MORSE–SMALE CIRCLE DIFFEOMORPHISMS AND MODULI OF ELLIPTIC CURVES

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To V. I. Arnold for his 65th birthday with admiration

Abstract. To any circle diffeomorphism there corresponds, by a classical construction of V. I. Arnold, a one-parameter family of elliptic curves. Arnold conjectured that, as the parameter approaches zero, the modulus of the corresponding elliptic curve tends to the (Diophantine) rotation number of the original diffeomorphism. In this paper, we disprove the generalization of this conjecture to the case when the diffeomorphism in question is Morse–Smale. The proof relies on the theory of quasiconformal mappings.


Key words and phrases. Circle diffeomorphism, rotation number, moduli of elliptic curves, quasiconformal mappings.

1. Statement of the main result

A complex perturbation of an analytic circle diffeomorphism is a byholomorphic map whose factor space is an elliptic curve. V. I. Arnold conjectured that the modulus of this curve tends to the Diophantine rotation number of the original diffeomorphism as the perturbation parameter tends to zero [Ar2, 27A]. Arnold’s conjecture is proved in [M]; in this paper, we show that, for any Morse–Smale diffeomorphism of a circle, the generalization of this conjecture fails.

In what follows, all maps of the circle are assumed to be homeomorphic and orientation-preserving. Let $F: S^1 \to S^1$ be an analytic diffeomorphism of a circle. Let us lift $F$ to a diffeomorphism $\tilde{F}: \mathbb{R} \to \mathbb{R}$ of the real line. We denote the mapping $\tilde{F} + i\alpha$ from the lower boundary of the strip $\Pi_\alpha = \{z \in \mathbb{C}: 0 \leq \text{Im } z \leq \alpha\}$ to its upper boundary by $F_\alpha$. Gluing together the points of $\Pi_\alpha$ that differ by an integer and attaching the bases of the cylinder obtained to each other by the map $F_\alpha$, we get an elliptic curve $T_\alpha$ with the complex structure inherited from the covering strip. By a well-known theorem from complex analysis, the curve $T_\alpha$ is byholomorphically equivalent to $\mathbb{C}/(\mathbb{Z} + b_\alpha\mathbb{Z})$, where $\text{Im } b_\alpha > 0$ and $b_\alpha$ is well-defined up to an integer
(we always assume that the gluing $z \to z + 1$ corresponds to the generator 1 of the lattice $\mathbb{Z} + b \mathbb{Z}$). The family $\{T_\alpha, \alpha > 0\}$ is called the family of elliptic curves corresponding to $F$. We set $\mu_\alpha := b \alpha \pmod{\mathbb{Z}}$; $\mu_\alpha$ is called the modulus of the elliptic curve $T_\alpha$. Arnold conjectured that this modulus tends to the rotation number of the original diffeomorphism if the rotation number is Diophantine (for instance, this is obvious in the case when $F$ is a rigid rotation). This conjecture was proved in [M]. It turns out, though, that the conjecture cannot be generalized to Morse–Smale diffeomorphisms. Namely, the following theorem holds.

**Theorem 1.1.** Let $F$ be an orientation-preserving analytic Morse–Smale diffeomorphism of a circle. Then there exist positive numbers $C$ and $\alpha_0$ such that the moduli $\mu_\alpha$ of the elliptic curves $T_\alpha$ determined by the diffeomorphism $F$ as above lie in the domain $\text{Im} \mu_\alpha > C$ for all $\alpha \in (0; \alpha_0]$.

2. Quasiconformal mappings and the Lipschitz equivalence

In this section, we formulate some auxiliary statements, which we use to prove Theorem 1.1 in Sections 3 and 4.

**Lemma 2.1** [M]. Consider elliptic curves $T_1$ and $T_2$, $T_i = \mathbb{C}/\Gamma_i$, where $\Gamma_i = \mathbb{Z} + \tau_i \mathbb{Z}$, $\text{Im} \tau_i > 0$. Assume that there exists a $K$-quasiconformal homeomorphism $f : T_1 \to T_2$. Then $d[\tau_1, \tau_2 + m] \leq \log K$ for some integer $m$, where $d[\cdot, \cdot]$ is the Lobachevsky (hyperbolic) distance in the upper half-plane.

This lemma, which is certainly known to specialists, is very close to Theorem III.3 [A]. It follows essentially from the fact that $\text{Teich}(\mathbb{T}) = \mathbb{H}$, where each point $\tau \in \mathbb{H}$ corresponds to the Riemann surface $\mathbb{C}/(\mathbb{Z} + \tau \mathbb{Z})$.

