THE LARGE $N$ LIMITS OF INTEGRABLE MODELS

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Dedicated to Sergey Novikov on the occasion of his 65th birthday

Abstract. We consider the large $N$ limits of Hitchin-type integrable systems. The first system is the elliptic rotator on $GL_N$ that corresponds to the Higgs bundle of degree 1 over an elliptic curve with a marked point. This system is gauge equivalent to the $N$-body elliptic Calogero–Moser system, which is obtained from the Higgs bundle of degree zero over the same curve. The large $N$ limit of the former system is the integrable rotator on the group of the non-commutative torus. Its classical limit leads to an integrable modification of 2D hydrodynamics on the two-dimensional torus. We also consider the elliptic Calogero–Moser system on the group of the non-commutative torus and consider the systems that arise after the reduction to the loop group.

2000 Math. Subj. Class. 37KXX, 70EXX.

Key words and phrases. Classical integrable systems, noncommutative geometry, 2D hydrodynamics.

1. Introduction

In this paper, we analyse two related integrable models, a modified integrable two-dimensional hydrodynamics on the torus and the large $N$ limit of the elliptic Calogero–Moser system (ECMS) with spin. The former system is also the large $N$ limit of the integrable $GL(N, \mathbb{C})$ elliptic rotator (ER) proposed in [22]. Integrable models are among the main scientific interests of Sergey Novikov, and his contribution to this subject is widely recognised.

It was established in [14] that, for finite $N$, the ECMS is gauge equivalent to the ER. Both systems are particular cases of the Hitchin construction [8], [16], [19]. Namely, they are derived from Higgs bundles of rank $N$ over an elliptic curve. The corresponding bundle for the ECMS has degree zero, while the bundle corresponding to the ER has degree one. The gauge equivalence is determined by an upper modification that transforms the trivial bundle into the bundle of degree one.

Received March 4, 2003.

Supported in part by the Grant NSh-1999-2003.2 of the program “Leading scientific schools”, by the RFBR Grant 03-02-17554, and by the CRDF Grant RM1-2545.
We analyse these two systems in a similar way, trying to reveal structures already known for the open finite-dimensional Toda chain. Namely, for the open finite-dimensional Toda chain, there exists an infinite chain limit \([27]\), a reduction of the infinite chain to a periodic chain, and a dispersionless version of the infinite chain \([10], [25]\).

Here we consider the special limit of \(N \to \infty\) that corresponds to the passage from GL\((N, \mathbb{C})\) to the infinite-dimensional group of the non-commutative torus (NCT). First, we consider the ER \(^1\). This system is an example of the integrable Euler–Arnold tops (EAT) on the group SL\((N, \mathbb{C})\). The EAT are Hamiltonian systems defined on coadjoint orbits of groups \([1]\). Particular examples of such systems are the Euler top on SO\((3)\), its integrable SO\((N)\) generalisation \([3], [15]\), and the hydrodynamics of the ideal incompressible fluid on a space \(M\). The corresponding group of the last system is SDiff\((M)\). We consider the case when \(M\) is the torus \(T^2\). The EAT are completely determined by their Hamiltonians, since the Poisson structure is related to the Kirillov–Kostant form on the coadjoint orbits. The Hamiltonians are determined by the inertia-tensor operator \(J\) mapping the Lie algebra \(g\) to the Lie coalgebra \(g^*\). Special choices of \(J\) lead to completely integrable systems (see the survey \([4]\)). In the case of 2D hydrodynamics, \(J\) has the form of the Laplace operator, and it turns out that the theory is non-integrable \([32]\). One of the subjects of this paper are integrable EAT related to SDiff. Some integrable models related to SDiff were considered in \([25], [6], [23]\).

Integrable models on SDiff\((M)\) can be described as the classical (dispersionless) limit of integrable models when the commutators in the Lax equations are replaced by Poisson brackets. This approach was proposed in \([13], [30]\) and later developed in numerous publications (see, e.g., the survey \([26]\)).

Here we use the same strategy, namely, define an integrable system on the non-commutative torus (NCT) and then take the SDiff\((T^2)\) classical limit. In analogy with the ER on GL\((N, \mathbb{C})\), we consider a special \(N \to \infty\) limit of GL\((N, \mathbb{C})\), which leads to the group \(G_\theta\) of the NCT, where \(\theta\) is the non-commutative parameter \(^2\). The Hamiltonian is defined by inertia-tensor operators depending on the module \(\tau\), \(\text{Im} \tau > 0\), of an elliptic curve. This curve is the basic spectral curve in the Hitchin description of the model. The group \(G_\theta\) is defined as the set of invertible elements of the NCT algebra \(A_\theta\). It can be embedded in GL\((\infty)\), and \(G_\theta\) can be described as a special limit of GL\((N, \mathbb{C})\). We define a family of EAT on \(G_\theta\) parametrised by \(\tau\). Then, we construct the Lax operator with spectral parameter on an elliptic curve with the same parameter \(\tau\).

In the classical limit of \(\theta \to 0\), \(G_\theta \to \text{SDiff}(T^2)\) and the inertia-tensor operator \(J\) takes the form \(\partial^2\). The conservation laws survive in this limit, while the commutators in the Lax hierarchy become Poisson brackets. It turns out that the classical limit is essentially the same as the rational limit of the basic elliptic curve, so the product of the Planck constant \(\theta\) and the half periods of the basic curve are constant.

\(^1\)This part is an extended version of the talk \([20]\).

\(^2\)For Manakov’s top \(\lim N \to \infty\) was considered in \([28]\).
We also construct the ECMS system related to the NCT. In both cases, we discuss the systems that arise after the reduction to the loop algebra $\hat{L}(\text{GL}(N, \mathbb{C}))$.

2. The Lie Algebra of the Non-Commutative Torus

Here we reproduce some basic results about the NCT and the Lie algebra $\mathfrak{sin}_\theta$ related to it [5].

1. The non-commutative torus. The NCT $\mathcal{A}_\theta$ is a unital algebra with two generators $(U_1, U_2)$ that satisfy the relation

$$U_1 U_2 = e(-\theta) U_2 U_1, \quad e(\theta) = e^{2\pi i \theta}, \quad \theta \in [0, 1). \quad (2.1)$$

The elements of $\mathcal{A}_\theta$ are the double sums

$$\mathcal{A}_\theta = \bigg\{ x = \sum_{m,n \in \mathbb{Z}} a_{m,n} U_1^m U_2^n, \quad a_{m,n} \in \mathbb{C} \bigg\},$$

where $a_{m,n}$ are elements of the ring

$$\mathfrak{S} = \big\{ a_{m,n}: \sup_{m,n \in \mathbb{Z}} (1 + m^2 + n^2)^k |a_{m,n}| < \infty \text{ for all } k \in \mathbb{N} \big\} \quad (2.2)$$

of rapidly decreasing sequences on $\mathbb{Z}^2$. The trace functional $\text{tr}(x)$ on $\mathcal{A}_\theta$ is defined as

$$\text{tr}(x) = a_{00}. \quad (2.3)$$

The dual space to $\mathfrak{S}$ is

$$\mathfrak{S}' = \bigg\{ s_{k,j}: \sum_{m+n \in \mathbb{Z}} a_{m,n} s_{k,j} < \infty, \quad a_{m,n} \in \mathfrak{S} \bigg\}. \quad (2.4)$$

The associative algebra $\mathcal{A}_\theta$ can be identified with the quantisation of the commutative algebra of smooth functions on the two-dimensional torus

$$T^2 = \{ \mathbb{R}^2 / \mathbb{Z} \oplus \mathbb{Z} \} \sim \{ 0 < x \leq 1, \ 0 < y \leq 1 \}. \quad (2.5)$$

by means of the identification

$$U_1 \rightarrow e(x), \quad U_2 \rightarrow e(y), \quad (2.6)$$

where the multiplication of functions on $T^2$ is the Moyal multiplication:

$$(f \ast g)(x, y) := fg + \sum_{n=1}^{\infty} \frac{(\pi \theta)^n}{n!} \varepsilon_{r_1,s_1} \cdots \varepsilon_{r_n,s_n} (\partial_{r_1 \cdots r_n} f)(\partial_{s_1 \cdots s_n} g). \quad (2.7)$$

The trace functional (2.3) in the Moyal identification is the integral

$$\text{tr} f = \int_{\mathcal{A}_\theta} f \, dx \, dy = f_{00}. \quad (2.8)$$

Another representation of $\mathcal{A}_\theta$ is defined by the operators that act on the space of Schwartz functions on $\mathbb{R}$:

$$U_1 \rightarrow e(-2\pi \theta \partial_x), \quad U_2 \rightarrow e^{i\phi}. \quad (2.9)$$
Finally, we can identify $U_1$ and $U_2$ with matrices from $\text{GL}(\infty)$. We define $\text{GL}(\infty)$ as the associative algebra of infinite matrices $a_{jk}E_{jk}$ such that

$$\sup_{j,k \in \mathbb{Z}} |a_{jk}|^2 |j - k|^n < \infty \quad \text{for all } n \in \mathbb{N}.$$ 

Consider the following two matrices from $\text{GL}(\infty)$:

$$Q = \text{diag}(e^{(j\theta)}), \quad \Lambda = E_{j,j+1}, \quad j \in \mathbb{Z}.$$ 

We have the identification

$$U_1 \rightarrow Q, \quad U_2 \rightarrow \Lambda. \quad (2.10)$$

2. The sin-algebra. Consider the following quadratic combinations of the generators:

$$T_{m,n} = \frac{i}{2\pi \theta} e^{\left(\frac{mn}{2}\theta\right)} U_1^m U_2^n. \quad (2.11)$$

