STRANGE FACTOR REPRESENTATIONS OF TYPE II$_1$
AND PAIRS OF DUAL DYNAMICAL SYSTEMS

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Abstract. Given a pair of dynamical systems, we construct a pair of commuting factors of type II$_1$. This construction is a generalization of the classical von Neumann–Murray construction of factors as crossed products and of the groupoid construction. The suggested construction provides natural examples of factors with non-unity coupling constant. First examples of this kind, related to actions of abelian groups and to the theory of quantum tori, were given by Connes and Rieffel and by Faddeev; our generalization includes these examples as well as new examples of factorizations related to lattices in Lie groups, the infinite symmetric group, etc.

Key words and phrases. Coupling constant, dynamical system, factor representation, Heisenberg group, pseudogroupoid, infinite symmetric group.

1. General Scheme and Definitions

Let $X$ be a locally compact non-compact separable space with a Borel $\sigma$-finite infinite continuous measure $\mu$. Assume that two arbitrary countable groups $G$ and $H$ act on the space $X$ by homeomorphisms and the following conditions are satisfied.

1. The actions of the groups $G$ and $H$ are free, commute with each other, and are transversal (i.e., the intersection of any $G$-orbit with any $H$-orbit is at most one-point). It is convenient to assume that the action of one group ($G$) is left and the action of the other group ($H$) is right.

2. Both actions preserve the measure $\mu$.

3. Both actions are totally disconnected, i.e., no orbit has limiting points in $X$.

In view of condition (3), we can define measurable fundamental domains for the actions of the groups $G$ and $H$ (i.e., measurable sets $F_G \subset X$ and $F_H \subset X$ such that $F_G$ contains exactly one point of each $G$-orbit and $F_H$ contains exactly one point of each $H$-orbit and restrict the measure $\mu$ to these sets $F_G$ and $F_H$.

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We can also define the spaces $G\backslash X$ and $X/H$ of orbits of the groups $G$ and $H$ as topological quotient spaces and define the quotient measures $\mu_G$ and $\mu_H$ by using their isomorphisms with the fundamental domains; if the measure of a fundamental domain is infinite, then the quotient measure is defined up to a positive constant.\(^1\)

According to the commutation condition (1), the group $G$ acts on the space $X/H$, i.e., on the space of orbits of the group $H$, and the group $H$ acts on the space $G\backslash X$ of orbits of the group $G$. We can transfer these actions to the fundamental domains.

Since the measure $\mu$ is infinite, we can multiply it by an arbitrary positive constant without affecting the invariance; however the ratio of the (finite) measures of two subsets does not change under this multiplication. For example, we can normalize the measure $\mu$ so that the measure of the fundamental domain $F_G$ be equal to one, provided that this measure is finite.

**Definition.** Assume that the measures of both fundamental domains are finite. The ratio $\lambda(G, H) = \frac{\mu(F_H)}{\mu(F_G)}$ is called the coupling constant of the pair of dynamical systems $(G, H)$; it is clear from the definition that $\lambda(G, H)\cdot\lambda(H, G) = 1$, and that these values do not depend on the normalization of the measure $\mu$.

Let $C_G (C_H)$ be the algebra of all bounded continuous functions on $X$ that are constant on the orbits of the action of the group $G$ (respectively, $H$). It follows from condition (3) that the algebra $C_G$ (respectively, $C_H$) is canonically isomorphic to the algebra $C(G\backslash X)$ (respectively, $C(X/H)$) of bounded continuous functions on the space $G\backslash X$ (respectively, $X/H$).

It is clear from condition (1) that the groups $G$ and $H$ act naturally on the algebras $C_H$ and $C_G$, respectively, by automorphisms. Condition (2) guarantees that the actions of the group $G$ on $X/H$ and of the group $H$ on $G\backslash X$ are measure-preserving.

Consider the complex Hilbert space $L^2(X, \mu)$ of square integrable functions and representations of all objects under consideration in this space, namely, the representation of the groups $G$ and $H$ by the unitary operators

$$[(U_g)f](x) = f(g^{-1}x), \quad g \in G; \quad [(V_h)f](x) = f(xh), \quad h \in H$$

(where $f \in L^2(X, \mu)$) and the representation of the commutative algebras $C_G$ and $C_H$ by the multiplicators

$$(M_\phi f)(x) = \phi(x) \cdot f(x), \quad \phi \in C_G, C_H.$$

Finally, we introduce the $W^*$-algebras $A_G (A_H)$ generated by the sets of all operators $\{U_g, g \in G\}$ and $\{M_\phi, \phi \in C_H\}$ (respectively, $\{U_h, h \in H\}$ and $\{M_\phi, \phi \in C_G\}$) in the Hilbert space $L^2(X, \mu)$ (more precisely, the weak closures of these sets of operators). We also consider the $W^*$-algebra $B$ generated by both algebras, i.e., by the operators of both groups $G$ and $H$ and by the multiplicators from $C_G$ and $C_H$.

\(^1\)Construction of the fundamental domains uses the total disconnectedness of the actions; however, in what follows, it is in fact used only that the partitions into the orbits of the group actions are measurable, which is a metric rather than topological condition; all further constructions are also purely metric. For simplicity, in this section, we use the language of topological dynamics instead of the metric language used in Section 6.1 in the definition of a measurable pseudogroupoid.
Recall of the notion of the coupling constant of finite factors from the classical work by Murray and von Neumann [8] (see also [7], [5]; a more detailed exposition can be found in [13]). Let $B$ be a finite factor, and let $B'$ be its commutant in a Hilbert space $K$. By $\text{tr}(\cdot)$ and $\text{tr}'(\cdot)$ we denote the canonical normal traces in the $W^*$-algebras $B$ and $B'$ (of finite type). The coupling constant of the factor $B$ is the positive number $\lambda(B) = \text{tr}(P_h)/\text{tr}'(P'_h)$, where $h \in K$ is an arbitrary unit vector and $P_h$ ($P'_h$) is the orthogonal projection from the factor $B$ (respectively, $B'$) to the cyclic subspace generated by the vector $h$ under the action of all operators from the factor $B'$ (respectively, $B$). It is known that this ratio does not depend on the choice of the unit vector $h$. There is an obvious relation between the coupling constants of a factor and its commutant, namely, $\lambda(B') \cdot \lambda(B) = 1$.