**Lemma 2.2.** Let $F_1$ and $F_2$ be circle diffeomorphisms conjugated by a Lipschitz homeomorphism $H$ such that $H \circ F_1 = F_2 \circ H$ and $H^{-1}$ is also Lipschitz. Then, for any $\alpha > 0$, the curves $T_\alpha^1$ and $T_\alpha^2$ corresponding to $F_1$ and $F_2$, respectively, are $K$-quasiconformally equivalent, where $K$ is the maximum of the Lipschitz constants of $H$ and $H^{-1}$.

**Proof.** The mapping $H_\alpha : \Pi_\alpha \to \Pi_\alpha$ defined by $H_\alpha(x + iy) = H(x) + iy$ transforms the gluing for $T_\alpha^1$ into the gluing for $T_\alpha^2$ and, therefore, descends to a mapping $H_\alpha : T_\alpha^1 \to T_\alpha^2$. The geometrical definition of quasiconformality [A] immediately implies that the quasiconformal dilatation of the mapping $H_\alpha$ does not depend on $\alpha$ and is equal to the maximum of the Lipschitz constants of $H$ and $H^{-1}$. This proves Lemma 2.2.

**Remark.** The authors gratefully acknowledge that the preceding lemma was communicated to them by V. Kleptsyn.

A pair of diffeomorphisms satisfying the assumption of Lemma 2.2, is called *Lipschitz equivalent*. This notion can be generalized in an obvious way to apply to diffeomorphisms of any metric space, in particular, to diffeomorphisms of an interval (which are considered in Proposition 2.4 below).
Corollary 2.3. Let \( F_1 \) and \( F_2 \) be two Lipschitz equivalent diffeomorphisms. If Theorem 1.1 holds for the diffeomorphism \( F_1 \), then it holds also for the diffeomorphism \( F_2 \).

Proof. By assumption, the moduli \( \mu^1_\alpha \) of the elliptic curves \( T^1_\alpha \) satisfy \( \text{Im} \mu^1_\alpha > C \) for some \( C > 0 \). At the same time, a combination of Lemmas 2.1 and 2.2 implies that \( d[\mu^1_\alpha, \mu^2_\alpha] < \log K \) for some \( K \) not depending on \( \alpha \). Hence \( \text{Im} \mu_2 > C/K \), and Theorem 1.1 holds for the diffeomorphism \( F_2 \).

Corollary 2.3 suggests that the Lipschitz equivalence might be useful for proving Theorem 1.1. Thus, it would be nice to have a tool which would have allowed us to establish the Lipschitz equivalence of particular circle diffeomorphisms. This tool is given by the following proposition.

Proposition 2.4. Let \( f_1 \) and \( f_2 \) be two diffeomorphisms of \([0; 1] \), each having exactly two fixed points 0 and 1, and let \( f'_1(0) = f'_2(0) = a \neq 1 \), \( f'_1(1) = f'_2(1) = b \neq 1 \). Then \( f_1 \) and \( f_2 \) are Lipschitz equivalent.

Proof. Let \( x \) and \( y \) be Sternberg parameters linearizing \( f_1 \) and \( f_2 \) near zero, and let \( \tilde{x} \) and \( \tilde{y} \) be similar parameters near 1. Consider the map \( h \) that transforms the chart \( x \) into \( y \):

\[
x(p) = y(h(p))
\]

In the charts \( x \) and \( y \), the maps \( f_1 \) and \( f_2 \) coincide with the same linear map \( x \mapsto ax \) (or \( y \mapsto ay \)). Hence \( h \) conjugates \( f_1 \) and \( f_2 \) near zero. The linearizing charts \( x \) and \( y \) are as smooth as \( f_1 \) and \( f_2 \); hence \( h \) is smooth near zero. The charts \( x \) and \( y \) can be extended over the whole interval \([0, 1] \) to \( x : [0, 1] \to \mathbb{R} \) and \( y : [0, 1] \to \mathbb{R} \) by means of the dynamics as

\[
x(f_1(p)) = ax(p), \quad y(f_2(p)) = ay(p).
\]

Thus \( h \) can be extended to the whole \([0, 1] \). Setting \( h(1) = 1 \) makes it homeomorphic on \([0, 1] \). The map \( h \) is smooth on \([0, 1] \); we need to prove that it is Lipschitz at 1.

