ON AVERAGING IN TWO-FREQUENCY SYSTEMS WITH SMALL HAMILTONIAN AND MUCH SMALLER NON-HAMILTONIAN PERTURBATIONS

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To V. I. Arnold on the occasion of his 65th birthday

Abstract. A system which differs from an integrable Hamiltonian system with two degrees of freedom by a small Hamiltonian perturbation and much a smaller non-Hamiltonian perturbation is considered. The unperturbed system is isoenergetically nondegenerate. The averaging method is used for an approximate description of solutions of the exact system on a time interval inversely proportional to the amplitude of the non-Hamiltonian perturbation. The error of this description (averaged over initial conditions) is estimated from above by a value proportional to the square root of the amplitude of the Hamiltonian perturbation.


Key words and phrases. Perturbation theory, averaging method.

1. Introduction

Consider a system which differs from an integrable Hamiltonian system with 2 degrees of freedom (DOF) depending on parameters by a small, \( \sim \varepsilon \), Hamiltonian perturbation and a much smaller, \( \sim \delta \ll \varepsilon \), non-Hamiltonian perturbation. The equations of motion for such a system have the form

\[
\dot{I} = -\varepsilon \frac{\partial H_1}{\partial \varphi} + \delta f, \quad \dot{z} = \delta g, \quad \dot{\varphi} = \frac{\partial H_0}{\partial I} + \varepsilon \frac{\partial H_1}{\partial I} + \delta l.
\]

(1.1)

Here \( I = (I_1, I_2) \) and \( \varphi = (\varphi_1, \varphi_2) \) are two-dimensional vectors, \( z \) is an \( m \)-dimensional vector, \( \varepsilon \) and \( \delta \) are small positive parameters, \( \delta \leq \varepsilon \), Hamiltonian perturbation and a much smaller, \( \delta \ll \varepsilon \), non-Hamiltonian perturbation. The components of the vector \( \varphi \). The components of the vectors \( I \) and \( z \) are called slow variables. The components of the vector \( \varphi \) are called fast variables or phases. The components of the vector \( \omega_0 = \partial H_0/\partial I \) are called frequencies.

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Along with the system (1.1), consider the following system that we call the averaged system:

\[
\dot{J} = \delta F(J, y), \quad \dot{y} = \delta G(J, y),
\]

(1.2)

where \(F(J, y)\) and \(G(J, y)\) are the mean values of \(f(J, y, \varphi, 0, 0)\) and \(g(J, y, \varphi, 0, 0)\) with respect to \(\varphi\):

\[
\begin{align*}
F(J, y) &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(J, y, \varphi, 0, 0) \, d\varphi_1 \, d\varphi_2, \\
G(J, y) &= \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} g(J, y, \varphi, 0, 0) \, d\varphi_1 \, d\varphi_2.
\end{align*}
\]

We will consider solutions \((I(t), z(t), \varphi(t))\) and \((J(t), y(t))\) of systems (1.1) and (1.2) with the same initial data \((I_0, z_0^0, \varphi_0^0)\):

\[
\begin{align*}
I(t) &= I(t, I_0^0, z_0^0, \varphi_0^0, \varepsilon, \delta), \quad z(t) = z(t, I_0^0, z_0^0, \varphi_0^0, \varepsilon, \delta), \\
J(t) &= J(\delta t, I_0^0, z_0^0), \quad y(t) = y(\delta t, I_0^0, z_0^0), \\
I(0) &= I_0^0, \quad z(0) = z_0^0, \quad \varphi(0) = \varphi_0^0, \quad J(0) = I_0^0, \quad y(0) = z_0^0.
\end{align*}
\]

The main result of the present paper (Theorem 1, which will be formulated in Section 2) is as follows. The error, averaged over initial conditions, in the description of the slow component \((I(t), z(t))\) of the solution to the exact system by the solution \((J(t), y(t))\) to the averaged system over a time interval of order \(1/\delta\) is estimated from above by a value of order \(\sqrt{\varepsilon}\). This estimate cannot be improved. The estimate is valid under rather general assumptions, which are also formulated in Section 2.

If the non-Hamiltonian perturbation is absent, then system (1.1) is a Hamiltonian 2 DOF system close to an integrable one, and this is the standard object of Kolmogorov–Arnold–Moser theory (see, e.g., [2], [4]). The averaged system becomes \(\dot{J} = 0, \quad \dot{y} = 0\). It is shown by Arnold [2] that, if the unperturbed system is isoenergetically non-degenerate (i.e., condition (2.1) given in Section 2 is satisfied), then the solutions \(J(t) = \text{const}\) of the averaged system eternally (i.e., over the time interval \(-\infty < t < \infty\)) describe the behavior of \(I(t)\) in the exact system with an error that tends to 0 as \(\varepsilon \to 0\). The result by Lazutkin [8] implies that this error can be estimated from above by a value of order of \(\sqrt{\varepsilon}\). Therefore, the error averaged over the initial conditions is also \(O(\sqrt{\varepsilon})\). This estimate follows from the estimate in Theorem 1 considered in the limit as \(\delta \to 0\) (the proof of Theorem 1 uses a construction from [8]).

If the non-Hamiltonian and Hamiltonian perturbations are of the same order \((\delta = \varepsilon)\), then system (1.1) is a standard two-frequency perturbed system of the averaging method, and system (1.2) is the corresponding averaged system (see, e.g., [6], [5]). The use of solutions to the averaged system for the approximate description of the behavior of the slow variables in the exact system is known as the averaging method. The error of the averaging method for two-frequency systems was obtained by Arnold [3] under the assumption that the ratio of frequencies changes.
at a non-zero rate along the solutions to the exact system (the estimate for the case in which the ratio of frequencies changes with non-zero velocity along solutions of the averaged system was obtained in [10]). Even without any assumptions about the behavior of the frequencies, the error of the averaging method on the time interval \(1/\varepsilon\) for a majority of initial conditions tends to 0 as \(\varepsilon \to 0\). This follows from Anosov’s theorem [1], which deals with systems of much more general form. A similar result was obtained by Kasuga [7]. In the case of \(\varepsilon = \delta\), Theorem 1 of the present paper is a particular case of a result obtained in [11].

Systems of the form (1.1) where the non-Hamiltonian perturbation is much smaller than the Hamiltonian one \((\delta \ll \varepsilon)\) appear, in particular, in evolutionary problems of celestial mechanics, as it was pointed out by Moltchanov in [9]. In this case, the Hamiltonian perturbations are due to the gravitation of planets, and the non-Hamiltonian perturbations are due, for example, to tidal friction, interplanetary media drag, or the non-Hamiltonian part of the solar pressure caused by the Pointing–Robertson effect.

