NEW SINGULARITIES AND PERESTROIKAS OF FRONTS OF LINEAR WAVES

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To V. I. Arnold on the occasion of his 65th birthday

ABSTRACT. The subject of the paper is the propagation of linear waves in plane and three-dimensional space. We describe some new (as compared with the ADE-classification) typical singularities and perestroikas of their fronts when the light hypersurface has conical singularities. Such singularities appear if the waves propagate in a non-homogeneous anisotropic medium and are controlled by a variational principle.

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Usually waves of different polarisation propagate independently of each other. In low dimensions, typical singularities and perestroikas (= metamorphoses, bifurcations, or transitions) of the fronts of such independent waves are described by the ADE classification due to V. I. Arnold (see, for example, [A1, Chapter 3]). However, sometimes the phase velocities of different waves coincide, which causes the so-called Hamilton conical refraction in anisotropic crystals. But the crystals are homogeneous and, therefore, non-generic from the point of view of singularity theory. What will happen if the wave properties of the medium depend not only on the direction but on the point as well? Apparently, this problem was first formulated by V. I. Arnold in [A1, A2]; we partially solve it in the present paper.

We investigate the propagation of discontinuities and short waves described by the Euler–Lagrange system of linear partial differential equations defined by some variational principle in plane and space. The Lagrangian is assumed to be a quadratic form with respect to the unknown variables and their time and space derivatives. The coefficients of this form have to be “generic enough” smooth functions of a point of the space or space-time; the corresponding genericity conditions can be checked in concrete cases.

The big front of a discontinuous solution is just the hypersurface of its discontinuities in space-time. A big front of a short wave approximation is a level
Figure 1. The new perestroika $\Theta_3$ of instant fronts in plane

hypersurface of its phase. The sections of big fronts with isochrones $t = \text{const}$ are called \textit{instant} fronts; they propagate with time. The instant front at the fixed time $t = 0$ is called \textit{initial}. The propagation of an instant front is determined by the initial front and a hypersurface in the projectivised cotangent bundle over space-time. This hypersurface is called \textit{light}; Section 2 describes how the original Lagrangian determines it.

An instant front propagating with time may experience perestroikas. For example, even if the initial front is smooth, the instant front may become singular after some time. If the light hypersurface is smooth, the initial front is smooth and generic, and the space dimension does not exceed five, then all singularities of instant and big fronts, as well as perestroikas of instant fronts, are described by the $ADE$ classification (its applicability in this situation was proved in [B1]). However, the light hypersurface can have singularities. In the case when the coefficients of the original variational principle depend generically on a point of space or space-time, the typical singularities of the light hypersurface are classified up to formal contact diffeomorphisms in [A2], [A1, Chapter 8], [Kh]. These singularities generate new singularities of instant fronts, new singularities of big fronts, and new perestroikas of instant fronts.

In the present paper, we classify some of such new singularities of instant and big fronts in plane and space up to diffeomorphisms provided that the smooth initial front is generic. The non-trivial perestroikas of instant fronts defined by our new singularities of big fronts are shown in Fig. 1 (plane) and in Figs. 2 and 3 (space).

In mathematical terms, the big front is the projection of the Legendre variety being the closure of the extension of the initial front along characteristics of the light hypersurface. If the initial front and the light hypersurface are smooth, then the Legendre variety is smooth as well. In this case, the $ADE$ classification is obtained. However, if the light hypersurface has singularities, then the Legendre variety becomes non-smooth, and new singularities of instant and big fronts appear.

The simplest singularity $A_1$ of the Legendre variety is a cylinder over the non-algebraic curve $q = p \ln p^2$. In the physically interesting cases of plane and space, its dimension does not exceed three. The fronts of such Legendre varieties are
locally classified up to diffeomorphisms in this paper provided that the Legendre projections are generic. We give a finite list of the new singularities of fronts: $\Theta_2$, $\Theta_3$, $\Theta_4^\pm$, and $\Xi_4$. All of them are simple, i.e., their normal forms do not contain continuous invariants. It should be mentioned that such a fine classification is a purely contact phenomenon; there is no Lagrangian counterpart, since the symplectic diffeomorphism group of the non-algebraic Lagrangian curve $q = p \ln p^2$ is trivial.

The key technical idea of our classification is to consider the non-singular fibres of a Legendre fibration as Legendre submanifolds to be classified up to $\Lambda_1$-preserving contact diffeomorphisms and the whole fibration as a versal deformation of such a fibre. It is important that, in the cases of plane and space, only simple fibres can occur generically.

Our Legendre variety has more complicated singularities $\Lambda_2$ described in [A1, Chapter 8]. In the case of plane, the corresponding new singularities of big fronts and perestroikas of instant fronts were classified in [B2], [B3], but the simplest case $\Lambda_1$ was missed there! The missed perestroika of an instant front is shown in Fig. 1.

On the Legendre variety, the singularities $\Lambda_1$ and $\Lambda_2$ have codimensions 1 and 2, respectively. According to [A1, Chapter 8] and [B2], in the case of plane and a generic smooth initial front, they exhaust all possible singularities of the Legendre
variety up to contact diffeomorphisms. In the case of space, other singularities may appear; their classification is an open problem.

**Terminology.** “Generic” means “from an open dense subset” of the space of objects under consideration. This space is assumed to be endowed with the fine Whitney topology; its definition can be found, for example, in [AGV, 2.3, Remark 2].

The term “smooth” means infinitely differentiable. All manifolds and submanifolds under consideration are smooth; the Legendre varieties are the closures of Legendre submanifolds.

**Distribution of the material.** In Section 1, we formulate our main theorem giving normal forms of the new singularities \( \Theta_2, \Theta_3, \Theta_4^\pm, \) and \( \Xi_4 \) of instant and big fronts with respect to diffeomorphisms. The main theorem asserts that, in the physically interesting cases of plane and space, all generic fronts of singularities \( \Lambda_1 \) of a Legendre variety are exhausted by this list up to diffeomorphisms.

