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SPACES OF HERMITIAN OPERATORS WITH SIMPLE SPECTRA AND THEIR FINITE-ORDER COHOMOLOGY

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Abstract. V. I. Arnold studied the topology of spaces of Hermitian operators with non-simple spectra in $\mathbb{C}^n$ in relation to the theory of adiabatic connections and the quantum Hall effect. (Important physical motivations of this problem were also suggested by S. P. Novikov.) The natural stratification of these spaces into the sets of operators with fixed numbers of eigenvalues defines a spectral sequence providing interesting combinatorial and homological information on this stratification.

We construct a different spectral sequence, also converging to homology groups of these spaces; it is based on the universal techniques of *topological order complexes* and *conical resolutions* of algebraic varieties, which generalizes the combinatorial inclusion-exclusion formula, and is similar to the construction of finite-order knot invariants.

This spectral sequence stabilizes at the term E_1, is (conjecturally) multiplicative, and it converges as $n \to \infty$ to a stable spectral sequence calculating the cohomology of the space of infinite Hermitian operators without multiple eigenvalues whose all terms $E_r^{p,q}$ are finitely generated. This allows us to define the finite-order cohomology classes of this space and apply well-known facts and methods of the topological theory of flag manifolds to problems of geometric combinatorics, especially to those concerning continuous partially ordered sets of subspaces and flags.

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1. Introduction

Let $\mathcal{H}(n)$ denote the space of Hermitian operators in $\mathbb{C}^n$; this is an n^2-dimensional real vector space. The *discriminant variety* $\Sigma \equiv \Sigma(n) \subset \mathcal{H}(n)$ consists of operators with at least one eigenvalue of multiplicity ≥ 2; this is a subvariety of codimension 3. For physical motivations of the study of the spaces $\mathcal{H}(n) \setminus \Sigma$, see [2], [4], [10], [11].

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The Alexander duality

$$\bar{H}_i(\Sigma(n)) \simeq \tilde{H}^{n^2-i-1}(\mathcal{H}(n) \setminus \Sigma(n)) \tag{1}$$

relates its Borel–Moore homology groups (i. e., the homology groups of locally finite chains) to the standard cohomology groups (reduced modulo a point) of the complementary space of matrices with simple spectra.

To study these homology groups, V. I. Arnold considered the natural filtration of the space $\Sigma(n)$ according to the numbers of different eigenvalues of operators [4]. Although the answer is known (there is a ring isomorphism

$$H^*(\mathcal{H}(n) \setminus \Sigma(n)) \cong H^*(\mathbb{CP}^{n-1} \times \mathbb{CP}^{n-2} \times \cdots \times \mathbb{CP}^1)\,), \tag{2}$$

this study provides interesting information on the topology of the discriminant set. In particular, the spectral sequence defined by this filtration stabilizes at the second term E^2 (see [13]), and its all groups $E^1_{p,q}$ consist of homology groups of certain complex flag manifolds.

We consider a different spectral sequence, also converging to the same group (1) and built of homology groups of flag manifolds, but based on a *conical resolution* of Σ. This sequence (and the related filtration in the ring (2)) seems to be interesting, because

1. it stabilizes at the *first* term $E^1 \equiv E^\infty$ (at least in characteristic 0);
2. it is compatible very much with the inclusions $\mathcal{H}(n) \hookrightarrow \mathcal{H}(n+1) \hookrightarrow \cdots$, thus stabilizing to a similar spectral sequence converging to the cohomology ring of the space of infinite Hermitian matrices without multiple eigenvalues whose all terms $E_r^{p,q}$ are finitely generated;
3. it is (conjecturally) multiplicative;
4. it is closely related to the shift operator of the spectrum, and the corresponding filtration in the cohomology is invariant under this operator;
5. it allows us to apply the (more or less easy or well-known) facts on the topology of flag manifolds to problems of geometrical combinatorics, especially to those concerning topological partially ordered sets and order complexes;
6. its algebraic presentation is very similar to the theory of finite-order invariants of knots (the stratum of operators with eigenvalues of multiplicities $a_1, \ldots, a_l$ corresponds to that of smooth maps $S^1 \to \mathbb{R}^3$ with l self-intersection points of the same multiplicities), which allows an application of useful structures and notions of this theory to the ring (2). In particular, it allows us to define orders of cohomology classes and, for any class of order p, its *symbol* (or *generalized residue*) at the strata of $\Sigma(n)$ of complexity p in the same way as it was done in [20], [17] for knot invariants and strata of singular knots.

The main results of this work were announced in [19].

2. Hermitian Matrices with Simple Spectra

If all eigenvalues λ_i of a Hermitian operator $\mathbb{C}^n \to \mathbb{C}^n$ are different, then they (and the corresponding one-dimensional complex eigenspaces) can be ordered as

$\lambda_1 < \cdots < \lambda_n$. These eigenspaces form n line bundles over the space $\mathcal{H}(n) \setminus \Sigma(n)$ of all Hermitian operators with different eigenvalues; let $c^1, \ldots, c^n$ be the first Chern classes of these bundles.

Proposition 1 (see [6]). *There is a canonical ring isomorphism*

$$H^*(\mathcal{H}(n) \setminus \Sigma(n)) \simeq \mathbb{Z}[c^1, \ldots, c^n]/\{\mathrm{Sym}\}, \tag{3}$$

where $\{\mathrm{Sym}\}$ *is the ideal generated by all symmetric polynomials of positive degrees.* □

It is convenient to deal with all such rings for all n simultaneously, i. e., consider the ring

$$\mathbb{Z}[[a^0, a^1, a^{-1}, a^2, a^{-2}, \ldots]]/\{\mathrm{Sym}\} \tag{4}$$

of formal power series in infinitely many (in both directions) two-dimensional variables a^j divided by the ideal spanned by the symmetric series of positive degrees. It is natural to call this ring *the cohomology ring of the space of infinite Hermitian matrices with simple spectra*.

The ring (3) can be identified in many ways with a quotient ring of (4): we can choose any number $i = 0, 1, -1, \ldots$ and factor (4) additionally modulo all elements a^j with $j \leq i$ or $j > i+n$. Identifying the variables $c^1, \ldots, c^n$ with $a^{i+1}, \ldots, a^{i+n}$ respectively, we get an isomorphism between this quotient ring and (3).

The obvious operator mapping each variable a^i to a^{i+1} acts on the algebra (4); it is called the *shift operator*.

3. Topological Posets and Conical Resolutions

Suppose that we have a stratified variety and wish to calculate its homology groups. There are two main approaches to this problem. The method of *open strata* (used, in particular, in [2], [4]), is as follows: we filter the variety by unions S_i of smooth strata of "complexity $\geq i$" and consider the corresponding spectral sequence (whose term $E^1_{p,q}$ is the group $H_{p+q}(S_p, S_{p+1})$ Poincaré dual to a cohomology group of the smooth manifold $S_p \setminus S_{p+1}$).

A different approach, modelling the combinatorial formula of inclusions and exclusions, is as follows: first, we consider a singularity resolution $\pi\colon \tilde{V} \to V$ of the entire variety V (thus changing it over the singular set), and then improve it consequently over the *closures* of strata of decreasing dimensions in such a way that, at the last step, we get a space V' with a *proper* projection onto V and contractible preimages of all points. The space V' is homotopy equivalent to the original space; in many important cases, this space V' has a very transparent topological structure, in particular, a natural filtration whose spectral sequence stabilizes very rapidly, see, e. g., [19], [15], [17], [18].

Example. The group $H^2(\mathcal{H}(n) \setminus \Sigma(n))$ is $(n-1)$-dimensional and consists of all sequences $(\alpha_1, \ldots, \alpha_n)$ of integer numbers (i. e., of the corresponding sums $\sum \alpha_i c^i$) factored by the space of constant sequences.

On the corresponding line $\{p+q=n^2-3\}$, the spectral sequence from [4] has a unique nontrivial term E_∞^{n-1,n^2-n-2} isomorphic to $\mathbb{Z}^{n-1}$ and canonically generated by linking numbers with all the $n-1$ smooth strata of maximal dimension in $\Sigma(n)$, i. e., by "δ'-like" sequences of the form $(0, \ldots, 0, 1, -1, 0, \ldots, 0)$. The spectral sequence defined below has $n-1$ one-dimensional groups on the same line, and the pth term of the corresponding filtration consists of all integer-valued polynomial sequences of degree $\leq p$ (modulo the constants). The stable filtration in the 2-dimensional component of (4) is also finitely generated: it is defined by polynomials of any degrees; in particular, it is invariant under the shift operators.

The explicit implementation of our method is based on the notion of a *topological order complex* and on the techniques of *conical resolutions*. It generalizes the method of *simplicial resolutions* (see, e. g., [14], [17]), wich applies to the case when all essential singularities of the variety V are finite (self-)intersections.