2. Formulation of assumptions and results

Suppose that the following assumptions about system (1.1) are fulfilled. In these assumptions \(\varepsilon_0, \delta_0, \rho, \) and \(K\) are positive constants, \(D\) and \(D_0\) are bounded domains in \(R^{m+2}\) with piecewise-smooth boundaries, and \(D_0 \subset D\).

1°. Functions \(H_0, H_1, f, g,\) and \(l\) are defined and smooth enough (for simplicity, we assume them to be \(C^\infty\)-smooth) with respect to \(I, z, \varphi, \varepsilon, \delta\); \(\varphi \mod 2\pi \in T^2, |\varepsilon| \leq \varepsilon_0, \) and \(|\delta| \leq \delta_0;\) and jointly \(C^1\)-smooth with respect to the set of variables \(I, z, \varphi, \varepsilon, \delta;\) for any integer \(r \geq 1,\) there exists a constant \(C_0 = C_0(r)\) that bounds the absolute values of the derivatives of \(H_0, H_1, f, g,\) and \(l\) with respect to \(I, z, \varphi, \varepsilon, \delta,\) up to the order \(r\) from above.

2°. The unperturbed Hamiltonian \(H_0\) is isoenergetically nondegenerate in the domain \(D,\) i.e.,

\[
\det \left( \frac{\partial^2 H_0}{\partial I^2} \frac{\partial H_0}{\partial I} \frac{\partial H_0}{\partial I} \frac{\partial H_0}{\partial I} \right) \neq 0. \tag{2.1}
\]

3°. For \((I^0, z^0) \in D_0\) and \(0 \leq \delta t \leq K,\) a solution \((J(t), z(t))\) to system (1.2) is defined, and the distance of its trajectory to the boundary of \(D\) is larger than \(\rho.\)

Consider the value \(d(I^0, z^0, \varphi^0, \varepsilon, \delta)\) defined as follows. If the solution \((I(t), z(t), \varphi(t)), I(0) = I^0, z(0) = z^0, \varphi(0) = \varphi^0\)
of (1.1) is well-defined for \(0 \leq t \leq K/\delta\) and if \(\sup_{0 \leq t \leq K/\delta} (|I(t) - J(t)| + |z(t) - y(t)|) < \rho/2,\) then

\[
d(I^0, z^0, \varphi^0, \varepsilon, \delta) = \sup_{0 \leq t \leq K/\delta} (|I(t) - J(t)| + |z(t) - y(t)|).
\]

Otherwise \(d(I^0, z^0, \varphi^0, \varepsilon, \delta) = d_1,\) where \(d_1\) is the diameter of \(D.\)

Let \(dI \ dz \ d\varphi\) denote the standard volume element in \(R^{2+m} \times T^2.\)
Theorem 1. There exist positive constants $\varepsilon_1$ and $C$ such that, for $0 < \delta \leq \varepsilon < \varepsilon_1$,

$$\int_{D_0 \times T^2} d(I^0, z^0, \varphi^0, \varepsilon, \delta) dI^0 d\varepsilon^0 d\varphi^0 < C\sqrt{\varepsilon}$$

(2.2)

(the integral is Lebesgue).

Corollary. For a given $\alpha > 0$ let $E(\alpha, \varepsilon, \delta)$ be the subset of $D_0 \times T^2$ on which

$$d(I^0, z^0, \varphi^0, \varepsilon, \delta) > \alpha.$$  

Then

$$\text{mes} E(\alpha, \varepsilon, \delta) < C\sqrt{\varepsilon}/\alpha.$$  

(2.3)

The example

$$\dot{I_1} = -\varepsilon \sin \varphi_1 + \delta/2, \quad \dot{\varphi}_1 = I_1, \quad 0 < \delta \leq \varepsilon$$

(2.4)

shows that the estimate of Theorem 1 cannot be improved. This system is equivalent to the equation

$$\ddot{\varphi}_1 = \varepsilon \left[-\sin \varphi_1 + \frac{1}{2}(\delta/\varepsilon)\right]$$

describing the motion of a pendulum with constant torque. The averaged equation corresponding to (2.4) is $J_1 = \delta/2$. The oscillatory domain of the pendulum corresponds to the set of initial data of measure of order $\sqrt{\varepsilon}$. For these initial data, $I_1(t) = O(\sqrt{\varepsilon})$, while $J_1(t)$ grows with the velocity $\delta/2$. Therefore, the difference between $J_1(1/\delta)$ and $J_1(1/\varepsilon)$ is of order 1. On each trajectory from the domain of rotation of the pendulum, there is a segment on which $I_1(t)$ decays by a value of order $\sqrt{\varepsilon}$ during time of order $1/\sqrt{\varepsilon}$, while $J_1(t)$ grows. Hence there is a set of measure of order 1 of initial data such that $d(I^0, z^0, \varphi^0, \varepsilon, \delta)$ in (2.2) is of order $\sqrt{\varepsilon}$. Therefore, for (2.4), the left-hand side in (2.2) is estimated from below by a value of order $\sqrt{\varepsilon}$.

The example of

$$\dot{I_1} = -\varepsilon \sin \varphi_1 + \delta a(z_1), \quad \dot{z}_1 = 0, \quad \dot{z}_2 = \delta \cos \varphi_1, \quad \dot{\varphi}_1 = I_1$$

(2.5)

shows that estimate (2.3) can not be improved in the following sense. For any smooth function $b(\cdot)$ whose value tends to 0 as its argument tends to 0, there exists a smooth function $a(\cdot)$ whose value tends to 0 as its argument tends to 0 and

$$\text{mes} E(\alpha, \varepsilon, \delta) > \sqrt{\varepsilon} a(\sqrt{\varepsilon})$$

for system (2.5). We omit the proof of this assertion, because it is too long.

Theorem 1 follows from a similar assertion (let us call it Theorem 1') for the case when $I$ and $\varphi$ are one-dimensional but the functions $H_1, f, g,$ and $l$ are additionally $2\pi$-periodic in time $t$, i.e., in system (1.1), $H_1 = H_1(I, z, \varphi, t, \varepsilon)$, $f = f(I, z, \varphi, t, \varepsilon, \delta)$, $g = g(I, z, \varphi, t, \varepsilon, \delta)$, $l = l(I, z, \varphi, t, \varepsilon, \delta)$, and $(I, z) \in D$.