Let \( z = x^{-\alpha}, w = y^{-\alpha}, \) and \( a^{-\alpha} = b, \) that is, \( \alpha = -\log b/\log a \). The charts \( z \) and \( w \) vanish at 1, and they conjugate \( f_1 \) and \( f_2 \) with the linear maps \( z \mapsto bz \) (and \( w \mapsto bw \)). In fact, \( z \) is not even smooth with respect to \( \tilde{x} \) at 1.

Proposition 2.5. The map \( z = h_1(\tilde{x}) \) is Lipschitz with respect to \( \tilde{x} \), and the map \( w = h_2(\tilde{y}) \) is Lipschitz with respect to \( \tilde{y} \).

Let us define \( h_0 \) as \( \tilde{x}(p) = \tilde{y}(h_0(p)) \). By (1), \( z(p) = w(h(p)) \). Thus

\[
h = h_2^{-1} \circ h_0 \circ h_1.
\]

The map \( h_0 \) is smooth; hence Proposition 2.5 implies Proposition 2.4.

Proof of Proposition 2.5. Both \( z \) and \( \tilde{x} \) conjugate \( f_1 \) with the same linear map, namely, multiplication by \( b \). Hence

\[
h_1(\tilde{b}x) = bh_1(\tilde{x}).
\]

Thus the graph of \( h_1 \) is invariant under \((\tilde{x}, \tilde{z}) \mapsto b(\tilde{x}, \tilde{z}) \). Hence \( \max_{[0, 1]} |h_1'| = \max_{[\tilde{b}x, \tilde{x}]} |h_1'| \); the same for \((h_1^{-1})' \). Thus \( h_1 \) is Lipschitz, as well as \( h^{-1} \).  \( \square \)
3. The case of rotation number 0

In this section, we prove Theorem 1.1 for diffeomorphisms with rotation number 0 by the following plan. In Section 3.1, we reduce the problem to the special case of an embeddable diffeomorphism, that is, a diffeomorphism of the form \( F = g_1^v(x) \), where \( v \) is a vector field on the circle and \( g_1^v \) is its time-one flow transformation. The idea of the proof of Theorem 1.1 for an embeddable diffeomorphism is to use the corresponding flow \( g_t^v \) in order to define a new family \( S_\alpha \) of elliptic curves, which stay quasiconformally close to \( T_\alpha \) and whose moduli could be calculated more or less explicitly. This idea is implemented in Section 3.2.

3.1. Reduction to an embeddable diffeomorphism. We call two circle diffeomorphisms \( F_1 \) and \( F_2 \) with rotation number 0 similar if they are conjugated by a homeomorphism which maps the fixed points of \( F_1 \) to those of \( F_2 \) in such a way that the derivatives at the corresponding fixed points are equal.

Lemma 3.1. Let \( F_1 \) and \( F_2 \) be two similar circle diffeomorphisms with rotation number 0. Then \( F_1 \) and \( F_2 \) are Lipschitz equivalent. In particular, any Morse–Smale circle diffeomorphism \( F \) with rotation number 0 is Lipschitz equivalent to some (analytic) embeddable diffeomorphism \( G \).

Proof. Let \( h : S^1 \to S^1 \) be the homeomorphism witnessing the similarity for \( F_1 \) and \( F_2 \). Take any arc \( \gamma_1 \) bounded by two neighboring fixed points of \( F_1 \) and let \( \gamma_2 = h(\gamma_1) \). After being properly rescaled, the diffeomorphisms \( f_i = F_i|_{\gamma_i} \) satisfy the conditions of Proposition 2.1 (the derivatives at the fixed points are not equal to 1 since the diffeomorphisms are Morse–Smale). Applying Proposition 2.4 to all pairs of corresponding arcs, we prove the first part of the lemma.

To prove the second part, it suffices to take an analytic vector field \( v \) on the circle whose multiplicators at the singular points equal \( \log F'(a_k) \), where \( a_k \) run through the fixed points of the diffeomorphism \( F \), and let \( G = g_1^v \). This completes the proof of the lemma. \( \square \)

Corollary 3.2. To prove Theorem 1.1 in the case of rotation number 0, it is sufficient to prove it for all embeddable diffeomorphisms.

Proof. Let \( F_1 \) be a circle diffeomorphism with rotation number 0. Take an embeddable diffeomorphism \( F_2 \) Lipschitz equivalent to \( F_1 \) (it exists by Lemma 3.1). Theorem 1.1 holds for \( F_2 \) by assumption. By Corollary 2.3, we conclude that Theorem 1.1 then holds also for \( F_1 \). \( \square \)

So it remains to prove Theorem 1.1 for an arbitrary embeddable diffeomorphism.