Before proceeding to formally define these notions, we give an illustration important for our further calculations.

3.1. The determinant variety and the homology of the group $U(n)$. Consider the space $\mathrm{Mat}(\mathbb{C}^n)$ of all complex linear operators $\mathbb{C}^n \to \mathbb{C}^n$; the variety in question is its *determinant subvariety* Det consisting of all degenerate operators. All possible kernels of degenerate operators are precisely all vector subspaces in $\mathbb{C}^n$; the union of all such kernels is the disjoint union of the Grassmann manifolds $G_1(\mathbb{C}^n), \ldots, G_{n-1}(\mathbb{C}^n), G_n(\mathbb{C}^n)$. The incidence of subspaces corresponding to their points makes the disjoint union of these Grassmannians a *poset* (:= partially ordered set) with unique maximal element $\{\mathbb{C}^n\} \subset G_n(\mathbb{C}^n)$. Take the join $G_1(\mathbb{C}^n) * \cdots * G_n(\mathbb{C}^n)$, i. e., roughly speaking, the union of all simplices (of various dimensions) whose vertices belong to different Grassmannians. Such a simplex is called *coherent* if all subspaces in $\mathbb{C}^n$ corresponding to its vertices form a flag. Finally, the *topological order complex* $\Theta(n)$ is the union of all coherent simplices endowed with the topological structure induced by that of the join. It is contractible, because it is a cone with vertex $\{\mathbb{C}^n\}$. Its *link* $\partial\Theta(n)$ is defined in a similar way, as the union of all coherent simplices in the join $G_1(\mathbb{C}^n) * \cdots * G_{n-1}(\mathbb{C}^n)$.

Proposition 2 (see [7], [15], [17], [12]). *The space $\partial\Theta(n)$ is PL-homeomorphic to the (n^2-2)-dimensional sphere.* □

For any subspace $L \subset \mathbb{C}^n$, we define the cone $\Theta(L)$ as the union of coherent simplices subordinate to L, i. e., such that all subspaces corresponding to their vertices are contained in L. In particular, $\Theta(n) \equiv \Theta(\{\mathbb{C}^n\})$. The subspace $\chi(L) \subset \mathrm{Mat}(\mathbb{C}^n)$ is defined as the space of all operators whose kernels contain L.

The desired *conical resolution* Δ_n of Det is a subset of $\mathrm{Mat}(\mathbb{C}^n) \times \Theta(n)$. Namely, for any subspace $L \subset \mathbb{C}^n$ of any positive dimension, we take the space

$$\chi(L) \times \Theta(L) \subset \mathrm{Mat}(\mathbb{C}^n) \times \Theta(n)$$

and define Δ_n as the union of all such spaces over all possible L. This space Δ_n is naturally filtered: the term F_i of the filtration is the union of the sets $\chi(L) \times \Theta(L)$ over all L of dimensions $\leq i$.

Proposition 3. *The obvious map* $\Delta_n \to \mathrm{Det}$ (*defined by the projection* $\mathrm{Mat}(\mathbb{C}^n) \times \Theta(n) \to \mathrm{Mat}(\mathbb{C}^n)$) *induces a homotopy equivalence of one-point compactifications of these spaces, in particular, an isomorphism of their Borel–Moore homology groups.* □

Now, consider the spectral sequence calculating these groups and generated by filtration $\{F_i\}$ in Δ_n. By definition,

$$E^1_{p,q} \simeq \bar{H}_{p+q}(F_p \setminus F_{p-1}); \tag{5}$$

$F_p \setminus F_{p-1}$ is the space of a fiber bundle whose base is $G_p(\mathbb{C}^n)$ and the fiber over the point $\{L\}$ is diffeomorphic to the direct product $\mathbb{C}^{n(n-p)} \times (\Theta(L) \setminus \partial\Theta(L))$. By Proposition 2, the last factor of this product is homeomorphic to the open disc of dimension $p^2 - 1$; thus, the group (5) is isomorphic to $H_t(G_p(\mathbb{C}^n))$, where $t = p + q - (2n^2 - 2np + p^2 - 1)$.

This spectral sequence stabilizes at the first step: $E^1 \equiv E^\infty$.

The Alexander dual cohomological spectral sequence, which is defined by the identity

$$E_r^{p,q} \equiv E^r_{-p,2n^2-q-1},$$

converges to the cohomology group of the complementary space $\mathrm{GL}(n, \mathbb{C}) \sim \mathrm{U}(n)$. Its stabilization gives us immediately the *Miller splitting formula*

$$H^m(U(n)) \cong \bigoplus_{p=0}^{n} H^{m-p^2}(G_p(\mathbb{C}^n)). \tag{6}$$

Similar splittings hold also for two other classical Lie groups, $\mathrm{O}(n)$ and $\mathrm{Sp}(n)$.

This cohomological sequence is multiplicative. Indeed, the ring $H^*(U(n))$ is an exterior algebra with n generators $\alpha_1, \alpha_3, \ldots, \alpha_{2n-1}$ of corresponding dimensions. Our spectral sequence induces a filtration in this ring; its i-th term coincides with the subalgebra generated by all products of $\le i$ elements α_j.

4. Construction of the Conical Resolution of $\Sigma(n)$. The Main Theorems

Consider again the discriminant variety $\Sigma(n) \subset \mathcal{H}(n)$. The corresponding conical resolution, order complexes, etc. are constructed as follows.

Definition 1. Let $A = (a_1 \ge a_2 \ge \cdots \ge a_l)$ be a sequence of naturals with $a_l \ge 2$ and $\sum a_i \le n$. We define the *complexity* of the multi-index A as $\sum_{i=1}^{l}(a_i - 1)$, the *length* $\#A$ as the number of its elements a_i (denoted above by l), and the *liberty* $\delta(A)$ as $n - \sum a_i$. We also set $|A| = \sum a_i$, so the complexity is equal to $|A| - \#A$.

By $\Gamma_A(n)$ we denote the space of *unordered* collections of $\#A$ pairwise Hermitian orthogonal subspaces in $\mathbb{C}^n$ of complex dimensions $a_1, \ldots, a_{\#A}$.

The space $\Gamma_A(n)$ can be identified with the flag manifold $\mathcal{F}_{a_1,a_1+a_2,\ldots,a_1+\cdots+a_{\#A}}$ if all numbers a_i are different and with the quotient space of this flag manifold by the (free) action of the corresponding permutation group $S(A)$ if some of the a_i coincide. In any case, it is an even-dimensional compact manifold.

The disjoint union of all spaces $\Gamma_A(n)$ is a partially ordered set: a point $\gamma \in \Gamma_A(n)$ is subordinate to a point $\gamma' \in \Gamma_{A'}(n)$ if each of the $\#A$ subspaces forming the point γ is contained in some of the $\#A'$ subspaces forming γ'. In this case, we say also that the points γ and γ' are *incident* to one another. This poset has a unique maximal element, the point $\{\mathbb{C}^n\} \subset \Gamma_{(n)}(n)$.

The corresponding topological order complex $\Xi(n)$ is defined as in the previous section. Namely, we consider the join of the spaces $\Gamma_A(n)$ over all possible multi-indices A with $|A| \le n$ and take the union of all coherent simplices in it, i. e., of simplices whose all vertices are incident to one another, thus forming a monotone sequence. This union $\Xi(n)$ is obviously a compact cone with vertex at the point $\{\mathbb{C}^n\} \subset \Gamma_{(n)}(n)$.

To each point $\gamma \in \Gamma_A(n)$ there corresponds the subcomplex $\Xi(\gamma) \subset \Xi(n)$ being the union of all coherent simplices whose all vertices are subordinate to γ. In particular, $\Xi(n) \equiv \Xi(\{\mathbb{C}^n\})$.

We define the *link* $\partial\Xi(\gamma)$ as the union of all coherent simplices forming $\Xi(\gamma)$ which *do not* contain the maximal vertex $\{\gamma\}$. The *open cone* $\breve{\Xi}(\gamma)$ is defined as the difference $\Xi(\gamma) \setminus \partial\Xi(\gamma)$. The homology groups of the link and the open cone are related by the boundary isomorphism

$$\partial\colon \bar{H}_*(\breve{\Xi}(\gamma)) \xrightarrow{\sim} \tilde{H}_{*-1}(\partial\Xi(\gamma)).$$

Theorem 1. *For any n, the reduced homology group $\tilde{H}_i(\partial\Xi(n), \mathbb{C})$ is trivial in all even dimensions if n is even and in all odd dimensions if n is odd.*

Examples. For $n = 2, 3, 4$, and 5, the Poincaré polynomials of the groups $\tilde{H}_*(\partial\Xi(n), \mathbb{C})$ are equal to t, $t^2(1+t^2)$, $t^3(1+t^4)(1+t^2+t^4)$, and $t^4(1+t^2+t^4+t^6)(1+t^2+t^4+t^6+t^8+t^{10})$, respectively. The last three groups are encoded also in the rightmost nontrivial columns of the three tables in Fig. 1.