The coupling constant plays the role of a measure for comparing the multiplicities of representations in the theory of factor representations of type II. Its finite-dimensional analogue is the ratio of the multiplicities of the left and right irreducible representations in the tensor product $\pi_1 \otimes \pi_2$ of two primary representations of groups or algebras.

Two factor representations $\rho_1$ and $\rho_2$ of type II of an algebra are called algebraically isomorphic if there exists an algebraic isomorphism of the corresponding factors that sends one representation (as a homomorphism of the (group) algebra to the algebra of operators of the Hilbert space) to the other; they are called spatially isomorphic if this isomorphism is generated by an isometry of the Hilbert spaces and quasiequivalent (in the sense of Mackey) if each of the representations is spatially isomorphic to a subrepresentation of the direct sum of some (maybe, infinite) number of copies of the other representation. For factors, the notions of algebraic isomorphism and quasiequivalence coincide; thus, trace is the only invariant of a factor representation of type II up to algebraic equivalence or quasiequivalence.

The most important fact is that the coupling constant is a complete invariant of spatial isomorphism in the class of a given algebraic type of factor representations; this means that two algebraically isomorphic factor representations of an algebra (group) of type II are spatially isomorphic if and only if their coupling constants coincide. Clearly, for applications of the representation theory, the most important isomorphisms are the spatial ones.

Recall also that a factor $B$ has a cyclic vector $f$ (this means that the cyclic subspace generated by $f$ is the whole Hilbert space) if $\lambda(B) \leq 1$; it has a separating vector $h$ (i.e., an $h$ such that $Uh \neq 0$ for all $U \in B$) if $\lambda(B) \geq 1$; and it has a bicyclic vector (i.e., a vector separating and cyclic simultaneously, or, equivalently, cyclic simultaneously for the factor and its commutant) if and only if $\lambda(B) = \lambda(B') = 1$.

All these facts have direct applications to the representation theory (see below).

During the preparation of this paper for publication, the author discovered a link between the notion of coupling constant in the sense of the general scheme described above and some definitions and problems which were discussed in a number of papers devoted to the theory of dynamical systems (by M. Gromov [6], D. Gaboriau [4], and D. Furman [3]) and to the theory of $C^*$-algebras and noncommutative geometry (by A. Connes [1], M. Rieffel [11], [12], and J. Renault [10]). Note that Gromov’s definition of measure equivalent (ME) groups in [6] is very similar to our definition of pairs of dynamical systems at the beginning of this article. We,
however, would like to stress the following: we consider pairs of topological actions of groups and try to emphasize the natural setting when such a pairing exists. For example, from the point of view of the theory of measure preserving transformations all ergodic action amenable groups are measure equivalent. But from topological, smooth, or algebraic point of view the constructions of the pairing of amenable actions of two groups in our sense (even if these groups coincide as abstract groups) have many interesting invariants including coupling constants and others. The same is true if we speak about von Neumann factors with commutant and with two given Cartan subalgebras. We hope to return to these links elsewhere.

2. A Theorem on the Coupling Constant

**Theorem 1.** Assume that conditions (1)–(3) hold, the measures \( \mu_G \) and \( \mu_H \) are finite, and the action of the group \( G \times H \) on the space \( (X, \mu) \) with invariant \( \sigma \)-finite measure \( \mu \) is free and ergodic. Then the following assertions are valid.

1. The \( W^* \)-algebra \( \mathcal{B} \) generated by the two algebras \( \mathcal{A}_G \) and \( \mathcal{A}_H \) is the algebra of all bounded operators in \( \mathcal{B}(L^2) \); in other words, the representation of the crossed product of the group \( G \times H \) and the commutative algebra generated by \( \mathcal{C}_G \) and \( \mathcal{C}_H \) on the space \( L^2(X, \mu) \) is irreducible.

2. The algebras \( \mathcal{A}_G \) and \( \mathcal{A}_H \) as subalgebras of \( \mathcal{B}(L^2) \) are factors of type \( \text{II}_1 \); they are mutual commutants in \( \mathcal{B}(L^2) \) and factorize the irreducible representation described in 1.

3. The weak closures of the abelian subalgebras \( \mathcal{C}_G \) and \( \mathcal{C}_H \) are Cartan (i.e., regular, maximal, self-adjoint, and abelian) subalgebras in the corresponding factors.

4. The coupling constants of the factors \( \mathcal{A}_G \) and \( \mathcal{A}_H \) are equal to the coupling constants of the dynamical systems defined above, i.e., \( \lambda(G, H) = \frac{\mu(F_H \cap F_G)}{\mu(F_G)} = \lambda(A_G) \).

**Proof.** The first claim follows from the ergodicity of the action of the group \( G \times H \) on \( (X, \mu) \), since the algebra \( \mathcal{B} \) is generated by the ordinary crossed product of the group \( G \times H \) and the function space. The same ergodicity implies that the actions of the group \( G \) on \( (X/H, \mu_H) \) and of the group \( H \) on \( (G \setminus X, \mu_G) \) are ergodic. The joint representation of both algebras in the space \( L^2(X, \mu) \) can be regarded as a representation of their tensor product

\[
\mathcal{A}_G \otimes \mathcal{A}_H \equiv \mathcal{A}_{G,H}
\]

in the space \( L^2(X, \mu) \). As we have seen, this representation is irreducible, hence the weak closure of the commutative algebra generated by the multiplicators \( M_\phi \) from both commutative algebras \( (\phi \in \mathcal{C}_G \vee \mathcal{C}_H) \) is a maximal commutative subalgebra of \( \mathcal{B}(L^2) \) because of the transversality of the actions of the groups \( G \) and \( H \). But the restriction of an irreducible representation of the tensor product of two commuting algebras to each factor is a factor representation (see [13]). Thus, each of the algebras \( \mathcal{A}_G \) and \( \mathcal{A}_H \) is a factor, and they are mutual commutants, that is, \( [\mathcal{A}_G]^' = \mathcal{A}_H \). Now, let us prove that they are factors of type \( \text{II}_1 \).