3.2. Proof of Theorem 1.1 for an embeddable diffeomorphism. Let \( F(x) \) be an embeddable diffeomorphism, i.e., \( F(x) = g_1^v(x) \) with an analytic \( v \). Then we can link the diffeomorphism \( F \) with yet another family \( S_\alpha \) of elliptic curves defined in the following way. Let us lift the vector field \( v \) to a vector field on \( \mathbb{R} \) and extend it to a holomorphic vector field in some neighborhood of the real axis, still denoted by \( v \). We define \( P_\alpha \) to be a curvilinear strip bounded by the real axis \( \mathbb{R} \) and the curve \( g_{1+i\alpha}(\mathbb{R}) \). Let us glue together the points of \( P_\alpha \) by the mappings \( z \to z + 1 \) (\( z \in P_\alpha \)) and \( G_\alpha : x \to g_{1+i\alpha}^1(x) \) (\( x \in \mathbb{R} \)). We denote the elliptic curve
thus obtained is by $S_\alpha$. It turns out that $S_\alpha$ and $T_\alpha$ are quasiconformally close uniformly in $\alpha$. To be more precise, the following lemma holds.

**Lemma 3.3.** There exist positive constants $K$ and $\alpha_0$ such that, for any $\alpha \in (0; \alpha_0)$, the elliptic curves $T_\alpha$ and $S_\alpha$ are $K$-quasiconformally equivalent.

**Proof.** The map $G_\alpha$ that generates the curve $S_\alpha$ is the restriction to $\mathbb{R}$ of the time-one phase flow transformation of the equation

$$\dot{z} = v(z) + i\alpha. \quad (2)$$

Therefore, $G_0 = g_0^t = F$ and $G_\alpha = F + \alpha f + O(\alpha^2)$, where the complex-valued function $f$ is the derivative of the solution to (2) with respect to the parameter $\alpha$ at $\alpha = 0$ for $t = 1$. Along the solution to $\dot{x} = v(x)$ with the initial condition $x(0) = x_0$, this derivative satisfies the equations

$$\dot{x} = v(x), \quad \dot{y} = v'(x)y + i, \quad y(0, x_0) = 0. \quad (3)$$

The function

$$X(t, x_0) = \frac{v(g_\alpha^t x_0)}{v(x_0)}$$

is a solution to the equation $\dot{X} = v'(g_\alpha^t x_0)X$ with the initial condition $X(0, x_0) = 1$. It is positive and smooth everywhere on $S^1$, including the singular points of $v$. Hence the function

$$g(t, x) = iX(t, x) \int_0^t \frac{d\tau}{X(\tau, x)}$$

solves (3.2). Therefore,

$$f(x) = iX(1, x) \int_0^1 \frac{dt}{X(t, x)}.$$ 

We need only the statement that $f/i$ is positive everywhere on $S^1$; it follows from the formula for $f$ above.

Now, consider the map $h_\alpha : \Pi_\alpha \to \mathbb{C}$ defined by

$$h_\alpha(z) = z + \frac{y}{\alpha}(G_\alpha(u) - F_\alpha(u)), \quad u \in F^{-1}(x);$$

let us check that it descends to a map $T_\alpha \to S_\alpha$. Indeed, $h(z + 1) = h(z) + 1$. Moreover, on the real axis, $h_\alpha \circ F_\alpha = G_\alpha \circ h_\alpha$. In more detail, $h_\alpha|_\mathbb{R} = x$. Hence, for $x \in \mathbb{R}$, we have $F_\alpha(x) = F(x)$ and

$$h_\alpha \circ F_\alpha(x) = x + i\alpha + (G_\alpha(x) - (x + i\alpha)) = G_\alpha(x) = G_\alpha \circ h_\alpha(x).$$

Thus $h_\alpha$ descends to a map $T_\alpha \to S_\alpha$.