A recursive method for calculating these homology groups will be described at the end of Section 5.4, but I do not know any compact expression for them.

For any multi-index A and any $\gamma \in \Gamma_A(n)$, we use $\chi(\gamma)$ to denote the subspace in $\mathcal{H}(n)$ consisting of the Hermitian operators whose eigenspaces contain all spaces of dimension a_i, forming the point γ. Thus, $\chi(\gamma)$ is a real vector space of dimension $\#A + (\delta(A))^2$.

The conical resolution $\sigma(n)$ of the discriminant variety $\Sigma(n) \subset \mathcal{H}(n)$ is defined as the subset of the direct product $\mathcal{H}(n) \times \Xi(n)$ being the union of the products $\chi(\gamma) \times \Xi(\gamma)$ over all possible indices A and all points $\gamma \in \Gamma_A(n)$.

Proposition 4. *The space $\sigma(n)$ is semialgebraic. The obvious map $\sigma(n) \to \Sigma(n)$ (defined by the projection $\mathcal{H}(n) \times \Xi(n) \to \mathcal{H}(n)$) is proper and induces a homotopy equivalence of one-point compactifications of these spaces, in particular, an isomorphism of their Borel–Moore homology groups.* □

There is the *standard filtration* $\sigma_1(n) \subset \cdots \subset \sigma_{n-1}(n) \equiv \sigma(n)$ in the space $\sigma(n)$; its term $\sigma_i(n)$ is the union of the products $\chi(\gamma) \times \Xi(\gamma)$ over all $\gamma \in \Gamma_A(n)$, where A is some multi-index of complexity $\le i$.

The *main spectral sequence* calculating the Borel–Moore homology group of $\sigma(n)$ is generated by this filtration; by definition, its term $E^1_{p,q}$ is isomorphic to $\bar{H}_{p+q}(\sigma_p(n) \setminus \sigma_{p-1}(n))$.

q \ p	1	2
5	$\mathbb{C}$	
4	0	$\mathbb{C}$
3	$\mathbb{C}$	0
2	0	$\mathbb{C}$
1	$\mathbb{C}$	0
0	0	

q \ p	1	2	3
12	$\mathbb{C}$		
11		$\mathbb{C}$	
10	$\mathbb{C}$		$\mathbb{C}$
9		$\mathbb{C}^2 + \mathbb{C}$	
8	$\mathbb{C}^2$		$\mathbb{C}$
7		$\mathbb{C}^2$	
6	$\mathbb{C}$		$\mathbb{C}^2$
5		$\mathbb{C}^2 + \mathbb{C}$	
4	$\mathbb{C}$		$\mathbb{C}$
3		$\mathbb{C}$	
2			$\mathbb{C}$
1		$\mathbb{C}$	

q \ p	1	2	3	4
21	$\mathbb{C}$			
20		$\mathbb{C}$		
19	$\mathbb{C}$		$\mathbb{C}$	
18		$\mathbb{C}^2 + \mathbb{C}$		$\mathbb{C}$
17	$\mathbb{C}^2$		$\mathbb{C}^2 + \mathbb{C}$	
16		$\mathbb{C}^3 + \mathbb{C}$		$\mathbb{C}^2$
15	$\mathbb{C}^2$		$\mathbb{C}^4 + \mathbb{C}^2$	
14		$\mathbb{C}^4 + \mathbb{C}^2$		$\mathbb{C}^3$
13	$\mathbb{C}^2$		$\mathbb{C}^5 + \mathbb{C}^3$	
12		$\mathbb{C}^4 + \mathbb{C}^2$		$\mathbb{C}^4$
11	$\mathbb{C}$		$\mathbb{C}^6 + \mathbb{C}^4$	
10		$\mathbb{C}^3 + \mathbb{C}^3$		$\mathbb{C}^4$
9	$\mathbb{C}$		$\mathbb{C}^5 + \mathbb{C}^4$	
8		$\mathbb{C}^2 + \mathbb{C}^2$		$\mathbb{C}^4$
7			$\mathbb{C}^4 + \mathbb{C}^3$	
6		$\mathbb{C} + \mathbb{C}^2$		$\mathbb{C}^3$
5			$\mathbb{C}^2 + \mathbb{C}^2$	
4		$\mathbb{C}$		$\mathbb{C}^2$
3			$\mathbb{C} + \mathbb{C}$	
2		$\mathbb{C}$		$\mathbb{C}$

FIGURE 1. The main spectral sequences for $n = 3$ (left top), $n = 4$ (left bottom), and $n = 5$ (right)

Theorem 2. *The main spectral sequence calculating the group* (1) *with complex coefficients stabilizes at the first term:* $E^1 \equiv E^\infty$.

These spectral sequences for $n = 3$, 4, and 5 are shown in Fig. 1.

Over the integers, the similar statement is not true: see the end of Section 5.3.

Now, let us describe the terms $E^1_{p,q}$ of this spectral sequence. For any point $\gamma = (\gamma_1, \dots, \gamma_{\#A}) \in \Gamma_A(n)$, we denote the Hermitian orthogonal complement of the linear span of all subspaces γ_i by $\gamma^\perp$. By definition, its complex dimension is equal to $\delta(A)$.

Proposition 5. *Any space $\sigma_p(n) \setminus \sigma_{p-1}(n)$ is a disjoint union (over all indices A of complexity exactly p) of the spaces of fiber bundles whose bases are the spaces $\Gamma_A(n)$ and the fibers over points $\gamma \in \Gamma_A(n)$ split into the direct sums*

$$\mathbb{R}^{\#A} \times \mathcal{H}(\delta(A)) \times \breve{\Xi}(\gamma). \tag{7}$$

Here the factor $\mathbb{R}^{\#A}$ is defined by the eigenvalues of operators $\Lambda \in \chi(\gamma)$ on all subspaces γ_i forming the point γ; the factor $\mathcal{H}(\delta(A))$ is defined by the restrictions of such operators to the orthogonal subspaces $\gamma^\perp$; and the sign $\breve{}$ over $\Xi(\gamma)$ indicates fact that the subspace $\chi(\gamma) \times \partial\Xi(\gamma)$ lies in the smaller term $\sigma_{p-1}(n)$ of the filtration.

We denote the space of this fiber bundle by $\beta_A(n)$. So, $\beta_A(n)$ is the space of a fibered product of three bundles over $\Gamma_A(n)$ whose fibers are the three factors in (7).

Proposition 6. *The second bundle over $\Gamma_A(n)$, which is formed by spaces isomorphic to $\mathcal{H}(\delta(A))$, is orientable.*

Indeed, this bundle is induced from a similar bundle over the simply connected Grassmannian manifold $G_{|A|}(\mathbb{C}^n)$ by the obvious map (sending any collection γ of subspaces to their linear span).

On the other hand, the bundle of the first factors in (7) is nonorientable if some of the numbers a_i coincide. Any such fiber splits into the sum of the one-dimensional oriented subspaces associated with all $\#A$ subspaces forming the basepoint γ; a path in $\Gamma_A(n)$ determining an odd permutation of these subspaces violates the orientation of this factor.

The following proposition reduces the homology group of the space $\breve{\Xi}(A)$ to similar groups for indices A' of length 1.

Recall that, rigorously, the join $X * Y$ of two topological spaces X and Y is defined as the quotient space of the product $X \times [-1, 1] \times Y$ by the equivalence relations $(x, -1, y) \sim (x, -1, y')$ and $(x, 1, y) \sim (x', 1, y)$ for any $x, x' \in X$ and any $y, y' \in Y$.

Proposition 7. *Suppose that a multi-index A is the disjoint union of two multi-indices A' and A'', $\gamma \in \Gamma_A(n)$, and the collection of subspaces in $\mathbb{C}^n$ defining the point γ is divided into subcollections defining some points $\gamma' \in \Gamma_{A'}(n)$ and $\gamma'' \in \Gamma_{A''}(n)$. Then there is a standard homeomorphism of the complex $\Xi(A)$ to the join $\Xi(\gamma') * \Xi(\gamma')$, which sends the vertex $\{\chi(\gamma)\}$ to the point $\{\chi(\gamma')\} \times 0 \times \{\chi(\gamma'')\}$ and the link $\partial\Xi(\gamma)$ to the image (under the factorization map mentioned above) of the boundary*

$$(\Xi(\gamma') \times [-1, 1] \times \partial\Xi(\gamma'')) \cup (\partial\Xi(\gamma') \times [-1, 1] \times \Xi(\gamma''))$$

of the prism

$$\Xi(\gamma') \times [-1, 1] \times \Xi(\gamma'').$$

The proof is immediate. □

Corollary 1. *Let $h_*(\gamma)$ be the graded group defined by $h_i(\gamma) \equiv \bar{H}_{i-1}(\breve{\Xi}(\gamma), \mathbb{C})$. Then, for any $A = (a_1, \ldots, a_{\#A})$ and $\gamma \in \Gamma_A(n)$, there is an* almost *canonical isomorphism*

$$h_*(\gamma) \simeq \bigotimes_{i=1}^{\#A} h_*(\gamma_i), \tag{8}$$

where all γ_i are subspaces of dimensions a_i forming the point γ considered as points of the corresponding Grassmannian manifolds $\Gamma_{(a_i)}(n)$.