The averaged system has form (1.2), where $F(J, y)$ and $G(J, y)$ are the mean values of $f(J, y, \varphi, t, 0, 0)$ and $g(J, y, \varphi, t, 0, 0)$ with respect to $\varphi$ and $t$. The condition of isoenergetical nondegeneracy should be replaced by the condition

$$\partial^2 H_0 / \partial I^2 \neq 0.$$  

(2.6)

In the case under consideration, we have
\[ I(t) = I(t, t^0, z^0, v^0, t^0, \varepsilon, \delta), \quad z(t) = z(t, t^0, z^0, v^0, t^0, \varepsilon, \delta), \]
\[ \varphi(t) = \varphi(t, t^0, z^0, v^0, t^0, \varepsilon, \delta), \]
\[ J(t) = J(\delta(t - t^0), t^0, z^0), \quad y(t) = y(\delta(t - t^0), t^0, z^0), \]
\[ I(t^0) = I^0, \quad z(t^0) = z^0, \quad \varphi(t^0) = \varphi^0, \quad J(t^0) = I^0, \quad y(t^0) = y^0. \]

As above, consider the value \( d(I^0, z^0, v^0, t^0, \varepsilon, \delta) \) defined as follows. If the solution \((I(t), z(t), \varphi(t))\), \(I(t^0) = I^0, z(t^0) = z^0, \varphi(t^0) = \varphi^0\) is well-defined for \( t^0 < t < t^0 + K/\delta \) and if
\[
\sup_{t^0 \leq t \leq t^0 + K/\delta} (|I(t) - J(t)| + |z(t) - y(t)|) < \rho / 2,
\]
then
\[
d(I^0, z^0, v^0, t^0, \varepsilon, \delta) = \sup_{t^0 \leq t \leq t^0 + K/\delta} (|I(t) - J(t)| + |z(t) - y(t)|).
\]

Otherwise \( d(I^0, z^0, v^0, t^0, \varepsilon, \delta) = d_1 \), where, as before, \( d_1 \) is the diameter of \( D \).

**Theorem 1'**. There exist positive constants \( \varepsilon_1 \) and \( C \) such that, for \( 0 < \delta < \varepsilon < \varepsilon_1 \),
\[
\int_{D_0 \times T^1} d(I^0, z^0, v^0, t^0, \varepsilon, \delta) dI^0 dz^0 d\varphi^0 < C \sqrt{\varepsilon}.
\]

If, for (1.1) with \( I = (I_1, I_2) \) and \( \varphi = (\varphi_1, \varphi_2) \), one of the frequencies does not vanish (for example, \( \omega_2 = \partial H_0 / \partial I_2 \neq 0 \) in \( D \)), then Theorem 1 can be reduced to Theorem 1' by means of an isoenergetic reduction of the Hamiltonian part of the system. The phase whose frequency does not vanish (\( \varphi_2 \) in the case under consideration) becomes the new time variable, and the value of the Hamiltonian \( h = H_0 + \varepsilon H_1 \) should be appended to the parameter vector \( z \) as a new slow variable. The value \( I_2 \) can be found from the equation \( h = H_0 + \varepsilon H_1 \). In the general case, any of the frequencies \( \partial H_0 / \partial I_1 \) and \( \partial H_0 / \partial I_2 \) can vanish on some surface in \( D \). However, because of the isoenergetic nondegeneracy condition (2.1), the frequencies do not vanish simultaneously. This makes it possible to reduce the general case to the case when one of the frequencies does not vanish by partitioning the domain \( D - \rho / 2 \)

\footnote{Here and in what follows \( D - \rho / 2 \) is a set of points whose closed \( \rho / 2 \)-neighborhoods belong to \( D \).}

\( D - \rho / 2 \) into a finite number of subdomains; every such subdomain is a neighborhood of a segment of an averaged system trajectory, and in every subdomain one of the frequencies does not vanish. We shall not dwell on the reduction of Theorem 1 to Theorem 1'; instead, we shall prove Theorem 1'.
Lemma 1. For any integer \( r \geq 2 \) and for \( c_1 \sqrt{\varepsilon} \leq \omega < c_2^{-1} \), there exists a \( C^r \)-smooth transformation of variables \( (I, z, \phi, t) \mapsto (\bar{I}, z, \varphi, t) \) in the domain \( \{ (\bar{I}, z, \varphi, t) \in D_1 \times T^2 \} \), where \( D_1 = D - \rho/8 \), such that

(a) the transformation \( (\bar{I}, \varphi) \mapsto (I, \varphi) \) is symplectic with generating function \( \bar{I} \varphi + \varepsilon S(I, z, \phi, t, \varepsilon, \delta, \omega) \);

(b) the function \( \mathcal{H} = \varepsilon \frac{\partial S}{\partial \varepsilon} + H_0 + \varepsilon H_1 \) has the form

\[
\mathcal{H} = H_0(\bar{I}, z, \varepsilon, \delta, \omega) + \varepsilon H_1(\bar{I}, z, \phi, t, \varepsilon, \delta, \omega) + \varepsilon H_2(\bar{I}, z, \phi, t, \varepsilon, \delta, \omega),
\]

where the absolute values of the function \( \mathcal{H}_2 \) and its first derivatives with respect to \( \bar{I} \) and \( \varphi \) are bounded from above by \( \delta \) and the function \( \mathcal{H}_1 \) vanishes identically in the subdomain of \( D_1 \times T^2 \) where

\[
\left| \frac{\partial H_0}{\partial \bar{I}} - \frac{p}{q} \right| > \omega |q|^{-3}, \quad 0 < |p| + |q| < N_1 \tag{3.1}
\]

for integer \( p \) and \( q \); here \( N_1 \) is an integer depending on \( \varepsilon, \delta, \) and \( \omega \);

(c)

\[
\left| \frac{\partial H_0}{\partial \bar{I}} \right| < c_3 \left( \varepsilon + \frac{\varepsilon^2}{\omega^3} \right), \quad \left| \frac{\partial^2 H_0}{\partial \bar{I} \partial \bar{z}} \right| < c_3 \left( \varepsilon + \frac{\varepsilon^2}{\omega^3} \right), \quad \left| \frac{\partial^2 H_0}{\partial \bar{I} \partial \bar{z}} \right| < c_3 \left( \varepsilon + \frac{\varepsilon^2}{\omega^3} \right),
\]

\[
\int_{D_1} \left| \frac{\partial^2 H_0}{\partial \bar{I} \partial \bar{z}} \right| d\bar{I} d\bar{z} < c_3 \left( \varepsilon + \frac{\varepsilon^2}{\omega^3} \right), \quad \int_{D_1} \left| \frac{\partial^2 H_0}{\partial \bar{I} \partial \bar{z}} \right| d\bar{I} d\bar{z} < c_3 \left( \varepsilon + \frac{\varepsilon^2}{\omega^3} \right),
\]

\[
c_3^{-1} < \frac{\partial^2 H_0}{\partial \bar{I}^2} < c_3, \quad \frac{\partial^2 H_0}{\partial \bar{I} \partial \bar{z}} < c_3 \tag{3.2}
\]