Their commutator has the form of the sin-algebra [5]

$$[T_{m,n}, T_{m',n'}] = \frac{1}{\pi \theta} \sin \pi \theta (mn' - m'n) T_{m+m', n+n'}. \quad (2.12)$$

We use $\text{sin}_\theta$ to denote the Lie algebra with generators $T_{m,n}$ over the ring $\mathcal{S}$ (2.2); its elements are

$$\psi = \sum_{m,n} \psi_{m,n} T_{m,n}, \quad \psi_{m,n} \in \mathcal{S}. \quad (2.13)$$

By $G_\theta$ we denote the group of invertible elements from $A_\theta$. The coalgebra $\text{sin}_\theta^*$ is the linear space

$$\text{sin}_\theta^* = \left\{ S = \sum_{j,k} s_{jk} T_{jk}, \ s_{jk} \in \mathcal{S'} \right\}. $$

In the Moyal representation (2.7), the commutator of $\text{sin}_\theta$ has the form

$$[f(x, y), g(x, y)] = \{f, g\}^* := \frac{1}{\theta} (f * g - g * f). \quad (2.14)$$

The algebra $\text{sin}_\theta$ has a central extension $\widehat{\text{sin}}_\theta$. The corresponding additional term in (2.12) has the form of the star-bracket

$$(am + bn) \delta_{m,-m'} \delta_{n,-n'}, \quad a, b \in \mathbb{C}. \quad (2.15)$$

In other words, the commutator in $\widehat{\text{sin}}_\theta$ takes the form

$$[f, g] = \{f, g\}^* + \frac{1}{4\pi^2} \int_{A_\theta} f(a \partial_x g + b \partial_y g). $$
3. The loop algebra \( \hat{\mathcal{L}}(\mathfrak{sl}(N, \mathbb{C})) \). Let \( \theta \) be a rational number \( \theta = p/N \), where \( p, N \in \mathbb{N} \) are coprime. In this case, \( \mathcal{A}_\theta \) has the ideal

\[
I_N = \left\{ \sum_{m,n} c_{m,n}(l) \left( U_1^m U_2^n - U_1^{m+N} U_2^n \right) = 0, \quad l = \frac{1}{N} \right\}.
\]

The factor algebra \( \mathcal{A}_\theta / I_N \) can be represented via embedding in \( \text{GL}(\infty) \). Let us represent an arbitrary element of \( \text{GL}(\infty) \) as

\[
\psi_{m,n} e \left( \frac{mn}{N} \right) U_1^m U_2^n.
\]

In the factor algebra, we have

\[
\psi_{N+k,n} = \psi_{k,n}.
\]

Thus, any element from \( \mathcal{A}_\theta / I_N \) takes the form

\[
\sum_{l \in \mathbb{Z}} a_{j,r} E_{j,j+Nl} + r t^{Nl+r}, \quad j = \frac{1}{N}, \quad r = -N + 1, N - 1,
\]

where \( a_{j,r} = \sum_{k=1}^N \psi_{k,Nl+r} e \left( \frac{rk}{N} \right) \). To this element we assign the current from \( \mathcal{L}(\mathfrak{sl}(N, \mathbb{C})) \) defined by

\[
g(t) = \sum_{l \in \mathbb{Z}} g_{j,r}^{(l)} E_{j,j+Nl+r} t^{Nl+r}.
\]

We apply the gauge transform by \( \text{diag}(1, t, \ldots, t^{N-1}) \) to kill the factor \( t^r \) and replace \( t^N \) by \( w \). We obtain a loop algebra with the principle gradation

\[
g(w) = \sum_{l \in \mathbb{Z}} g^{(l)} w^l.
\]

The central extension

\[
\oint \text{Tr}(g_1(w) \partial_w g_2(w) \frac{dw}{w})
\]

is proportional to the cocycle \( (2.15) \) for \( a = 0 \) and \( b = 1 \). Here \( \text{Tr} \) is the trace in the fundamental representation of \( \mathfrak{sl}(N, \mathbb{C}) \).

3. 2D Hydrodynamics on \( \mathcal{A}_\theta \)

1. 2D hydrodynamics. Let \( \mathbf{v} = (V_x, V_y) \) be the velocity of the ideal incompressible fluid on a compact manifold \( M \) (\( \dim M = 2 \), \( \text{div} \mathbf{v} = 0 \)), and let \( \text{curl} \mathbf{v} = \partial_y V_x - \partial_x V_y \) be its vorticity \(^3\)

The Euler equation for 2D hydrodynamics takes the form \([1]\)

\[
\partial_t \text{curl} \mathbf{v} = \text{curl}[\mathbf{v}, \text{curl} \mathbf{v}].
\]

We define the stream function \( \psi(x, y) \) as the Hamiltonian function generating the vector field \( \mathbf{v} \), i.e., such that

\[
i_\mathbf{v} \, dx \wedge dy = d\psi.
\]

In other words,

\[
V_x = \partial_y \psi, \quad V_y = -\partial_x \psi.
\]

Let \( \mathfrak{g} \) be the Poisson algebra of the stream functions \( \mathfrak{g} = \{ \psi \} \) on \( M \) defined as

\[
\mathfrak{g} \sim C^\infty(M) / \mathbb{C} \quad \text{with} \quad \{ \psi_{v_1}, \psi_{v_2} \} = -i_{v_1} d\psi_{v_2}.
\]

\(^3\)For simplicity, we assume that the measure on \( M \) is \( dx \wedge dy \), though all expressions can be written in a covariant way.
Consider the Lie algebra SVect\(M\) of vector fields with \(\text{div} \mathbf{v} = 0\) on \(M\). We have the following interrelation between the Lie algebras \(\mathfrak{g}\) and SVect(\(M\)):

\[
\{\psi_{\mathbf{v}_1}, \psi_{\mathbf{v}_2}\} = \psi_{[\mathbf{v}_1, \mathbf{v}_2]} + c(\mathbf{v}_1, \mathbf{v}_2),
\]

where \(c(\mathbf{v}_1, \mathbf{v}_2)\) is a central extension of SVect(\(M\)). In other words, we have the exact sequence

\[
0 \rightarrow \mathfrak{g} \rightarrow \text{SVect}(M) \rightarrow H^1(M) \rightarrow 0,
\]

where the map \(\text{SVect}(M) \rightarrow H^1(M)\) is generated by the two fluxes \((c_1 \partial_1, c_2 \partial_2)\).

Let \(\mathfrak{g}^*\) be the dual space of distributions on \(M\). The vorticity \(S = \text{curl} \mathbf{v}\) of the vector field \(\mathbf{v}\), which is

\[
S = -\Delta \psi,
\]

can be considered as an element from \(\mathfrak{g}^*\). The Euler equation (3.1) in terms of the Poisson bracket has the form

\[
\partial_t S = \{S, \psi\}, \quad \text{or} \quad \partial_t S = \{S, \Delta^{-1} S\}. \tag{3.3}
\]

We can view (3.3) as the Euler–Arnold equation for the rigid top related to the Lie algebra \(\mathfrak{g}\), where the Laplace operator is the map

\[
\Delta: \mathfrak{g} \rightarrow \mathfrak{g}^*
\]

which plays the role of the inertia-tensor. The phase space of the system is a coadjoint orbit of the group of the canonical transformations SDiff(\(M\)). The equation (3.3) takes the form

\[
\partial_t S = \text{ad}_{\nabla H}^* S, \tag{3.4}
\]

where \(\nabla H = \frac{\partial H}{\partial S} = \psi\) is the variation of the Hamiltonian

\[
H = -\frac{1}{2} \int_M S \Delta^{-1} S = \int_M \psi \Delta \psi. \tag{3.5}
\]

There is an infinite set of Casimirs defining the coadjoint orbits:

\[
C_k = \int_M S^k. \tag{3.6}
\]

Consider the particular case when \(M\) is the two-dimensional torus (2.5) equipped with the measure \(-\frac{dx dy}{4\pi^2}\). In terms of the Fourier modes \(s_{m,n}\) of the vorticity

\[
S = \sum_{m,n} s_{m,n} e(-mx - ny),
\]

the Hamiltonian (3.5) is

\[
H = -\frac{1}{2} \sum_{m,n} \frac{1}{m^2 + n^2} s_{m,n} s_{-m,-n}, \tag{3.7}
\]

and we come to the equation

\[
\partial_t s_{m,n} = \sum_{j,k} \frac{j n - k m}{j^2 + k^2} s_{j k} s_{m-j,n-k}. \tag{3.8}
\]
2. **2D hydrodynamics on non-commutative torus.** We can consider the similar construction where the Poisson bracket is replaced by the Moyal bracket (2.14) [31], [9]. We introduce the vorticity \( S \) as the element of \( \sin^\theta \) defined by

\[
S = \sum_{m,n} s_{m,n} T_{-m,-n}.
\]

The equation (3.3) takes the form

\[
\partial_t S(x, y) = \{ S(x, y), \Delta^{-1} S(x, y) \}^*,
\]

or, for the Fourier modes,

\[
\partial_t s_{m,n} = \frac{1}{8\pi^3 \theta} \sum_{j,k} \sin(\pi \theta (jn-km)) \frac{j^2 + k^2}{s_jk s_{m-j,n-k}}.
\]

(3.10)

This system is the EAT on the group \( G_\theta \) of invertible elements of \( \mathcal{A}_\theta \), and the coadjoint orbits are defined by the same Casimirs (3.6) as for SDiff(\( T^2 \)). In the limit of \( \theta \to 0 \), (3.10) reproduces (3.8).