Section 2 describes constructing the light hypersurface from a quadratic Lagrangian and contains the normal forms of typical singularities of the light hypersurface with respect to formal contact diffeomorphisms. These normal forms were found in [A2] for light hypersurfaces satisfying additional conditions of non-degeneracy. According to the transversality theorem from [Kh], these conditions hold if the coefficients of the original variational principle depend generically on a point of space-time.

In Section 3, we formulate and prove Theorem 1 describing the singularities \( \Lambda_1 \) and \( \Lambda_2 \) of the Legendre variety being the closure of the extension of a generic smooth initial front along characteristics of the light hypersurface which has the singularities described in [A2] (see also Section 2). In the case of plane, Theorem 1 was proved in [A1, Chapter 8]. In this case, the singularities \( \Lambda_1 \) and \( \Lambda_2 \) exhaust all possible singularities of our Legendre variety up to the contact diffeomorphisms preserving the light hypersurface. For higher dimensions, other singularities can appear; their classification is an open problem.

The main theorem is a corollary of Theorem 2 stated in Section 4 and describing the normal forms of generic Legendre fibrations up to the so-called weak \( \Lambda_1 \)-equivalence for \( n \leq 3 \), where \( n + 1 \) is the dimension of the base, which is just space-time. We start the proof of Theorem 2 in Section 5 with finding normal forms for separate fibres with respect to the contact diffeomorphisms preserving the normal form \( \Lambda_1 \) itself. The corresponding results are formulated as Theorem 3, which is reduced to Lemma 2 in Section 5. Apparently, Theorem 3 and the ADE classification describe all simple smooth Legendre submanifolds up to the local contact diffeomorphisms preserving \( \Lambda_1 \). Theorem 3 classifies the simple germs of Legendre submanifolds at the singular points of the Legendre variety \( \Lambda_1 \), and the ADE classification, at its smooth points.

We prove Lemma 2 in Section 6 with the help of the standard homotopy method applied to the group of contact diffeomorphisms preserving the normal form \( \Lambda_1 \). It turns out that the Legendre fibrations mentioned in Theorem 2 are versal deformations of the separate fibres from Theorem 3 in the class of all smooth Legendre submanifolds. We develop the corresponding versality theory in Section 6. It turns out, for example, that the infinitesimal versality in our situation is nothing but the
Givental criterion for the stability of the Legendre mapping of the singularity $\Lambda_1$ (see [G, 3.3]).

1. Singularities and Perestroikas of Fronts

**Plane.** In this case, the instant front at a typical moment of time can have cusps $A_2$ and new stable singularities $\Theta_2$; the latter are discontinuities of the third derivative and propagate along rays. At separate moments of time, the instant front can experience the new perestroika shown in Fig. 1. The singularities $\Theta_2$ of the instant fronts propagate along a smooth ray, and their cusps run through a pair of smooth curves with infinite order of tangency. Our perestroika is described by a big front lying in 3D space-time and called $\Theta_3$. The front $\Theta_3$ looks like the usual swallowtail, but its cuspidal edge consists of two smooth curves with infinite order of tangency.

**Space.** In this case, the instant front at a typical moment of time can have cuspidal edges $A_2$, swallowtails $A_3$, and new stable singularities $\Theta_2$ and $\Theta_3$. The singularities $\Theta_2$ propagate along rays. The new singularities of the big front are $\Theta_2$, $\Theta_3$, $\Theta_4^\pm$, and $\Xi_4$. The two possible perestroikas $\Theta_3$ are shown in Fig. 2, the perestroika $\Xi_4$ is shown in Fig. 3, and the perestroikas $\Theta_4^\pm$ are topologically trivial: before and after these perestroikas, the instant fronts have the singularity $\Theta_3$ and are locally homeomorphic to the instant fronts at the moments of time when the perestroikas occur. During the perestroika $\Xi_4$, the singularities $\Theta_2$ run through the Whitney umbrella.

**Normal forms of fronts.** The normal forms of the singularities $\Theta_2$, $\Theta_3$, $\Theta_4^\pm$, and $\Xi_4$ are given in local coordinates $(y, z) = (y_1, \ldots, y_n, z)$ by the equations

$$z = F(s, y), \quad F_k(s, y) = 0,$$

where $s = (s_1, s_2)$ are parameters and

$$\begin{align*}
(\Theta_2) \quad F &= -s_1^2 \ln(s_1^2/e) + y_1 s_1; \\
(\Theta_4^\pm) \quad F &= -s_1^2 \ln(s_1^2/e) + s_1 s_2 \pm s_2^2 + y_1 s_2 + \cdots + y_{k-1}s_{k-1}, \quad n + 1 \geq k \geq 3; \\
(\Xi_4) \quad F &= -s_1^2 \ln(s_1^2/e) + s_2^2 + y_1 s_1 + y_2 s_2 + y_3 s_1 s_2.
\end{align*}$$

**Remarks.** 1. The singularity $\Theta_2$ is given by the equation $z = \varphi(y_1) = y_1^2 / \ln y_1^2 + o(y_1^2 / \ln y_1^2)$ as $y_1 \to 0$, where $\varphi$ is the Legendre transform of $-s_1^2 \ln(s_1^2/e)$.

2. The change $s \mapsto -s$ shows that the singularities $\Theta_4^\pm$ and $\Theta_k$ are diffeomorphic if $k$ is odd.

3. Removing the terms of degree 3 and more from $F$, we get the following asymptotic normal form for the singularities $\Theta_k^\pm$ with $k \geq 3$:

$$F = -s_1^2 \ln(s_1^2/e) + s_1 s_2 + y_1 s_2 + y_2 s_2.$$

It shows that these singularities are homeomorphic to each other.

**Definition.** A smooth fibration $\pi: E \to B$ is called Legendre if its space $E$ is a contact manifold and the fibres are Legendre submanifolds. The image $\pi(\Lambda)$ of a Legendre variety $\Lambda \subset E$ is called its front.