\(^2\)Here the term “factorization” has the initial meaning of a decomposition into factors; it was in this sense that the term was used by von Neumann, and this gave rise to the term “factor”. In what follows, we consider factorization of irreducible representations. Note that these decompositions are not decompositions of operator algebra into tensor factors.
We define traces on $\mathcal{A}_G$ and $\mathcal{A}_H$ treated as von Neumann algebras in the standard way, as on crossed products; the trace of a linear combination is defined by the formula

$$\text{tr}_{\mathcal{A}_G}(M_\phi \otimes U_g) = \delta(e, g) \int_{X/H} \phi(x) d\mu,$$

$$\text{tr}_{\mathcal{A}_H}(M_\phi \otimes U_h) = \delta(e, h) \int_{X/G} \phi(x) d\mu.$$  

The trace is obviously continuous and finite, hence the factors are of type II$_1$. It is clear from the construction that these factors are algebraically isomorphic to the factors of regular (von Neumann) representations of type II$_1$; however, as we shall see, the constructed factors are not spatially isomorphic to standard von Neumann factors.

Now, let us compute the coupling constants of the constructed factors. To this end, we choose a unit vector $\chi \in L^2(X, \mu)$ and find the ratio of the traces of the factors $\mathcal{A}_G$ and $\mathcal{A}_H$ on the corresponding cyclic subspaces of the vector $\chi$. Recall that the result does not depend on the choice of the vector. So, take $\chi = \chi_{F_H}$, where $\chi_{F_H} \in \mathcal{C}_G \subset \mathcal{A}_G$ is the characteristic function of the fundamental domain $F_H$. Assume that the measure $\mu$ is normalized so that $\mu(F_H) = 1$. It is known from the theory of crossed products that the trace of the projection $P_\chi$ corresponding to the multiplicator by a characteristic function $\chi_E \equiv \chi$ from a maximal regular abelian subalgebra of the factor is equal to the measure of the set.

Thus, $\text{tr}(P_{\chi_{F_H}}) = \mu(F_H) = 1$. Applying the same argument to the second factor $\mathcal{A}_H$ shows that the value of the trace at the similar projection generated by the same characteristic function in the factor $\mathcal{A}_H$ is equal to the measure of this set, but under another normalization of the measure $\mu$, in which $\mu(F_G) = 1$. But this means that the ratio of the trace values under the same (for example, first) normalization of the measure $\mu$ is equal to the ratio of the measures of the fundamental domains. Thus, the coupling constants of the factors in the sense of the theory of factors coincide with the coupling constants of the dynamical systems defined above:

$$\lambda(\mathcal{A}_G) = \frac{\mu(F_H)}{\mu(F_G)}, \quad \lambda(\mathcal{A}_H) = \frac{\mu(F_G)}{\mu(F_H)}.$$

Remarks. 1. As mentioned, the value of the coupling constant provides a criterion for the existence of a bicyclic vector, i.e., vector which is cyclic for the factor and its commutant simultaneously; this condition is $\lambda = 1$, which means in our terminology that the measures of the fundamental domains coincide. However, it is not so easy to find this vector explicitly (except in the case when there exists a common fundamental domain); this requires analyzing the structures of orbits of both dynamical systems.

2. From the viewpoint of group representation theory, the most interesting case is when the coupling constant is less than or equal to one; only in this case, there exists a positive definite function on the group (a vector state) that generates the representation. In the case of $\lambda > 1$, such a vector state does not exist, and we have the complicated problem of explicitly describing the linear combination of vector states that generates this representation.
In relation to this construction, the following problem arises: What pairs of dynamical systems \((Y, G, \nu)\) and \((Z, H, \phi)\) with countable groups \(G\) and \(H\) acting freely on the measure spaces \((Y, \mu)\) and \((Z, \nu)\), respectively, do admit “coupling” within the scheme described at the beginning of this section? In other words, when do there exist a space \(X\) with a \(\sigma\)-finite measure \(\mu\) and actions of the groups \(G\) and \(H\) on \((X, \mu)\) such that the given dynamical systems are isomorphic to the actions of the groups on fundamental domains, as defined above?

3. Regular Representations of Dynamical Systems

Let us fit the well-known von Neumann–Murray construction of a factor representation of the algebra generated by a dynamical system with an invariant measure into our scheme. Suppose that a countable group \(G\) acts freely and ergodically on a Lebesgue space \((X_0, \mu_0)\) with a finite or \(\sigma\)-finite continuous measure (following precisely the scheme of the previous section, we should assume that the group acts by homeomorphisms of a locally compact space \(X_0\) with a \(\sigma\)-finite measure \(\mu_0\); but here this restriction does not affect the exposition at all). Thus, we start with one action of the group \(G\) on \(X_0\): \(x \mapsto gx\), and define two actions of the group \(G\), the left \((L)\) and the right \((R)\), on the space \(X = X_0 \times G\) by

\[ R_g(x, q) = (xg, qg), \quad L_g(x, q) = (x, g^{-1}q), \quad g, q \in G, \quad x \in X_0, \quad (x, q) \in X. \]

It is easy to check that these actions commute with each other and satisfy all the conditions of the scheme of the first section. In this sense, the groups coincide, that is, \(G = H\) (to be more precise, \(H\) is the opposite group to \(G\), i.e., \(H = G^0\); see below), and have a common fundamental domain \(X_0 \equiv X_0 \times \{e\} \subset X\) (where \(e\) is the identity element of the group \(G\)), which can be identified with both spaces of orbits. At the same time, these two actions do not coincide.

Both commutative algebras \(C_G\) and \(C_H\) can be identified with \(C(X_0)\), but they differ as subalgebras of \(C(X)\), since they are embedded differently in this algebra (according to the orbits of their actions). The left orbit of a point \((x, e), x \in X_0\), is the set \(\{(x, g)\}, g \in G\), and the right orbit of the same point is the set \(\{gx, g\}\), \(g \in G\). Thus, the right action of the group \(G\) induces the initial action \(x \mapsto gx\) of the group on the fundamental domain \(X_0\) of the left action

and the action of \(G\) induced by the left action on the fundamental domain of the right action is opposite to the initial one and has the same orbits: an element \(g \in G\) acts as \(g^{-1}\).