4. The \( \text{SL}(N, \mathbb{C}) \)-Elliptic Rotator

1. **The elliptic rotator (ER) on \( \text{SL}(N, \mathbb{C}) \).** In this section, we consider differential equations related to \( \text{SL}(N, \mathbb{C}) \) that are not a priori related to hydrodynamics. The elliptic \( \text{SL}(N, \mathbb{C}) \)-rotator is an example of the EAT [1]. It is defined on \( \mathfrak{sl}(N, \mathbb{C})^* \), and its phase space is a coadjoint orbit of \( \text{SL}(N, \mathbb{C}) \):

\[
R_{\text{rot}} = \{ S \in \mathfrak{sl}(N, \mathbb{C}), \ S = g^{-1} S^{(0)} g \}.
\]

(4.1)

The phase space \( R_{\text{rot}} \) is equipped with the Kirillov–Kostant symplectic form

\[
\omega_{\text{rot}} = \text{tr}(S^{(0)} Dgg^{-1} \wedge Dgg^{-1}).
\]

(4.2)

The Hamiltonian has the form

\[
H_{\text{rot}} = -\frac{1}{2} \text{Tr}(SJ(S)),
\]

(4.3)

where \( J \) is a linear operator on \( \mathfrak{sl}(N, \mathbb{C}) \). The inverse operator is called the inertia tensor. The equation of motion takes the form

\[
\partial_t S = [S, J(S)].
\]

(4.4)

The special form of \( J \) ensures the integrability of the system [22], [14]. Let us represent \( S \) as

\[
S = -\frac{4\pi^2}{N^3} \sum_{m,n} S_{m,n} T_{-m,-n},
\]

where \( T_{m,n} \) form a basis of \( \mathfrak{sl}(N, \mathbb{C}) \) similar to the basis of the sin-algebra (2.11); namely,

\[
T_{m,n} = \frac{iN}{2\pi} e \left( \frac{mn}{N} \right) Q^m_N A^n_N,
\]

\((m, n) \in \tilde{\mathbb{Z}}^{(2)}_N = \{(\mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}) \setminus (0, 0)\}.
\]
Here $\Lambda_N$ and $Q_N$ are defined by (A.7). Let

$$J(S) = \sum_{m,n \in \mathbb{Z}} J_{m,n} S_{m,n} T_{-m,-n},$$

where

$$J = \{J_{m,n}\} = \{\varphi\left[\begin{matrix} m \\ n \end{matrix}\right]\}, \quad \varphi\left[\begin{matrix} m \\ n \end{matrix}\right] = \varphi\left(\frac{m + n\tau}{N}; \tau\right). \quad (4.5)$$

Then (4.4) takes the form

$$\partial_t S_{m,n} = \sum_{k,l \in \mathbb{Z}} S_{k,l} S_{m-k,n-l} \varphi\left[\begin{matrix} k \\ l \end{matrix}\right] \sin\left(\frac{\pi M}{N}(ml - kn)\right). \quad (4.6)$$

It was observed in [16] that the ER is the Hitchin system corresponding to the vector bundle $E$ of rank $N$ and degree 1 over the elliptic curve $E_{\tau}$ with the marked point $z = 0$. To prove this fact, we first demonstrate that (4.6) is equivalent to the Lax equation

$$\partial_t L^{\text{rot}}(z) = [L^{\text{rot}}(z), M^{\text{rot}}(z)];$$

then we show that $L^{\text{rot}}(z)$ is the Higgs field in corresponding bundle (see Appendix A). The Lax matrices in the basis $T_{m,n}$ of $\mathfrak{gl}(N, \mathbb{C})$ are represented as

$$L^{\text{rot}} = \sum_{m,n \in \mathbb{Z}} S_{m,n} \varphi\left[\begin{matrix} m \\ n \end{matrix}\right](z) T_{m,n}, \quad \varphi\left[\begin{matrix} m \\ n \end{matrix}\right](z) = e\left(\frac{nz}{N}\right) \phi\left(\frac{m + n\tau}{N}; z\right), \quad (4.7)$$

$$M^{\text{rot}} = -\sum_{m,n \in \mathbb{Z}} S_{m,n} f\left[\begin{matrix} m \\ n \end{matrix}\right](z) T_{m,n}, \quad f\left[\begin{matrix} m \\ n \end{matrix}\right](z) = e\left(\frac{nz}{N}\right) \partial_u \phi\left(u; z\right)|_{u = \frac{m + n\tau}{N}}. \quad (4.8)$$

They lead to the Lax equation

$$\partial_t S_{m,n} \varphi\left[\begin{matrix} m \\ n \end{matrix}\right](z) = \sum_{k,l \in \mathbb{Z}} S_{m-k,n-l} S_{k,l} \varphi\left[\begin{matrix} m-k \\ n-l \end{matrix}\right](z) f\left[\begin{matrix} k \\ l \end{matrix}\right](z) \sin\left(\frac{\pi M}{N}(nk - ml)\right).$$

for the matrix elements. Using the Calogero functional equation (B.14), we rewrite it in the form (4.6).

The phase space $R^{\text{rot}}(4.1)$ is the result of the Hamiltonian reduction of the GL$_N$ Higgs bundle of degree 1. In this case, there are no moduli degrees of freedom except the Jacobian of the determinant bundle (A.4). In fact, the determinant bundle coincides with the theta-bundle and, therefore, has degree 1. This implies that the gauge fixing is complete, and the reduced phase space is the orbit $\mathcal{O}$ (see (A.9)). In the symplectic form (A.10), only the last term, which coincides with (4.2), survives. For a generic orbit, $\text{dim } R^{(1)} = N(N - 1)$. The transition functions can be chosen in the form (A.6). The transition functions (A.6) allow us to define a Lax operator depending only on the orbit variables. It can be checked directly that the Lax operator (4.7) is a meromorphic $\mathfrak{gl}(N, \mathbb{C})$-valued 1-form on $E_{\tau}$ such that

$$\text{Res } L^{\text{rot}}|_{z=0} = S,$$

and it satisfies the quasi-periodicity conditions with the transition functions (A.7):

$$L^{\text{rot}}(z) - Q_N L^{\text{rot}}(z + 1) Q_N^{-1} = 0, \quad L^{\text{rot}}(z) - \Lambda_N L^{\text{rot}}(z + \tau) \Lambda_N^{-1} = 0.$$
It follows from the general prescription that we have \( \frac{N(N-1)}{2} \) independent integrals of motion \((A.20)\). In particular,

\[
\frac{1}{2} \text{tr}(L_{\text{rot}}(z))^2 = - \frac{i\pi}{N} H_{\text{rot}} + \text{tr} S^2 \nu(z).
\]

The equations of motion corresponding to the higher integrals have the Lax form \((A.19)\). The properties of \( M_{s,j}(z) \) can be determined from the equation of motion \((A.17)\) restricted to \( R_{\text{red}} \):

\[
M_{s,j}(z) - Q_N M_{s,j}(z + 1) Q_N^{-1} = 0.
\]

For \( s = 0 \), we have

\[
M_{0,j}(z) - \Lambda_N M_{0,j}(z + \tau) \Lambda_N^{-1} = 2\pi i (L^j)^{-1}(z).
\]

If \( s \neq 0 \), then \( M_{s,j}(z) \) is quasi-periodic, i.e.,

\[
M_{s,j}(z) - \Lambda_N M_{s,j}(z + \tau) \Lambda_N^{-1} = 0,
\]

and its singular part is defined by the singular part of \( L_N^{j-1} z^s \), i.e.,

\[
(M_{s,j}(z))_- = (L_N^{j-1} z^s)_-.
\]

5. THE ELLIPTIC ROTATOR ON \( A_\theta \)

1. Description of the system. It follows from \((2.10)\) that the non-commutative torus \( A_\theta \) corresponds to a special \( N \to \infty \) limit of \( \text{SL}(N, \mathbb{C}) \). Consider the ER related to the NCT group \( G_\theta \); we assume that \( \theta \) is a irrational number. We replace the inverse inertia-tensor \( \Delta^{-1} \) of hydrodynamics by the operator \( J : S \to \psi \) acting diagonally on the Fourier coefficients \((3.9)\):

\[
J : s_{m,n} \to \phi \begin{bmatrix} m \\ n \end{bmatrix} \cdot s_{m,n} = \psi_{m,n}, \quad s_{00} = 0, \quad \phi \begin{bmatrix} m \\ n \end{bmatrix} = \phi((m + n\tau)\theta; \tau). \quad (5.1)
\]

Consider the EAT on the group \( G_\theta \) with the inertia-tensor defined by the inverse \( J^{-1} \) to \((5.1)\). The corresponding coadjoint orbit is

\[
\mathcal{O}_{S_0} = \{ S \in \sin_\theta^* ; S = h^{-1} S_0 h, \ h \in G_\theta, \ S_0 \in \sin_\theta^* \} \quad (5.2)
\]

equipped with the Kirillov–Kostant symplectic form

\[
\omega_\theta = \int_{A_\theta} S^0 Dhh^{-1} \wedge Dhh^{-1}.
\]

The Poisson structure on the coalgebra \( \sin_\theta^* \) is defined by the Moyal bracket

\[
\{ S, S' \} = \{ S, S' \}^*.
\]

Let

\[
S = -4\pi^2 \theta^3 \sum_{m,n \in \mathbb{Z}} s_{m,n} T_{-m,-n} \in \sin_\theta^*, \quad s_{00} = 0,
\]

and

\[
J(S) = \sum_{m,n \in \mathbb{Z}} s_{-m,-n} \phi \begin{bmatrix} m \\ n \end{bmatrix} T_{m,n} \in \sin_\theta.
\]
The Hamiltonian is determined by an integral over \( A_\theta \) (2.8) as
\[
H_\theta = -\frac{1}{2} \int_{A_\theta} S J(S) = -\frac{1}{2} \sum_{m,n \in \mathbb{Z}} \varphi \left[ \frac{m}{n} \right] s_{m,n} s_{-m,-n}.
\]
(5.3)

