AN INTEGRAL GENERALIZATION 
OF THE GUSEIN-ZADE–NATANZON THEOREM

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To V. I. Arnold with admiration

Abstract. Several years ago N. A’Campo invented a construction of a 
link from a real curve immersed into the disk. In the case of the curve 
obtained by the real morsification method of singularity theory the link 
is isotopic to the link of the corresponding singularity. S. M. Gusein-
Zade and S. M. Natanzon proved that the Arf invariant of the obtained 
knot equals $J^-/2 \pmod{2}$ of the corresponding curve. Here we describe 
the Casson invariant of A’Campo knots as a $J^\pm$-type invariant of the 
immersed curve. Thus we get an integral generalization of the Gusein-
Zade–Natanzon theorem. It turns out that this $J^2\pm$-invariant is a second 
order invariant of the mixed $J^+\,$- and $J^-\,$-types. To the best of my 
knowledge, nobody has yet tried to study the mixed $J^\pm\,$-type invariants. 
It seems that our invariant is one of the simplest such invariants.


Key words and phrases. Knots, A’Campo’s divides, immersed curves, Cas-
son invariant, $J^\pm\,$-type invariants.

One of the useful methods of singularity theory, the method of real morsifications
[AC0], [GZ] (see also [AGZV]), reduces the study of discrete topological invariants 
of a critical point of a holomorphic function in two variables to the study of certain 
real plane curves immersed into a disk with only simple double points of self-
intersection. For closed real immersed plane curves, Arnold [Ar] found the three 
simplest first-order invariants $S_t$ and $J^\pm$. Arnold’s theory can be easily adapted 
to curves immersed into a disk. In [GZN], Gusein-Zade and Natanzon proved that 
the Arf invariant of a singularity is equal to $J^-/2 \pmod{2}$ of the corresponding 
immersed curve. They used the definition of the Arf invariant in terms of the 
Milnor lattice and the intersection form of the singularity. However, it is equal to 
the Arf invariant of the link of singularity, the intersection of the singular complex 
curve with a small sphere in $\mathbb{C}^2$ centered at the singular point. In knot theory, it is 
well known (see, for example, [Ka]) that the Arf invariant is the mod 2 reduction 
of an integer-valued invariant, the second coefficient of the Conway polynomial, or 
the Casson invariant [PV]. Arnold’s $J^-/2$ is also an integer-valued invariant. So 
one might expect a relationship between these integral invariants.
A few years ago, A’Campo [AC2], [AC1] invented the construction of a link from a real curve immersed into a disk. In the case of the curve originating from the real morsification method, the link is isotopic to the link of the corresponding singularity. But there are some curves which do not occur in singularity theory. In this article, we describe the Casson invariant of A’Campo’s knots as a $J^{±}$-type invariant of the immersed curves. Thus, we get an integral generalization of the Gusein-Zade–Natanzon theorem. It turns out that this $J^{±}$ invariant is a second-order invariant of mixed $J^{+}$- and $J^{-}$-types. To the best of my knowledge, so far nobody tried to study mixed $J^{±}$-type invariants. It seems that our invariant is one of the simplest such invariants. The problem of describing all second-order $J^{±}$-type invariants is open.

In Section 1, we describe the A’Campo construction and list the properties of the links obtained. In Sections 2 and 3, we introduce the Casson invariant and Arnold type invariants of curves immersed into a disk. In Section 4, we formulate our main result. The proof is based on Hirasawa’s construction [Hir] of a Seifert surface for the A’Campo links. We describe this construction in Section 5.

After submitting this paper, I have learned that the problem of description of the Casson invariant as an invariant of plane curves was posed by S. M. Gusein-Zade at his seminar “Topology of singularities” at the Moscow State University and was solved by its participant Sergei Shadrin. He got the same description as that presented in this paper. Also, Shimakovitch [Sh] found a formula for the Casson invariant in terms of Arnold’s invariants of some pieces of the curve.

