix} \cdot \det \begin{pmatrix} A & B \\ C & D \end{pmatrix}
\]
\[
= \det A \cdot \det \begin{pmatrix} E_l & A^{-1}B \\ C & D \end{pmatrix}
\]
\[
= \det A \cdot \det \begin{pmatrix} E_l & A^{-1}B \\ C - CE_l & D - CA^{-1}B \end{pmatrix}
\]
\[
= \det A \cdot \det \begin{pmatrix} E_l & A^{-1}B \\ 0 & D - CA^{-1}B \end{pmatrix}
\]
\[
= \det A \cdot \det (D - CA^{-1}B). \quad \square
\]

**Lemma 4.** Let $A$ be an invertible $l \times l$ matrix, $B$ be an $l \times m$ matrix, $C$ be an $m \times l$ matrix, and $D = (d_{ij})$ be an $m \times m$ matrix. Let $c^i$ denote the $i$-th row of $C$ and $b_j$ the $j$-th column of $B$. Finally, let $H$ be the matrix
\[
\begin{pmatrix} A & b_j \\ c^i & d_{ij} \end{pmatrix}_{i,j=1,\ldots,m}.
\]

Then
\[
\det H = (\det A)^{m-1} \begin{vmatrix} A & B \\ C & D \end{vmatrix}.
\]
\[
\square
\]
Proof. By Lemma 3,

\[ H = (\det A(d_{ij} - c^i A^{-1} b_j))_{i,j=1,...,m}. \]

Another application of Lemma 3 yields

\[ \det H = (\det A)^m \det (D - CA^{-1} B) = (\det A)^{m-1} \begin{vmatrix} A & B \\ C & D \end{vmatrix}. \]

For any \( \kappa \in \mathbb{C}^{k+M+n} \) and any singular point \( P \) of the 1-form \( \Omega \) on the (possibly singular) level variety \( V_{\mathbb{C},\kappa} \) (i.e., for a singular point of \( V_{\mathbb{C},\kappa} \) or for a zero of the 1-form on its smooth part), let \( I_P \) be the ideal of \( \mathcal{O}_{P,x} \) generated by the functions \( f_{\varepsilon_1} - \varepsilon_1 \) \( (i = 1, \ldots, k) \) and by the \((k+1) \times (k+1)\) minors of the matrix

\[
\begin{pmatrix}
\frac{\partial f_{\varepsilon_1}}{\partial x_1} & \cdots & \frac{\partial f_{\varepsilon_1}}{\partial x_n} \\
\vdots & \ddots & \vdots \\
\frac{\partial f_{\varepsilon_k}}{\partial x_1} & \cdots & \frac{\partial f_{\varepsilon_k}}{\partial x_n} \\
A_1 - \alpha_1 & \cdots & A_n - \alpha_n
\end{pmatrix}.
\]

Let \( J_P = \mathcal{O}_{P,x}/I_P \). From the proof of Theorem 1 it follows that:

(1) \( \sum_{P \in V_{\mathbb{C},\kappa}} \dim_{\mathbb{C}} J_P = \text{const} = \dim_{\mathbb{C}} J_0; \)

(2) if \( \varphi_1, \ldots, \varphi_L \) are real elements of the ring \( \mathcal{O}_{\mathbb{C},0} \) which are representatives of a basis of the factor-algebra \( J_0 \) (considered as analytic functions defined in a common neighbourhood of the origin in \( \mathbb{C}^n \)), then, for sufficiently small \( \kappa \), the multigerms of the functions \( \varphi_1, \ldots, \varphi_L \) are representatives of a basis of the algebra \( J_{\kappa} = \bigoplus_{P \in V_{\mathbb{C},\kappa}} J_P \).

Thus the algebras \( J_{\kappa} \) form a trivial vector bundle over a neighbourhood of the origin in \( \mathbb{C}^{k+M+n} \), and the choice of (real) representatives of the elements of the basis of the algebra \( J_0 \) fixes a trivialization of this bundle compatible with the real structure.

As it is usual in similar situations, a quadratic form \( Q_{\kappa} \) on the algebra \( J_{\kappa} \) is defined by the formula

\[ Q_{\kappa}(\psi_1, \psi_2) = \ell_{\kappa}(\psi_1 \psi_2), \]

where \( \ell_{\kappa} \) is a linear function on the vector space \( J_{\kappa} \), and \( \psi_1 \psi_2 \) means the product in the algebra \( J_{\kappa} \).