Below, we define the phase under the assumption that the Hamiltonian \( H_\theta \) is finite. The equation of motion has the standard form of the Moyal bracket, which is the commutator in \( \text{GL}(\infty) \):
\[
\partial_t S = \{ S, J(S) \}^* = [ S, J(S) ].
\]
(5.4)

In terms of the Fourier components, it takes the form
\[
\partial_t s_{m,n} = \frac{1}{\pi \theta} \sum_{j,k \in \mathbb{Z}} s_{jk} s_{m-j,n-k} \times \varphi \left[ \frac{j}{k} \right] \sin(\pi \theta(jn - km)).
\]
(5.5)

2. Integrability of the elliptic rotator on \( A_\theta \). We shall prove that the Hamiltonian system of ER (5.4), (5.5) has an infinite set of involutive integrals of motion in addition to the Casimirs (3.6). This will follow from the Lax form
\[
\partial_t L_\theta = [ L_\theta, M_\theta ]
\]
(5.6)
of the equations (5.4) and (5.5). The Lax operators are similar to the corresponding Lax matrices for the elliptic rotator (4.7), (4.8), which are
\[
L_\theta = \sum_{m,n \in \mathbb{Z}} s_{m,n} \varphi \left[ \frac{m}{n} \right] (z) T_{m,n}, \quad M_\theta = -\sum_{m,n \in \mathbb{Z}} s_{m,n} f \left[ \frac{m}{n} \right] (z) T_{m,n},
\]
(5.7)

where
\[
\varphi \left[ \frac{m}{n} \right] (z) = e^{n \theta z} \phi((m + n \tau) \theta, z),
\]
(5.8)
\[
f \left[ \frac{m}{n} \right] (z) = e^{n \theta z} \partial_u \phi(u, z) \big|_{u=(m+n\tau)\theta},
\]
(5.9)

and \( \phi(u, z) \) is defined by (B.8). The equivalence of (5.6) and (5.5) follows from the Calogero functional equation (B.14).

Consider the holomorphic vector bundle \( E \) of infinite rank over \( E_\tau \) with structure group \( G_\theta \). We assume that it is similar to the \( \text{GL}(N, \mathbb{C}) \) bundle of degree one (see Appendix A), where \( \text{GL}(N, \mathbb{C}) \) is replaced by \( G_\theta \). This means that the transition functions \( g_\alpha, \alpha = 1, 2, \) have the form
\[
g_1 = Q, \quad g_2 = \Lambda = e \left( -\frac{1}{2} \tau + z \right) \theta \Lambda.
\]
The Higgs bundle is \((T^* E, \mathcal{O}_{E^\theta})\), where the coadjoint orbit \( \mathcal{O}_{E^\theta} \) is defined by (5.2). The cotangent bundle \( T^* E \) is described by the Higgs field \( \Phi = f^{-1}(z)L_\theta f(z) \). The Lax operator \( L_\theta \) satisfies the moment constraint equation
\[
\partial L_\theta = 0, \quad \text{Res} L_\theta |_{z=0} = S,
\]
\[
L_\theta(z + 1) = Q L_\theta(z) Q^{-1}, \quad L_\theta(z + \tau) = \Lambda L_\theta(z) \Lambda^{-1}.
\]
(5.10)
The reduced phase space is described by solutions to (5.10), (5.11) such that

$$I_{s,j} = \int_{E_r} \int_{A_0} (\theta)^j \mu_{s,j} < \infty \quad (s \leq j, \ j \in \mathbb{N}),$$

and $\mu_{s,j}$ are defined by (A.14) and (B.22). The integrals $I_{s,j}^\alpha$ can be extracted from the expansion over the basis of the elliptic functions (B.20)

$$\int_{A_0} (\theta)^j (z) = I_{0,j} + \sum_{r=2}^j I_{r,j} \wp^{(r-2)} (z) \quad (j = 2, \ldots).$$

In particular,

$$\int_{A_0} (\theta)^2 (z) = I_{0,2} + \wp (z) \int_{A_0} S^2, \quad I_{0,2} = 2\pi^2 \theta^2 H\theta.$$

Note that

$$I_{j,j} \sim C_j = \int_{A_0} S^j$$

are the Casimirs (3.6).

Consider, for example, the integrals that have the third order in the field $S$. It follows from (B.18) that, in terms of the Fourier modes $S = \{s_{m,n}\}$, the integrals take the form

$$I_{2,3} = \sum_{m_j = \sum n_j = 0}^3 \prod_{j=1}^3 s_{m_j,n_j} \left( \zeta \left[ \begin{array}{c} m_1 \\ n_1 \end{array} \right] + \zeta \left[ \begin{array}{c} m_2 \\ n_2 \end{array} \right] + \zeta \left[ \begin{array}{c} m_3 \\ n_3 \end{array} \right] \right),$$

$$I_{0,3} = \sum_{m_j = \sum n_j = 0}^3 \prod_{j=1}^3 s_{m_j,n_j} \left( -\frac{1}{2} \wp \left[ \begin{array}{c} m_3 \\ n_3 \end{array} \right] - \wp \left[ \begin{array}{c} m_2 \\ n_2 \end{array} \right] + \zeta \left[ \begin{array}{c} m_3 \\ n_3 \end{array} \right] \right).$$

The functionals (5.12) play the role of Hamiltonians for the infinite hierarchy on the phase space $O_{S^0}$ (5.2), namely,

$$\partial_{s,j} S = \{\nabla I_{s,j}, S\}^* \quad \left( \partial_{s,j} = \partial_{s,j}, \ \nabla I_{s,j}^\alpha = \frac{\delta I_{s,j}^\alpha}{\delta S} \right).$$

This contains (5.4) and (5.5) for $I_{0,2}$.

3. The classical limit. In the classical limit, $S$ becomes a function on $T^2$:

$$S = \sum_{m,n \in \mathbb{Z}} s_{m,n} \exp(2\pi i(-mx - ny)).$$

In the case under consideration, the classical limit is essentially the same as the rational limit of the basic spectral curve $E_r$. Replace for a moment the half periods $(\frac{1}{2}, \frac{1}{2})$ of $E_r$ by $\omega_1, \omega_2$. The rational limit is that of $\omega_1, \omega_2 \to \infty$. The Weierstrass function degenerates as

$$\wp(u) \to \frac{1}{u^2}.$$
Consider the double limit as $\theta \to 0$ and $\omega_1, \omega_2 \to \infty$ such that $\lim \omega_1 \theta = 1$ and $\lim \omega_2 \theta = \tau$. We have

$$\varphi \left[ \frac{m}{n} \right] \to \frac{1}{(m + n\tau)^2}. $$

The quadratic Hamiltonian in the double limit takes the form

$$H = -\frac{1}{2} \int_{\mathcal{A}_\theta} \psi(\bar{\partial})^2 \psi = -\frac{1}{2} \sum_{m,n \in \mathbb{Z}} \frac{s_{m,n}s_{m,n}}{(m + n\tau)^2}, \quad (5.16)$$

where $\bar{\partial} = \frac{1}{2\pi i} (\partial_x + \tau \partial_y)$. The operator $\bar{\partial}^2$ plays the role of the inertia-tensor. It replaces the Laplace operator $\Delta \sim \partial \bar{\partial}$ of the standard hydrodynamics. We call this system the modified hydrodynamics.

In the classical limit, the Lax equation assumes the form

$$\partial_t L(x_1, x_2; z) = \{L(x_1, x_2; z), M(x_1, x_2)\},$$

where

$$L(x, y; z) = \bar{\partial}^{-1} S(x, y) + \frac{1}{z} S(x, y) \quad (5.17)$$

and

$$M(x, y) = -\bar{\partial}^{-2} S(x, y). \quad (5.18)$$

The integrals of motion (5.12) survive in this limit. We have already specified the form of the Hamiltonian $H$ (5.16). The third order integrals take the forms

$$I_{2,3} = \sum_{\sum m_j = \sum n_j = 0} \prod_{j=1}^{3} s_{m_j,n_j} \sum_{j} \frac{1}{m_j + n_j \tau},$$

$$I_{0,3} = \sum_{\sum m_j = \sum n_j = 0} \prod_{j=1}^{3} s_{m_j,n_j} \left( -\frac{1}{(m_3 + n_3 \tau)^3} \right) \left( \frac{1}{m_1 + n_1 \tau} + \frac{1}{m_2 + n_2 \tau} + \frac{1}{m_3 + n_3 \tau} \right).$$

4. Reduction to the loop algebra. Let $\theta$ be the rational number $\theta = p/N$. As was explained in Section 2.3, we can pass to the factoralgebra $L(gl(N, \mathbb{C}))$ or its central extension $\hat{L}(gl(N, \mathbb{C}))$. In the former case, we have a family of non-interacting ER parametrised by $w \in S^1$. If the central charge is nonzero, the situation changes drastically [14], [11]. The Lax operator is no longer a 1-form but a connection on $S^1$:

$$\partial_w + L(z, w) \quad (w \in S^1). \quad (5.19)$$

The integrals of motion can be calculated from the expansion of the trace of the monodromy matrix for the linear system

$$(\partial_w + L(z, w))\Psi(z, w) = 0.$$
They define the hierarchy of the ER on the coadjoint orbits of $\hat{\mathcal{L}}(\text{GL}(N, \mathbb{C}))$. For $N = 2$, this top is the Landau–Lifshitz equation

$$
\partial_t S = \frac{1}{2}[S, J(S)] + \frac{1}{2}[S, \partial_{ww}S].
$$

Here $S \in L^*(\mathfrak{sl}(2, \mathbb{C})).$

6. The Elliptic Calogero–Moser System on $A_\theta$

1. The SL($N, \mathbb{C}$)-Elliptic Calogero–Moser system. The SL($N, \mathbb{C}$)-elliptic Calogero–Moser system (CM$_N$) system was first introduced in the quantum setting [2] and then in the classical setting [17]. We consider its generalisation, the CM$_N$ system with spin. The elliptic CM$_N$ corresponds to the trivial Higgs bundle over the elliptic curve $E_\tau$ [7]. Its phase space is

$$
\mathcal{R}^{CMN} = \{C^{2N}, \hat{\mathcal{O}}\},
$$

where $\hat{\mathcal{O}} = \mathcal{O}/D$ is the symplectic quotient of the coadjoint orbit of SL($N, \mathbb{C}$)

$$
\mathcal{O} = \{p \in \mathfrak{sl}(N, \mathbb{C}) : p = h^{-1}p^0h, \ h \in \text{SL}(N, \mathbb{C}), \ p^0 \in D\}
$$

with respect to the action of the diagonal subgroup $D$ of SL($N, \mathbb{C}$). The moment constraint imply that the diagonal matrix elements of the orbit vanish, i.e., $p_{jj} = 0$.