(d)

\[
\frac{\varepsilon}{\bar{I}} \frac{\partial H_1}{\partial \bar{I}} < c_4 \frac{\varepsilon}{\omega}, \quad \frac{\varepsilon}{\phi} \frac{\partial H_1}{\partial \phi} < c_4 \varepsilon,
\]

\[
\int_{D_1} \left| \frac{\varepsilon}{\bar{I}} \frac{\partial H_1}{\partial \bar{I}} \right| d\bar{I} d\bar{z} < c_4 \varepsilon, \quad \int_{D_1} \left| \frac{\varepsilon}{\phi} \frac{\partial H_1}{\partial \phi} \right| d\bar{I} d\bar{z} < c_4 \varepsilon \omega;
\]

(e) the transformation \( (\bar{I}, \varphi) \mapsto (I, \varphi) \) can be represented in the form \( I = \bar{I} + \varepsilon u(\bar{I}, z, \phi, t, \varepsilon, \delta, \omega) \), \( \varphi = \phi + \varepsilon v(\bar{I}, z, \phi, t, \varepsilon, \delta, \omega) \), where

\[
|\varepsilon u| < c_5 \frac{\varepsilon}{\omega^2}, \quad |\varepsilon v| < c_5 \frac{\varepsilon}{\omega^2}, \quad \frac{\varepsilon}{\bar{I}} \frac{\partial u}{\partial \bar{I}} < c_5 \frac{\varepsilon}{\omega^2}, \quad \frac{\varepsilon}{\phi} \frac{\partial u}{\partial \phi} < c_5 \frac{\varepsilon}{\omega^2}, \quad \frac{\varepsilon}{\bar{I}} \frac{\partial v}{\partial \bar{I}} < c_5 \frac{\varepsilon}{\omega^2}, \quad \frac{\varepsilon}{\phi} \frac{\partial v}{\partial \phi} < c_5 \frac{\varepsilon}{\omega^2}, \tag{3.3}
\]
We obtain the system of equations that are valid:

\[ \int_{D_1} |\varepsilon u| \, dI \, dz < c_5 \varepsilon |\ln w|, \quad \int_{D_1} |\varepsilon v| \, dI \, dz < c_5 \frac{\varepsilon}{w}, \]

\[ \int_{D_1} \left( |\varepsilon \frac{\partial u}{\partial I}| + |\varepsilon \frac{\partial u}{\partial z}| \right) \, dI \, dz < c_5 \frac{\varepsilon}{w}, \]

\[ \int_{D_1} \left( |\varepsilon \frac{\partial v}{\partial \varphi}| + |\varepsilon \frac{\partial v}{\partial \varphi}| \right) \, dI \, dz < c_5 \frac{\varepsilon}{w}. \]

This lemma is essentially proved as an intermediate result in [8]. The results of [8] are obtained for the case when the parameters \( z \) are absent; adding these parameters does not cause additional difficulties. The estimates for the integrals over \( D_1 \) are not contained in [8], but they follow immediately from the formulas for the transformation of variables in question that are given there. In [8], a symplectic map is considered rather than a Hamiltonian system; we can regard this map as a Poincaré return map for a Hamiltonian system. In [8], the stronger assertion that we can take \( \mathcal{H}_2 = 0 \) and \( N_1 = \infty \) in (3.1) is also proved. The results of [12] imply that the transformation of variables whose existence is asserted in Lemma 1 can be chosen \( C^\infty \)-smooth.

Let us apply transformation of variables from Lemma 1 to system (1.1) with \( r = 7 \). We obtain the system

\[
\dot{I} = -\varepsilon \frac{\partial \mathcal{H}_1}{\partial \varphi} + \delta f(\bar{I}, z, \varphi, t, \varepsilon, \delta) + \delta \varepsilon f_1, \\
\dot{z} = \delta g(\bar{I}, z, \varphi, t, \varepsilon, \delta) + \delta \varepsilon g_1, \\
\dot{\varphi} = \frac{\partial \mathcal{H}_0}{\partial I} + \varepsilon \frac{\partial \mathcal{H}_1}{\partial I} + \delta \ell(\bar{I}, z, \varphi, t, \varepsilon, \delta) + \delta \varepsilon l_1. 
\]

Here \( f_1, g_1 \) and \( l_1 \) are \( C^{r-1} \)-smooth functions in \( D_1 \times T^2 \). The following estimates are valid:

\[ |\varepsilon f_1| \leq c_6 \left( \varepsilon + |\varepsilon u| + |\varepsilon v| + \left| \varepsilon \frac{\partial u}{\partial z} \right| + \left| \varepsilon \frac{\partial v}{\partial \varphi} \right| \right), \]

\[ |\varepsilon g_1| \leq c_6 (|\varepsilon u| + |\varepsilon v|), \]

\[ |\varepsilon l_1| \leq c_6 \left( \varepsilon + |\varepsilon u| + |\varepsilon v| + \left| \varepsilon \frac{\partial u}{\partial I} \right| + \left| \varepsilon \frac{\partial v}{\partial I} \right| + \left| \varepsilon \frac{\partial v}{\partial z} \right| \right). \]

Now, we replace \( \bar{I} \) by the new variable \( \bar{\vartheta} \) defined by the equation \( \bar{\vartheta} = \partial \mathcal{H}_0 / \partial \bar{I} \). We obtain the system of equations

\[
\dot{\bar{\vartheta}} = -\varepsilon \frac{\partial^2 \mathcal{H}_0}{\partial I^2} \frac{\partial \mathcal{H}_1}{\partial \varphi} + \delta \frac{\partial^2 \mathcal{H}_0}{\partial I^2} f(\bar{I}, z, \varphi, t, \varepsilon, \delta) + \delta \frac{\partial^2 \mathcal{H}_0}{\partial I \partial z} g(\bar{I}, z, \varphi, t, \varepsilon, \delta) + \delta \varepsilon f_1, \\
\dot{z} = \delta g(\bar{I}, z, \varphi, t, \varepsilon, \delta) + \delta \varepsilon g_1, \\
\dot{\varphi} = \bar{\vartheta} + \varepsilon \frac{\partial \mathcal{H}_1}{\partial I} + \delta \ell(\bar{I}, z, \varphi, t, \varepsilon, \delta) + \delta \varepsilon l_1. 
\]
The function \( f'_1 \) satisfies the estimate
\[
|\varepsilon f'_1| < 2c_3c_6 \left( \varepsilon + |\varepsilon u| + |\varepsilon v| + \left| \varepsilon \frac{\partial u}{\partial z} \right| + \left| \varepsilon \frac{\partial u}{\partial \varphi} \right| + \left| \varepsilon \frac{\partial v}{\partial \varphi} \right| \right).
\]

In the right-hand sides of (3.6), we must express \( \bar{I} \) through \( \bar{\omega} \) and \( z \) in accordance with the equation \( \bar{\omega} = \partial H_0 / \partial \bar{I} \).