"Almost" here means the following. The isomorphism (8) depends on the order of subspaces γ_i forming the collection γ. A reordering of such subspaces (of equal dimensions a_i) multiplies this isomorphism by $(-1)^s$, where s is the parity of the permutation. In particular, we cannot realize the isomorphism (8) by a geometrical construction depending continuously on the point $\gamma \in \Gamma_A(n)$, because a path in $\Gamma_A(n)$ determining such an odd permutation would violate this construction.

It is convenient to consider, together with the bundle $\beta_A(n) \to \Gamma_A(n)$, its "universal covering"

$$\beta!_A(n) \to \Gamma!_A(n). \tag{9}$$

Here $\Gamma!_A(n)$ is the universal covering space of $\Gamma_A(n)$; its points are *ordered* collections of Hermitian-orthogonal spaces of dimensions a_i. This is a principal covering whose group $S(A)$ is the direct product of permutation groups S_μ, where μ are the multiplicities of the numbers a_i in the sequence A. The space $\Gamma!_A(n)$ is canonically diffeomorphic to the flag manifold $\mathcal{F}_{a_1,a_1+a_2,\ldots,|A|}$. The bundle (9) is again the fibered product of three bundles with fibers (7); the first bundle with fibers $\mathbb{R}^{\#A}$ is trivial, and the decomposition (8) of the homology of the third bundle can be realized so that it depend continuously on the basepoint $\gamma! \in \Gamma!_A(n)$. The group of the covering $\Gamma!_A(n) \to \Gamma_A(n)$ acts naturally on the space $\beta!_A(n)$ by permutation of the corresponding fibers, and $\beta_A(n)$ can be considered as the quotient space under this action.

5. Examples

In this section, we calculate the spectral sequences for $n \leq 5$ and determine the meaning of our filtration in the groups $H^2(\mathcal{H}(n) \setminus \Sigma(n))$.

5.1. The marginal columns of the spectral sequence. For any n, the first term $\sigma_1(n)$ of the natural filtration of $\sigma(n)$ is the *canonical resolution* of $\Sigma(n)$, i. e., the space of pairs {a 2-plane in $\mathbb{C}^n$; an operator whose restriction to this plane is scalar}.

Its Borel–Moore homology group coincides with the usual homology group of the Grassmann manifold $G_2(\mathbb{C}^n)$ with grading shifted by $(n-2)^2+1$; see the columns $\{p = 1\}$ of all three tables in Fig. 1.

Now, consider the rightmost columns $\{p = n-1\}$. The unique multi-index A of complexity $n-1$ is equal to (n). The space $\Gamma_A(n)$ in this case consists of one point $\{\mathbb{C}^n\}$; thus, by Proposition 5, we have the isomorphism of homology groups

$$\bar{H}_i(\sigma_{n-1}(n) \setminus \sigma_{n-2}(n)) \equiv \bar{H}_i(\breve{\Xi}(n) \times \mathbb{R}^1) \simeq \tilde{H}_{i-2}(\partial\Xi(n)). \tag{10}$$

5.2. The cases of $n = 2$ and 3. If $n = 2$, then $\Sigma(n)$ consists of the scalar matrices $\left(\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right)$, where $\lambda \in \mathbb{R}$, and $\mathcal{H}(2) \setminus \Sigma(2) \sim S^2$ (see [4]). The ingredients of the construction of Section 4 are as follows. The unique admissible multi-index A consists of one number 2, the corresponding space $\Gamma_A(2)$ is a point O, the space $\Xi(O) \equiv \breve{\Xi}(O)$ is also a point, and $\chi(O)$ is a real line. Thus, the (homological) spectral sequence consists of the unique element $E_{1,0} \simeq \mathbb{Z}$.

Now, let $n = 3$. Both columns corresponding to this case are described in Section 5.1. Namely, the column $\{p = 1\}$ again contains the homology groups of the canonical resolution $\sigma_1(n) \to \Sigma(n)$. This resolution is a local diffeomorphism over all points of $\Sigma(n)$ other than the scalar operators, and the scalar operators are "blown up" to copies of the space $G_2(\mathbb{C}^3) \equiv \partial\Xi(3)$. The term $\sigma_2(n) \setminus \sigma_1(n)$ consists of open cones $\breve{C}(G_2(\mathbb{C}^3)) \sim \breve{\Xi}(3)$ spanning all these fibers $\partial\Xi(3)$. Thus, we get the left top table in Fig. 1.

5.3. The calculations for $n = 4$. Here we justify the left bottom table in Fig. 1. Its first column $\{p = 1\}$ again follows immediately from the considerations of Section 5.1; let us calculate the column $\{p = 2\}$.

For $n = 4$, there are exactly two multi-indices A of complexity 2, namely, (3) and (2, 2). The corresponding blocks $\beta_A(4)$ are as follows.

For $A = (3)$, the space $\Gamma_{(3)}(n)$ is the Grassmannian manifold $G_3(\mathbb{C}^4) \cong \mathbb{CP}^3$, and for any point γ of this space, the space $\partial\Xi(\gamma)$ is equal to $\mathbb{CP}^2$. Thus, the spectral sequence of the fiber bundle $\beta_{(3)}(4) \to \Gamma_{(3)}(4)$ has term $\mathcal{E}^2_{a,b}$ isomorphic to $H_a(\mathbb{CP}^3, \tilde{H}_{b-3}(\mathbb{CP}^2))$, where the number 3 in the subscript $b - 3$ is the sum of the three numbers $\#A = 1$, $(\delta(A))^2 = 1$, and the loss in dimension under the boundary homomorphism, which is also equal to 1. This spectral sequence obviously stabilizes in the term $\mathcal{E}^2$ and gives us the direct summand of the column $\{p = 2\}$ of the left bottom table in Fig. 1 written on the left-hand side of this column.

For $A = (2, 2)$, the space $\Gamma_A(4)$ is the quotient space of the Grassmannian manifold $G_2(\mathbb{C}^4)$ under the involution sending each 2-plane into its Hermitian-orthogonal plane. It is easy to show that the complex homology group of this quotient manifold is isomorphic to $\mathbb{C}$ in dimensions 0, 4, and 8 and is trivial in the other dimensions. The three factors of the fiber (7) of the bundle $\beta_A(4) \to \Gamma_A(4)$ in this case are equal to $\mathbb{R}^2$, one point, and an open interval, respectively. (For any $\gamma \in \Gamma_{(2,2)}(4)$, this interval consists of the two segments joining the point γ with the two subordinate points of $G_2(\mathbb{C}^4)$. Their endpoints lying in $G_2(\mathbb{C}^4)$ are of smaller filtration and, hence, are removed.) The generator of the group $\pi_1(\Gamma_A(4)) \sim \mathbb{Z}_2$ violates the orientations of both the bundle of spaces $\mathbb{R}^2$ and that of open intervals; hence $\bar{H}_i(\beta_A(4)) \simeq H_{i-3}(\Gamma_A(4))$, which gives us the right-hand side of the column $\{p = 2\}$.

Finally, let us calculate the column $\{p = 3\}$, i. e., the homology group of the link $\partial\Xi(4)$.

Consider the subset Ω of this link swept out by all coherent segments connecting points of manifolds $G_2(\mathbb{C}^4)$ and $G_3(\mathbb{C}^4)$.

Lemma 1. *The group $\tilde{H}_a(\Omega)$ is isomorphic to $\mathbb{Z}$ in dimensions 7, 9, and 11 and is trivial in all other dimensions.*

Proof. Consider the space $\Theta(4)$ mentioned in Proposition 2 (and, according to this proposition, homeomorphic to the sphere S^{14}). The space Ω is canonically embedded in this sphere and, by construction, *Spanier–Whitehead dual* (see [15]) to the space $\mathbb{CP}^3$ (also embedded in S^{14}). Hence their reduced homology groups are related by the Alexander duality in S^{14}, which proves the lemma. $\square$

Further, the space $\partial\Xi(4) \setminus \Omega$ is the space of a nonorientable fiber bundle over $\Gamma_{(2,2)}(4)$ whose fibers are open intervals.