The Poisson structure has the form

$$
\{v_j, u_k\} = \delta_{j,k}, \quad \{p_{k,l}, p_{j,n}\} = \delta_{j,l}p_{k,n} - \delta_{n,k}p_{j,l},
$$

where $v = (v_1, \ldots, v_N)$ and $u = (u_1, \ldots, u_N)$ are canonical coordinates on $C^{2N}$.

The Hamiltonian, which has the second order with respect to the momenta $v$, has the form

$$
H^{CMN}_2 = \frac{1}{2} \sum_{j=1}^{N} v_j^2 + \sum_{j>k} p_{jk}p_{kj}\phi(u_j - u_k; \tau).
$$

It describes the interaction of $N$ particles with complex coordinates $u_1, \ldots, u_N$ on the elliptic curve $E_\tau$ (B.1). The pair-wise potential is defined by the Weierstrass function. The spin degrees of freedom $p_{jk}$ look similar to the EAT with the inertia-tensor determined by $\phi(u_j - u_k; \tau)$, but the corresponding phase subspace is the symplectic quotient $\mathcal{O}/D$, as opposed to the standard EAT.

The equation of motion with respect to $H^{CMN}_2$ has the Lax form $\partial_t L^{CMN} = [L^{CMN}, M^{CMN}]$ with

$$
L^{CMN} = P + X, \quad \text{where} \quad P = \text{diag}(v_1, \ldots, v_N), \ X_{jk} = p_{jk}\phi(u_j - u_k, z) \quad (6.5)
$$

($\phi$ is defined by (B.8)), and

$$
M^{CMN} = -D + Y, \quad \text{where} \quad D = \text{diag}(Z_1, \ldots, Z_N), \ Y_{jk} = y(u_j - u_k, z), \quad (6.6)
$$

$$
Z_j = \sum_{k \neq j} \phi(u_j - u_k), \quad y(u, z) = \frac{\partial\phi(u, z)}{\partial u}.
$$
The equivalence of the Lax equation and the equations of motion is based again on (B.14) and (B.12).

We use relations from Appendix A to derive the elliptic CM\(_N\) system and its Lax representation from the Hitchin construction [7]. If \(d = \text{degree}(E_{\mathcal{N}}) = 0\), then the transition functions \(g_\alpha\) can be gauge transformed to the constant matrices (A.5).

The Lax operator is a meromorphic matrix-valued 1-form. Its quasi-periodicity properties are defined by the transition functions (A.5)

\[
L_{\text{CM}_N}(z + 1) = L_{\text{CM}_N}(z), \quad L_{\text{CM}_N}(z + \tau) = \exp(\vec{u})L_{\text{CM}_N}(z)\exp(-\vec{u}).
\]

It has a simple pole at \(z = 0\) such that

\[
\text{Res}_{z=0}(L_{\text{CM}_N}(z)) = L_{\text{CM}_N}^{-1} = p \in \tilde{\mathcal{O}}.
\] (6.7)

The integrals of motion \(I_{s,j}\) (A.20) produce the CM\(_N\) hierarchy

\[
\partial_{s,j}L_{\text{CM}_N} = [L_{\text{CM}_N}, M_{s,j}^{\text{CM}_N}].
\] (6.8)

The following properties of \(M_{s,j}^{\text{CM}_N}\) can be extracted from the equations of motion (A.17):

\[
M_{s,j}^{\text{CM}_N}(z + 1) = M_{s,j}^{\text{CM}_N}(z),
\]

\[
M_{0,j}^{\text{CM}_N}(z) - \exp(\vec{u})M_{0,j}^{\text{CM}_N}(z + \tau)\exp(-\vec{u}) = 2\pi i(L_{\text{CM}_N})^j_{-1} - \partial_{0,j}\vec{u}.
\] (6.9)

For \(s \neq 0\), we have

\[
M_{s,j}^{\text{CM}_N}(z) - \exp(\vec{u})M_{s,j}^{\text{CM}_N}(z + \tau)\exp(-\vec{u}) = -\partial_{s,j}\vec{u},
\]

and the singular part of \(M_{s,j}^{\text{CM}_N}(z)\) has the form

\[
(M_{s,j}^{\text{CM}_N}(z))_- = (L_{\text{CM}_N}(z))^{j-1}z^s_-.
\]

In particular, \(I_{0,2} = H_2^{\text{CM}_N}\) and \(M_{0,2}^{\text{CM}_N} = M_{\text{CM}_N}^{6.6}\).

Let \(f(z)\) be the gauge transformation that diagonalises \(g_2\). It is defined up to the conjugation by a constant diagonal matrix. This remnant gauge freedom is responsible for the symplectic reduction of the orbit \(\tilde{\mathcal{O}} = \mathcal{O}/D\).

2. Equilibrium configuration. In this section, we prove that the following configuration of particles and spins is an equilibrium set with respect to the Hamiltonian \(H_2^{\text{CM}_N}\) (6.4). Consider \(N = n^2\) particles and orbit variables enumerated by the pair of integer numbers \(a, b = 1, \ldots, n:\)

\[
p_{a,b,c,d} = \nu, \quad v_{a,b} = 0, \quad u_{a,b} = \frac{a + b\tau}{N}, \quad a, b = 1, \ldots, n.
\] (6.10)

The identity

\[
\varphi(Nz|\tau) = \frac{1}{N^2} \left[ \varphi(z|\tau) + \sum_{j=1}^{N} \left( \sum_{k=1}^{N-1} \varphi \left( z + \frac{j + k\tau}{N} | \tau \right) \right) + \varphi \left( z + \frac{j}{N} | \tau \right) \right]
\] (6.11)

implies

\[
\sum_{j=1}^{N} \left( \sum_{k=1}^{N-1} \varphi \left( \frac{j + k\tau}{N} | \tau \right) \right) + \varphi \left( \frac{j}{N} | \tau \right) = 0.
\] (6.12)
It follows from (B.13) and (6.11) that
\[
\wp'\left(\frac{j + k\tau}{N}\right) + \sum_{m \neq j, n \neq k} \wp'\left(\frac{j + m\tau - n\tau}{N}\right) = 0. \tag{6.13}
\]
Relation (6.13) implies that (6.10) is the equilibrium set in \( R_2^{CM} \) with respect to \( H_2^{CM} \) (6.4). Moreover, (6.12) means that the Hamiltonian (6.4) vanishes at this point:
\[
H_2^{CM} = 0. \tag{6.14}
\]
Note that the configuration (6.10) is preserved by the action of the higher integrals \( I_{s,k} \).

3. The symplectic Hecke correspondence. There exists a canonical transformation (symplectic Hecke correspondence) that defines a passage from the CM\(_N\) model related to the Higgs bundle of degree zero to the ER on GL\((N, \mathbb{C})\) related to the Higgs bundle of degree 1 [14]. This is a singular gauge transformation \( \Xi \) with kernel of a special form. The eigen-vector of the residue \( L_1^{CM}\) (6.7) is annihilated by the kernel. Thus, this gauge transform
\[
L^e = \Xi^{-1} L_2^{CM} \Xi \tag{6.15}
\]
preserves the order of the pole. The matrix \( \Xi \) has the following form. Let \( p^0 = \text{diag}(p_1, \ldots, p_N) \) be the diagonal matrix defining the coadjoint orbit (6.2) in the elliptic CM\(_N\) system. Then \( \Xi = \Xi(p_l) \) depends on the choice of the eigen-value \( p_l \).