1. A’Campo Divides and their Links

1.1. Definition [AC2], [AC1]. A divide $D$ is the image of a generic immersion of a finite number of copies of the unit interval $I = [0, 1]$ into the unit disk $B \subset \mathbb{R}^2$ such that $\partial I \subset \partial B$. Here the word “generic” means that double points are the only singularities allowed and $D$ is transversal to $\partial B$.

Similar objects are known as long curves [Ta], [GZN].

We consider divides up to an isotopy of the disk $B$. The isotopy is not assumed to be identical on the boundary $\partial B$.

1.2. Example. The curve $x^3 + y^4 = 0$ has a singularity of type $E_6$ at the origin [AGZV]. A small perturbation of it is a divide which looks as follows.

1.3. Definition [AC2], [AC1]. Let $x$ be the horizontal coordinate on the disk $B$, and let $y$ be the vertical coordinate. A divide link $\mathcal{L}_D$ is a link in the 3-sphere $S^3 = \{(x, y, u, v) \in \mathbb{R}^4; x^2 + y^2 + u^2 + v^2 = 1\}$ such that $(x, y)$ is a point on $D$ and $u, v$ are the coordinates of a tangent vector to $D$ at the point $(x, y)$.

So, each interior point of $D$ has two corresponding points on $\mathcal{L}_D$, whereas a boundary point of $D$ gives a single point on $\mathcal{L}_D$. 
The number of components of $\mathcal{L}_D$ equals the number of branches of the divide $D$, which is the number of the copies of the unit interval $I$ in Definition 1.1. In particular, if $D$ consists of only one branch (like in Example 1.2), then $\mathcal{L}_D$ is a knot.

1.4. Properties of $\mathcal{L}_D$.

1.4.1. The topological type of a divide link does not change under a regular transversal isotopy of the disk $B$. So it does not depend on the choice of coordinates in Definition 1.3. Also, it does not change under moving of a piece of the curve $D$ through a triple point [CP]. In particular, the following two divides have the same knot type as in Example 1.2.

1.4.2. The link $\mathcal{L}_D$ has a natural orientation. Indeed, choose any orientation on every branch of $D$. Let $(u, v)$ be the tangent vector to $D$ at $(x, y)$ directed according to the chosen orientation of $D$. Then the orientation of $\mathcal{L}_D$ is given by the vector $(\dot{x}, \dot{y}, \dot{u}, \dot{v})$. It is easy to see that this orientation of $\mathcal{L}_D$ does not depend on the choice of orientations of branches of $D$.

1.4.3. In [AC2], A’Campo proved that any singularity link is a divide link. But the divide in the picture on the right does not occur in singularity theory. Indeed, by the real morsification method, we can find that the Milnor number of the corresponding singularity would be 4. However there are only two singularities with Milnor number 4, $A_4$ and $D_4$. But their intersection forms are different from the one which corresponds to this divide. The knot arising from the divide is 10_{145} (see [AC1], [Ch]).

It has been known for a long time [Bu] that all singularity knots are classified by the Alexander polynomial. A’Campo ([AC3]) found two different divide knots with the same Alexander polynomial. Morton [Mor], to whom I have shown A’Campo’s example, found that these knots are mutant. So they cannot be distinguished by any classical polynomial invariants (Jones, HOMFLY, Kauffman). He distinguished them by a quantum invariant coming from the Lie algebra $\mathfrak{gl}_n$ in a certain higher (non-standard) representation.
1.4.4. A’Campo [AC1] showed that the links \( \mathcal{L}_D \) corresponding to a connected divide \( D \) are fibered and computed their monodromy in terms of the combinatorics of the divide \( D \). Not all fibered links have the form \( \mathcal{L}_D \). Figure eight knot \( 4_1 \) is not a divide knot. It is not clear how large the class of divide links in the class of all fibered links is.

1.4.5. Considering \( \mathbb{R}^4 \) with coordinates \( x, y, u, v \) as a complex plane \( \mathbb{C}^2 \) with coordinates \( z_1 = x + iu \) and \( z_2 = y + iv \), we see that every tangent space to the unit sphere \( S^3 \) contains a unique complex line which is a two-dimensional real subspace in the tangent space. The distribution of these 2-planes forms the standard contact structure on \( S^3 \). The divide knots are transversal knots with respect to this contact structure. This fact was noticed in [AC1], but the arguments given there should be modified. For a one-branch divide \( D \), the Bennequin number (self-linking number) of the knot \( \mathcal{L}_D \) is equal to \( 2\delta - 1 \), where \( \delta \) is the number of double points of \( D \).