Now, let us check that the maps $h_\alpha$ have uniformly bounded quasiconformal dilatations $\mu_\alpha$. Indeed, let $f(F^{-1}(x)) = if_1(x)$; as shown above, $f_1 > 0$. We have

$$h_\alpha(x + iy) = x + iy + \frac{y}{\alpha}(x + \alpha f_1(x) + O(\alpha^2)) - x - i\alpha = x + iy f_1(x) + O(\alpha).$$

Hence

$$\mu_\alpha = \frac{h_\alpha}{h_\alpha} = \frac{1 - f_1(x) + iy f'_1 + O(\alpha)}{1 + f_1(x) + iy f'_1 + O(\alpha)}.$$
For some $\alpha_0$ and any $\alpha \in (0, \alpha_0)$ and $y \in [0, \alpha]$, $\mu$ is close to $(1 - f_1(x))/(1 + f_1(x))$. The absolute value of this fraction is bounded from above by $1 - \varepsilon$ for some $\varepsilon > 0$, because $f_1$ is positive on $S^1$. Consequently, $h_\alpha$ is quasiconformal uniformly in $\alpha$, and we are done.

Unlike the modulus of $T_\alpha$, the modulus of $S_\alpha$ can be calculated explicitly.

**Lemma 3.4.** The modulus $\nu_\alpha$ of the elliptic curve $S_\alpha$ is equal to $1/I_\alpha$, where $I_\alpha = \int_0^1 \frac{dx}{v(x) + i\alpha}$. Furthermore, $\text{Im} \, \nu_\alpha > C$ for some $C > 0$ not depending on $\alpha$ for a sufficiently small $\alpha$.

**Proof.** The change of variables $z \mapsto t$, $dt = \frac{dz}{v(z) + i\alpha}$, which rectifies the vector field $v + i\alpha$, serves as a uniformization chart for the elliptic curve $S_\alpha$. Namely, $S_\alpha = \mathbb{C}/\Gamma$, where $\Gamma$ is generated by $I_\alpha$ and 1. The generator $I_\alpha$ corresponds to the gluing $z \mapsto z + 1$ in the original construction of the elliptic curve $S_\alpha$, whereas the generator 1 corresponds to the gluing $x \mapsto g_v^1(x)$; therefore, $\nu_\alpha = 1/I_\alpha$.

Our goal is to estimate $I_\alpha$. We shall do this by finding $I_\alpha = \lim_{\alpha \to 0} I_\alpha$. Choose a $\beta > 0$ such that $v$ extends as a holomorphic function to the domain $\Pi_\beta$ and does not have there any zeroes except for those on the real line. For $0 < \alpha < \beta$, consider the contour integral $I_\alpha := \int_\gamma \frac{dz}{v(z) + i\alpha}$, where $\gamma$ is the boundary of the rectangle $D_\beta = \{0 < \text{Re} \, z < 1; 0 < \text{Im} \, z < \beta\}$ oriented counterclockwise. Since $v$ is 1-periodic, we have $I_\alpha = I_\alpha - J_\alpha$, where $J_\alpha = \int_0^{2\pi} \frac{dx}{v(x + i\beta) + i\alpha}$. On the other hand, by the Cauchy theorem, $L_\alpha = 2\pi i \Sigma_\alpha$, where $\Sigma_\alpha = \Sigma_{D_\beta} \text{res}(\frac{1}{v + i\alpha})$. Thus,

$$I_\alpha = J_\alpha + 2\pi i \Sigma_\alpha,$$

and $I_0 = J_0 + 2\pi i \Sigma_0$, where $J_0 := \lim_{\alpha \to 0} J_\alpha = \int_0^{2\pi} \frac{dx}{v(x + i\beta)}$ and $\Sigma_0 = \lim_{\alpha \to 0} \Sigma_\alpha$.

By the Cauchy theorem, the integral $J_0$ is equal to the integral $\int_{R_\varepsilon} \frac{dx}{v(x)}$, where the path $R_\varepsilon$ coincides essentially with the path $[0; 1]$ but does not pass through the zeroes of $v$, moving around them along the (clockwise oriented) semicircles of radius $\varepsilon$ in the upper halfplane. Note that $v$ has only simple real zeros; hence $1/v$ has nonzero residues on $[0, 1)$. Denote the sum of all (all negative, all positive) residues of $1/v$ on $[0, 1)$ by $\Sigma$ (respectively, by $\Sigma^-$, $\Sigma^+$).