The complex Borel–Moore homology group of this space M participates in the (splitting) Smith exact sequence of the double covering $G_2(\mathbb{C}^4) \to \Gamma_{(2,2)}(4)$, which is

$$\xrightarrow{0} \bar{H}_{i+1}(M, \mathbb{C}) \to H_i(G_2(\mathbb{C}^4), \mathbb{C}) \to H_i(\Gamma_{(2,2)}(4), \mathbb{C}) \xrightarrow{0} \bar{H}_i(M, \mathbb{C}) \to \cdots,$$

and hence can easily be calculated: it is equal to $\mathbb{C}$ in dimensions 3, 5, and 7 and trivial in all other dimensions.

Since $\bar{H}_*(M) \equiv \tilde{H}_*(\partial\Xi(4), \Omega)$, the homology group of $\partial\Xi(4)$ is obtained from the exact sequence of this pair. The result is encoded in the right-hand column of the left bottom table in Fig. 1.

Remark. The similar integer-valued spectral sequence does not stabilize at the term E^1, since the right-hand side of its column $\{p = 2\}$ contains a 2-torsion, which disappears in the limit homology group.

5.4. The case $n = 5$. Here we prove that the term E^1 of the main spectral sequence calculating $\bar{H}_*(\Sigma(5), \mathbb{C})$ is as shown in the right-hand table of Fig. 1. The column $\{p = 1\}$ is already justified: it coincides with the homology group of $G_2(\mathbb{C}^5)$ up to a shift of dimensions.

The left-hand side of the column $\{p = 2\}$ corresponds to the block $\beta_{(3)}(5)$; its term of level q coincides with the $(q-4)$-dimensional component of the graded group $H_*(G_3(\mathbb{C}^5)) \otimes \tilde{H}_*(G_2(\mathbb{C}^3))$. (Here the number 4 in $q-4$ is $\#A + (\delta(A))^2 + 1 - p$.)

The right-hand side of the same column contains the homology of the space $\beta_{(2,2)}(5)$. Its base space $\Gamma_{(2,2)}(5)$ is fibered over $\mathbb{CP}^4$ with fiber $\Gamma_{(2,2)}(4)$. The spectral sequence of this fibration obviously stabilizes at the second term, so $H_*(\Gamma_{(2,2)}(5)) \simeq H_*(\mathbb{CP}^4) \otimes H_*(\Gamma_{(2,2)}(4))$. The fiber (7) of the bundle $\beta_{(2,2)}(5) \to \Gamma_{2,2}(5)$ is the direct product of $\mathbb{R}^3$ and an open interval, and this fibration is orientable. Therefore, the Thom isomorphism gives the right-hand side of the column $\{p = 2\}$.

The left-hand side of the column $\{p = 3\}$ contains the homology groups of the space $\beta_{(4)}(5)$, which is fibered over $\Gamma_{(4)}(5) \equiv \mathbb{CP}^4$ with fiber $\mathbb{R}^2 \times \breve{\Xi}(4)$. The homology groups of the space $\breve{\Xi}(4)$ are calculated in the preceding subsection (see the column $\{p = 3\}$ of the corresponding spectral sequence). It follows from the calculation that the terms $\mathcal{E}^2_{a,b}$ of the spectral sequence of the fibration $\beta_{(4)}(5) \to \mathbb{CP}^4$ can be nontrivial only if a is even and b is odd. Therefore, this sequence stabilizes and gives us the left-hand side of the column $\{p = 3\}$.

The right-hand side of the same column contains the homology groups of the space $\beta_{(3,2)}(5)$. Its base space $\Gamma_{(3,2)}(5)$ coincides with $G_3(\mathbb{C}^5)$, and the three factors of the fiber (7) over its some point are equal to $\mathbb{R}^2$, one point, and the open cone over the suspension $\Sigma(\mathbb{CP}^2)$, respectively (cf. Proposition 7). Therefore the column $\{p = 3\}$ is also justified.

Finally, the last column $\{p = 4\}$ is obtained by combining formula (2) (which provides all the groups $\bar{H}_i(\Sigma(n), \mathbb{C})$), Theorem 2 (which says that any such group is the direct sum of groups $E^1_{p,i-p}$), and the columns calculated above (which provides all the other elements of these sums).

Similarly, we can calculate the groups $H_*(\partial\Xi(n))$ with arbitrary n from groups $H_*(\partial\Xi(m))$ with $m < n$ and the homology groups of the spaces $\beta_A(n)$ with A of complexities $\leq n - 2$. Concerning the calculation of these groups, see Section 6 below.

5.5. The 2-dimensional cohomology classes of $\mathcal{H}(n)\setminus\Sigma(n)$. It is convenient to replace the homological spectral sequence $E^r_{p,q} \to \bar{H}_{p+q}(\Sigma(n))$ defined in Section 4 by its "Alexander dual" cohomological sequence obtained by formal inversion of indices:

$$E_r^{p,q} \equiv E^r_{-p,n^2-q-1}. \tag{11}$$

This spectral sequence lies in the second quadrant in the wedge $\{p \leq 0,\ p+q \geq 0\}$ and converges exactly to the group $\tilde{H}^*(\mathcal{H}(n) \setminus \Sigma(n))$.

Proposition 8. *For any n, the integer-valued cohomological spectral sequence (11) has exactly $n - 1$ nontrivial terms $E_1^{p,q}$ with $p + q = 2$, namely, $E^{-1,3}$, $E^{-2,4}$, ..., $E^{-n+1,n+1}$, all of which are isomorphic to $\mathbb{Z}$. The corresponding filtration in the group $H^2(\mathcal{H}(n) \setminus \Sigma(n))$ is as follows: its term F_p consists of all sums $\sum_{i=1}^n \alpha_i c^i$, where the sequence $\{\alpha_i\}$ coincides with a polynomial of degree $\leq p$ taking integer values at integer points.*

So, the quotient group $E^{-p,p+2}$ is generated by the basic polynomial sequence of degree p, $\alpha_i = i(i-1)\cdots(i-p+1)/p!$, or, if we are interested only in $\mathbb{C}$-valued cohomology, by the monomial i^p.

Proposition 9. *For any n, the group $E^{-1,5}$ or first-order elements of the group $H^4(\mathcal{H}(n)\setminus\Sigma(n), \mathbb{Z})$ is one-dimensional, and it is generated by the series $\sum_{i=1}^n i(c^i)^2$.*

The proofs of these two propositions are immediate. □

6. Proof of Theorems 1 and 2

We shall prove these theorems by induction on n.

Lemma 2. *Suppose that Theorem 1 is true for all $n \leq n_0$, and all elements a_i of the multi-index $A = (a_1, \ldots, a_{\#A})$ do not exceed n_0; let $\beta_A(N)$ be the corresponding block of the resolution of the discriminant space $\Sigma(N) \subset \mathcal{H}(N)$, and let $\beta!_A(N)$ be its universal covering space (see the end of Section 4). Then*

(a) *the spectral sequence of the fiber bundle $\beta!_A(N) \to \Gamma!_A(N)$ calculating the group $\bar{H}_*(\beta!_A(N))$ stabilizes at the second term: $E^2 \equiv E^\infty$;*

(b) *the groups $\bar{H}_*(\beta!_A(N), \mathbb{C})$ and $\bar{H}_*(\beta_A(N), \mathbb{C})$ are trivial in all even (odd) dimensions if N is even (respectively, odd).*

Proof. The space $\beta!_A(N)$ is the space of a fiber bundle over the simply connected manifold $\Gamma!_A(N)$ (whose all odd-dimensional homology groups are trivial) with fiber (7), whose all $(N - \text{even})$-dimensional Borel–Moore homology groups are also trivial by our assumption. This implies statement (a) of the lemma and the assertion of statement (b) concerning the group $\bar{H}_*(\beta!_A(N), \mathbb{C})$. The assertion concerning $\bar{H}_*(\beta_A(N), \mathbb{C})$ follows from it, because the complex homology group of the base of a finite covering is "not greater" than that of its total space. □

Proposition 10. *If Theorem 1 is true for all $n < N$, then Theorem 2 is true for $n = N$, i. e., the spectral sequence calculating the group $\bar{H}_*(\Sigma(N), \mathbb{C})$ stabilizes at the first term.*

This proposition implies the assertion of Theorem 1 for $n = N$ and thus completes the induction step. Indeed, by Proposition 10 the group $\bar{H}_i(\Sigma(N), \mathbb{C})$ splits into a direct sum of groups $E^1_{p,i-p}$. By formulas (1) and (2), this group is trivial in all dimensions congruent to N mod 2. Hence the group

$$E^1_{N-1,i-(N-1)} \equiv \bar{H}_i(\beta_{(N)}(N), \mathbb{C}) \equiv \bar{H}_i(\mathbb{R}^1 \times \breve{\Xi}(N), \mathbb{C}) \cong \tilde{H}_{i-2}(\partial\Xi(N), \mathbb{C})$$

is also trivial for such i.