**Remark.** In the case under consideration, $E = PT^* B$ and $\dim B = n + 1$. 
Main Theorem. Suppose that \((p, q, u) = (p_1, \ldots, p_n, q_1, \ldots, q_n, u)\) are coordinates in \(E\) such that the contact structure is given by the form \[
\theta = du - (pdq - qdp)/2, 
\]
\[
A_1 = \{2p_1 \ln p_1^2 + q_1 = 0, \ p_2 = \cdots = p_n = 0, \ u + p_1^2 = 0\} \subset E
\]
is a Legendre variety such that its smooth part is defined by the specified equations, \(\pi: E \to B\) is a generic Legendre fibration, and \(n \leq 3\).

Then the fronts of the germs of \(A_1\) at its singular points are diffeomorphic to the normal forms \(\Theta_2, \Theta_3, \Theta^E_4\), and \(\Xi_4\).

2. Singularities of the Light Hypersurface

Let us start with the variational principle
\[
\delta \int_B L(x, v, v_x) \, dx_0 \ldots dx_n = 0,
\]
where \(x_0 = t\) is time, \(x = (x^0, x^1, \ldots, x^n) \in B\) are independent variables, \(v(x) = (v^1(x), \ldots, v^n(x))\) are unknown functions of a point of space-time \(B\), and the Lagrangian is a quadratic form of the unknown variables and their first derivatives, i.e.,
\[
L(x, v, v_x) = \frac{1}{2} A_{ij}^k(x)v^i_{x^k}v^j_{x^k} + B_{ij}^k(x)v^i_{x^k}v^j + \frac{1}{2} C_{ij}(x)v^i v^j,
\]
where the coefficients \(A, B, C\) are smooth functions of the independent variables such that
\[
A_{ij}^k(x) = A^k_{ij}(x), \quad C_{ij}(x) = C_{ji}(x).
\]
(We use the tensor notation, which assumes summation over repeated indices.) More generally, the unknown functions are smooth sections of a vector bundle \(V \to B\) over space-time.

Our variational principle defines the Euler–Lagrange system of \(m\) linear partial differential equations
\[
\frac{\partial}{\partial x_k} \left( \frac{\partial L}{\partial v^i_{x^k}} \right) - \frac{\partial L}{\partial v^i} = 0, \quad i = 1, \ldots, m;
\]
its principal matrix symbol is a section of the vector bundle \(S^2V^* \otimes S^2(TB)\) and can be represented as the symmetric matrix function defined by the formula
\[
\sigma_{ij}(x, p) = \sigma_{ji}(x, p) = A^k_{ij}(x)p_k p_l,
\]
where \(p = (p_0, \ldots, p_n)\) is dual to \(x\). This function is homogeneous with respect to \(p\), and the equation \(\det \sigma = 0\) determines the light hypersurface \(\Sigma \subset PT^n B\).

Let \(\Sigma^{2n} \subset PT^* \mathbb{R}^{n+1}\) be the light hypersurface of our variational principle. According to [A1, Chapter 8], [A2], and [Kh], if its coefficients depend generically on a point of space-time and \(n > 1\), then, in neighbourhoods of typical singular points, \(\Sigma^{2n}\) is reduced by formal contact diffeomorphisms to one of the two normal forms
\[
q^1_1 + p_2 q_2 = 0, \quad p_1^2 = p_2^2 + q_2^2
\]
which are called hyperbolic and elliptic, respectively. Here \((p, q, u)\) are coordinates in \(PT^* \mathbb{R}^{n+1}\) in which the contact structure has the form \(\theta = du - (pdq - qdp)/2\).
In contrast to the elliptic case, for each hyperbolic singularity of $\Sigma^{2n}$, there are two characteristics passing through it. Let $H^{2n-2} \subset \Sigma^{2n}$ be the manifold of all hyperbolic singularities of the light hypersurface, and let $\tilde{H}^{2n-1} \subset \Sigma^{2n}$ be the union of all its characteristics passing through $H^{2n-2}$. In the above local coordinates, $\tilde{H}^{2n-1} = \{p_1 = p_2 = 0\} \cup \{p_1 = q_2 = 0\}$.

**Example.** The propagation of perturbations in a three-dimensional elastic medium is a good model example of the situation described above, where $\nu = (u, v, w)$ is a shift vector of a point of the medium and the number of unknown variables is equal to the space dimension ($m = n = 3$). For instance, if

$$L(t, x, y, z) = \frac{u_t^2 + v_t^2 + w_t^2}{2} - \frac{u_x^2 + (1 + ax)v_y^2 + (1 + bx)(u_y + v_x)^2 + w_z^2}{2},$$

then

$$\sigma(t, x, y, z; s, p, q, r) = \begin{pmatrix} s^2 - p^2 - (1 + bx)q^2 \\ -(1 + bx)pq \\ s^2 - (1 + bx)p^2 - (1 + ax)q^2 \\ s^2 - r^2 \end{pmatrix}.$$

Setting $s = 1$, we obtain an equation of the light hypersurface $\Sigma$ in the affine part of $PT^*\mathbb{R}^4$. At the points where $x = 0$ and $s = q = 1$, the light hypersurface has conical singularities if $ab(a - b) \neq 0$; their quadratic parts are

$$\left| \begin{array}{cc} 2(q - 1) + bx & p \\ p & 2(q - 1) + ax \end{array} \right|.$$

The case of $ab < 0$ is elliptic and of $ab > 0$ hyperbolic. The contact structure is defined by the formula $dt + p dx + q dy + r dz = 0$.

### 3. Singularities of Legendre Varieties

An initial front defines a Legendre submanifold $L^n \subset PT^*\mathbb{R}^{n+1}$ consisting of all contact elements which are tangent to the initial front. The transversal intersection $L^{n-1} = L^n \cap \Sigma^{2n}$ is an integral submanifold of $PT^*\mathbb{R}^{n+1}$, and the closure of the union of all characteristics of $\Sigma^{2n}$ beginning on $L^{n-1}$ is a Legendre variety; it is denoted by $\hat{L}^n \subset PT^*\mathbb{R}^{n+1}$. Its projection is the big front in space-time describing the propagation of the instant fronts defined by the initial one. If some characteristic beginning on $L^n$ ends at a hyperbolic singularity of $\Sigma^{2n}$, then, after this moment of time, the Legendre variety $\hat{L}^n$ acquires the singularities described by Theorem 1, which was proved in [A1, Chapter 8] for $n = 2$.