Consider two von Neumann algebras \(A_G\) and \(A_G^r\) in the Hilbert space

\[ L^2(X, \mu) = L^2(X_0, \mu_0) \otimes l^2(G), \]

constructed in the first section. We obtain a classical representation of the crossed product, which is called the regular or von Neumann representation generated by the dynamical system. Since the action is free and ergodic, the corresponding von Neumann algebras are factors of type \(\Pi_1\) or \(\Pi_{\infty}\), depending on whether the measure \(\mu\) is finite or infinite, and they are mutual commutants. Usually, only one factor (the right one) is considered; of course, it determines the commutant, but we consider also an embedding of the dual dynamical system into the commutant; more precisely, we establish an anti-isomorphism of the factor and its commutant.
Note that the representation of the tensor product of algebras is irreducible and corresponds to the so-called Koopmans representation of the action of $G \times G^0$ on $(X, \mu)$ in $L^2(X, \mu)$. One might say that we have factorized the Koopmans representation. If the measure $\mu_0$ is finite, then the coupling constants of both factors are equal to one, since the fundamental domains coincide, and the characteristic function of the set $X_0$ is a bicyclic vector.

Von Neumann’s construction, as well as its version described above, can be easily extended to actions of locally compact groups and to the important case of non-free actions (see below). In this context, it is more convenient to consider groupoids or equivalence relations rather than about group actions, but this requires no significant modifications of the construction. However, this construction does not provide examples of a non-unity coupling constant. Below we consider examples of realization of the scheme described in the first section.

4. Factorization of Irreducible Representations of the Heisenberg Group

In this section, we study the simplest realization of our scheme that differs from the classical von Neumann’s construction. This example was one of the reasons why this author begun to study the role played by non-unity coupling constants in the theory of factors. It seems that this is the simplest example of a factor of type $\text{II}_1$, and it is surprising that it was not included in textbooks and remained unnoticed until the most recent times. It appeared in a paper by Faddeev [2] in relation to the so-called quantum double and quantum groups, and in less explicit form, in papers by Rieffel and Connes [1], [11]. On the whole, no instances where coupling constants arose naturally in the representation theory have appeared in the literature so far.

We shall describe the simplest example from several points of view: as an direct example realizing the scheme of a pair of dynamical system from Section 1, as an example of decomposition of an irreducible representation of the Heisenberg group into factors, and as an example of a representation of the rotation algebra (the quantum torus). We shall also illustrate the role of coupling constants in this example by a natural problem concerning the Fourier transform.

4.1. The action of the group $\mathbb{Z}$ on $\mathbb{R}$ and the Heisenberg group. Let $\lambda_1$ and $\lambda_2$ be two real nonzero numbers. Consider the action of the groups $G = \mathbb{Z}\lambda_1$ and $H = \mathbb{Z}\lambda_2$ on the real line $X = \mathbb{R}$ by the shifts $x \to x + \lambda_1$, $x \to x + \lambda_2$; let $\mu$ be the Lebesgue measure on $X = \mathbb{R}$. Clearly, all conditions from Section 1 are satisfied, and the joint action of $\mathbb{Z}^2$ on $X = \mathbb{R}$ is ergodic if the ratio $\lambda_1/\lambda_2$ is irrational. We may assume that $\lambda_1, \lambda_2 > 0$. Take the half-open intervals $[0, \lambda_1)$ and $[0, \lambda_2)$ as the fundamental domains; the spaces of orbits are the circles $\mathbb{R}/\mathbb{Z}\lambda_1$ and $\mathbb{R}/\mathbb{Z}\lambda_2$. The coupling constants of the two systems are, obviously, the ratios $\{\lambda_2/\lambda_1\}$ and $\{\lambda_1/\lambda_2\}$. The algebras of functions $C_G$ and $C_H$ are the algebras of periodic functions with periods $\lambda_1$ and $\lambda_2$, respectively. The action of the first group $\mathbb{Z}$ on the second circle and the action of the second group $\mathbb{Z}$ on the first circle are, of course, rotations; more precisely, these actions are isomorphic to the
rotations of the unit circle by angles equal to the same fractional parts \( \{\lambda_2/\lambda_1\} \) and \( \{\lambda_1/\lambda_2\} \).

Thus, we have representations in the space \( L^2(\mathbb{R}, \mu) \) of two \( W^* \)-algebras: one of them is generated by the pairs of operators

\[
(V_1f)(x) = \exp\{i2\pi\lambda_1^{-1}x\}f(x) \quad \text{and} \quad (U_1f)(x) = f(x + \lambda_2)
\]

and the second one, by the pair of operators

\[
(V_2f)(x) = \exp\{i2\pi\lambda_2^{-1}x\}f(x) \quad \text{and} \quad (U_2f)(x) = f(x + \lambda_1).
\]

We could assume, without loss of generality, that \( \lambda_1 = 1 \) (using an appropriate normalization of the Lebesgue measure on \( \mathbb{R} \)), denote \( \lambda' = \lambda_2 \), and obtain the pairs of operators

\[
(V'_1f)(x) = \exp\{i2\pi x\}f(x) \quad \text{and} \quad (U'_1f)(x) = f(x + \lambda')
\]

and

\[
(V'_2f)(x) = \exp\{i2\pi\lambda'^{-1}x\}f(x) \quad \text{and} \quad (U'_2f)(x) = f(x + 1).
\]

In this case, reduction to rotations of the unit circle gives two rotations by angles \( \{\lambda\} \) and \( \{\lambda^{-1}\} \) (in \([2], -\lambda^{-1}\) is used instead of \(\lambda^{-1}\), but these two versions are equivalent up to change of generator in the group). But it is more convenient for us not to specialize the parameters \( \lambda_1 \) and \( \lambda_2 \).

Now, assume that the number \( \lambda \equiv \frac{\lambda_2}{\lambda_1} \) is irrational.

The main statement concerning these dynamical systems follows immediately from Theorem 1.

**Theorem 2.** Consider an irrational positive number \( \lambda \in \mathbb{R} \) and two pairs of operators \((U_1, V_1)\) and \((U_2, V_2)\) defined above. Then the following assertion hold.