It is easy to see that $\int_{R_\varepsilon} \frac{dx}{v(x)} = \text{V.p.} \int_0^{2\pi} \frac{dx}{v(x)} - 2\pi i \Sigma + O(\varepsilon)$. Since $J_0$ does not depend on $\varepsilon$, we conclude that

$$J_0 = \text{V.p.} \int_0^{2\pi} \frac{dx}{v} - 2\pi i \Sigma.$$  \hfill (4)

To find $\Sigma_0$, note that the poles of $1/(v + i\alpha)$ in $D_\beta$ correspond to the zeroes of $v + i\alpha$ near the real line in the upper halfplane. If $\alpha$ is small enough, each of the latter arises from a zero $a_i$ of $v$ with $v'(a_i) < 0$. (Note that the perturbation $v + i\alpha$ moves the zeroes of $v$ at which the derivative is positive (negative) to the lower (upper) halfplane.) This observation immediately implies

$$\Sigma_0 = \Sigma^-.$$  \hfill (5)
Combining (4) and (5), we get
\[ I_0 = \text{V.p.} \int_0^{2\pi} \frac{dx}{v} + \pi i \Sigma^- - \pi i \Sigma^+. \]  
\[ (6) \]

From (6), we see that \( \text{Im} \, I_0 < 0 \). Thus, for \( C_1 := |I_0|, \ C_2 := \text{Im} \, I_0 \), and a sufficiently small \( \alpha \), we have

\[ |I_\alpha| < 2C_1, \quad \text{Im} \, I_\alpha < C_2/2 < 0, \]

and, consequently,

\[ \text{Im} \, \nu_\alpha = \text{Im} \, 1/I_\alpha = -\frac{\text{Im} \, I_\alpha}{|I_\alpha|^2} > C = C_2/8C_1^2 > 0. \]

\[ \square \]

**Corollary 3.5.** Theorem 1.1 holds for any embeddable diffeomorphism.

**Proof.** Given an embeddable diffeomorphism \( F \), we have proved in Lemma 3.4 that the moduli \( \nu_\alpha \) of elliptic curves \( S_\alpha \) constructed for the diffeomorphism \( F \), satisfy \( \text{Im} \, \nu_\alpha > C \). But, by Lemma 3.3, the elliptic curve \( T_\alpha \) remains \( K \)-quasiconformally close to \( S_\alpha \), whence, by Lemma 2.1, \( \text{Im} \, \mu_\alpha \geq (\text{Im} \, \nu_\alpha)/K > C/K \); thus Theorem 1.1 holds for \( F \).

\[ \square \]

4. The case of a rational nonzero rotation number

In this section, we prove Theorem 1.1 in the case when the diffeomorphism \( F \) has a rational nonzero rotation number, say \( p/q \). The plan of the proof is completely analogous to that in the case of zero rotation number. In Section 4.1, we reduce the problem to the special case of pseudoembeddable diffeomorphism, that is, a diffeomorphism that can be represented in the form \( F = g^v(x) + p/q \), where \( v \) is a vector field on the circle equivariant with respect to the \( p/q \)-shift. Mimicking the proof from Section 3 in Section 4.2, we define a new family \( S_\alpha \) of elliptic curves which remain quasiconformally close to \( T_\alpha \) and whose moduli can be calculated more or less explicitly.

4.1. Reduction to a pseudoembeddable diffeomorphism. We say that two circle diffeomorphisms \( F_1 \) and \( F_2 \) with rotation number \( p/q \) (here and in what follows, we assume that \( \gcd(p, q) = 1 \)) are similar if \( F_1^q \) and \( F_2^q \) are similar in sense of Section 3.1.

**Lemma 4.1.** Let \( F_1 \) and \( F_2 \) be two similar Morse–Smale diffeomorphisms with rotation number \( p/q \). Then \( F_1 \) and \( F_2 \) are Lipschitz equivalent. In particular, any Morse–Smale circle diffeomorphism with rational rotation number is Lipschitz equivalent to some pseudoembeddable diffeomorphism.

**Proof.** The first part of the lemma is proved as in the case of zero rotation number. Namely, let \( h: S^1 \to S^1 \) be the map that establishes the Lipschitz equivalence of \( F_1^q \) and \( F_2^q \), i.e., such that

\[ h \circ F_1^q = F_2^q \circ h; \]

its existence is guaranteed by Lemma 3.1. The mapping \( h \) does not necessarily conjugate \( F_1 \) and \( F_2 \); however, we can deform \( h \) into a (bi-Lipschitz) mapping \( h_1: S^1 \to S^1 \) which conjugates \( F_1 \) and \( F_2 \) as follows. Take an
arc $\gamma$ between two successive fixed points of $F_1^q$. We set $h_1|_{\gamma} = h|_{\gamma}$ and transfer $h_1$ to the image of $\gamma$ under the action of $F_1$ by the formula