**Theorem 1.** Suppose that $L^n$ does not pass through singular points of $\Sigma^{2n}$ and transversally intersect $\tilde{H}^{2n-1}$ at a point $O$. Let $P \in H^{2n-1}$ be the endpoint of the characteristic beginning at $O$. Then, in a neighbourhood of $P$, the Legendre variety $\hat{L}^n$ is reduced to the normal form

$$\Lambda_2 = \{2p_1u_2 + q_1 = 0, \ p_1^2 + p_2q_2 = 0, \ p_3 = \ldots = p_n = 0, \ u + p_2^2/2 = 0\}.$$
by a contact diffeomorphism preserving the light hypersurface $p_1^2 + p_2 q_2 = 0$. (The normal form $\Lambda_2$ is the Legendre variety with smooth part defined by the specified equations.)

**Corollary.** The Legendre variety $\tilde{\mathcal{L}}^n$ described by Theorem 1 has singularities when $p_1 = \cdots = p_n$, $q_1 = u = 0$ and $q_2 \leq 0$. If $q_2 < 0$, then, in neighbourhoods of these singularities, $\tilde{\mathcal{L}}^n$ is reduced to the normal form

$$\Lambda_1 = \{2p_1 \ln p_1^2 + q_1 = 0, \quad p_2 = \cdots = p_n = 0, \quad u + p_1^2 = 0\}$$

by a local contact diffeomorphism reducing the light hypersurface to the form $p_2 = 0$. (The normal form $\Lambda_1$ is the Legendre variety with smooth part defined by the specified equations.)

**Proof.** The diffeomorphism is $p_2 \mapsto (p_1^2 - p_2)e^{-q_2}, \quad q_1 \mapsto q_1 + 2p_1 q_2, \quad q_2 \mapsto -e^{q_2}$, $u \mapsto u + (p_1^2 - p_2 + p_2 q_2)/2$. $\square$

**Proof of Theorem 1.** Our proof is analogous to that given in [A1, Chapter 8] for $n = 2$. In the above coordinates, the characteristics of $\Sigma^{2n}$ are described by the equations

$$\dot{p}_1 = 0, \quad \dot{p}_2 = -p_2, \quad \dot{q}_1 = 2p_1, \quad \dot{q}_2 = q_2, \quad \dot{u} = 0, \quad \dot{u} = 0,$$  \hspace{1cm} (1)

where $p_* = (p_3, \ldots, p_n)$ and $q_* = (q_3, \ldots, q_n)$. Consider the characteristic $\mathcal{OP}$ given by the equations $p_1 = 0$, $p_2 \geq 0$, $q_1 = q_2 = 0$, $p_* = q_* = 0$, and $u = 0$ (maybe after an obvious change of the variables). On $\Sigma^{2n}$, the intersection

$$\tilde{\mathcal{L}}^{n-1} = \{p_2 = 1\} \cap \tilde{\mathcal{L}}^n$$

is transversal to $\tilde{\mathcal{H}}^{2n-1}$, whose equation on $\Sigma^{2n}$ is $p_1 = 0$. So $(p_1, q_*)$ are coordinates on $\tilde{\mathcal{L}}^{n-1}$ obtained by several contact changes preserving $\Sigma^{2n}$ and having the form

$$(p_1, q_1) \mapsto (q_1, -p_1), \quad \text{where } i = 3, \ldots, n.$$  

Therefore,

$$\tilde{\mathcal{L}}^{n-1} = \{p_2 = 1, \quad p_* = f_*(p_1, q_1), \quad q_1 = -f_*(p_1, q_*), \quad q_2 = -p_1^2, \quad u = g(p_1, q_*)\}.$$  

Indeed, the function $f$ exists because $d((-q_1 dp_1 + p_1 dq_1)|_{\tilde{\mathcal{L}}^{n-1}}) = dp \wedge dq|_{\tilde{\mathcal{L}}^{n-1}} = -d\theta|_{\tilde{\mathcal{L}}^{n-1}} = 0$. The sympletic change

$$(p_1, p_2, p_*, q_1, q_2, q_*) \mapsto (p_1, p_2, p_* - f_*(p_1, q_*), q_1 + f_*(p_1, q_*), q_2, q_*)$$

kills $f$, preserving $\Sigma^{2n}$. The corresponding contact change reduces $g$ to $-p_1^2/2$, because, on $\tilde{\mathcal{L}}^{n-1}$, we have $du = (p_2 dq_2 - q_2 dp_2)/2 = -p_1 dp_1$ if $f = 0$. Thus

$$\tilde{\mathcal{L}}^{n-1} = \{p_2 = 1, \quad p_* = 0, \quad q_1 = 0, \quad q_2 = -p_1^2, \quad u = -p_1^2/2\}.$$  

But the functions $2p_1 \ln p_2 + q_1, \quad p_1^2 + p_2 q_2, \quad p_*, \quad \text{and } u + p_1^2/2$ are constant along the characteristics (1) and vanish on $\tilde{\mathcal{L}}^{n-1}$. $\square$

### 4. NORMAL FORMS OF LEGENDRE FIBRATIONS

The main theorem follows from Theorem 2 formulated in this section and proved in Section 6.
Definition. Let $\Lambda \subset E$ be a Legendre variety. Two Legendre fibrations $\pi, \pi' : E \to B$ are called weakly $\Lambda$-equivalent if $\pi' \circ h = f \circ \pi$, where $f$ is a diffeomorphism of $B$ and $h$ is a diffeomorphism of $E$ which preserves $\Lambda$.

Remarks. 1. If $\pi$ and $\pi'$ are weakly $\Lambda$-equivalent, then the fronts $\pi(\Lambda)$ and $\pi'(\Lambda)$ are diffeomorphic.