In the proof of Proposition 10, we shall use the following version of the Poincaré duality in singular semialgebraic sets like $\overline{\beta_A(N)}$, $\overline{\beta!_A(N)}$, and $\overline{\sigma(N)}$ (cf. [8]).

Given a compact Whitney stratified semialgebraic variety V, we embed it into some space $\mathbb{R}^M$ as an absolute retract of its some bounded neighborhood $U \subset \mathbb{R}^M$ whose boundary ∂U is a smooth manifold. A *quasicycle* in V is any semialgebraic relative cycle of the pair $(\mathbb{R}^M, \mathbb{R}^M \setminus U)$ generic with respect to the stratification of V. Any complex-valued cohomology class of the space V can be realized as the intersection index of V and some such quasicycle. The *support* of a quasicycle is the intersection set of this quasicycle and V.

Proof of Proposition 10. We shall prove the dual (and thus equivalent) statement concerning the cohomological spectral sequence calculating the Borel–Moore *cohomology* of $\Sigma(N)$ (i. e., the reduced cohomology of its one-point compactification $\overline{\Sigma(N)}$). Its term $e_1^{p,q}$ is equal to

$$\tilde{H}^{p+q}\big(\overline{(\sigma_p(N) \setminus \sigma_{p-1}(N))}, \mathbb{C}\big) \equiv \bigoplus_{|A|-\#A=p} \tilde{H}^{p+q}\big(\overline{\beta_A(N)}, \mathbb{C}\big).$$

So, we need only to prove that, for any element ω of such a group $\tilde{H}^{p+q}\big(\overline{\beta_A(N)}, \mathbb{C}\big)$, all its differentials $d^i(\omega) \in e^{p+i,q-i+1}$ are trivial.

Let $\omega!$ be the preimage of ω in the cohomology group of $\beta!_A(N)$, and let $[\omega!]$ be an arbitrary compact quasicycle in $\beta!_A(N)$ realizing this class via the Poincaré duality. Its projection $[\omega] \subset \beta_A(N)$ is Poincaré dual to some nonzero integer multiple of the class ω.

Consider the open subset $\operatorname{reg}\beta!_A(N) \subset \beta!_A(N)$ consisting of the triples

$$\big(\text{a Hermitian operator } \Lambda;\ \text{a point } \gamma! \in \Gamma!_A(N);\ \text{a point } \xi \in \breve{\Xi}(\gamma!)\big)$$

such that the restriction of Λ to the orthogonal plane $(\gamma!)^\perp \subset \mathbb{C}^N$ is an operator with simple (i. e., $\delta(A)$-element) spectrum.

Lemma 3. *For any class* $\omega! \in \bar{H}^*(\beta!_A(N), \mathbb{C})$*, the Poincaré dual quasicycle* $[\omega!]$ *can be chosen so that its support lies in* $\operatorname{reg}\beta!_A(N)$*.*

Proof. This assertion is equivalent to the statement that the homomorphism

$$\bar{H}^*(\beta!_A(N), \mathbb{C}) \to \bar{H}^*(\operatorname{reg}\beta!_A(N), \mathbb{C}) \tag{12}$$

induced by the identity embedding is monomorphic. To prove it, consider the total space $\Upsilon_A(N)$ of the quotient bundle of the fiber bundle $\beta!_A(N) \to \Gamma!_A(N)$ whose fibers are the products of only first and third factors in (7). We have the commutative diagram of fiber bundles

$$\begin{array}{ccc} \beta!_A(N) & \leftarrow & \operatorname{reg}\beta!_A(N) \\ \downarrow & & \downarrow \\ \Upsilon_A(N) & \stackrel{\mathrm{Id}}{\longleftarrow} & \Upsilon_A(N) \end{array} \tag{13}$$

with fibers $\mathcal{H}((\gamma!)^\perp)$ and $\mathcal{H}((\gamma!)^\perp) \setminus \Sigma((\gamma!)^\perp)$, respectively.

The left-hand bundle in (13) is an orientable $(\delta(A))^2$-dimensional vector bundle; hence the second term E_2 of the corresponding spectral sequence calculating the Borel–Moore cohomology of $\beta!_A(N)$ has only one nontrivial row $q = (\delta(A))^2$, which coincides with the graded group $\bar{H}^*(\Upsilon_A(N), \mathbb{C})$. In particular, this sequence stabilizes at this term. Corollary 1 to Proposition 7 and Theorem 1 (recall that, by the induction hypothesis, it is assumed to be true for all values of n equal to elements a_i of the multi-index A) imply that this graded group is trivial in the dimensions congruent to $|A| \bmod 2$.

The similar spectral sequence for the right-hand bundle in (13) also stabilizes at the term E_2: this again follows from dimension considerations, because the group $\bar{H}^*(\mathcal{H}(\gamma!^\perp) \setminus \Sigma(\gamma!^\perp), \mathbb{C})$ is nontrivial only in dimensions congruent to $\delta(A) \bmod 2$.

The homomorphism of these terms E_2 of the two spectral sequences defined by diagram (13) is an isomorphism of the rows $\{q = (\delta(A))^2\}$, and it is zero for all other q. This implies the lemma. □

Further, consider the subset $\operatorname{perf}\beta!_A(N) \subset \operatorname{reg}\beta!_A(N)$ consisting of triples

$$\big(\text{an operator } \Lambda;\ \text{a point } \gamma! = (\gamma_1, \ldots, \gamma_{\#A});\ \text{a point } \xi \in \breve{\Xi}(\gamma!)\big)$$

such that the eigenvalues of Λ on the spaces

$$\gamma_1, \gamma_2, \ldots, \gamma_{\#A}, \ \text{ and } \gamma!^\perp$$

lie, respectively, in the intervals

$$(0, 1), (2, 3), \ldots, (2\#A - 2, 2\#A - 1), \ \text{ and } (2\#A, +\infty).$$

Lemma 4. *For any class* $\omega! \in \bar{H}^*(\beta!_A(N), \mathbb{C})$*, the Poincaré dual quasicycle* $[\omega!]$ *can be chosen so that its support lies in* $\operatorname{perf}\beta!_A(N)$*.*

Proof. On the space $\beta!_A(N)$, the group $\mathbb{R}^{\#A} \oplus \mathbb{R}$ acts; $\mathbb{R}^{\#A}$ acts by shifts along the first (trivial) factor in (7) and $\mathbb{R}$ acts by adding operators whose restrictions to the spaces $\gamma!^\perp$ are scalar operators and the restrictions to all spaces γ_i are trivial. (If $\delta(A) = 0$, then the action of $\mathbb{R}$ is trivial.) This action leaves invariant all fibers of the bundle $\beta!_A(N) \to \Gamma!_A(N)$ and the subspace $\operatorname{reg}\beta!_A(N)$, and it takes quasicycles to quasicycles.

Any compact subset in $\beta!_A(N)$ can be moved to the domain $\operatorname{perf}\beta!_A(N)$ by the action of this group and the additional action of the group $\mathbb{R}_+$ of dilations multiplying operators by non-zero constants. $\square$

So, our class $\omega \in H^*(\beta_A(N), \mathbb{C})$ can be realized by the projection $[\omega]$ of some compact quasicycle $[\omega!] \subset \operatorname{perf}\beta!_A(N)$. But the projection of the set $\operatorname{perf}\beta!_A(N)$ to $\beta_A(N)$ does not meet the closures of any other blocks $\beta_{A'}(N)$ of greater filtration (i. e., with $|A'| - \#A' > |A| - \#A$). Thus, our quasicycle $[\omega]$ is a quasicycle in the entire space $\overline{\sigma(n)}$ and the dual Borel–Moore cohomology class in $\sigma(n)$ is well defined. This completes the proof of Proposition 10. $\square$

The following notion is an analogue of the *weight systems* from the knot theory.

Definition 2. Given a class $\omega \in H^*(\mathcal{H}(n) \setminus \Sigma(n))$ of order p and a multi-index A of complexity p, the corresponding *symbol* $s(A, \omega) \in \bar{H}_*(\beta_A(n))$ is the restriction of the corresponding homology class in $\bar{H}_*(\Sigma)$ to the block $\beta_A(n)$. (In more detail, we realize the latter class by a cycle in $F_p(\sigma(n))$ and then reduce it modulo the union of σ_{p-1} and all the other blocks $\beta_{A'}(n)$ of complexity p with $A' \neq A$.) Similarly, the *!-symbol* $s!(A, \omega) \in \bar{H}_*(\beta!_A(n))$ is the homology class of the full preimage of this cycle in the covering space $\beta!_A(n)$.

By Lemma 2, any such !-symbol is an element of the cohomology group of the flag manifold $\Gamma!_A(n)$ with coefficients in the homology group of the topological order complex $\partial\Xi(A)$. Similarly, the usual symbol $s(A, \omega)$ is an element of the cohomology group of the manifold $\Gamma_A(n)$ with coefficients in a *local system* of groups with the same fibers.