$$h_1(F_1^k(x)) = F_2^k(h_1(x)), \quad (8)$$

where $x \in \gamma$ and $k = 1, \ldots, q - 1$. From (4.2) it follows immediately that $h_1 \circ F_1 = F_2 \circ h_1$ on $\{F_1^k(\gamma), k = 0, \ldots, q - 2\}$, and from (7), (8) imply that $h_1 \circ F_1 = F_2 \circ h_1$ on $F_1^q (\gamma)$. Further, since $h$ is bi-Lipschitz, $h_1(F_1^k(x)) = F_2^k(h(x))$, and $F_1$ and $F_2$ are diffeomorphisms, $h_1$ is bi-Lipschitz on $\{F_1^k(\gamma); k = 0, \ldots, q - 1\}$. Repeating this procedure for arcs between successive fixed points of $F_1^q$ lying in $S^1 - \bigcup_1^{q-1} F_1^k(\gamma)$ while there are arcs where $h_1$ is not yet defined, we get a mapping $h_1 : S^1 \to S^1$ bi-Lipschitz and satisfying $h_1 \circ F_1 = F_2 \circ h_1$, which proves the first part of the lemma.

Let us prove the second part. Let $A$ denote the set of fixed points of $F_1$ and $B$ the set of $q$th roots $b_k$ of 1: $b_k^q = 1$; here $n$ is the number of periodic orbits of $F$ (certainly, they have period $q$). Let $H$ be a homeomorphism of the circle that maps $B$ onto $A$ and conjugates the action of $F$ on $A$ with the action of the rotation $Q : x \to x + \frac{p}{q}$ on $B$. Let $\nu_k$ be the multiplier of the periodic orbit of $a_k$ under $F$: $\nu_k = (F_1^q)'(a_k)$, and let $\lambda_k = \frac{1}{q} \log \nu_k$. Consider arbitrary analytic vector field $w$ on $S^1$ having singular points $b_k$ with eigenvalues $w'(b_k) = \lambda_k$. Let $v = \frac{1}{q} \sum_0^{q-1} Q_k^* w$. Then the field $v$ is analytic, has the same singular points and eigenvalues as $w$, and is invariant under $Q$. Let $G = g_\beta + \frac{p}{q}$. Then $G$ is a pseudoembeddable diffeomorphism similar to $F$. By the first statement of the lemma, it is Lipschitz equivalent to $F$.

To complete the proof of Lemma 4.1, it remains to note that $G^q$ is similar to $F^q$.

**Corollary 4.2.** To prove Theorem 1.1 in the case of rational nonzero rotation number, it is sufficient to prove it for all the pseudoembeddable diffeomorphisms.

The proof of this corollary is quite similar to that of Corollary 3.2.

So it remains to prove Theorem 1.1 for an arbitrary pseudoembeddable diffeomorphism.

### 4.2. Proof of Theorem 1.1 for a pseudoembeddable diffeomorphism.

Let $F = g_\beta(x) + p/q$, where $v$ is a vector field on the circle which is invariant under the shift by $p/q$. As in the case of zero rotation number, we define a family of elliptic curves $S_\alpha$ for the curvilinear strip on the plane bounded by the real axis $\mathbb{R}$ and by the curve $g_{\beta+\alpha}(\mathbb{R})$ with the use of the gluings $z \to z + 1$ and $x \to G_\alpha(x) = g_\beta^{1+i\alpha}(x) + p/q$ ($x \in \mathbb{R}$).

**Lemma 4.3.** There exist positive constants $K$ and $\alpha_0$ such that, for $\alpha \in (0; \alpha_0)$, the elliptic curves $T_\alpha$ and $S_\alpha$ are $K$-quasiconformally equivalent.

**Proof.** Let $F^0 = F - p/q$. Then $F^0$ has rotation number 0 and is embeddable, i.e., $F^0 = g_\beta$. Starting from $F^0$, we define $\tilde{F}^0$, $T_\alpha^0$, $G_\alpha^0$, and $S_\alpha^0$ as in Sections 1 and 3.2. Consider the family of mappings $h_\alpha : \Pi_\alpha \to \mathbb{C}$ defined by the formula

$$h(x + iy) = x + iy + \frac{y}{\alpha}(G_\alpha(u) - \tilde{F}^0_\alpha(u)), \quad u = (F^0)^{-1}(x).$$
We are in the situation of the proof of Lemma 3.2 with \( F^0_\alpha, C^0_\alpha, T^0_\alpha, S^0_\alpha \), and \( h_\alpha \) playing the role of \( F_\alpha, G_\alpha, T_\alpha, S_\alpha \), and \( h_\alpha \). In the proof of Lemma 3.2, it is established that

(i) the maps \( h_\alpha \) are uniformly quasiconformal in \( \alpha \) and satisfy \( \tilde{h}_\alpha(z + 1) = \tilde{h}_\alpha(z) + 1 \) for any \( z \in \Pi_\alpha \);

(ii) \( \tilde{h}_\alpha \circ F^0_\alpha(x) = G^0_\alpha \circ \tilde{h}_\alpha(x) \) for any \( x \in \mathbb{R} \).