2. The diffeomorphism $g$ may be non-contact.

**Definition.** Let $W : \mathbb{R}^n \times \mathbb{R}^{n+1} \to \mathbb{R}$ be a smooth function of $p_1, q_J$, and let $(y_1, \ldots, y_n, z) \in \mathbb{R}^{n+1}$ satisfy the non-degeneracy condition

\[
\det \begin{vmatrix}
W_{p_1y} & W_{p_2z} \\
W_{q_Jy} & W_{q_Jz}
\end{vmatrix} \neq 0
\]

where $I \cap J = \emptyset$ and $I \cup J = \{1, \ldots, n\}$. Then $W$ is called a generating family of the Legendre fibration $\pi : (p, q, u) \mapsto (y, z)$ whose contact structure and fibres are given by the formulas

\[
dw - (pdq - qdp)/2 = 0,
\]

\[
\pi^{-1}(y, z) = \{p_J = W_{q_J}, q_I = -W_{p_I}, u = W - p_1 W_{p_1}/2 - q_J W_{q_J}/2\}.
\]

**Theorem 2.** If $n \leq 3$, then, in a neighbourhood of a singular point of the Legendre variety $\Lambda_1$, a generic Legendre fibration is weakly $\Lambda_1$-equivalent to one of the normal forms given in a neighbourhood of the origin by the following generating families:

- $(\Theta_2)$ $W(p, y, z) = y_1 p_1 + \cdots + y_n p_n - z$;
- $(\Theta_k)$ $W(q_1, p_2, \ldots, p_n, y, z) = \pm q_1^k + y_1 q_1 + \cdots + y_{k-1} q_1^{k-1} + y_2 p_2 + \cdots + y_n p_n - z$, where $n + 1 \geq k \geq 3$;
- $(\Xi_4)$ $W(p_1, q_2, p_3, \ldots, p_n, y, z) = q_2^2 + y_1 p_1 + y_2 q_2 + y_3 p_1 q_2 + y_3 p_3 + \cdots + y_n p_n - z$, where $n \geq 3$.

**Remarks.** 1. The change $(p_1, q_1) \mapsto (-p_1, -q_1)$ shows that the singularities $\Theta_k^\pm$ and $\Theta_k^-\pm$ are reduced to each other if $k$ is odd.

2. It can be shown that all simple stable mappings of the Legendre variety $\Lambda_1$ are exhausted by the singularities $\Theta_k^\pm$ and $\Xi_4$.

**Proof of the main theorem.** A direct verification shows that the fronts of $\Lambda_1$ mentioned in Theorem 2 give the normal forms required in the main theorem. □

5. **Normal Forms of Fibres**

We start the proof of Theorem 2 with finding normal forms for separate fibres which pass through the singular points of the Legendre variety $\Lambda_1$ with respect to contact diffeomorphisms preserving $\Lambda_1$ itself. The results are collected in Theorem 3.

**Definition.** Let $\Lambda \subset E$ be a Legendre variety. Two Legendre submanifolds are called $\Lambda$-equivalent if they coincide up to a contact diffeomorphism preserving $\Lambda$. 
If the contact structure is \( du - (p dq - q dp)/2 = 0 \), then in the coordinates \((p, q, u)\) any Legendre submanifold \( L \) is locally determined by at least one of the 2\(^n\) generating functions \( w(p_I, q_J) \) according to the formulas

\[
p_I = w_{q_J}, \quad q_I = -w_{p_I}, \quad u = w - p_I w_{p_I}/2 - q_J w_{q_J}/2,
\]

where \( I \cap J = \emptyset \) and \( I \cup J = \{1, \ldots, n\} \). On the other hand, if \((p_I, q_J)\) are local coordinates on \( L \), then

\[
w(p_I, q_J) = \psi_I|_L, \quad \text{where} \quad \psi_I(p, q, u) = u - (p_I q_I - p_J q_J)/2,
\]
is its generating function. For example, \( w(p_1, q_2, \ldots, q_n) = p_1^2 \ln(p_1^2/e) \) is a generating function of the Legendre submanifold \( \Lambda_1 \).

**Theorem 3.** Let \( L_b \) be a generic family of Legendre submanifolds depending on a point \( b \in B \), where \( \dim B \leq 4 \), and let \( L_b \) be any Legendre submanifold from \( L_b \) which intersects the Legendre variety \( \Lambda_1 \) at its singular point. Then, for any \( n \geq 1 \), in a neighbourhood of this point \( L_b \) is \( \Lambda_1 \)-equivalent to one of the normal forms given in a neighbourhood of the origin by the following generating functions:

\[
(\Theta_2) \quad w(p) = 0;
\]

\[
(\Theta_k^k) \quad w(q_1, p_2, \ldots, p_n) = \pm q_1^k, \quad \text{where} \quad \dim B \geq k \geq 3;
\]

\[
(\Xi_4) \quad w(p_1, q_2, p_3, \ldots, p_n) = q_2^4, \quad \text{where} \quad \dim B \geq 4.
\]

**Remarks.**

1. The fibres of a generic Legendre fibration form a family of Legendre submanifolds depending generically on a point \( b \in B \), where \( \dim B = n + 1 \).

2. The change \((p_1, q_1) \mapsto (-p_1, -q_1)\) shows that the singularities \( \Theta_k^+ \) and \( \Theta_k^- \) are reduced to each other if \( k \) is odd.

3. It can be shown that all simple germs of Legendre submanifolds at singular points of the Legendre variety \( \Lambda_1 \) are exhausted by the singularities \( \Theta_k^\pm \) and \( \Xi_4 \) up to \( \Lambda_1 \)-preserving contact diffeomorphisms.

**Proof.** Let \( L_b \) intersect \( \Lambda_1 \) at its singular point \( p = 0, q = q^0 = (0, q^0_2, \ldots, q^0_n), u = 0 \). The contact diffeomorphism \( p \mapsto p, q \mapsto q - q^0, u \mapsto u + pq^0/2 \) moves this point to 0 and preserves \( \Lambda_1 \). So the Legendre submanifold obtained from \( L_b \) can be locally specified by a generating function \( w \) such that \( w(0) = w_{p_1}(0) = w_{q_1}(0) = 0 \).