7. Stabilization

Definition. The *order* of an element of the cohomology group (1) is the filtration of the Alexander dual homology class in $\Sigma(n)$, i. e., the smallest number i such that it can be realized by a locally finite chain inside σ_i.

Consider again the cohomological spectral sequences

$$E_r^{p,q}(n) \to H^{p+q}(\mathcal{H}(n) \setminus \Sigma(n))$$

(see (11)).

Let $n < N$ be any two naturals, s be one of the numbers $0, \dots, N-n$, and $i_s \colon \mathcal{H}(n) \to \mathcal{H}(N)$ be an arbitrary smooth embedding sending each operator $\Lambda \colon \mathbb{C}^n \to \mathbb{C}^n$ to an operator $\Lambda + \Lambda'$, where Λ' acts on the orthogonal complement of $\mathbb{C}^n$ in $\mathbb{C}^N$, depends smoothly on Λ, and has simple spectrum exactly s (respectively, $N-n-s$) elements of which are below (respectively, above) the spectrum of Λ. In particular, $\Sigma(n) = i_s^{-1}(\Sigma(N))$.

Proposition 11. *Any such embedding i_s induces a homomorphism of cohomological spectral sequences*

$$E_r^{p,q}(N) \to E_r^{p,q}(n), \tag{14}$$

where $r \geq 1$, which does not depend *on s and on the embedding i_s, is epimorphic for any p and q, and determines an isomorphism of the terms $E_r^{p,q}$ if n is sufficiently large with respect to $|p| + |q|$.*

Thus, we can define the *stable cohomological spectral sequence*, whose term $E_r^{p,q}$ is the inverse limit of the groups $E_r^{p,q}(n)$ over all possible homomorphisms induced by such embeddings. By the last statement of Proposition 11, any such stable term is finitely generated. This sequence converges to the entire ring (4), thus defining the orders of the elements of this ring. In simple terms, an element of the ring (4) is *of order* i if, for any sufficiently large n it becomes an element of order i under any of the projections of (4) onto $H^*(\mathcal{H}(n) \setminus \Sigma(n), \mathbb{C})$ described in Section 2. Certainly, not all elements of the ring (4) have finite order; however, the spaces of finite-order elements *weakly converge* to this ring in the following sense: for each n, any cohomology class on the space $\mathcal{H}(n) \setminus \Sigma(n)$ coincides with the restriction of some stable class of finite order. In the theory of knot invariants, the similar statement (on the completeness of finite-order invariants) is not proved.

Let us estimate the rate of stabilization of the groups $E_r^{p,q}(n)$ as n increases. For any multi-index A, the obvious inclusions of flag manifolds

$$\Gamma!_A(n) \hookrightarrow \Gamma!_A(n+1) \hookrightarrow \cdots,$$

induce maps of their cohomology groups. For every i, let $\operatorname{stab}(A, i)$ denote the smallest number n at which all these cohomology groups of dimensions $\leq i$ stabilize, i. e., all further inclusions induce isomorphisms in these dimensions.

Proposition 12. *For any bi-index (p, q) with $p \leq 0$ and $p + q \geq 0$, the groups $E_1^{p,q}(n)$ stabilize no later than at*

$$n = \max_{|A| - \#A = -p} \operatorname{stab}(A, p + q - 2\#A).$$

The precise construction of the homomorphism mentioned in Proposition 11 is as follows (cf. [16], [17]). The embedding $i_s \colon (\mathcal{H}(n), \Sigma(n)) \to (\mathcal{H}(N), \Sigma(N))$ can be tautologically lifted to the filtration-preserving map of resolutions $i_s \colon \sigma(n) \to \sigma(N)$. Its image has an open neighborhood U in $\sigma(N)$ homeomorphic to $\sigma(n) \times \mathbb{R}^{N^2 - n^2}$, where the image of any space $* \times \mathbb{R}^{N^2-n^2}$ lies entirely in one block $\beta_A(N)$. Consider the map $\bar{H}_*(\sigma(N)) \to \bar{H}_{*-(N^2-n^2)}(\sigma(n))$ being the composition of the restriction homomorphism $\bar{H}_*(\sigma(N)) \to \bar{H}_*(U)$ and the Künneth isomorphism $\bar{H}_*(\sigma(n) \times \mathbb{R}^{N^2-n^2}) \to \bar{H}_{*-(N^2-n^2)}(\sigma(n))$. By construction, it is compatible with the map of Alexander dual groups $i_s^* \colon H^*(\mathcal{H}(N) \setminus \Sigma(N)) \to H^*(\mathcal{H}(n) \setminus \Sigma(n))$ induced by the embedding i_s. The homomorphism of the terms E_1 of our spectral sequences is defined by the restricting this construction to any block $\beta_A(N)$.

For any A, we set $U_A \equiv U \cap \beta_A(N)$. The homeomorphism $\beta_A(n) \times \mathbb{R}^{N^2-n^2} \to U_A$ is compatible with the structure of a fiber bundle in the spaces $\beta_A(\cdot)$ mentioned in Proposition 5. Namely, the embedding $\mathbb{C}^n \hookrightarrow \mathbb{C}^N$ defines embeddings $\Gamma_A(n) \hookrightarrow \Gamma_A(N)$ and $\Gamma!_A(n) \hookrightarrow \Gamma!_A(N)$; there is a tubular neighborhood W of $\Gamma_A(n)$ in $\Gamma_A(N)$ whose bundle is orientable and fibers are homeomorphic to $\mathbb{R}^{2|A|(N-n)}$. We get the diagrams of bundles

$$\begin{array}{ccccc} \beta!_A(n) & \hookrightarrow & U! & \subset & \beta!_A(N) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma!_A(n) & \hookrightarrow & W! & \subset & \Gamma!_A(N), \end{array} \qquad \begin{array}{ccccc} \beta_A(n) & \hookrightarrow & U & \subset & \beta_A(N) \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_A(n) & \hookrightarrow & W & \subset & \Gamma_A(N). \end{array} \tag{15}$$

For any point $\gamma \in \Gamma_A(n)$ or $\gamma! \in \Gamma!_A(n)$, the fibers $\mathbb{R}^{\#A}$ and $\breve{\Xi}(\gamma)$ of the first and third factors of the fibered products $\beta_A(n) \to \Gamma_A(n)$ and $\beta!_A(n) \to \Gamma!_A(n)$ over this point (see (7)) are mapped identically to the corresponding fibers of the bundles over $\Gamma_A(N)$ and $\Gamma!_A(N)$, while the fibers $\mathcal{H}(n-|A|)$ of the second (vector) bundle are embedded into the fibers $\mathcal{H}(N-|A|)$ of the second bundle over $\Gamma!_A(N)$ as subspaces whose quotient spaces form an orientable $(N^2-n^2-2|A|(N-n))$-dimensional vector bundle over the image of $\Gamma_A(n)$ or $\Gamma!_A(n)$. The permutation group $S(A)$ acts on the left-hand diagram in (15), and the right-hand diagram is formed by the spaces of its orbits; thus, all complex homology homomorphisms induced by the arrows in the latter diagram are determined by the homomorphisms induced by the arrows in the former one.

But these homomorphisms in the complex Borel–Moore homology groups are very simple: the bottom homomorphism

$$\bar{H}_*(\Gamma!_A(n), \mathbb{C}) \leftarrow \bar{H}_{*+2|A|(N-n)}(\Gamma!_A(N), \mathbb{C}) \tag{16}$$

is nothing else but the standard cohomology map induced by the obvious embedding and rewritten in terms of Poincaré dual groups. The Borel–Moore homology groups of the spaces of both bundles of this diagram are just the tensor products of the corresponding groups (16) of their bases and the homology groups of fibers, which coincide up to a shift of dimensions by $N^2-n^2-2|A|(N-n)$. The corresponding map $\bar{H}_*(\beta!_A(n), \mathbb{C}) \leftarrow \bar{H}_{*+(N^2-n^2)}(\beta!_A(N), \mathbb{C})$ of these tensor products decomposes into these actions on their factors.

The map (16) is always epimorphic, thus so is the map (14).

Proof of Proposition 12. Let us fix some multi-index A and dimension i and estimate the greatest dimension $J = J(A, i)$ such that the group $\bar{H}_J(\beta!_A(n))$ depends on the group $H^i(\Gamma!_A(n))$. By Proposition 5 and formula (8), this dimension J does not exceed the sum of four numbers: $\dim \Gamma!_A(n) - i$, $\#A$, $(n-|A|)^2$, and $\#A - 1 + \sum_{i=1}^{\#A} \dim_h \breve{\Xi}(a_i)$, where $\dim_h(\cdot)$ is the maximal dimension i such that $\bar{H}_i(\cdot)$ is nontrivial.