In the case under consideration,

\[ \tilde{h}_\alpha(z) - z = \frac{y}{\alpha}(\tilde{c}_\alpha(u) - \tilde{F}_\alpha(u)) \]

has the additional virtue of being \( p/q \)-periodic, since so are \( F^0 \) and \( G^0 \). Thus,

\[ \tilde{h}_\alpha \circ (F^0_\alpha + p/q) = G^0_\alpha \circ h_\alpha + p/q, \]

which means exactly that

\[ \tilde{h}_\alpha \circ \tilde{F} = \tilde{G} \circ \tilde{h}_\alpha, \]

where \( \tilde{G} = g^1_{\nu + i\alpha} + p/q \). So \( \tilde{h}_\alpha \) descends to the mapping \( h_\alpha : T_\alpha \rightarrow S_\alpha \). An argument similar to that used in the proof of Lemma 3.2 proves that the mappings \( h_\alpha \) are quasiconformal uniformly in \( \alpha \), which completes the proof of the lemma. \( \square \)

**Lemma 4.4.** The modulus \( \nu_\alpha \) of the elliptic curve \( S_\alpha \) is equal to \( p/q + 1/I_\alpha \), where \( I_\alpha = \int_1^1 dx/(v(x) + i\alpha) \). Furthermore, \( \text{Im} \nu_\alpha > C \) for some \( C > 0 \) not depending on \( \alpha \).

**Proof.** Let us normalize the vector field \( v + i\alpha \) by the change of variables \( dt = \frac{dz}{v(z) + i\alpha} \). The points of the real axis that differ on \( p/q \) are mapped by this change to points which differ by \( \int_x^{x+1} \frac{dz}{v(z) + i\alpha} = pI_\alpha/q \) due to the equivariance of \( v(x) \). Consequently, the points \( x \in \mathbb{R} \) and \( g^1_{\nu + i\alpha}(x) + p/q = g^1_{\nu + i\alpha}(x + p/q) \) are mapped to points which differ by \( pI_\alpha/q + 1 \). This means that the change of coordinate \( dt = \frac{dz}{v(z) + i\alpha} \) uniformizes the elliptic curve \( S_\alpha \), namely, \( S_\alpha = \mathbb{C}/\Gamma \), where \( \Gamma \) is generated by \( I_\alpha \) and \( pI_\alpha/q + 1 \). Here the generator \( I_\alpha \) corresponds to the gluing \( z \rightarrow z + 1 \) in the original construction of the elliptic curve \( S_\alpha \), whereas the generator \( pI_\alpha/q + 1 \) corresponds to the gluing \( x \rightarrow g^1_{\nu + i\alpha}(x) + p/q \). Hence, arguing as in Lemma 3.4, we get that the modulus of the elliptic curve \( S_\alpha \) is equal to \( \nu_\alpha = p/q + 1/I_\alpha \). The second part of the lemma, stating that \( \text{Im} \nu_\alpha > C \) for some \( C > 0 \) not depending on \( \alpha \), is a direct consequence of Lemma 3.4, since \( \text{Im}(p/q + 1/I_\alpha) = 1/I_\alpha \). This completes the proof of Lemma 4.4. \( \square \)

**Corollary 4.5.** Theorem 1.1 holds for any pseudoembeddable diffeomorphism.

**Proof.** Given a pseudoembeddable diffeomorphism \( F \), we have proved in Lemma 4.4 that the moduli \( \nu_\alpha \) of the elliptic curves \( S_\alpha \) determined by the diffeomorphism \( F \) satisfy \( \text{Im} \nu_\alpha > C \). But, by Lemma 4.3, the elliptic curve \( T_\alpha \) remains \( K \)-quasiconformally close to \( S_\alpha \); by Lemma 2.1, we have \( \mu_\alpha \geq (\text{Im} \nu_\alpha)/K > C/K \), and Theorem 1.1 holds for \( F \). \( \square \)

**Acknowledgment.** The authors are grateful to the referee for many useful suggestions.
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