Consider the following \( N \times N \) matrix \( \widetilde{\Xi}(z, u; \tau) \):
\[
\widetilde{\Xi}_{ij}(z, u; \tau) = \theta \left[ \frac{N}{2} - \frac{i}{2} \right] (z - Nu_j, \tau),
\]
where \( \theta_{1b}^{[a]}(z, \tau) \) is the theta function with characteristics (B.24). We have
\[
\Xi(z, u; p_l; \tau) = \widetilde{\Xi}(z) \times \text{diag} \left( \frac{(-1)^l}{p_l} \prod_{j<k, j,k \neq l} \vartheta(u_k - u_j, \tau) \right). \tag{6.16}
\]
Consider the case \( N = 2 \). The phase space has dimension two, since the orbit variables (6.2) are gauged away. Let \( \nu^2 \) be the value of the Casimir of the orbit. Then the transformation takes the form
\[
\begin{align*}
S_1 &= -v \frac{\theta_{10}(0) \theta_{10}(2u)}{\vartheta'(0) \vartheta(2u)} - \nu \frac{\theta_{01}(0)}{\vartheta(0)} \frac{\theta_{00}(2u) \theta_{01}(2u)}{\vartheta(2u)}, \\
S_2 &= -v \frac{\theta_{10}(0) \theta_{01}(2u)}{\sqrt{-1} \vartheta'(0) \vartheta(2u)} - \nu \frac{\theta_{00}(0)}{\sqrt{-1} \vartheta(0)} \frac{\theta_{10}(2u) \theta_{01}(2u)}{\vartheta(2u)}, \\
S_3 &= -v \frac{\theta_{01}(0) \theta_{10}(2u)}{\vartheta'(0) \vartheta(2u)} - \nu \frac{\theta_{00}(0)}{\vartheta(0)} \frac{\theta_{00}(2u) \theta_{10}(2u)}{\vartheta(2u)},
\end{align*} \tag{6.17}
\]
where \( \nu^2 = \frac{1}{2} (S_1^2 + S_2^2 + S_3^2) \).
4. The elliptic CM system on \( \mathcal{A}_\theta \). Consider the \( N \to \infty \) limit \( \text{CM}_\infty \) of the CM\(_N\) system corresponding to \( \mathcal{A}_\theta \). We identify the coordinates of infinitely many particles in \( E_\tau \) with the diagonal matrix

\[
\vec{u} = \text{diag}(\ldots, u_{-N}, \ldots, u_{-1}, u_0, u_1, \ldots, u_N, \ldots),
\]
in \( \text{GL}_\infty \) and let \( \vec{v} = \text{diag}(\ldots, v_{-N}, \ldots, v_{-1}, v_0, v_1, \ldots, v_N, \ldots) \) be their momenta.

The Hamiltonian of \( \text{CM}_\infty \) has the form

\[
H_{\text{CM}_\infty} = \frac{1}{2}(\vec{v}, \vec{v}) + \sum_{j < k, j, k \in \mathbb{Z}} p_{jk} p_{kj} \wp(u_j - u_k; \tau),
\]
(6.18)

where the orbit elements \( p_{jk} \) are written in terms of the generators \( E_{jk} \). Although the number of the particles on the torus \( E_\tau \) is infinite, the Hamiltonian \( H_{\text{CM}_\infty} \) remains finite around the equilibrium configuration (6.10), (6.14).

The phase space \( \mathcal{R}^{\text{CM}_\infty} \) of \( \text{CM}_\infty \) is similar to that in the finite-dimensional case (6.1):

\[
\mathcal{R}^{\text{CM}_\infty} = \{ \mathbb{C}^\infty \oplus \mathbb{C}^\infty; \tilde{\mathcal{O}}_{\infty} \}.
\]

Here \( \tilde{\mathcal{O}}_{\infty} = \mathcal{O}_{\infty}/D \) is the symplectic quotient of the coadjoint orbit modulo the Cartan subgroup \( D \subset \text{SIN}_\theta \), generated by \( T_{m, 0}, m \in \mathbb{Z} \). The coadjoint orbit \( \mathcal{O}_{\infty} \subset \text{sin}_\theta^* \) of \( \text{SIN}_\theta \subset \text{GL}_\infty \) is

\[
\mathcal{O}_{\infty} = \{ p \in \text{sin}_\theta^*; p = h^{-1} p^\theta h, \ h \in \text{SIN}_\theta \}.
\]

We assume that \( \vec{v} \in \mathbb{C}^\infty \) satisfies (6.23). There are additional constraints coming from the finiteness of the integrals (6.33) defined below.

In terms of the coordinates on \( \mathfrak{g}_\infty^* \) the Poisson bracket is given by the following formulae similar to (6.3):

\[
\{ v_j, u_k \} = \delta_{j,k},
\]
(6.19)

\[
\{ p_{j,l}, p_{j,n} \} = \delta_{j,l} p_{k,n} - \delta_{n,k} p_{j,l}.
\]
(6.20)

We express the coordinates of the particles in terms of the coordinates \( \vec{u} = \sum_{j \neq 0} \tilde{u}_j T_{j,0} \) on \( \mathcal{A}_\theta \):

\[
u_j = \frac{i}{2\pi \theta} \sum_{k \in \mathbb{Z}} \tilde{u}_k e(jk\theta), \quad \tilde{u}_k \in \mathcal{G}.
\]
(6.21)

Evidently, the \( u_j \) are represented by convergent series. Similarly,

\[
v_j = -2\pi i \theta^2 \sum_{k \in \mathbb{Z}} \tilde{v}_k e(jk\theta),
\]
(6.22)

where \( \tilde{v}_k \in \mathcal{G}' \) (2.4) and it is assumed that

\[
\sum_{k \in \mathbb{Z}} (v_k)^j < \infty, \quad j = 2, 3, \ldots.
\]
(6.23)

Consider the generating functions

\[
u(x) = \frac{i}{2\pi \theta} \sum_{m \in \mathbb{Z}} \tilde{u}_m e(x)^m
\]
(6.24)
and
\[ v(x) = -2\pi i \theta^2 \sum_{m \in \mathbb{Z}} \tilde{v}_m e(x)^{-m}, \quad (6.25) \]
where \( e(x) \) is identified with the generator \( U_1 \). In terms of the coordinates \((\tilde{v}_m, \tilde{u}_n)\), the Poisson bracket takes the form
\[ \{\tilde{v}_m, \tilde{u}_n\} = \delta_{m,n}, \quad (6.26) \]
or
\[ \{v(x), u(x')\} = \delta(x - x'), \quad (6.27) \]
where \( \delta(x) = \sum_{m \in \mathbb{Z}} e(m\theta x) \).

We define the orbit variables in the basis \( T_{m,n} \) as
\[ S(x, y) = -2\pi i \theta^2 \sum_{m,n} \left( \frac{mn\theta}{2} \right) s_{m,n} e(x)^{-m} * e(y)^{-n} \quad (U_1 \sim e(x), \ U_2 \sim e(y)). \]

It follows from (2.11) that the orbit variables are expanded in the coordinates \( s_{m,n} \) on the NCT as
\[ p_{j,j+n} = -2\pi i \theta^2 \sum_{m \in \mathbb{Z}} e \left( m \theta \left( \frac{n}{2} - j \right) \right) s_{m,n}. \quad (6.29) \]

Since \( p_{j,j} = 0 \), we have \( s_{m,0} = 0 \) and \( S(x, 0) \equiv 0 \). The bracket (6.20) takes the form
\[ \{s_{m,n}, s_{m',n'}\} = \frac{1}{\pi \theta} \sin \pi \theta (mn' - m'n) s_{m+m',n+n'}. \quad (6.30) \]

The Hamiltonian (6.18) can be rewritten in terms of the NCT variables. Using (6.21) and (6.24), we obtain
\[ \varphi(u_j - u_{j+n}; \tau) = \varphi(u(\theta j) - u(\theta(j + n)); \tau). \]

Similarly to (6.29), we define the coefficients \( r_{m,n} \) by
\[ \varphi(u(\theta j) - u(\theta(j + n))) = \sum_{m \in \mathbb{Z}} e \left( m \theta \left( \frac{n}{2} - j \right) \right) r_{m,n} \]
and the corresponding function on \( A_\theta \) by
\[ P(x, y) = \sum_{m,n} e \left( \frac{mn\theta}{2} \right) r_{m,n} e(x)^{-m} * e(y)^{-n}. \]

Thus,
\[ \varphi(u_j - u_{j+n}; \tau)p_{j,j+n} = -2\pi \theta^2 \sum_{m} e \left( m \theta \left( \frac{n}{2} + j \right) \right) \sum_{k} s_{k-m,n} r_{k,n}. \]

This makes it possible for us to establish a correspondence between the product \( \varphi(u_j - u_{j+n}; \tau)p_{j,j+n} \) and the “convolution”
\[ (P \odot S)(x, y) := -2\pi \theta^2 \sum_{m,n} e \left( \frac{mn\theta}{2} \right) \left( \sum_{k} r_{k,n} s_{k-m,n} \right) e(x)^{-m} * e(y)^{-n}. \]
Along with (6.25) and (6.28), this leads to the following expression for the Hamiltonian (6.18):

\[ H_{CM\infty} = \frac{1}{2} \int_{A_\theta} v(x)^2 dx + \int_{A_\theta} (\mathcal{P} \odot \mathcal{S})(x, y) \ast S(x, y). \]

The CM\infty comes from the trivial infinite rank Higgs bundle over \( E_\tau \) with transition functions \( g(z) \in \text{SIN}_\theta \). The whole procedure is similar to that in the finite-dimensional case. In particular,

\[ L_{CM\infty} = P + X. \]

Here

\[ P = \text{diag}(\ldots, v_{-N}, \ldots, v_0, v_1, \ldots, v_N, \ldots), \quad (6.31) \]

\[ X_{jk} = p_{jk} \phi(u_j - u_k, z), \quad p_{jk} \in \tilde{O}_\infty. \quad (6.32) \]

We define the coefficients \( \tilde{\phi}_{m,n} \) by the expansion

\[ \phi(u_j - u_{j+n}; z) \equiv \phi(u(\theta j) - u(\theta(j+n)); z) = \sum_m e\left( m\theta \left( \frac{n}{2} - j \right) \right) \tilde{\phi}_{m,n}(u, z) \]

and construct the generating function

\[ F(u, x, y; z) = i \frac{1}{2\pi\theta} \sum_{m,n} \tilde{\phi}_{m,n}(u; z) e\left( \frac{mn\theta}{2} \right) e(x)^{-m} e(y)^{-n}. \]

In terms of the NCT, \( L_{CM\infty} \) has the form

\[ L_{CM\infty}(x, y) = \mathcal{V}(x) + (\mathcal{S} \odot \mathcal{F}(u, x, y; z))(x, y), \]

where \( \mathcal{V} \) and \( \mathcal{S} \) are defined by (6.25) and (6.28).