1. Each pair of operators generates a \( W^* \)-algebra in \( L^2(\mathbb{R}, m) \); we denote these algebras by \( \mathcal{A}_\lambda \) and \( \mathcal{A}_{\lambda^{-1}} \), respectively. The algebras \( \mathcal{A}_\lambda \) and \( \mathcal{A}_{\lambda^{-1}} \) are mutual commutants and factors of type \( \Pi_1 \) in the algebra of all bounded operators \( \mathcal{B}(L^2(\mathbb{R})) \).

2. The coupling constant of the first factor is equal to \( \lambda \), and the coupling constant of the second factor is equal to \( \lambda^{-1} \), where \( \lambda = \frac{\lambda_2}{\lambda_1} \).

3. Four operators \( V_1, U_1, V_2 \) and \( U_2 \) generate an irreducible infinite-dimensional representation of the Heisenberg group (see below), and the factors defined above factorize this representation.

**Proof.** Assertions 1 and 2 follow immediately from Theorem 1; we shall prove assertion 3 and discuss its remarkable relation to the Heisenberg group.

Let \( \mathcal{H} \) be the quotient of the three-dimensional Heisenberg group \( H^R \) of upper-triangular unipotent matrices over \( \mathbb{R} \) modulo the subgroup \( \mathbb{Z} \) of its centre; topologically, this group is the product of the real plane and the unit circle.

We shall denote the elements of the group \( \mathcal{H} \) by triples

\[
\mathcal{H} = \{(a, b, \alpha); \ a, b \in \mathbb{R}, \ \alpha \in S^1 = \mathbb{R}/\mathbb{Z}\},
\]

multiplication in this group is defined by

\[
(a, b, \alpha) \cdot (a', b', \alpha') = (a + a', b + b', \alpha \cdot \alpha' \cdot \exp\{2\pi iab'\})
\]

(where we use the additive notation for the first two coordinates and the multiplicative notation for the third coordinate).
For nonzero real numbers $\lambda_1$ and $\lambda_2$, we define a class of discrete subgroups $\Gamma(\lambda_1, \lambda_2)$ of the group $H$ by
$$\Gamma(\lambda_1, \lambda_2) = \left\{ \left( m\lambda_1, n\lambda_2^{-1}, \exp \left\{ \frac{2\pi i r}{\lambda_2} \right\} \right), \ m, n, r, \in \mathbb{Z} \right\}.$$

We shall consider pairs of such groups $\Gamma(\lambda_1, \lambda_2)$ and $\Gamma(\lambda_2, \lambda_1)$. We have
$$\Gamma(\lambda_2, \lambda_1) = \left\{ \left( m'\lambda_2, n'\lambda_1^{-1}, \exp \left\{ \frac{2\pi i r'}{\lambda_1} \right\} \right), \ m', n', r', \in \mathbb{Z} \right\}.$$

A simple computation shows that the groups $\Gamma(\lambda_1, \lambda_2)$ and $\Gamma(\lambda_2, \lambda_1)$ commute with each other (moreover, each subgroup is the centralizer of the other in the group $H$ modulo the centre). In the case where the number $\lambda \equiv \frac{\lambda_1}{\lambda_2}$ is irrational, two groups together generate topologically the whole group $H$.

Note that, for all pairs $(\lambda_1, \lambda_2)$ with irrational $\lambda$, the group $\Gamma(\lambda_1, \lambda_2)$ is isomorphic to the discrete Heisenberg group $H^Z = \{(m, n, p), m, n, p \in \mathbb{Z} \}$, which is a lattice in $H^R$. $\Box$

Now, let us consider the canonical irreducible representations of the group $H$ in the Hilbert space $L^2(\mathbb{R}, \mu)$ (where $\mu$ is the Lebesgue measure). They are indexed by the values of the Planck constant; for the group $H$, it assumes only integral values $n \in \mathbb{Z} \setminus \{0\}$, which correspond to non-identical characters of the centre; the corresponding unitary operators are of the form
$$(U_{a,b,\alpha} f)(x) = \alpha^n \exp\{2\pi i a \cdot x\} f(x+b), \quad x \in \mathbb{R}, \ f \in L^2(\mathbb{R}), \ a, b \in \mathbb{R}, \ \alpha \in \mathbb{R}/\mathbb{Z}.$$ We denote this representation by $\rho_n$.

**Lemma 3.** Assume that the number $\frac{\lambda_1}{\lambda_2}$ is irrational.

The restrictions of the irreducible representation $\rho_n$ of the group $H$ to the subgroups $\Gamma(\lambda_1, \lambda_2)$ and $\Gamma(\lambda_2, \lambda_1)$ are factor representations of type $\Pi_1$, and the corresponding factors are mutual commutants in $B(L^2)$. The coupling constant is equal to $\frac{\lambda_1}{\lambda_2}$ for the first factor and to $\frac{\lambda_2}{\lambda_1}$ for the second one.

For $n = 1$, the representation $\rho_1$ obviously coincides with the representation defined in the statement of Theorem 2.

All the assertions of Lemma 3 can be checked directly. Thus, we have “factorized” the irreducible representation of the complete Heisenberg group with an integral Planck constant into two factors of type $\Pi_1$, each being generated by the restriction of this representation to a pair of commuting sublattices of the Heisenberg group.

For different $\lambda_1$ and $\lambda_2$ we obtain different factorizations of the irreducible representation; the ratios $\lambda$ and $\lambda^{-1}$ are the coupling constants, a complete spatial invariant of the factorization. It is interesting that these constants are determined by subgroups; it is not clear whether there is a natural way to obtain other values of the constant for a given subgroup. The case of a nonintegral Planck constant can be reduced to the case under consideration by renormalizing the subgroups $\Gamma$. We shall not dwell on this.

All the above considerations can be directly transferred to the case of a pair of actions of the group $\mathbb{Z}^d$ on $\mathbb{R}^d$ by shifts. We will obtain a pair of factors generated by shifts on tori with a non-unity coupling constant. As above, their description
reduces to factor representations of the \((2d + 1)\)-dimensional Heisenberg group \(H^{\mathbb{R}^d}\) and its sublattices. Within the framework of the same approach, we can consider an arbitrary Abelian locally compact non-compact group \(A\) and its two discrete cocompact subgroups \(G, H\) that generate topologically the whole group \(A\). The action of these subgroups on \(A\) by shifts fits in the same scheme; the role of the Heisenberg group is played by the central \(S^1\)-extension \(H\) of the group \(A \times \hat{A}\) (where \(\hat{A}\) is the group of characters of \(A\)) with 2-cocycle \(\alpha((x, \chi), (x', \chi')) = \langle\chi, x'\rangle\), and the corresponding two factors factorize the irreducible representation of the group \(H\) in the space \(L^2(A, \mu)\). The details are completely similar to the case \(d = 1\) considered above.