The first equation in (3.6) can be rewritten in the form
\[
\dot{\bar{\omega}} = -\varepsilon \frac{\partial^2 H_0}{\partial \bar{I}^2} \frac{\partial H_1}{\partial \bar{\varphi}} + \delta b(\bar{I}, z, \bar{\varphi}, t, \varepsilon, \delta) + \delta \varepsilon f''_1,
\]
where
\[
b(\bar{I}, z, \bar{\varphi}, t, \varepsilon, \delta) = \frac{\partial^2 H_0}{\partial \bar{I}^2} f(\bar{I}, z, \bar{\varphi}, t, \varepsilon, \delta) + \frac{\partial^2 H_0}{\partial \bar{I} \partial z} g(\bar{I}, z, \bar{\varphi}, t, \varepsilon, \delta),
\]
and
\[
|\varepsilon f'_1| < 2c_3c_6 \left( \varepsilon + |\varepsilon u| + |\varepsilon v| + \left| \varepsilon \frac{\partial u}{\partial z} \right| + \left| \varepsilon \frac{\partial u}{\partial \varphi} \right| + \left| \varepsilon \frac{\partial v}{\partial \varphi} \right| \right) \varepsilon c_7 \left( \left| \frac{\partial^2 H_0}{\partial \bar{I}^2} - \frac{\partial^2 H_0}{\partial \bar{I} \partial z} \right| + \left| \frac{\partial^2 H_0}{\partial \bar{I} \partial z} - \frac{\partial^2 H_0}{\partial \bar{I}^2} \right| \right). \tag{3.8}
\]

Let us denote
\[
B(\bar{I}, z) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} b(\bar{I}, z, \varphi, t, 0, 0) \, d\varphi \, dt.
\]

**Lemma 2.** For any \( \alpha \geq c_8 \sqrt{\varepsilon} \), where \( c_8 > c_1 \), there exist functions \( w_{1,2} = w_{1,2}(\bar{\omega}, z, \bar{\varphi}, t, \varepsilon, \delta, \alpha) \) which are \( C^1 \)-smooth with respect to \( \bar{\omega}, z, \bar{\varphi}, \) and \( t \) and a number \( N_2 \geq N_1 \) such that

a) if \( |\bar{\omega} - \frac{p}{q}| > \alpha |q|^{-3} \) for integer \( p \) and \( q \) satisfying \( 0 < |p| + |q| < N_2 \), then
\[
\left| \frac{\partial w_1}{\partial \varphi} \bar{\omega} + \frac{\partial w_1}{\partial t} + b(\bar{I}, z, \bar{\varphi}, t, 0, 0) - B(\bar{I}, z) \right| < \sqrt{\varepsilon},
\]
\[
\left| \frac{\partial w_2}{\partial \varphi} \bar{\omega} + \frac{\partial w_2}{\partial t} + g(\bar{I}, z, \bar{\varphi}, t, 0, 0) - G(\bar{I}, z) \right| < \sqrt{\varepsilon};
\]

b) \[
\int_{D_1} \left| \frac{\partial w_1}{\partial \varphi} \bar{\omega} + \frac{\partial w_1}{\partial t} + b(\bar{I}, z, \bar{\varphi}, t, 0, 0) - B(\bar{I}, z) \right| d\bar{I} \, dz < c_9 \alpha \varepsilon,
\]
\[
\int_{D_1} \left| \frac{\partial w_2}{\partial \varphi} \bar{\omega} + \frac{\partial w_2}{\partial t} + g(\bar{I}, z, \bar{\varphi}, t, 0, 0) - G(\bar{I}, z) \right| d\bar{I} \, dz < c_9 \alpha \varepsilon;
\]
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c) \( |w_i| < \frac{c_{10}}{\varepsilon} \),

\[
\begin{align*}
\left| \frac{\partial w_i}{\partial \bar{\omega}} \right| & < \frac{c_{10}}{\varepsilon^2}, \\
\left| \frac{\partial w_i}{\partial z} \right| & < \frac{c_{10}}{\varepsilon}, \\
\left| \frac{\partial w_i}{\partial \bar{\phi}} \right| & < \frac{c_{10}}{\varepsilon}, \\
\left| \frac{\partial w_i}{\partial t} \right| & < \frac{c_{10}}{\varepsilon},
\end{align*}
\]

\[
\int_{D_1} |w_i| \, d\bar{I} \, dz < c_{10} |\ln \varepsilon|,
\]

\[
\int_{D_1} \left| \frac{\partial w_i}{\partial \bar{\omega}} \right| \, d\bar{I} \, dz < \frac{c_{10}}{\varepsilon},
\]

\[
\int_{D_1} \left( \left| \frac{\partial w_i}{\partial z} \right| + \left| \frac{\partial w_i}{\partial \bar{\phi}} \right| + \left| \frac{\partial w_i}{\partial t} \right| \right) \, d\bar{I} \, dz < c_{10} |\ln \varepsilon|,
\]

\[
\int_{D_1} \left| \frac{\partial w_i}{\partial \bar{\omega}} \right| \frac{\partial H_1}{\partial \bar{\phi}} \, d\bar{I} \, dz < c_{10} \frac{\varepsilon}{\bar{\omega}},
\]

\[
\int_{D_1} \left| \frac{\partial w_i}{\partial \bar{\phi}} \right| \frac{\partial H_1}{\partial \bar{\phi}} \, d\bar{I} \, dz < c_{10} \frac{\varepsilon}{\bar{\omega}},
\]

\(\text{for } i = 1, 2\).

In resonant domains, the functions \(w_i\) smooth the functions involved in the standard formulas for variable transformations of the averaging method. Explicit formulas for \(w_i\) are contained in [11]; the estimates from Lemma 2 follow from these formulas.