We have the infinite set of integrals of motion

\[ I_{s,j} = \int_{E_\tau} \int_{A_\theta} (L_{CM\infty})^j \mu_{s,j}, \quad (6.33) \]

and we assume that they are finite, i.e., \( I_{s,j} < \infty \). In particular,

\[ \int_{A_\theta} (L_{CM\infty})^2(z) = I_{0,2} + \varphi(z)I_{2,2}, \quad I_{2,2} = \int_{A_\theta} S^2, \quad H_{CM\infty} = \frac{1}{2} I_{0,2}. \]

The integrals (5.12) give rise to the hierarchy of commuting flows \( \partial_{s,j} \sim I_{s,j} \).

5. Reduction to the loop algebra. For a rational number \( \theta = p/N \), we can pass to \( \hat{L}(\mathfrak{g}(N, \mathbb{C})) \). The Lax operator which is a 1-form on \( S^1 \) (see (5.19)), acquires the form [14], [11]

\[ L_{CM} = -\frac{\delta_{ij}}{2\pi\sqrt{-1}} \left( \frac{v_i}{2} + \sum_{\alpha} p_\alpha^i E_i(z - w_\alpha) \right) - \frac{1 - \delta_{ij}}{2\pi\sqrt{-1}} \sum_{\alpha} p_\alpha^i \phi(u_{ij}, z - w_\alpha). \]

The integrals of motion can be calculated from the expansion of the trace of the monodromy matrix for the linear system

\[ (\partial_w + L(z, w))\Psi(z, w) = 0. \]
They define the hierarchy of the elliptic CM field theory. For $N = 2$, the first non-trivial integral has the form

$$H = \oint \frac{dw}{w} \left( -\frac{v^2}{16\pi^2} \left( 1 - \frac{u_w^2}{h} \right) + (3u_w^2 - h)(2u) - \frac{u_w^4}{4v^2} \right), \quad (6.34)$$

where $h$ is the Casimir corresponding to the co-adjoint orbit of $\hat{L}(\text{GL}(N, \mathbb{C}))$ and $v^2 = h - u_w^2$. For an arbitrary $N$, the quadratic Hamiltonians of type $I_{0,2}$ were calculated in $[11]$.

Let $L_{\text{LL}}$ be the Lax operator for the Landau–Lifshitz equation, and let $L_{\text{CM}}^{2D}$ be the Lax operator corresponding to (6.34). Then (see (6.15))

$$L_{\text{LL}} = \Xi^{-1} \partial_w \Xi + \Xi^{-1} L_{\text{CM}}^{2D} \Xi,$$

where $\Xi$ is defined by (6.16) for $N = 2$. The explicit relations between the phase space variables are given by (6.17).

7. Conclusion

There are four related subjects that are not covered here.

- We have not considered the classical limit of the CM$_{\infty}$ model. Possibly, it can be described independently as the Hitchin system with structure group $\text{SDiff}(T^2)$.
- Expectedly, the symplectic Hecke correspondence survives in the limit of $N \to \infty$. If this is so, then the CM system on the non-commutative torus $A_\theta$ and the ER on $A_\theta$ are symplectomorphic. This means, in particular, that the former system is not far from the non-commutative modification of 2D hydrodynamics. The symplectic Hecke correspondence merely transfers the particles degrees of freedom to the orbit variables. Presumably, the correspondence survives in the classical limit.
- It will be interesting to define both systems on the central extended algebra $\widehat{\text{sin}}_\theta$ (2.15). The central charge produces an additional dimension, and the corresponding systems cover the CM field theory and the Landau–Lifshitz model.
- Two different tori are incorporated in our construction, the gauge NCT $A_\theta$ and the basic spectral curve $E_\tau$. In the classical limit, they become dual. It seems natural to replace $E_\tau$ by another NCT $A_{\theta'}$. In the general setting, this means a generalisation on the Higgs bundles over the non-commutative base. The categories of holomorphic vector bundles on the non-commutative torus were constructed in the recent paper $[21]$. An effort in this direction was made in $[24]$.

Appendix A. Hitchin Systems on an Elliptic Curve

Let $E_N^\theta$ be a rank $N$ stable holomorphic vector bundle over the elliptic curve $E_\tau$ (B.1). It can be described by the holomorphic $\text{GL}(N, \mathbb{C})$-valued transition functions

$$g_1(z): z \rightarrow z + 1, \quad g_2(z): z \rightarrow z + \tau, \quad \alpha = 1, 2$$

$$g_\alpha \in \Omega^{0}(\mathcal{U}_\alpha, \text{Aut}^*E_N),$$
(\mathcal{U}_1 \text{ is a neighborhood of } [0, \tau] \text{ and } \mathcal{U}_2 \text{ is a neighborhood of } [0, 1]) \text{.} \) They satisfy the cocycle conditions
\[ g_1(z)g_2(z+1)g_1^{-1}(z+1+\tau)g_2^{-1}(z+\tau) = \text{Id}. \]

We define the action of the gauge group \( \mathcal{G}_N = \{ f(z) \} \) as
\[ g_1(z) \rightarrow f(z)g_1(z)f^{-1}(z+1), \quad g_2(z) \rightarrow f(z)g_2(z)f^{-1}(z+\tau). \quad (A.2) \]

The moduli space of the stable holomorphic bundles \( \mathcal{M}_N(E_r) \) is defined as the quotient
\[ \mathcal{M}_N(E_r) = \mathcal{G}_N \backslash E_N. \quad (A.3) \]

The space \( \mathcal{M}_N(E_r) \) is the disjoint union of the components labeled by the corresponding degrees \( d = c_1(\det E_N); \quad \mathcal{M}_N(E_r) = \bigsqcup \mathcal{M}_N^{(d)}. \) The tangent space to \( \mathcal{M}(E_r) \) is isomorphic to \( h^1(E_r, \text{End}_{\mathcal{E}} E_N^\ast). \) Its dimension can be determined from the Riemann–Roch theorem; it is equal to
\[ \dim h^1(E_r, \text{End}_{\mathcal{E}} E_N^\ast) = \dim h^0(E_r, \text{End}_{\mathcal{E}} E_N). \]

As a result, we have
\[ \dim \mathcal{M}_N^{(d)} = \text{g. c. d.}(N, d) \quad (A.4) \]

The generic stable bundles can be transformed by \( (A.2) \) to the constant diagonal form. For the trivial bundles \((d = 0)\), we have
\[ g_1^{(0)} = \text{Id}, \quad g_2^{(0)} = f^{-1}(z) \text{ diag } \exp uf(z). \quad (A.5) \]

For \( d = 1 \), the transition functions can be chosen in the form
\[ g_1^{(1)} = f^{-1}(z)Q_N f(z), \quad g_2^{(1)} = f^{-1}(z)\tilde{A}_N f(z), \quad \tilde{A}_N = e \left( -\frac{\tau + z}{N} \right) A_N, \quad (A.6) \]

where
\[ Q_N = \text{diag} \left( 1, e \left( \frac{1}{N} \right), \ldots, e \left( \frac{1}{N-1} \right) \right), \quad \Lambda_N = \sum_{j=1,N \text{ (mod } N)} E_{j,j+1}. \quad (A.7) \]

Consider the cotangent bundle \( T^*E_N^\ast \). We choose the transition functions in the form
\[ \eta_\alpha \in \Omega^{(1,0)}(\mathcal{U}_\alpha, \text{End}_{\mathcal{E}} E_N^\ast), \quad \alpha = 1, 2 \]

(\mathcal{U}_1 \text{ is a neighborhood of } [0, \tau] \text{ and } \mathcal{U}_2 \text{ is a neighborhood of } [0, 1]). The bundle \( T^*E_N \) is called the Higgs bundle over \( E_r \).

To the marked point \( z = 0 \) we assign a coadjoint orbit of \( \text{SL}(N, \mathbb{C}) \):
\[ \mathcal{O} = \{ p \in \text{sl}(N, \mathbb{C}): p = h^{-1}p^0h, \quad h \in \text{SL}(N, \mathbb{C}), \quad p^0 \in \text{sl}(N, \mathbb{C}) \}. \quad (A.8) \]

The unreduced phase space is the pair
\[ \mathcal{R} = (T^*E_N^\ast, \mathcal{O}) \quad (A.9) \]

with the symplectic form
\[ \omega = \sum_{\alpha=1,2} \oint_{\gamma_\alpha} \text{tr}(D(g_{\alpha}^{-1}\eta_\alpha) \wedge Dg_{\alpha}) + \text{tr}(D(h^{-1}p^0) \wedge Dh). \quad (A.10) \]

Here the integrals \( \oint \) are taken over \( \gamma_1 \sim [0, \tau) \) and \( \gamma_2 \sim [0, 1) \). We assume that the marked point \( z = 0 \) lies inside the closed contour \( \gamma_1(z)\gamma_2(z+\tau)\gamma_1^{-1}(z+1)\gamma_2^{-1}(z) \).
The space $\mathcal{R}$ (A.9) is called the Higgs bundle with quasi-parabolic structure at the marked point $z = 0$.