The singularities of \( \Lambda_1 \) form a submanifold of codimension \( n + 2 \) in \( E \). So the germs of Legendre submanifolds from \( L_b \) at singular points of \( \Lambda_1 \) form a family depending generically on \( \dim B + \dim L_b - (n + 2) = \dim B - 2 \leq 2 \) parameters. Any germ from such a family can be specified by a generating function \( w(p_I, q_J) \), where \( \#J = 0 \) or 1. Indeed, the condition \( \dim TL_b \cap \{dp = du = 0\} \geq 1 \) for the tangent plane \( TL_b \) to the germ of \( L_b \) involves at least three parameters. It remains to apply Lemma 1 proved below.

**Notation.** For two germs \( w \) and \( w' \) of generating functions, we write \( w \sim_\Lambda w' \) if the corresponding germs of Legendre submanifolds are \( \Lambda \)-equivalent.

**Lemma 1.** If \( w(p_I, q_J) \) is the germ at 0 of a generating function such that \( w(0) = w_{p_I}(0) = w_{q_J}(0) = 0 \), then

(a) \( J = \emptyset \implies w \sim_\Lambda 0 \);

(b) \( J = \{1\}, \partial_{q_1} w(0) = \cdots = \partial_{q_1}^{-1} w(0) = 0, \partial_{q_1}^k w(0) \neq 0, k \geq 3 \implies w \sim_\Lambda q_1^k. \)
(c) \( J = \{2\} \), \( \partial_p, \partial_q, w(0) = \partial^2_{q_2} w(0) = 0, \partial^3_{q_2} w(0) \neq 0 \implies w \sim_{\Lambda_1} q^3_2 \).

Proof. It is sufficient to prove (a) and (b) for \( n = 1 \) and (c) for \( n = 2 \), which follow from the equivalence \( w \sim_{\Lambda_1} w_0 \), where \( w_0(p_1, q_2) = w|_{p''=0} \). \( I'' = I \cap \{2, \ldots, n\} \). The equivalence is established by the contact diffeomorphism

\[
(p_1, p_2, q_1, q_2, u) \mapsto (p_1, p_2 - \hat{w}q_2, q_1 + \hat{w}p_1, q_2, u - \hat{w} + p_1 \hat{w}p_1/2 + q_2 \hat{w}q_2/2),
\]

where \( \hat{w} = w - w_0 \). This diffeomorphism preserves \( \Lambda_1 \), because it shifts the plane \( p_1'' = 0 \) only along \( q_1'' \) (preserving \( p_1, p_2, q_1, q_2, \) and \( u \)), which follows from the equality \( \hat{w}|_{p_{1''}=0} = 0 \).

The infinite chains \( a_2 \Rightarrow a_3 \Rightarrow \ldots, b^k \Rightarrow b^k_{k+1} \Rightarrow b^k_{k+2} \Rightarrow \ldots, \) and \( c^6 \Rightarrow c^6_1 \Rightarrow c^6_2 \Rightarrow \ldots \) of assertions of the following Lemma 2 prove (a), (b), and (c), respectively, on the level of formal series. To prove Lemma 1 in the smooth case, it is sufficient to apply the finite-determinacy theorem from [AGLV, Chapter 3, 2] to the nice geometric group of \( \Lambda \)-equivalence [AGLV, Chapter 3, 2.5]. □

Lemma 2. Let \( \alpha_1 = \deg p_1 \) and \( \beta_1 = \deg q_1 \) be positive integer quasides, and let \( A_0 \supseteq A_1 \supseteq \ldots \) be the corresponding quasihomogeneous filtration in the algebra of the germs at 0 of smooth functions of \( p_1 \) and \( q_1 \). Then

(a) \( n = 1, J = \emptyset, \alpha_1 = 1, w_2 \in A_d, d \geq 2 \Rightarrow w_2 \sim_{\Lambda_1} 0 \) (mod \( A_{d+1} \));

(b) \( n = 1, J = \{1\}, \beta_1 = 1, w_k \in A_k, \partial^k_{q_1} w_k(0) \neq 0, k \geq 3 \implies \)

(c) \( n = 2, J = \{2\}, \alpha_1 = 3, \beta_2 = 2, w_6 \in A_d, \partial^3_{q_2} w_6(0) \neq 0 \implies \)

Lemma 2 is proved in Section 6.

6. CONTACT VECTOR FIELDS AND \( \Lambda \)-VERSALITY

In this section, we prove Theorem 2 and Lemma 2. Theorem 2 follows from Theorem 3, which was reduced to Lemma 2 in Section 5.

A vector field preserving a contact structure on a manifold is called contact. It is well known that any contact field is uniquely determined by its generator. If the contact structure is given as the null subspaces of a 1-form \( \theta \), then the generator of a contact vector field \( v \) is the function \( K = \theta(v) \). In our case, \( \theta = du - (pdq - qdp)/2 \) and \( v \) is defined by the formulas

\[
\hat{p} = K_q + pK_u/2, \quad \hat{q} = -K_p + qK_u/2, \quad \hat{u} = K - (pK_p + qK_q)/2.
\]

Let \( L(w) \) be the Legendre submanifold determined by a generating function \( w(p_1, q_1) \), i.e.,

\[
L(w) = \{p_1 = w_{q_1}, q_1 = -w_{p_1}, \ u = w - p_1w_{p_1}/2 - q_1, w_{q_1}/2\}
\]

and let \( K(w) \) denote the derivative of the generating function when the Legendre submanifold is subjected to the action of the contact vector field \( v \) with generator \( K \).
Lemma 3. \( K(w) = K|_{L(w)} \).

Proof. Indeed, \((p_I, q_J)\) are local coordinates on \(L(w)\), and \(w(p_I, q_J) = \psi_I|_{L(w)}\), where \(\psi_I(p, q, u) = u - (pqI - p^2qJ)/2\). So \(K(w) = \psi_I|_{L(w)} - (p_I|_{L(w)} w_{p_I} + q_J|_{L(w)} w_{q_J}) = (u + \dot{p}q/2 - \dot{p}q/2)|_{L(w)} = K|_{L(w)} \). \(\Box\)

Definition. Let \(\Lambda \subset E\) be a Legendre variety. Two families \(L_\ast, L_\ast'\) of Legendre submanifolds depending on a point \(b \in B\) are called \(\Lambda\)-equivalent if \(L_{\ast(b)} = g_\ast(L_b)\), where \(f\) is a diffeomorphism of \(B\) and \(g_\ast\) is a family of contact diffeomorphisms of \(E\) preserving \(\Lambda\) and depending on a point \(b \in B\).