Let us make the transformation of variables \((\bar{\omega}, z) \mapsto (\tilde{\omega}, \tilde{z})\) in system (3.6), (3.7) defined by

\[
\tilde{\omega} = \bar{\omega} + \delta w_1(\bar{\omega}, z, \bar{\phi}, t, \varepsilon, \delta, \omega),
\]

\[
\tilde{z} = z + \delta w_2(\bar{\omega}, z, \bar{\phi}, t, \varepsilon, \delta, \omega).
\]

The estimates of Lemma 2 imply that, if \(\varepsilon \geq c_{11} \sqrt{\varepsilon}\), where \(c_{11} > c_8\), then this transformation of variables is well-defined for \((\tilde{I}, \tilde{z}, \bar{\phi}) \in D_2 \times T^1\), where \(D_2 = D - \rho/4\) and \(\tilde{\omega} = \partial H_0(\tilde{I}, \tilde{z})/\partial \tilde{I}\). This transformation satisfies

\[
c_{12}^{-1} < \frac{D(\tilde{\omega}, \tilde{z})}{D(\bar{\omega}, z)} < c_{12}.
\]

In the new variables, the equations of motion have the form (we write down the formulas for \(\dot{\tilde{\omega}}\) and \(\dot{\tilde{z}}\) only):

\[
\begin{align*}
\dot{\tilde{\omega}} & = -\varepsilon \frac{\partial^2 H_0(\bar{I}, z, \varepsilon, \delta, \omega)}{\partial \bar{I}^2} \left( \frac{\partial H_1(\bar{I}, z, \bar{\phi}, t, \varepsilon, \delta, \omega)}{\partial \bar{\phi}} \right) + \delta B(\bar{I}, \tilde{z}) \\
& \quad + \delta \left[ \frac{\partial w_1}{\partial \bar{\phi}} \dot{\bar{\phi}} + \frac{\partial w_1}{\partial t} + b(\bar{I}, z, \bar{\phi}, t, 0, 0) - B(\bar{I}, z) \right] + \delta \varepsilon f_2 + \delta^2 f_3, \\
\dot{\tilde{z}} & = \delta G(\bar{I}, \tilde{z}) + \delta \left[ \frac{\partial w_2}{\partial \bar{\phi}} \dot{\bar{\phi}} + \frac{\partial w_2}{\partial t} + g(\bar{I}, z, \bar{\phi}, t, 0, 0) - G(\bar{I}, \tilde{z}) \right] + \delta \varepsilon g_2 + \delta^2 g_3,
\end{align*}
\]
Proof. \[ |\delta f_2| \leq c_{13}\delta |f'_1| + c_{14}\delta \left( \left| \frac{\partial w_1}{\partial \omega} \right| + \left| \frac{\partial w_1}{\partial \varphi} \right| \right), \]
\[ |\delta^2 f_3| \leq c_{15}\delta^2(|w_1| + |w_2|) + c_{16}\delta^2 \left( \left| \frac{\partial w_1}{\partial \omega} \right| + \left| \frac{\partial w_1}{\partial t} \right| \right), \]
\[ |\delta g_2| \leq c_{13}\delta |g_1| + c_{14}\delta \left( \left| \frac{\partial w_2}{\partial \omega} \right| + \left| \frac{\partial w_2}{\partial \varphi} \right| \right), \]
\[ |\delta^2 g_3| \leq c_{15}\delta^2(|w_1| + |w_2|) + c_{16}\delta^2 \left( \left| \frac{\partial w_2}{\partial \omega} \right| + \left| \frac{\partial w_2}{\partial z} \right| \right). \]

We denote a solution \((I(t), z(t), \varphi(t))\) written in the variables \((I, z, \varphi), (\tilde{\omega}, z, \varphi)\), or \((\bar{\omega}, \bar{z}, \bar{\varphi})\) by \((\hat{I}(t), z(t), \varphi(t)), (\tilde{\omega}(t), z(t), \varphi(t))\), or \((\bar{\omega}(t), \bar{z}(t), \bar{\varphi}(t))\) respectively. The initial data at \(t = t^0\) for such a solution is denoted by \((\hat{I}^0, z^0, \varphi^0)\), \((\tilde{\omega}^0, z^0, \varphi^0)\), or \((\bar{\omega}^0, \bar{z}^0, \bar{\varphi}^0)\), respectively.

A solution \((J(t), y(t))\) written in the variables \((\Omega, y)\), where \(\Omega = \partial H_0(J, y)/\partial J\), is denoted by \((\Omega(t), y(t))\).

**Lemma 3.** The absolute values of the determinants

\[
\begin{align*}
\frac{D(I(t), z(t), \varphi(t))}{D(I^0, z^0, \varphi^0)}, & \quad \frac{D(\hat{I}(t), z(t), \varphi(t))}{D(\hat{I}^0, z^0, \varphi^0)}, & \quad \frac{D(\tilde{\omega}(t), z(t), \varphi(t))}{D(\tilde{\omega}^0, z^0, \varphi^0)}, & \quad \frac{D(\bar{\omega}(t), \bar{z}(t), \bar{\varphi}(t))}{D(\bar{\omega}^0, \bar{z}^0, \bar{\varphi}^0)}
\end{align*}
\]

are bounded from above by \(c_{17}\) and from below by \(c_{17}^{-1}\).

**Proof.** We have

\[
\frac{D(I(t), z(t), \varphi(t))}{D(I^0, z^0, \varphi^0)} = \exp \left( \delta \int_{t^0}^t \left( \frac{\partial f}{\partial I} + \text{tr} \frac{\partial g}{\partial \omega} + \frac{\partial b}{\partial \varphi} \right) d\tau \right). \]

This implies the required estimate for the first determinant in (3.11). The estimates for the other determinants follow from (3.2) and (3.9) with taking into account the determinant of a symplectic transformation is equal to 1.

For brevity, we write the first equation from (3.6) in the form

\[
\dot{\omega} = -\varepsilon \frac{\partial^2 H_0}{\partial I^2} \frac{\partial \varphi_1}{\partial \varphi} + \delta b. \tag{3.12}
\]

Along a solution, the value \(\partial H_1/\partial \varphi\) in (3.12) can non-vanish only at time moments such that

\[
|\dot{\omega}(t) - \frac{p}{q}| < a|q|^{-3}, \quad 0 \leq |p| + |q| \leq N_2 \tag{3.13}
\]

for some integer \(p, q\). Therefore, along a solution,

\[
-\varepsilon \frac{\partial^2 H_0}{\partial I^2} \frac{\partial \varphi_1}{\partial \varphi} = (\dot{\omega})_r - \delta(b)_r; \tag{3.14}
\]

here, for a function \(\beta\) considered along a solution, the symbol \((\beta)_r\) denotes the function of time which takes the same values as \(\beta\) at time moments when condition (3.13) is satisfied along the solution and vanishes at the other time moments (the subscript “\(r\)” is an abbreviation for “resonant”).
For brevity, we write system (3.10) in the form
\[ \dot{\omega} = -\xi \frac{\partial^2 H_0}{\partial T^2} + \delta B(\bar{I}, \hat{z}) + \delta \hat{b}, \]
\[ \ddot{z} = \delta G(\bar{I}, \hat{z}) + \delta \hat{g}. \]  
(3.15)