For non-abelian groups, the situation is essentially different; this case will be considered below.

4.2. An analytical consequence in the theory of Fourier transform. Let us exemplify the application of the theorem on the coupling constant to a natural problem of Fourier analysis.

**Theorem 4** (A cyclic vector for the coordinate and impulse operators). Consider the following unitary operators \(V\) and \(U\) in the space \(L^2(\mathbb{R}, \mu)\) (where \(\mu\) is the normalized Lebesgue measure on the interval \([0, 1]):\n
\[(Vf)(x) = \exp\{i2\pi x\}f(x)\quad\text{and}\quad(Uf)(x) = f(x + \gamma)\].

Assume that the number \(\gamma\) is irrational. Then the following two conditions are equivalent.

1. \(\gamma < 1\).
2. There exists a function \(f \in L^2(\mathbb{R}, \mu)\) of norm 1 such that the orbit of \(f\) under the action of the group generated by the operators \(V\) and \(U\) is a total set in \(L^2(\mathbb{R}, \mu)\) (that is, its linear hull is everywhere dense in \(L^2(\mathbb{R}, \mu)\)). In other words, \(f\) is a cyclic element for the algebra generated by these operators.

**Proof.** Let us use the above computation of the coupling constant. The commutant of the \(W^*\)-algebra generated by the operators \(V\) and \(U\) is the \(W^*\)-algebra generated by the operators \(V'\) and \(U'\), where

\[(V'f)(x) = \exp\{2\pi i \gamma^{-1}\}f(x),\quad(U'f)(x) = f(x + 1)\].

Hence the coupling constant of the first factor is equal to \(\gamma\), and by the above-mentioned theorem from the theory of factors (see [13], [7]), the existence of a cyclic vector is equivalent to the inequality \(\gamma < 1\).

It is not difficult to prove this equivalence directly. Nevertheless, preliminary consultations with specialists did not give at once the right answer; moreover, there are partial results on non-existence of a cyclic element in a given class of functions.\(^3\)

\(^3\)Perhaps, this natural problem has not been studied until now. Its multidimensional analogues (see the discussion of the multidimensional Heisenberg group below) are apparently not so accessible to elementary methods, though the absence of a cyclic vector follows immediately from the general coupling constant theorem. A. Stepin.
4.3. Relation to the quantum torus, rotation algebra, and discrete Heisenberg group. Consider the C*-algebra (= quantum 2-torus) $A_\theta$, where $\theta$ is an irrational positive number less than one. This algebra is generated topologically by two unitary elements $u$ and $v$ with the relation

$$UV = \exp\{2\pi i \theta\}VU.$$ 

It is well known that this algebra has a unique nontrivial trace which generates the regular representation of this algebra as a crossed product; see Section 2. The $K_0$-functor of this algebra treated as an Abelian group is the sum $\mathbb{Z} + \mathbb{Z}$, i.e., $K_0(A_\theta) = \mathbb{Z}^2$, and the set of positive elements in this group is $\{(m, n): m\theta + n > 0\}$.

But if we classify factor representations up to spatial (rather than algebraic) isomorphism, then we obtain continuum many factor representations indexed by positive numbers (the values of the coupling constant). The question arises as to how to construct a natural realization of the representation with a given value of the coupling constant. The above construction gives a realization of the factor representations of the rotation algebras $A_\theta$ and $A_{\theta^{-1}}$ with coupling constants $\theta$ and $\theta^{-1}$. The problem of realizing representations of the same algebra with arbitrary coupling constants remains open. Since all the factor representations in question are representations of the discrete Heisenberg group, the above considerations apply to this group too. Although for many groups (e.g., for $S_\infty$) the problem of describing finite characters and realizing the corresponding representations is solved (or, at least, is a smooth problem), the problem of describing representations with all possible values of coupling constants apparently has not been studied.

5. Factor Representations of Dual Pairs of Non-commutative Dynamical Systems

The case of actions of Abelian groups reduces to consideration of commuting lattices in the generalized Heisenberg group and subsequent factorization of an irreducible representation of this group. However, not all groups have commuting lattices that generate the whole group like the subgroups $\Gamma(\lambda_1, \lambda_2)$ in the Heisenberg group. For example, obviously, semisimple groups have no commuting cocompact lattices that generate topologically the whole group.

However, for general groups, we can apply our scheme to the right and left actions of subgroups and factorize an irreducible representation not of the group itself but of the crossed product of the group and an algebra of functions on a homogeneous space. Let us consider examples of this type.

Given a locally compact noncompact unimodular group $A$, choose two lattices $\Gamma_1$ and $\Gamma_2$ in $A$ with intersection consisting of the identity element and consider the left action of $\Gamma_1$ and the right action of $\Gamma_2$ on the group $A$ with the Haar measure $m$. Since the left and right actions commute with each other, the conditions of our scheme are satisfied. For the action to be ergodic, it suffices to require that the two lattices $\Gamma_1$ and $\Gamma_2$ generate topologically the whole group $A$, which means that the subgroup generated by $\Gamma_1$ and $\Gamma_2$ is everywhere dense in $A$.

In the Hilbert space $L^2(G, m)$, we obtain two factors $B_r$ and $B_l$ that are generated by the multiplicators by bounded functions on the homogeneous space $G/\Gamma_2$. 
(respectively, on $\Gamma_1 \backslash G$) and by the operators of left shifts by elements $\gamma \in \Gamma_1$ (respectively, by right shifts by elements $\gamma \in \Gamma_2$). These four families of operators determine a factorized irreducible representation of the crossed product $(\Gamma_1 \times \Gamma_2) \ltimes C_b(G)$ (where $C_b(g)$ is the space of all bounded measurable functions on $G$). If the lattices are not amenable groups, then, by a known theorem, the factors are not hyperfinite. As above, the coupling constant of the factors is equal to the ratio of the volumes of the fundamental domains.