Lemma 4. Two Legendre fibrations \(\pi, \pi': E \to B\) are weakly \(\Lambda\)-equivalent if the families of their fibres are \(\Lambda\)-equivalent.

Proof. Because the families \(\pi^{-1}(\ast)\) and \(\pi^{-1}(\ast)\) are \(\Lambda\)-equivalent, \(\pi^{-1}(f(b)) = g_b(\pi^{-1}(b)) = h(\pi^{-1}(b))\), where \(h(e) = g_{\pi(e)}(e)\) for \(e \in E\). The mapping \(h\) preserves \(\Lambda\) (but not the contact structure in general) and establishes diffeomorphisms between the fibres \(\pi^{-1}(b)\) and \(\pi'^{-1}(f(b))\) for all \(b \in B\). Therefore, \(h\) is a diffeomorphism such that \(\pi' \circ h = f \circ \pi\). \(\Box\)

Let \(\mathcal{E}\) be the algebra of smooth functions on \(E\) and \(\mathcal{I}_\Lambda \subset \mathcal{E}\) be the ideal consisting of all functions vanishing on \(\Lambda\). The contact vector fields with generators from \(\mathcal{I}_\Lambda\) are tangent to \(\Lambda\). Then, according to Lemma 3, the tangent space to the \(\Lambda\)-equivalence orbit of the Legendre submanifold \(L(w)\) is the restriction \(\mathcal{I}_\Lambda|_{L(w)}\). Let \(W(p_I, q_J, b)\) be a smooth deformation of the generating function \(W(p_I, q_J, 0) = w(p_I, q_J)\) and \(\dot{W} = \partial_b W|_{b = 0} \in \mathcal{E}_{p_Iq_J}\) be its initial velocities, which are elements of the algebra \(\mathcal{E}_{p_Iq_J}\) of smooth functions of \(p_I\) and \(q_J\).

Definition. The deformation \(W\) of the generating function \(w\) is called infinitesimally \(\Lambda\)-versal if it is transversal to the \(\Lambda\)-equivalence orbit of \(L(w)\):

\[ \langle \dot{W}\rangle_{\mathcal{E}} + \mathcal{I}_\Lambda|_{L(w)} = \mathcal{E}_{p_Iq_J}. \]

Remarks. 1. If \(W\) is a generating family of a Legendre fibration \(\pi: E \to B\), then the \(\Lambda\)-versality of \(W\) is nothing but the Givental stability criterion for the Legendre mapping \(\Lambda \to E \xrightarrow{\pi} B\) \([G, 3.3]\).

2. In order to define \(\Lambda\)-versality without coordinates, consider a deformation \(L_\ast = g_\ast(L)\) of a Legendre submanifold \(L_0 = g_0(L)\), where \(g_\ast\) is a family of contact diffeomorphisms depending smoothly on a point \(b \in B\) and \(L\) is a fixed Legendre submanifold. Let \(\dot{g}_0(\dot{g}_0(e)) = \partial_b g_0(e)|_{b = 0}\) be the initial velocities of the deformation \(g_\ast\), which are contact vector fields on \(E\). In this situation, the deformation \(L_\ast\) is called infinitesimally \(\Lambda\)-versal if

\[ \langle \theta(\dot{g}_0)\rangle_{\mathcal{E}} + \mathcal{I}_\Lambda + \mathcal{I}_{L_0} = \mathcal{E}, \]

where \(\mathcal{I}_{L_0} \subset \mathcal{E}\) is the ideal consisting of all functions vanishing on \(L_0\). The point is that the contact vector fields with generators from \(\mathcal{I}_{L_0}\) move \(L_0\) along itself.

Proof of Theorem 2. The generating families from Theorem 2 are infinitesimally \(\Lambda_1\)-versal deformations of the generating functions from Theorem 3. Indeed, \(\mathcal{I}_{\Lambda_1} = (u + p_1^2, p_2, \ldots, p_n), b = (y, z)\), and
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(\Theta_2) \ w(p) = 0, L(w) = \{ q = 0, \ u = 0 \}, \mathcal{I}_{\Lambda_1}|_{L(w)} = (p_1^2, p_2, \ldots, p_n), (\dot{W})_{\mathbb{R}} = \\
\{1, p\}_{\mathbb{R}};

(\Theta_1^k) \ w(q_1, p_2, \ldots, p_n) = \pm q_1^k, \ \mathcal{I}_{\Lambda_1}|_{L(w)} = \{ p_1 = \pm kq_1^{-1}, \ q_2 = \cdots = q_n = 0, \\ u = \pm (1 - k/2)q_1^k \}, \ \text{where} \ k \geq 3, \ \mathcal{I}_{\Lambda_1}|_{L(w)} = (q_1^k, p_2, \ldots, p_n), (\dot{W})_{\mathbb{R}} = \\
\{1, q_1, p_2 + q_1^2, \ldots, p_{k-1} + q_1^{k-1}, p_k, \ldots, p_n\}_{\mathbb{R}}, \ \text{where} \ n \geq k - 1; \\
(\Xi_k) \ w(p_1, q_2, p_3, \ldots, p_n) = q_1^2, \ L(w) = \{ p_2 = 3q_2^2, \ q_1 = q_2 = \cdots = q_n = 0, \\ u = -q_2^2/2 \}, \mathcal{I}_{\Lambda_1}|_{L(w)} = (p_1^2, q_2^2, p_3, \ldots, p_n), (\dot{W})_{\mathbb{R}} = \{1, p_1, q_2, p_3 + p_1q_2, \\ p_4, \ldots, p_n\}_{\mathbb{R}}, \ \text{where} \ n \geq 3.