Let $s_2(A)$ be the second symmetric polynomial in $a_1, \dots, a_{\#A}$, i. e.,

$$s_2(A) = a_1a_2 + a_1a_3 + \dots + a_{\#A-1}a_{\#A};$$

then $\dim \Gamma!_A(n) = 2(s_2(A) + |A|(n-|A|))$. According to (10) and Theorem 2, the group $\bar{H}_i(\breve{\Xi}(a), \mathbb{C})$ is a direct factor of the group $\bar{H}_{i+1}(\Sigma(a))$. Since $\dim \Sigma(a) = a^2-3$, we have $\dim_h \breve{\Xi}(a) \le a^2 - 4$. Thus, the sum of the four above numbers does not exceed

$$2(s_2(A) + |A|(n-|A|)) - i + \#A + (n-|A|)^2 + \#A - 1 + \sum a_i^2 - 4\#A \\ \equiv n^2 - 1 - 2\#A - i.$$

Therefore, by (11), any cell $E_1^{p,q}$ depends only on groups $H^i(\Gamma!_A(n), \mathbb{C})$ such that $|A| - \#A = -p$ and $p + q \ge i + 2|A|$. □

8. Problems

Problem 1. Our cohomological spectral sequence defines filtrations ("orders") in the rings (3) and (4). The problem is to find an explicit expression for the elements of this filtration. I hope very much that it coincides with or is closely related to some classical structure in these rings.

In some sense, the term F_{-p} of this filtration should consist of power series whose coefficients are "constructive functions of complexity $\leq p$" of the corresponding exponentials (cf. Section 5.5). What is the precise sense of this complexity?

Does there exist any natural (and physically motivated) representation of the elements of this filtration in terms of differential forms in $\mathcal{H}(n) \setminus \Sigma(n)$ (cf. [4])?

Here are two related subproblems, which can also be investigated independently.

Problem 2. The shift operator $\mathbf{S}$ (see the end of Section 2) preserves our filtration: $\mathbf{S}(F_p) = F_p$. Is it true that the corresponding derivative, which sends the cohomology class α to $\mathbf{S}(\alpha) - \alpha$, maps any F_p to F_{p-1}?

Problem 3. Is the following multiplicativity conjecture true?
Multiplicativity conjecture: *Our cohomological spectral sequence is multiplicative, i. e.,*

$$F_{p'} \smile F_{p''} \subset F_{p'+p''}.$$

If it is true, the next problem is to define the corresponding maps

$$E_1^{p',q'} \otimes E_1^{p'',q''} \to E_1^{p'+p'',q'+q''}$$

explicitly.

There is an obvious multiplication of this sort (in some sense, it models the Kontsevich formula for multiplication of knot invariants in terms of chord diagrams; see, e. g., [17]). Namely, consider any multi-indices $A' = (a'_1, \ldots, a'_{\#A'})$ and $A'' = (a''_1, \ldots, a''_{\#A''})$ and the set $A = A' \sqcup A''$. Below we define natural maps

$$\bar{H}_{n^2-i-1}(\beta_{A'}(n), \mathbb{C}) \otimes \bar{H}_{n^2-j-1}(\beta_{A''}(n), \mathbb{C}) \to \bar{H}_{n^2-i-j+1}(\beta_A(n), \mathbb{C}); \tag{17}$$

the desired multiplication can be obtained from these ones by applying the Alexander duality (11) and the decomposition

$$\sigma_p(n) \setminus \sigma_{p-1}(n) = \bigsqcup_{|A|-\#A=p} \beta_A(n).$$

Indeed, let us define the group $\bar{H}_*(\breve{\Xi}(A))$ as $\bar{H}_*(\breve{\Xi}(\gamma))$ for an arbitrary $\gamma \in \Gamma_A(n)$. Then the map (17) is the composition of the following maps of complex homology groups:

$$\begin{array}{ccccccccc}
\bar{H}_*(\beta_{A'}(n)) & \xrightarrow{\Delta} & \bar{H}_*(\beta!_{A'}(n)) & \xrightarrow{\sim} & H_*(\Gamma!_{A'}(n)) & \otimes & \bar{H}_*(\mathbb{R}^{2\#A'-1+(n-|A'|)^2}) & \otimes & \bar{H}_*(\breve{\Xi}(A')) \\
\otimes & & \otimes & & \otimes & & \otimes & & \otimes \\
\bar{H}_*(\beta_{A''}(n)) & \xrightarrow{\Delta} & \bar{H}_*(\beta!_{A''}(n)) & \xrightarrow{\sim} & H_*(\Gamma!_{A''}(n)) & \otimes & \bar{H}_*(\mathbb{R}^{2\#A''-1+(n-|A''|)^2}) & \otimes & \bar{H}_*(\breve{\Xi}(A'')) \\
\cline{1-4}
 & & & & \downarrow & & \downarrow & & \downarrow \\
\bar{H}_*(\beta_A(n)) & \xleftarrow{\nabla} & \bar{H}_*(\beta!_A(n)) & \xleftarrow{\sim} & H_*(\Gamma!_A(n)) & \otimes & \bar{H}_*(\mathbb{R}^{2\#A-1+(n-|A|)^2}) & \otimes & \bar{H}_*(\breve{\Xi}(A)).
\end{array}$$

Here the maps $\xrightarrow{\Delta}$ are the liftings representing cycles in $\beta_A(n)$ as projections of some cycles in $\beta!_A(n)$. All the isomorphisms $\xrightarrow{\sim}$ and $\xleftarrow{\sim}$ follow from Proposition 5 and statement (a) of Lemma 2. The vertical map

$$H_*(\Gamma!_{A'}(n)) \otimes H_*(\Gamma!_{A''}(n)) \to H_*(\Gamma!_A(n))$$

(it does not preserve degrees!) is the composition of
(a) Poincaré isomorphisms in $\Gamma!_{A'}(n)$ and $\Gamma!_{A''}(n)$,
(b) the Künneth isomorphism

$$H^*(\Gamma!_{A'}(n)) \otimes H^*(\Gamma!_{A''}(n)) \to H^*(\Gamma!_{A'}(n) \times \Gamma!_{A''}(n)),$$

(c) the map of cohomology groups induced by the tautological embedding of $\Gamma!_A(n)$ into this product, and
(d) the Poincaré isomorphism in $\Gamma_A(n)$.
The vertical map

$$\bar{H}_*(\mathbb{R}^{2\#A'-1+(n-|A'|)^2}) \otimes \bar{H}_*(\mathbb{R}^{2\#A''-1+(n-|A''|)^2}) \to \bar{H}_*(\mathbb{R}^{2\#A-1+(n-|A|)^2})$$

(which also does not presere the grading) takes the product of the canonical generators of the first two groups to the canonical generator of the last one.
The vertical maps

$$\bar{H}_*(\breve{\Xi}(A')) \otimes \bar{H}_*(\breve{\Xi}(A'')) \to \bar{H}_*(\breve{\Xi}(A))$$

are described in Proposition 7 (they increase the grading by 1); finally, the map $\xleftarrow{\nabla}$ is the obvious projection. It is easy to check that this composition maps an (n^2-i+1)-dimensional cycle in $\beta_{A'}(n)$ and an (n^2-j+1)-dimensional cycle in $\beta_{A''}(n)$ into an $(n^2-i-j-1)$-dimensional homology class in $\beta_A(n)$ and does not depend on the choice of the liftings $\xrightarrow{\Delta}$ of these cycles to the corresponding spaces $\beta!_\cdot(n)$.

Problem 4. If the multiplication conjecture is true, how is the usual multiplication in the spectral sequence related to that described above? What are the multiplicative generators of this spectral sequence?

In addition to our increasing filtration (by the orders), the ring (4) admits a decreasing filtration; its element is of *confinement* d if, for any $n < d$, it lies in the kernels of all surjections onto the ring (3) described in Section 2.

Is it possible to characterize the primitive (i. e., indecomposable) elements of our spectral sequence in terms of these two filtrations?

Problem 5. Find a compact general formula for the homology groups of the space $\partial\Xi(n)$ for any n.

This space is not a topological manifold (note that the right-hand columns of the tables in Fig. 1 present the *reduced* homology of these spaces).

Therefore, their intersection homology groups are also very interesting.

9. The Last Remark

All results of this work can be directly transferred to the theory of *hyper-Hermitian* forms (see [5]), i. e., of quadratic forms on the realification $\mathbb{R}^{4n}$ of the quaternionic space $\mathcal{H}^n$ invariant under the left multiplications by unitary quaternions. In particular, the main spectral sequence calculating the rational Borel–Moore homology groups of the discriminant space of such forms with $< n$ eigenvalues stabilizes at the term E^1, and the reduced homology groups of the quaternionic analogue of the order complex $\partial\Xi(n)$ are trivial in all dimensions not congruent to $2n^2 - n + 1$ modulo 4.

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