The canonical transformations of (A.10) are (A.2) along with
\[
\eta_\alpha \rightarrow f(z) \eta_\alpha(z, \bar{z}) f^{-1}(z), \quad h \rightarrow h f(0).
\]
(A.11)
The transformations are generated by the following first-class constraints. Let $\Phi(z)$ be a meromorphic 1-form on $E_\tau$. Then
\[
\eta_\alpha = \Phi(z), \quad \text{Res}(\Phi(z))|_{z=0} = p, \quad p \in \mathcal{O}.
\]
(A.12)

The form $\Phi$ is the so-called Higgs field. Moreover, the constraints imply the quasi-periodicity of $\eta_\alpha$:
\[
\eta_1(z, \bar{z}) = g_1(z) \eta_1(z+1, \bar{z}+1) g_1^{-1}(z), \quad \eta_2(z, \bar{z}) = g_2(z) \eta_2(z+\tau, \bar{z}+\bar{\tau}) g_2^{-1}(z).
\]
(A.13)

Let $\mu_s, j d\bar{z} \in \Omega^{-j,1}(E_\tau)$ be $(-j, 1)$-differentials on $E_\tau$. We choose representatives from $\Omega^{-j,1}(E_\tau)$ that form a basis in the cohomology space $h^1(E_\tau, \Gamma^j)$ (dim $h^1 = j$):
\[
\mu_{s,j} = (\mu_{0,j}, \partial_z^{j-1}, \mu_{2,j} \partial_z^{j-1}, \ldots, \mu_{j,j} \partial_z^{j-1}).
\]
(A.14)
The coefficients $\mu_{s,j}$ coincide with the basis (B.22) ($\mu_{s,j} = f_s$). The integrals
\[
I_{s,j} = \int_{E_\tau} \text{tr}(\Phi^j) \mu_{s,j} d\bar{z}
\]
(A.15)
are gauge invariant. They play the role of Hamiltonians in the integrable hierarchy. The equations of motion on the phase space $\mathcal{R}$ with respect to the Hamiltonian $I_{s,j}$ take the form
\[
\partial_{s,j} \Phi = 0,
\]
(A.16)
\[
(\partial_{s,j} g_{a}) g_{a}^{-1} = \Phi^{j-1} \mu_{s,j},
\]
(A.17)
\[
\partial_{s,j} p = 0.
\]
(A.18)

Consider the symplectic quotient $\mathcal{R}^{\text{red}} = \mathcal{R} // \mathcal{G}_N$. Let $f(z)$ be the gauge transform that brings the transition function to the standard form ((A.5) for $d = 0$ and (A.6) for $d = 1$). The Lax operator is the corresponding gauge transform of the Higgs field $\Phi$
\[
L_N(z) = f(z) \Phi(z) f^{-1}(z).
\]
(A.18)

In this case, the first equation (A.16) is equivalent to the Lax equation
\[
\partial_{s,j} L_N = [L_N, M_{N,s,j}],
\]
(A.19)
where $M_{N,s,j} = f^{-1} \partial_{s,j} f$.

In terms of the Lax matrix, the integrals (A.15) have the form
\[
I_{s,j} = \oint_{E_\tau} \text{tr}(L_N^j) \mu_{s,j}.
\]
(A.20)

They can be found from the basis expansion of the elliptic functions
\[
\text{tr}(L_N^j)(z) = I_{0,j} + \sum_{s=2}^{N} I_{s,j} (s-2)(z) \quad (s^k)(z) = \partial_z^k \psi(z).
\]
(A.21)
We summarize the basic formulae for elliptic functions, which are borrowed mainly from [29] and [18]. Consider the elliptic curve
\[ E_\tau = \mathbb{C}/\mathbb{Z} + \tau \mathbb{Z}, \quad q = e^{2\pi i\tau}. \] (B.1)

The basic element is the theta function
\[ \vartheta(z|\tau) = q^{1/8} \sum_{n \in \mathbb{Z}} (-1)^n e^{\pi i (n(n+1)\tau + 2nz)} = q^{1/8} e^{-i\pi/4} (e^{i\pi z} - e^{-i\pi z}) \prod_{n=1}^{\infty} (1 - q^n)(1 - q^n e^{2i\pi z})(1 - q^n e^{-2i\pi z}). \] (B.2)

The Weierstrass functions are
\[ \sigma(z|\tau) = \exp(\eta z^2) \frac{\vartheta(z|\tau)}{\vartheta(0|\tau)}, \] (B.3)
where
\[ \eta(\tau) = -\frac{1}{6} \frac{\partial^6}{\partial\vartheta^6}(0|\tau). \] (B.4)
\[ \zeta(z|\tau) = \partial_z \log \vartheta(z|\tau) + 2\eta(\tau)z, \quad \zeta(z|\tau) \sim \frac{1}{z} + O(z^3), \] (B.5)
\[ \wp(z|\tau) = -\partial_z \zeta(z|\tau), \] (B.6)
\[ \wp(u; \tau) = \frac{1}{u^2} + \sum_{j,k} \left( \frac{1}{(j+k\tau+u)^2} - \frac{1}{(j+k\tau)^2} \right). \] (B.7)

The next important function is
\[ \phi(u, z) = \frac{\vartheta(u+z)\vartheta'(0)}{\vartheta(u)\vartheta(z)}. \] (B.8)

It has a pole at \( z = 0 \) and the following properties:

(i) \[ \phi(u, z) = \frac{1}{z} + \zeta(u|\tau) + 2\eta(\tau)u + \frac{z}{2}((\zeta(u|\tau) + 2\eta(\tau)u)^2 - \varphi(u)) + \ldots; \] (B.9)

(ii) (relation to the Weierstrass functions)
\[ \phi(u, z)^{-1} \partial_u \phi(u, z) = \zeta(u+z) - \zeta(u) + 2\eta(\tau)z; \] (B.10)
\[ \phi(u, z) = \exp(-2\eta_1 uz) \frac{\sigma(u+z)}{\sigma(u)\sigma(z)}; \] (B.11)
\[ \phi(u, z)\phi(-u, z) = \wp(z) - \wp(u). \] (B.12)

(iii) (particular values)
\[ \wp'(z) = 0 \quad \text{for} \quad z = \frac{1}{2}, \frac{\tau}{2}, \frac{1+\tau}{2}; \] (B.13)

(iv) (addition formula, or Calogero functional equation)
\[ \phi(u, z)\partial_u \phi(v, z) - \phi(v, z)\partial_u \phi(u, z) = (\wp(v) - \wp(u))\phi(u+v, z). \] (B.14)
In fact, \(\phi(u, z)\) satisfies the following more general relation, which follows from the Fay three-section formula:

\[
\phi(u_1, z_1)\phi(u_2, z_2) - \phi(u_1 + u_2, z_1)\phi(u_2, z_2 - z_1) - \phi(u_1 + u_2, z_2)\phi(u_1, z_1 - z_2) = 0.
\]  

(B.15)

Particular cases of this formula are (B.12) and

\[
\phi(u_1, z)\phi(u_2, z) - \phi(u_1 + u_2, z)(\zeta(u_1) + \zeta(u_2) - 2\eta(\tau)(u_1 + u_2)) + \partial_2 \phi(u_1 + u_2, z) = 0.
\]  

(B.16)

It follows from (B.10), (B.12) and (B.16) that, for \(u_1 + u_2 + u_3 = 0\),

\[
\phi(u_1, z)\phi(u_2, z)\phi(u_3, z) = [\wp(z) - \wp(u_3)] [\zeta(u_1) + \zeta(u_2) + \zeta(u_3) - z + \zeta(z)].
\]

(B.17)

Therefore,

\[
\phi(u_1, z)\phi(u_2, z)\phi(u_3, z)|_{z = 0} = \frac{1}{z^3} + \frac{1}{2} \left[ \zeta(u_1) + \zeta(u_2) + \zeta(u_3) \right] - \frac{1}{2} \wp'(u_3) - \wp(u_3) \left[ \zeta(u_1) + \zeta(u_2) + \zeta(u_3) \right] + O(z).
\]

(B.18)

A basis of elliptic functions on \(E_\tau\). Consider elliptic functions on \(E_\tau\) with poles at \(z = 0\). Any elliptic meromorphic function \(F(z)\) is represented in the form

\[
F(z) = \sum_{j=0,2,3...} c_j e^j,
\]

where

\[
e^0 = 1, \quad e^j = \wp^{(j-2)}(z).
\]

(B.20)

The dual basis \(\{f_k\}\) with respect to the pairing

\[
\langle \ast | \ast \rangle = \int_{E_\tau}, \quad \langle f_k | e^j \rangle = \delta_k^j
\]

has the form

\[
f_0 = (\bar{z} - z)(1 - \chi(z, \bar{z})), \quad f_k = z^{k-1}\chi(z, \bar{z}), \quad k > 1,
\]

(B.21)

where \(\chi(z, \bar{z})\) is the characteristic function of a small neighborhood \(U_0\) of \(z = 0\), that is,

\[
\chi(z, \bar{z}) = \begin{cases} 1, & z \in U_0, \quad U_0' \supset U_0, \\ 0, & z \in E_\tau \setminus U_0'. \end{cases}
\]

(B.22)

Theta functions with characteristics. For \(a, b \in \mathbb{Q}\), we put

\[
\theta_{\left[\frac{a}{b}\right]}(z, \tau) = \sum_{j \in \mathbb{Z}} e \left( (j + a)\frac{z}{2} + (j + a)(z + b) \right).
\]

(B.24)

In particular, the function \(\theta\) defined by (B.2) is the theta function with the characteristic

\[
\vartheta(x, \tau) = \theta_{\left[\frac{1}{2}\right]}(x, \tau).
\]

(B.25)

For simplicity, we denote \(\theta_{\left[\frac{a}{b}\right]} = \theta_{ab} (a, b = 0, 1)\).
References


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