1.4.6. N. A’Campo [AC1] proved that the unknotting number of a one-branch divide knot \( \mathcal{L}_D \) and the genus of \( \mathcal{L}_D \) are equal to the number \( \delta \) of double points of \( D \). For a two-branch divide \( D \), the link \( \mathcal{L}_D \) has two components. Their linking number is equal to the number of common double points of the two branches of \( D \).

2. Casson’s Invariant of Knots

2.1. The Conway polynomial. The subject of this section is well known (see, for example, [Ka], [PV]). The Conway polynomial \( C(\mathcal{L}) \) of a link \( \mathcal{L} \) is a polynomial in a single variable \( z \). This is one of the simplest invariants of links in \( \mathbb{R}^3 \); it is by the two conditions: it is equal to 1 on the unknot and satisfies the skein relation

\[
C(\begin{array}{c}
\includegraphics{link1}
\end{array}) - C(\begin{array}{c}
\includegraphics{link2}
\end{array}) = z \cdot C(\begin{array}{c}
\includegraphics{link3}
\end{array}),
\]

where the three links are identical outside a small ball in \( \mathbb{R}^3 \) and look as shown inside the ball.

If \( \mathcal{L} \) is a knot, then the Conway polynomial is even:

\[
C(\mathcal{L}) = 1 + C_2(\mathcal{L}) \cdot z^2 + C_4(\mathcal{L}) \cdot z^4 + \cdots + C_{2n}(\mathcal{L}) \cdot z^{2n}.
\]

2.2. Casson’s invariant. The coefficient \( C_2(\mathcal{L}) \) is called Casson’s knot invariant (see [PV]). It can be defined ([Ka]) by the initial condition \( C_2(\text{unknot}) = 0 \) and the skein relation:

\[
C_2(\begin{array}{c}
\includegraphics{link4}
\end{array}) - C_2(\begin{array}{c}
\includegraphics{link5}
\end{array}) = \text{lk}(\begin{array}{c}
\includegraphics{link6}
\end{array}),
\]

where \( \text{lk} \) means the linking number of the two component link on the right-hand side of the relation.

The mod 2 reduction of Casson’s invariant \( C_2(\mathcal{L}) \) is called the Arf invariant of a knot \( \mathcal{L} \).
3. Arnold’s Invariants of Immersed Plane Curves

3.1. Arnold’s invariants of divides. In [Ar], Arnold defined the three basic invariants $J^+$, $J^-$, and $St$ of a generic closed immersed plane curve. A survey of various explicit formulas for these invariants is contained in [CD]. Following [GZN], we define these invariants for a one-branch divide $D$ as the arithmetic mean of the corresponding Arnold invariants on the two closed curves obtained by smoothing the union of $D$ with either arc of the unit circle. The invariants of divides thus defined will be denoted by the same symbols.

3.2. Example. Let us compute $J^-$ for the divide from Example 1.2:

$$J^-(\text{ divide }) = \frac{J^-(\text{ curve } 1) + J^-(\text{ curve } 2)}{2}.$$

According to Arnold’s table [Ar, p. 14], the first term on the right-hand side equals $-4$ and the second term equals $-8$. So the result is $(-4 - 8)/2 = -6$.

3.3. $J^\pm$ as invariants of order 1. We are going to use a description of Arnold’s $J^\pm$ invariants as invariants of order 1 in the sense of the theory of finite-type invariants.

Let us fix the images of the endpoints of the interval $I$ at the boundary of the disk $\partial B$. A version of the classical theorem of H. Whitney states that the space of all smooth immersions of the interval $I$ into the disk $B$ mapping $\partial I$ into two fixed points consists of a countable number of connected components differing in the absolute value of the rotation number.