Let \( \pi: E \to B \) be a generic Legendre fibration. Then its fibres form a family \( L_\ast = \pi^{-1}(*) \) of Legendre submanifolds depending generically on a point \( b \in B \). Therefore, provided that \( n = \dim B - 1 \leq 3 \), the germs of the family \( L_\ast \) at the singular points of the Legendre variety \( \Lambda_1 \) are infinitesimally \( \Lambda_1 \)-versal deformations of Legendre submanifolds such as in Theorem 3. So, according to the general versality theorem from [AGLV, Chapter 3, 2.3, 2.5], they are \( \Lambda_1 \)-equivalent to deformations described by Theorem 2. Lemma 4 implies the required weak \( \Lambda_1 \)-equivalence of the corresponding germs of Legendre fibrations. \( \square \)

**Proof of Lemma 2.** We use the standard homotopy method. Namely, let \( \hat{w}_\tau \) be a family of generating functions depending smoothly on a parameter \( \tau \), and let \( K_\tau \) be a smooth family of generators satisfying the homological equation

\[
K_\tau(\hat{w}_\tau) + \partial_\tau \hat{w}_\tau = 0
\]
on a segment \([\tau_0, \tau_1]\). Besides, we assume that the corresponding contact vector fields \( v_K \) are tangent to the Legendre variety \( \Lambda_1 \) and preserve \( 0 \), i.e., \( v_{K_\tau}(0) = 0 \). For the generators \( K_\tau \), these conditions mean that \( K_\tau|_{\Lambda_1} = 0 \) and \( K_\tau(0) = \partial_\tau K_\tau(0) = 0 \), respectively. Now, solving the Cauchy problem

\[
\dot{g}_\tau(p, q, u) = v_K(g_\tau(p, q, u)), \quad g_{\tau_0}(p, q, u) = (p, q, u)
\]
with respect to a family of diffeomorphisms \( g_\tau \) on the segment \([\tau_0, \tau_1]\) for small \((p, q, u)\), we obtain the equivalence \( \hat{w}_{\tau_0} \sim_{\Lambda_1} \hat{w}_{\tau_1} \) established by the local contact diffeomorphism \( g_{\tau_1} \) preserving \( \Lambda_1 \) and \( 0 \).

\((a_d)\) In this case, \( w_d = ap_1^d \). Let \( \tilde{w}_\tau = \tau p_1^d \) and \([\tau_0, \tau_1] = [0, a] \). Then

\[
K_\tau = -(u + p_1^2)p_1^{d-2}
\]
is the required solution of the homological equation. Indeed, using Lemma 3, we obtain

\[
K_\tau(\tilde{w}_\tau) + \partial_\tau \tilde{w}_\tau = -(\tau(1 - d/2)p_1^d + p_1^2)p_1^{d-2} + p_1^d = 0 \pmod{\mathcal{A}_d+1}
\]
provided that \( d \geq 2 \).

\((b_k)\) In this case, \( w_k = aq_1^k \), where \( a \neq 0 \). Let \( \tilde{w}_\tau = \tau q_1^k \) and \([\tau_0, \tau_1] = [a, \text{sign}(a)] \). Then

\[
K_\tau = \frac{u + p_1^2}{\tau(k/2 - 1)}
\]
is the required solution of the homological equation. Indeed, using Lemma 3, we get
\[ K_\tau(\mathcal{w}) + \partial_\tau \mathcal{w} = \frac{\tau(1-k/2)a_d q^d_1 + \tau^2 k d^2 q_1^{2k-2}}{\tau(k/2-1)} + q_1^d = 0 \pmod{A_{k+1}} \]
provided that \( k \geq 3 \).

(b) In this case, \( w_d = a q^d_1 \). Let \( \mathcal{w}_\tau = \pm q^d_1 + \tau q^d_1 \) and \([\tau_0, \tau_1] = [0, a]\). Then
\[ K_\tau = \pm \frac{u + p_1^d}{k/2 - 1} q_1^{d-k} \]
is the required solution of the homological equation. Indeed, Lemma 3 gives
\[ K_\tau(\mathcal{w}_\tau) + \partial_\tau \mathcal{w}_\tau = \pm \frac{(1-k/2)a_d q^d_1 + (k^2 q_1^{2k-1} + \tau^2 d q_1^{d-1})^2}{k/2 - 1} q_1^{d-k} = 0 \pmod{A_{d+1}} \]
provided that \( d > k \geq 3 \).

(c) In this case, \( w_6 = ap_1^2 + bq_1^3 \), where \( b \neq 0 \). The contact change \( p_2 \mapsto b^{1/3} p_2 \), \( q_2 \mapsto b^{-1/3} q_2 \) preserves \( \Lambda_1 \) and reduces \( w_6 \mapsto ap_1^2 + q_1^3 \). Let \( \mathcal{w}_\tau = \tau p_1^2 + q_1^3 \) and \([\tau_0, \tau_1] = [0, a]\). Then
\[ K_\tau = -(u + p_1^2) - p_2 q_2/6 \]
is the required solution of the homological equation. Indeed, Lemma 3 gives
\[ K_\tau(\mathcal{w}_\tau) + \partial_\tau \mathcal{w}_\tau = -(\tau(1-2/2)p_1^2 + (1-3/2)q_1^3 + p_1^2) - 3q_1^3/6 + p_1^2 = 0. \]

(d) In this case, \( w_d = ad_6(p_1, q_2) p_1^2 + b_{d-4}(p_1, q_2) q_1^2 \), where \( a_{d-6} \in A_{d-6} \) and \( b_{d-4} \in A_{d-4} \). Let \( \mathcal{w}_\tau = q_1^2 + \tau w_d \) and \([\tau_0, \tau_1] = [0, 1]\). Then
\[ K_\tau = -a_{d-6}(p_1, q_2)(u + p_1^2 + p_2 q_2/6) - b_{d-4}(p_1, q_2)p_2/3 \]
is the required solution of the homological equation. Indeed, Lemma 3 gives
\[ K_\tau(\mathcal{w}_\tau) + \partial_\tau \mathcal{w}_\tau = -a_{d-6}(p_1, q_2)(p_1^2 + \tau(w_d - p_1 q_1 w_d/2 - q_2 q_1 p_2 w_d/3)) - b_{d-4}(p_1, q_2)(q_1^2 + \partial_2 w_d/3) + w_d = 0 \pmod{A_{d+1}} \]
provided that \( d > 6 \).

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References


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