Below we give simple examples of this situation.

(a) Consider again the Heisenberg group $H^R$ and two its lattices, the discrete Heisenberg group $H^Z$ and the subgroup $\Gamma \equiv \Gamma(\lambda_1, \lambda_2)$ (see Section 3.1). Then, according to the above observations, we have two factors in the space $L^2(H^R)$, which factorize the irreducible representation of the crossed product of the group $H^R$ and the algebra of bounded measurable functions on the group.

In this example, the subgroup $\Gamma$ acts on the compact nilmanifold $H^R/H^Z$, and the group $H^Z$ acts isomorphically on the nilmanifold $H^R/\Gamma$; the coupling constant is easy to compute. The factors are hyperfinite.

(b) Consider the semisimple group SL(2, $R$) and two its lattices $\Gamma_i, i = 1, 2$:

$$\Gamma_i = \left\{ \gamma = \begin{pmatrix} m & n \lambda_i \\ p \lambda_i^{-1} & q \end{pmatrix} \right\},$$

where $\lambda_i, i = 1, 2$, are fixed real numbers with irrational ratio (for example, one of them is an integer greater than one and the second is irrational) and the matrix

$$\begin{pmatrix} m & n \\ p & q \end{pmatrix}$$

is an arbitrary element of the unimodular group SL(2, $Z$). Then our scheme gives two nonhyperfinite factors with a generally non-unity coupling constant generated by the actions of these lattices; they factorize the irreducible representation of the crossed product of SL(2, $R$) and the space of bounded measurable functions on this group.

6. Pseudogroupoids and Factor Representations of the Infinite Symmetric Group

6.1. Definition of a measurable pseudogroupoid. Let us generalize our construction to the case of a non-free group action, or, equivalently, let us restate it in terms of (orbit) partitions. This passage is similar to passage from group actions to groupoids and equivalence relations. Just as the scheme of Section 1 generalizes the classical construction of factor representations of crossed products (Section 2), the following construction generalizes the notion of groupoid and ergodic equivalence relations and their representations. We call the corresponding object a pseudogroupoid. By applying this construction, we can obtain a wider class of examples of factor representations with non-unity coupling constant, and we use it to construct the factor representations of the infinite symmetric group.

Let $(X, \mu)$ be a standard space with $\sigma$-finite infinite continuous measure (isomorphic in the measure-theoretic sense to the real line $R$ with the Lebesgue measure),
and let $\xi$ and $\eta$ be two measurable partitions of the space $(X, \mu)$ satisfying the following conditions.

1. **Homogeneity.**

Almost all elements of $\xi$ and $\eta$ are countable sets, and for any measurable subpartition with finite elements, the conditional measures of its elements are uniform measures (in short, $\xi$ and $\eta$ are homogeneous partitions with countable elements); this condition is satisfied automatically if the partitions are the orbit partitions of measure-preserving group actions.

Let $\Gamma_{\xi}$ ($\Gamma_{\eta}$) denote the group of all measure-preserving transformations that leave the partition $\xi$ (respectively, $\eta$) invariant and the partition $\eta$ (respectively, $\xi$) fixed (that is, all $\eta$- ($\xi$-) measurable subsets are invariant mod 0).

2. **Commutation.**

**Definition.** Partitions $\xi$ and $\eta$ are called commuting if the automorphism groups $\Gamma_{\xi}$ and $\Gamma_{\eta}$ commute.

3. **Ergodicity.**

The group generated by the groups $\Gamma_{\xi}$ and $\Gamma_{\eta}$ acts ergodically. In particular, the measurable intersection of $\xi$ and $\eta$ is trivial.

**Remarks.** 1. The ergodicity assumption is made only to exclude the trivial case when the groups $\Gamma_{\xi}$ and $\Gamma_{\eta}$ are too small. The definition still makes sense without condition (3).

2. Conditions (1)–(3) from Section 1 together with the ergodicity of the group actions of $G$ and $H$ are a special case of conditions (1)–(3) above; namely, the partitions $\xi$ and $\eta$ are the orbit partitions for the groups $G$ and $H$; here the actions are no longer assumed to be free.

By a fundamental domain of the partition $\xi$ (respectively, $\eta$) we mean an arbitrary measurable one-layer (i.e., intersecting almost every element of the partition in exactly one point) subset of maximum measure. It is not difficult to prove that such subsets exist and have equal (maybe infinite) measures.

Consider the algebras $C(\xi)$ and $C(\eta)$ of measurable functions that are constant on the elements of the partitions $\xi$ and $\eta$, respectively. We can again consider the crossed products of the group $\Gamma_{\xi}$ ($\Gamma_{\eta}$) and the algebra $C(\xi)$ (respectively, $C(\eta)$). These crossed products are naturally represented in the space $L^2(X, \mu)$ and generate two $W^*$-algebras that commute with each other and are factors under condition (3). If the measures of the fundamental domains are finite, then the factor is of type $\text{II}_1$. Theorem 1 remains true in this case too: the coupling constant is equal to the ratio of the measures of the fundamental domains; the proof remains the same. It is easy to see how the scheme of Section 1 fits into this construction: $\xi$ and $\eta$ are the orbit partitions of the groups $G$ and $H$.

Assume that the fundamental domain is the same for both partitions and has a finite measure. In this case, we can define actions of both groups $\Gamma_{\xi}$ and $\Gamma_{\eta}$ on this fundamental domain which determine the same ergodic equivalence relation. Thus, we have a (principal) groupoid, our construction turns into a groupoid construction,
and the common fundamental domain becomes the diagonal. Conversely, a measurable (principal) groupoid with measure is determined by an ergodic equivalence relation and eventually reduces to our construction in which the initial measure space is the common fundamental domain (diagonal) for the right and left actions of the groupoid (see [10]). This provides motivation for the following definition.

**Definition.** A space with a σ-finite measure and two measurable partitions with countable elements satisfying conditions (1)–(3) is called a *measurable pseudogroupoid*. If the partitions have a common fundamental domain, then the pseudogroupoid is a principal measurable groupoid (with countable elements).