Choose a standard divide $D_i$ for each nonnegative value of the rotation number $i$:

$D_0 \quad D_1 \quad D_2 \quad D_3 \quad D_4 \quad D_5 \quad \ldots$

Every one-branch divide can be deformed into one of the $D_i$’s in the space of immersions.

During the deformation, non-divides may occur, at certain moments because singularities of the corresponding curves are not allowed. For a generic deformation two types of such non-divides can occur: either a triple point on the curve or a self-tangency of the curve. In fact, the self-tangency case can be split into two types: direct self-tangency, where the two tangent strings have coherent orientations (for any of the two possible orientations of the curve), and inverse self-tangency, where the two tangent strings have opposite orientations. A first-order invariant is defined by its jumps at the singularities of the three types above and by its values at the standard divides.
In particular, $J^-$ and $J^+$ are defined by the following relations:

\begin{align*}
(i)^-J^- \left( \begin{array}{c}
\circlearrowleft
\end{array}\right) - J^- \left( \begin{array}{c}
\circlearrowright
\end{array}\right) & = 2; \\
(i)^+ J^- \left( \begin{array}{c}
\circlearrowleft
\end{array}\right) & = J^+ \left( \begin{array}{c}
\circlearrowright
\end{array}\right); \\
(i)^+ J^- \left( \begin{array}{c}
\circlearrowleft
\end{array}\right) & = J^+ \left( \begin{array}{c}
\circlearrowright
\end{array}\right); \\
(i)^+ J^- \left( \begin{array}{c}
\circlearrowright
\end{array}\right) & = J^+ \left( \begin{array}{c}
\circlearrowleft
\end{array}\right); \\
\text{(ii)} \quad J^- (D_i) & = -2i; \\
\text{(ii)} \quad J^+ (D_i) & = -i;
\end{align*}

In each equality, we mean that the two divides are identical outside the small fragment shown explicitly.

3.4. **Actuality tables for $J^\pm$**. According to Vassiliev [Va], actuality tables provide a way to organize the data necessary for the computation of a single finite-order invariant.

For the first-order invariants $J^\pm$, the actuality tables are essentially the same thing as the set of equations (i)$^*$–(ii). To describe the top row of an actuality table, we use chord diagrams. Since in the $J^\pm$ theory we have two types of singular events, the inverse and direct self-tangencies, we need two types of chords. We use a dashed chord to depict the inverse self-tangency and a solid chord to depict the direct self-tangency.

The actuality tables for $J^\pm$ look as follows:

| (i)$^-$ $J^- \left( \begin{array}{c}
\circlearrowleft
\end{array}\right)$ | (i)$^-$ $J^- \left( \begin{array}{c}
\circlearrowright
\end{array}\right)$ | (i)$^-$ $J^- \left( \begin{array}{c}
\circlearrowright
\end{array}\right)$ |
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(ii) $J^- (D_i)$</td>
<td>(ii) $J^+ (D_i)$</td>
<td></td>
</tr>
</tbody>
</table>

\begin{align*}
(i)^-J^- \left( \begin{array}{c}
\circlearrowleft
\end{array}\right) & = 2; \\
(i)^+ J^- \left( \begin{array}{c}
\circlearrowleft
\end{array}\right) & = J^+ \left( \begin{array}{c}
\circlearrowright
\end{array}\right); \\
(i)^+ J^- \left( \begin{array}{c}
\circlearrowright
\end{array}\right) & = J^+ \left( \begin{array}{c}
\circlearrowleft
\end{array}\right); \\
(i)^+ J^- \left( \begin{array}{c}
\circlearrowright
\end{array}\right) & = J^+ \left( \begin{array}{c}
\circlearrowleft
\end{array}\right); \\
\text{(ii)} \quad J^- (D_i) & = -2i; \\
\text{(ii)} \quad J^+ (D_i) & = -i;
\end{align*}

3.5. **Example.** Let us compute $J^-$ again for Example 1.2, using now only relations (i)$^*$–(iv). We introduce orientation as shown in order to distinguish between direct (i)$^+$ and inverse (i)$^-$ self-tangencies.