Let \( \kappa \in (\mathbb{C}^{k+M+n}, 0) \) be such that \( \Delta(P) \neq 0 \) for all singular points \( P \) of the 1-form \( \Omega|_{V_{\mathbb{C},\kappa}} \). In particular, this means that \( \kappa \) is not contained in the discriminant \( \Sigma \) of the deformation and the Hessian \( h \) is defined in a neighbourhood of each singular point \( P \). For a singular point \( P \) of the 1-form \( \Omega|_{V_{\mathbb{C},\kappa}} \), let \( h_P := h \cdot \Delta(P)^2 \). For \( \psi \in J_{\kappa} \), we define \( \ell_{\kappa}(\psi) \) as

\[ \ell_{\kappa}(\psi) = \sum_{P \in V_{\mathbb{C},\kappa}} \ell_P(\psi), \]

where \( \ell_P(\psi) = 0 \) for \( \psi \) from the summand \( J_P \) with \( P' \neq P \),

\[ \ell_P(\psi) = \lim_{\beta \to 0} \sum_{\beta P(a_i)} \psi(a_i), \]

\[ \beta P(a_i) = P(a_i) - \beta P. \]
for \( \psi \in J_P \), \( \beta = \beta_{k+1} dx_{k+1} + \cdots + \beta_n dx_n \), and the sum is over all simple (i.e., nondegenerate) zeros of the 1-form \( \Omega - \beta \) which emerge from the (generally, degenerate) zero \( P \) for a generic \( \beta \). The expression under the limit sign is defined for generic \( \beta \) (for a generic \( \beta \), the form \( \Omega - \beta \) has only nondegenerate zeros). The known properties of the similar objects for the smooth case (see, e.g., [AGV, § 5]) imply that this expression tends to a finite limit as \( \beta \) approaches zero (as an element of \( \mathbb{C}^{n-k} \)), this limit depends analytically on \( \kappa \), the corresponding quadratic form is nondegenerate, it becomes real when \( \kappa \) is real, and, in the last case, the signature of the corresponding real quadratic form is equal to

\[
\sum_{P \in V_c} \text{ind}_P \omega = \text{ind}_0 \omega + (\chi(V_c) - 1).
\]

At the moment, the required linear functions \( \ell_\kappa \) (and thus the quadratic forms \( Q_\kappa \)) are defined for \( \kappa \) outside the discriminant \( \Sigma \) and the set \( \Xi \) of those \( \kappa \) for which \( V_{C, \kappa} \) contains a zero \( P \) of the form \( \Omega|_{V_{C, \kappa}} \) with \( \Delta(P) = 0 \) (both \( \Sigma \) and \( \Xi \) are hypersurfaces in \( \mathbb{C}^{k+M+n} \)). Let us show that, in fact, this family of linear functions can be continued analytically to these two subsets as well. We should control the continuation to the last subset so that the quadratic form \( Q_\kappa \) does not degenerate there.

To show that the constructed linear functions \( \ell_\kappa \) have analytic continuations to the set \( \Xi \setminus \Sigma \) of those \( \kappa \in \mathbb{C}^{k+M+n} \setminus \Sigma \) for which there exists a zero \( P \) of the form \( \Omega|_{V_{C, \kappa}} \) with \( \Delta(P) = 0 \), it is sufficient to prove that, for a nondegenerate zero \( P \) of the form \( \Omega|_{V_{C, \kappa}} \) with \( \Delta(P) = 0 \), the linear function \( \tilde{\ell}_P(\psi) \) tends to a finite limit different from zero as \( P' \) approaches \( P \), where \( P' \) is a singular point of the 1-form \( \Omega|_{V_{C, \ell(P')}}, \) such that \( F(P') \notin \Xi \). In this case, \( \dim_{\mathbb{C}} J_P = \dim_{\mathbb{C}} J_{P'} = 1 \), the vector spaces \( J_P \) and \( J_{P'} \) are generated by one element \( \varphi \equiv 1 \), and \( \tilde{\ell}_P(1) = \frac{1}{\tilde{h}_P(P')} \); so it is sufficient to show that \( \tilde{h}_P(P') \) tends to a finite nonzero limit as \( P' \to P \). Let \( \sigma = \{ j_1, \ldots, j_n \} \) be a permutation of the indices \( 1, \ldots, n \) such that the Jacobian \( \Delta' \) of the functions \( f_1, \ldots, f_n \) with respect to the variables \( x_{j_1}, \ldots, x_{j_n} \) at the point \( P \) (and, thus, at all points \( P' \) close to \( P \)) is different from zero. In this case, \( x_{j_{k+1}}, \ldots, x_{j_n} \) are local coordinates on the manifold \( V_{C, \kappa} \) at the point \( P \) and, thus, at all points \( P' \) close to \( P \) (on the corresponding level manifold). At a point \( P' \) close to \( P \) and such that \( \Delta(P') \neq 0 \) (where \( x_{k+1}, \ldots, x_n \) are also local coordinates), the Jacobian of the coordinate change \( x_{k+1}, \ldots, x_n \to x_{j_{k+1}}, \ldots, x_{j_n} \) is equal to \( \text{sgn}(\sigma) \cdot \frac{\Delta'(P')}{\Delta(P')} \). Thus the value of the Hessian of the restriction of the form \( \Omega \) to the corresponding level manifold in the coordinates \( x_{j_{k+1}}, \ldots, x_{j_n} \) at the point \( P' \) is equal to

\[
\frac{h(P')\Delta(P')^2}{\Delta'(P')^2}
\]

and, therefore, differs from \( \tilde{h}(P') \) by a nonzero analytic factor. This finishes the proof in this case.