Substituting (3.14) into the first equation (3.15), we get
\[ \dot{\omega} = (\dot{\omega})_r + \delta B(\bar{I}, \hat{z}) - \delta (\hat{b})_r + \delta \hat{b}, \]
or, in the form of an integral equation,
\[ \bar{\omega}(t) - \bar{\omega}(t_0) = \int_{t_0}^{t} (\dot{\omega}(\tau))_r d\tau + \int_{t_0}^{t} (\delta B(\bar{I}, \hat{z}) - \delta (\hat{b})_r + \delta \hat{b}) d\tau. \]
For the solution under consideration, we set
\[ a(t) = \int_{t_0}^{t} (\omega)(\tau, d\tau). \]
The value \( a(t) \) is the total variation of \( \bar{\omega} \) on the part of the time segment \([t_0, t] \) that corresponds to the motion inside zones (3.13). Thus,
\[ |a(t)| < c_{19} \delta \omega. \]  
(3.16)

Setting \( \dot{\bar{\omega}}(t) = \bar{\omega}(t) - a(t) \), we obtain
\[ \dot{\bar{\omega}}(t) - \dot{\bar{\omega}}(t_0) = \int_{t_0}^{t} \delta B(\bar{I}, \hat{z}) d\tau + \int_{t_0}^{t} (-\delta (\hat{b})_r + \delta \hat{b}) d\tau, \]
\[ \bar{z}(t) - \bar{z}(t_0) = \int_{t_0}^{t} \delta G(\bar{I}, \hat{z}) d\tau + \int_{t_0}^{t} (\delta \hat{g} + \delta \hat{g}) d\tau, \]  
(3.17)

where \( \bar{I} = \bar{I}(\bar{\omega}, \hat{z}) \) is defined by the relation \( \dot{\bar{\omega}} = \partial H_0(\bar{I}, \hat{z})/\partial \bar{I} \) and
\[ |\delta \hat{b}| < c_{19} \delta \omega, \quad |\delta \hat{g}| < c_{19} \delta \omega. \]  
(3.18)

Let
\[ \bar{a}(t) = |\dot{\bar{\omega}}(t) - \Omega(t)| + |\bar{z}(t) - y(t)|. \]
Formulas (3.17) imply
\[ \bar{a}(t) \leq c_{20} \bar{a}(t_0) + c_{21} \int_{t_0}^{t} (|\delta (\hat{b})_r| + |\delta \hat{b}| + |\delta \hat{g}| + |\delta \hat{g}|) d\tau \]
for \( t_0 \leq t \leq t_0 + K/\delta \). Therefore,
\[ \sup_{t_0 \leq t \leq t_0 + K/\delta} \bar{a}(t) \leq c_{20} \bar{a}(t_0) + c_{21} \int_{t_0}^{t_0 + K/\delta} (|\delta (\hat{b})_r| + |\delta \hat{b}| + |\delta \hat{g}| + |\delta \hat{g}|) d\tau. \]
Hence the value
\[ \bar{d} = \int_{D_0 \times T^1} \sup_{t_0 \leq t \leq t_0 + K/\delta} \bar{a}(t) d\bar{I} d\bar{z} d\varphi \]
satisfies the inequality
\[ \bar{d} \leq c_{20} \bar{d}_0 + c_{21} \bar{d}_1, \]
where

\[ \tilde{d}_0 = \int_{D_0 \times T^1} \alpha(0) \, dI^0 \, dz^0 \, d\varphi^0, \]

\[ \tilde{d}_1 = \int_{D_0 \times T^1} \left( \int t^0 + K/\delta \left( |\delta(b)_t| + |\delta h| + |\delta t| + |\delta \tilde{g}| + |\delta g| \right) \, d\tau \right) \, dI^0 \, dz^0 \, d\varphi^0. \]  \tag{3.19}

The estimates given by Lemmas 1, 2, and 3 and estimate (3.16) imply

\[ \tilde{d}_0 < c_{22}(\alpha + \delta |\ln \alpha| + \varepsilon |\ln \alpha|) \leq c_{23} \alpha. \]

To obtain an estimate for \( \tilde{d}_1 \), we change the order of integration in the definition (3.19) of \( \tilde{d}_1 \) (we integrate first over \( D_0 \times T^1 \) and then over \( [t^0, t^0 + K/\delta] \)). In the inner integral, for a fixed \( \tau \), we pass from integration with respect to \( I^0 \), \( \varphi^0 \), and \( z^0 \) to integration with respect to \( I(\tau), z(\tau) \), and \( \varphi(\tau) \) and, then, to integration with respect to \( I(\tau), z(\tau), \) and \( \varphi(\tau) \). Using Lemma 3 and estimate (3.18), we obtain

\[ \tilde{d}_1 \leq \tilde{d}_{1,1} + \tilde{d}_{1,2}, \]

\[ \tilde{d}_{1,1} \leq c_{24} \int_{D_1 \times T^1} (|\tilde{b}| + |\tilde{g}|) \, dI \, dz \, d\varphi + c_{25} \alpha, \]

\[ \tilde{d}_{1,2} \leq c_{26} \int_{(D_1 \times T^1)_w} |\tilde{b}| \, dI \, dz \, d\varphi, \]

where \((D_1 \times T^1)_w \) is the set of points \((I, z, \varphi) \in D_1 \times T^1 \) for which there exist integer \( p \) and \( q \) such that \( 0 < |p| + |q| < N_2 \) and \( \tilde{\omega} - p/q < \alpha |q|^{-3} \). The estimates of Lemmas 1 and 2 and formulas (3.15), (3.12), (3.10), and (3.8) imply

\[ \tilde{d}_{1,1} < c_{27} \left( \alpha + \varepsilon + \varepsilon |\ln \alpha| + \frac{\varepsilon}{\alpha} + \frac{\varepsilon^2}{\alpha^3} + \delta |\ln \alpha| + \frac{\delta}{\alpha} \right), \]

\[ \tilde{d}_{1,2} < c_{28} \alpha. \]