\begin{align*}
J^- \left( \begin{array}{c}
\circlearrowleft
\end{array}\right) & = J^- \left( \begin{array}{c}
\circlearrowright
\end{array}\right) + J^- \left( \begin{array}{c}
\circlearrowright
\end{array}\right) (i)^+ J^- \left( \begin{array}{c}
\circlearrowleft
\end{array}\right) (i)^+ J^- \left( \begin{array}{c}
\circlearrowright
\end{array}\right) \\
& = -J^- \left( \begin{array}{c}
\circleddash
\end{array}\right) + J^- \left( \begin{array}{c}
\circlearrowright
\end{array}\right) (i)^- - 2 + J^- \left( \begin{array}{c}
\circlearrowright
\end{array}\right) \\
& = -2 + J^- (D_2) (ii)
\end{align*}

\begin{align*}
& = -2 + J^- (D_2) (ii)
& = -2 - 2 \cdot 2 = -6.
\end{align*}
4. Main Result

4.1. Theorem. For a one-branch divide $D$, the Casson invariant $C_2(L_D)$ is equal to the invariant $J_2^\pm(D)$ defined below in Section 4.2.

Corollary (Gusein-Zade–Natanzon [GZN]). For a one-branch divide $D$, $\text{Arf}(L_D) = J_2^-(D)/2 \pmod{2}$.

4.2. Definition of the invariant $J_2^\pm$. The invariant $J_2^\pm$ is a second-order $J^\pm$-type invariant. In particular, it does not change under the triple point move. To define it, we need to specify its values on the chord diagrams with two chords and on canonical divides with at most one self-tangency point. We have chosen the canonical divides without self-tangencies in Section 3.3. Now, we must choose the canonical divides with one self-tangency in such a way that any divide with a single self-tangency can be deformed to a canonical one if we allow passing through codimension-two strata in the space of immersions.

4.2.1. Canonical divides with self-tangency points. Our choice of the canonical divides is

$$D_{m,n}^{-} = \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} \quad D_{m,n}^{+} = \begin{array}{c}
\text{m curls} \\
\ldots
\end{array}$$

Here we omit the boundary circle (depicted above as a dashed circle) of the unit disk containing the divide. The parameters $m$ and $n$ in these divides range over all integral numbers. Under a negative curl, we mean a curl going clockwise (rather than counterclockwise, as above). Here is a couple of examples:

$$D_{-3,2}^- = \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} \quad D_{2,-3}^+ = \begin{array}{c}
\text{m curls} \\
\ldots
\end{array}$$

4.2.2. Actuality table for $J_2^\pm$. The actuality table for $J_2^\pm$ looks as follows.

<table>
<thead>
<tr>
<th>$J_2^\pm$</th>
<th>$D_{m,n}^-$</th>
<th>$J_2^\pm$</th>
<th>$D_{m,n}^+$</th>
<th>$J_2^\pm$</th>
<th>$D_n$</th>
</tr>
</thead>
</table>
| $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} | $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} | $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} |
| $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} | $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} | $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} |
| $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} | $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} | $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} |
| $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} | $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} | $J_2^\pm$ | \begin{array}{c}
\text{m curls} \\
\ldots
\end{array} |

$$J_2^\pm(D_{m,n}^-) = \begin{cases}
-6m - 3 \text{ for } m \geq 0, \\
2m - 3 \text{ for } m < 0
\end{cases}$$

$$J_2^\pm(D_{m,n}^+) = \begin{cases}
-6m - 4 \text{ for } m \geq 0, \\
2m - 4 \text{ for } m < 0
\end{cases}$$

$$J_2^\pm(D_n) = n$$
4.2.3. Comparing this actuality table with that given in Section 3.4, we see that the mod 2 reduction of $J^\pm_2$ is an invariant of the first order and $J^\pm_2 = J^-/2 \pmod{2}$. This proves the corollary from Section 4.1.

4.2.4. It would be interesting to find a Polyak–Viro style formula (see [PV], [CD]) for the invariant $J^\pm_2$, in terms of Gauss diagrams of the curve.