To show that the constructed linear functions \( \ell_\kappa \) have analytic continuations to the discriminant \( \Sigma \), it is sufficient to prove the following. Let \( P \) be a point at which the corresponding level variety \( V_{C, \kappa} \) has a singularity of type \( A_1 \), the 1-form
$\Omega$ (as a 1-form on $\mathbb{C}^{n+M+n}$) does not tend to zero, and the zero hyperplane of it is in general position with respect to the tangent cone of the variety $V_{\xi,\kappa}$ at the point $P$. In this case, $\dim_{\mathbb{C}} J_P = 2$. In a neighborhood of $x$ take a $\tilde{\kappa}$ of the form $(\tilde{\epsilon}_1, \ldots, \tilde{\epsilon}_k, \xi_{k+1}, \ldots, \xi_{k+M}, \alpha_1, \ldots, \alpha_n)$ ($\tilde{\kappa}$ differs from $x$ only by the first $k$ coordinates, which are the values of the functions $f_1, \ldots, f_k$) such that $\tilde{\kappa} \notin \Sigma$. The 1-form $\Omega|_{V_{\xi,\kappa}}$ has two nondegenerate zeros $P_1 = P_1(\tilde{\kappa})$ and $P_2 = P_2(\tilde{\kappa})$. It is sufficient to show that the linear function $\tilde{\ell}_{P_1} + \tilde{\ell}_{P_2}$ tends to a finite limit as $\tilde{\kappa} \to x$.

Without loss of generality, we can suppose that $k = 1$, $n \geq 2$, $P$ is the origin in $\mathbb{C}^n$, $f_1 = x_1^3 + \cdots + x_n^3$, $x = 0$, and $\omega(0) = dx_1$. The last equality means that $\omega = (1 + C_1)dx_1 + \cdots + C_n dx_n$, where $C_i \in \mathfrak{m}$, i.e., $C_i(0) = 0$. Let $\tilde{\kappa} = \epsilon^2$. As a basis of the algebra $J_P$ (treated as a vector space) and, thus, of the algebra $J_{P_1} \oplus J_{P_2}$ we can take $\varphi_1 \equiv 1$ and $\varphi_2 = x_1$. For the coordinates of the points $P_1$ and $P_2$, we have $x_1 = \pm \epsilon + o(\epsilon)$ and $x_i = o(\epsilon)$ for $i \geq 2$. Here and later on, all series are power series in $\epsilon$; thus, for example, $o(\epsilon)$ means $a_2 \epsilon^2 + \text{terms of higher degree}$. From Proposition 4, it follows that $h(P_1) = (-1)^{n-1}(\pm \epsilon)^{1-n} + \text{terms of higher degree}$ and, thus, $h(P_2) = (-1)^{n-1}(\pm \epsilon)^{3-n} + \cdots$. Therefore, $\tilde{\ell}_{P_1}(1) + \tilde{\ell}_{P_2}(1) = (-1)^{n-1}(\epsilon^{3-n} - (-\epsilon)^{3-n}) + \cdots$, $\tilde{\ell}_{P_1}(x_1) + \tilde{\ell}_{P_2}(x_1) = (-1)^{n-1}(\epsilon^{n-2} - (-\epsilon)^{n-2}) + \cdots$. Both expressions tend to finite limits as $\epsilon$ approaches zero (for $n = 2$, the terms $\epsilon^{-1}$ and $(-\epsilon)^{-1}$ sum up to zero).

**Remarks.** (1) To prove Theorem 3, actually, it was not necessary to make precise computations for the $A_1$ case. It is clear that the components of the linear function $\ell_\kappa$ (its values on the basis elements) can have only power asymptotics when $\epsilon \to 0$. Thus, multiplying the constructed functions by a suitably high power of the equation of the discriminant, we obtain the required family.

(2) The above calculations for the $A_1$-singularity show that, for $n - k = 1$ (i.e., for curves), the constructed family of quadratic forms does not degenerate anywhere (including the discriminant); in particular, the quadratic form $Q_0$ (defined on the algebra $J_0$) is nondegenerate, and its signature is equal to the same expression (2).

(3) In [Sz], a somewhat similar statement was proved; being reformulated in our terminology, it can be considered as the particular case where the 1-form $\omega$ is the differential $df_{k+1}$ of a function. However, the family of quadratic forms defined in [Sz] on the target space of the map $f : \mathbb{C}^n \to \mathbb{C}^k$ could degenerate at points $\epsilon \in \mathbb{C}^k$ for which there exists a zero $P$ of the 1-form $\omega$ on the level manifold $V_{\xi,\kappa}$ with $\Delta(P) = 0$ (and, maybe, somewhere else). (By the way, in some particular cases, all points $\epsilon$ from the target $\mathbb{C}^k$ may possess this property.) Thus our result somewhat improves the result of [Sz] for $\omega = df_{k+1} + \kappa$ as well.

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**References**

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