Therefore,

\[ \tilde{d} < c_{29} \alpha. \]  \tag{3.20}

Let

\[ \alpha(t) = |\omega(t) - \Omega(t)| + |z(t) - y(t)|, \]

where \( \omega(t) = \partial H_0(I(t), z(t))/\partial I \). We have

\[ \alpha(t) \leq \tilde{\alpha}(t) + |\delta w_1| + |\delta w_2| + |a(t)| + c_{30}(|\varepsilon u| + |\varepsilon v|) \]

\[ + \left| \frac{\partial H_0(I(t), z(t))}{\partial I} - \frac{\partial H_0(I(t), z(t))}{\partial I} \right|. \]

Using the estimates of Lemmas 1 and 2, we obtain

\[ \alpha(t) < \tilde{\alpha}(t) + c_{31} \left( \frac{\delta}{\alpha} + \frac{\varepsilon}{\alpha} + \frac{\varepsilon^2}{\alpha^3} \right). \]

The combination of this relation with (3.20) gives

\[ \int_{D_0 \times T^1} \sup_{t^0 \leq t \leq t^0 + K/\delta} \alpha(t) \, dI^0 \, dz^0 \, d\varphi^0 < c_{32} \alpha. \]
Choosing \( \alpha = c_{11} \sqrt{\varepsilon} \), we obtain
\[
\int_{D_0 \times T^1} \left( \sup_{\varphi^0 \leq t \leq \varphi^0 + K/\delta} \gamma(t) \right) dI^0 dz^0 d\varphi^0 < c_{33} \sqrt{\varepsilon}.
\]

Let \( \gamma(t) = |I(t) - J(t)| + |z(t) - y(t)| \). Condition (2.6) implies
\[
\gamma(t) \leq c_{34} \alpha(t).
\]

Therefore,
\[
\int_{D_0 \times T^1} \left( \sup_{\varphi^0 \leq t \leq \varphi^0 + K/\delta} \gamma(t) \right) dI^0 dz^0 d\varphi^0 < c_{35} \sqrt{\varepsilon}.
\]

This implies, in particular,
\[
\text{mes} \left\{ \left( I^0, z^0, \varphi^0 \right) : \sup_{\varphi^0 \leq t \leq \varphi^0 + K/\delta} \gamma(t) \geq \rho/2 \right\} < 2c_{35} \sqrt{\varepsilon}/\rho.
\]

This inequality and the definition of the function \( d(I^0, z^0, \varphi^0, t^0, \varepsilon, \delta) \) imply
\[
\int_{D_0 \times T^1} d(I^0, z^0, \varphi^0, t^0, \varepsilon, \delta) dI^0 dz^0 d\varphi^0 < C \sqrt{\varepsilon}.
\]

This is the required estimate of Theorem 1'.

Above, it was assumed for simplicity that the solution \( (I(t), z(t), \varphi(t)) \) with initial data \( (I^0, z^0) \in D_0 \) is well-defined for \( t^0 \leq t \leq t^0 + K/\delta \) and \( (I(t), z(t)) \in D - \rho/2 \). Now, consider the case when this assumption is not satisfied. The above estimates remain valid when we replace (i) the time interval \([t^0, t^0 + K/\delta]\) by the time interval \([t^0, t^0 + T]\), where \( 0 \leq T \leq K/\delta \), and (ii) the set of initial data \( D_0 \times T^1 \) by some measurable set \( \Phi \) such that the solution \( (I(t), z(t), \varphi(t)) \) with initial data \( (I^0, z^0, \varphi^0) \in \Phi \) is well-defined for \( t^0 \leq t \leq t^0 + T \) and \( (I(t), z(t)) \in D - \rho/2 \).

Take \( \alpha = c_{11} \sqrt{\varepsilon} \). Below we construct finite sequences of sets \( D_0 \times T^1 = \Phi_0 \supset \Phi_1 \supset \cdots \supset \Phi_m \) and time moments \( 0 = T_0 < T_1 < \cdots < T_m \), where \( K/\delta \leq T_m \leq K/\delta + O(1) \), and \( m = O(1) \). These sequences have the following property: the solution \( (I(t), z(t), \varphi(t)) \) with initial data \( (I^0, z^0, \varphi^0) \in \Phi_k \), where \( k = 1, \ldots, m \), is well-defined for \( t^0 \leq t \leq t^0 + T_k \) and \( (I(t), z(t)) \in D - 3/4 \rho \), and
\[
\int_{\Phi_k} \sup_{t^0 \leq t \leq t^0 + T_k} \left( |I(t) - J(t)| + |z(t) - y(t)| \right) dI^0 dz^0 d\varphi^0 < c_{35} \sqrt{\varepsilon}.
\]

We construct the sequences of \( \Phi_k \) and \( T_k \) by induction as follows. Equations (3.17) imply that rates of change of \( \hat{\varphi} \) and \( \hat{z} \) are of order \( \delta \). Let us define \( \hat{I} \) by the equation \( \partial H_0(\hat{I}, \hat{z})/\partial \hat{I} = \hat{\varphi} \). Then the rate of change of \( \hat{I} \) is of order \( \delta \). The formulas for the transformation of variables imply that \( |\hat{I} - I| + |\hat{z} - z| = O(\sqrt{\varepsilon}) \). Therefore, if we have \( (I(t), z(t)) \in D - 3/4 \rho \) for initial data \( (I^0, z^0, \varphi^0) \in \Phi_k \) and for \( t^0 \leq t \leq t^0 + T_k \), then we have also \( (I(t), z(t)) \in D - \rho/2 \) for \( t^0 \leq t \leq t^0 + T_{k+1} \), where \( T_{k+1} = T_k + c_{36} \delta/\delta \).

Thus,
\[
\int_{\Phi_k} \sup_{t^0 \leq t \leq t^0 + T_{k+1}} \left( |I(t) - J(t)| + |z(t) - y(t)| \right) dI^0 dz^0 d\varphi^0 < c_{35} \sqrt{\varepsilon}.
\]
Let
\[\Psi_k = \left\{ (I^0, z^0, \varphi^0) : (I^0, z^0, \varphi^0) \in \Phi_k, \sup_{t^0 \leq t \leq t^0 + T_{k+1}} |I(t) - J(t)| + |z(t) - y(t)| \geq \rho/4 \right\}.\]

Estimate (3.22) implies
\[\text{mes} \Psi_k < 4c_{35} \sqrt{\varepsilon}/\rho.\]

Now, we set \(\Phi_{k+1} = \Phi_k \setminus \Psi_k\), and so on. After we perform \(m = [c_{36}K]\) + 1 steps of this inductive procedure (3.21) with \(k = m\) and the estimate
\[\text{mes}((D_0 \times T^4) \setminus \Phi_m) < 4mc_{35} \sqrt{\varepsilon}/\rho = c_{37} \sqrt{\varepsilon}\]

imply the result of Theorem 1’.

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