4.2.5. Example. Let us compute $J^\pm_2$ for the divide from Example 1.2, as it was done in Section 3.5. The first step is pretty much the same:

$$J^\pm_2 \left( \begin{array}{c}
\circlearrowleft \\
\cdot
\end{array} \right) = J^\pm_2 \left( \begin{array}{c}
\circlearrowleft \\
\cdot
\end{array} \right) + J^\pm_2 \left( \begin{array}{c}
\circlearrowright \\
\cdot
\end{array} \right)$$

Now, we compute separately the two terms on the right-hand side:

$$J^\pm_2 \left( \begin{array}{c}
\circlearrowleft \\
\cdot
\end{array} \right) = J^\pm_2 \left( \begin{array}{c}
\circlearrowleft \\
\cdot
\end{array} \right) = J^\pm_2 \left( \begin{array}{c}
\circlearrowleft \\
\cdot
\end{array} \right) + J^\pm_2 \left( \begin{array}{c}
\circlearrowright \\
\cdot
\end{array} \right)$$

$$= J^\pm_2 \left( \begin{array}{c}
\circlearrowleft \\
\cdot
\end{array} \right) + J^\pm_2 \left( \begin{array}{c}
\circlearrowright \\
\cdot
\end{array} \right)$$

$$= 2 - J^\pm_2 \left( \begin{array}{c}
\circlearrowright \\
\cdot
\end{array} \right) + J^\pm_2 \left( \begin{array}{c}
\circlearrowright \\
\cdot
\end{array} \right)$$

$$= 2 - J^\pm_2 \left( \begin{array}{c}
\circlearrowright \\
\cdot
\end{array} \right) + J^\pm_2 (D^+_{0,1}) = -4.$$ 

For the second term, we have

$$J^\pm_2 \left( \begin{array}{c}
\circlearrowright \\
\cdot
\end{array} \right) = J^\pm_2 \left( \begin{array}{c}
\circlearrowleft \\
\cdot
\end{array} \right) = -J^\pm_2 \left( \begin{array}{c}
\circlearrowright \\
\cdot
\end{array} \right) + J^\pm_2 \left( \begin{array}{c}
\circlearrowright \\
\cdot
\end{array} \right)$$

$$= -J^\pm_2 \left( \begin{array}{c}
\circlearrowright \\
\cdot
\end{array} \right) + J^\pm_2 (D_2) = -J^\pm_2 (D^-_{2,0}) + 2 = 9.$$ 

Summing up, we get $J^\pm_2 \left( \begin{array}{c}
\circlearrowleft \\
\cdot
\end{array} \right) = 5$.

4.2.6. Idea of the proof. In the next section, we describe how to draw a diagram of $\mathcal{L}_D$ from the picture of the divide $D$. This allows us to trace what happens with the knot $\mathcal{L}_D$ when the curve $D$ changes by a direct (inverse) self-tangency move. Applying the skein relation from Section 2.2 gives us the corresponding changes of the Casson invariant. All this information can be summarized in the actuality table from Section 4.2.2.

5. Hirasawa’s Seifert Surface for $\mathcal{L}_D$

5.1. Hirasawa [Hir] suggested a procedure for drawing a picture of the minimal genus Seifert surface for the link $\mathcal{L}_D$, thus drawing the diagrams of the link $\mathcal{L}_D$. Other ways to draw the diagrams are suggested in [CP], [Ch], [Gi]. In this section, we describe Hirasawa’s construction.
First, let us prepare the divide $D$ for drawing the Seifert surface as follows. Choose an orientation for all branches of $D$. Then, deform $D$ inside the unit disk so that

- both branches are oriented from left to right at every double point;
- the $x$ coordinates of the double points and of the points of $D$ with vertical tangents are pairwise different.

Now, to draw the Seifert surface, we thicken every arc of $D$ to form a band and, then, modify the bands near the double points and near the points with vertical tangents where $D$ is oriented downward as shown in the pictures below.
5.2. Example. For the divide $D$ with a single curl (corresponding to the singularity $A_2$), this construction gives the following Seifert surface:

The corresponding knot $L_D$ is the trefoil:

\[ \sim = 3_1. \]

REFERENCES


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