A topological pseudogroupoid that agrees in a natural sense with the introduced notion of a measurable pseudogroupoid will be defined elsewhere.

### 6.2. A realization of the factor representations of the infinite symmetric group.

One of the new effects of the groupoid method and its pseudogroupoid generalization suggested above (in comparison with von Neumann’s construction) is that, sometimes, the group action itself (rather than the crossed product with a commutative algebra) generates a factor representation. This is possible because the action of the corresponding groups is non-free. Such a possibility often arises in the representation theory of inductive limits of classical groups; the effect was first observed in constructing factor representations of the infinite symmetric group described in this section.

Now, let us consider the problem of constructing a realization of the factor representations of the infinite symmetric group with nontrivial coupling constants. Simultaneously, we present the known realization [14] of these representations from a more general point of view.

The interest in factor representations of type II$_1$ is based on the fact that traces on these factors are finite characters of the group, and for many groups the set of finite characters is sufficiently large for the purposes of harmonic analysis on the group.

Since a trace is determined by an element of the space of the representation (in short, the trace is spatial) if and only if the coupling constant is equal to one, characters are not suitable for description of other factor representations. As mentioned above, these factor representations are *quasiequivalent*, or, which is the same thing in the case under consideration, *algebraically equivalent* to representations with traces; nevertheless, it is of interest to find their direct realization, especially since it is not easy to establish explicitly the quasiequivalence of factor representations. The complete list of factor representations of type II$_1$ (up to spatial isomorphism) is indexed by the points of the set $\Psi \otimes \mathbb{R}$, where $\psi \in \Psi$ is a finite character of the group (a trace of the algebra), and $\lambda \in \mathbb{R}$ is a positive number that determines the coupling constant.

For the infinite symmetric group, the problem of explicit description of all factor representations of type II$_1$ with spatial trace is solved long ago. A non-tautological (i.e., non-GNS-) realization of these representations was given in [14]. In this realization, as in the case of other classical series of groups, the groupoid (or orbit) construction of the crossed product was used. This construction was based on the
above-mentioned important fact that, in a nondegenerate case, the factor representation of the group itself gives the same $W^*$-algebra as the representation of the crossed product.

Let us first reproduce the groupoid construction of these representations of the group $S_\infty$ with spatial trace suggested in [14] in a slightly different form; on the one hand, we use a more convenient notation which was also applied in [9] for other purposes (to prove Thoma’s theorem); on the other hand, which is the main thing, it is suitable for subsequent realization of factors with nontrivial coupling constant, i.e., in the pseudogroupoid case.

Let $X_0$ be the set of \textit{eventually symmetric} sequences of symbols from a finite alphabet $A$:

$X_0 = \{ x = \{ x_i \}_{i \in \mathbb{Z} \setminus 0}, \ x_i \in A, \ i \in \mathbb{Z} \setminus 0 \}$,

and for every $x$, $x_{-i} = x_i$ for sufficiently large $i = i(x) > 0$,

and the compositions (i.e., the number of symbols of each kind) of finite intervals $x_1, \ldots, x_N$ and $x_{-1}, \ldots, x_{-N}$ of the positive and negative parts of a sequence coincide for sufficiently large $N$.

We endow the subset of symmetric sequences $X_{00} = \{ \{ x_i \}: x_i = x_{-i}, \ i = 1, 2, \ldots \}$ with the Bernoulli (product) measure $\mu_0$, which is determined by a probability measure $\nu$ on $A$. Two copies of the group $S_\infty$ act separately on the positive and negative parts of the sequences; obviously, these actions commute with each other. Consider the partitions $\xi$ and $\eta$ of the space $X_0$ into orbits: an element of the partition $\xi$ (respectively, $\eta$) is the set of all sequences with equal positive (respectively, negative) coordinates and lying in the same orbit of the left (respectively, right) group. We endow the space $X_0$ with a $\sigma$-finite measure that extends the measure $\mu_0$ on $X_{00}$ to a measure on $X_0$ invariant with respect to the action of $S_\infty \times S_\infty$; obviously, there such an extension is unique. The set $X_{00}$ is a common fundamental set for both partitions, and the conditions of the scheme of Section 1 are satisfied.

Now, consider the representation of the group $S_\infty \times S_\infty$ in the Hilbert space $L^2(X_0, \mu)$ by substitution operators.

**Theorem** [14]. \textit{If the values of the measure $\nu$ at one-point subsets of a countable (or finite) set $A$ are pairwise distinct, then the representation of each of the two symmetric groups is a factor representation of type $\text{II}_1$, and these representations are mutual commutants; thus, they factorize the irreducible representation of the group $S_\infty \times S_\infty$.}

**Remark.** In [14], the theorem was stated in terms of groupoid theory. As mentioned above, the nontrivial fact, which stems from the nonfree character of the group actions, is that the $W^*$-closure of the representation of the crossed product of each of the two symmetric groups with the corresponding space of functions coincides with the $W^*$-closure of the images of the group algebras; somehow the structure of a crossed product appears in the representation automatically!\footnote{This bears no relation to the well-known fact that each $\text{AF}$-algebra has the structure of a crossed product; here we have a completely different structure of such product, which is due to specific properties of the group algebra of the infinite symmetric group.}
The trace is generated by the characteristic function of the set $X_{00}$, which is a bicyclic element; the coupling constant is equal to one. The value of the character at any element of the two subgroups is the $\mu_0$-measure of the set of fixed points for the action of this element on $X_{00}$. The simplest example is $A = \{0, 1\}$, $\nu = \{\alpha, 1 - \alpha\}$, where $0 < \alpha < 1/2$.

In the case where the measure $\nu$ has multiple values at the points of the set $A$ (in the above example, this case corresponds to $\alpha = 1/2$), there is an additional symmetry, and the element $\chi$ is not cyclic: the commutant of each factor is wider, and the representations of $S_\infty \times S_\infty$ are reducible. However, it is easy to describe the additional decomposition into irreducible representations.

Let us proceed to the construction of a factor representation of the group $S_\infty$ with non-unity coupling constant; we shall use the pseudogroupoid construction of the previous section. Let $r$ be a positive integer, and let $X_r$ be the set of two-sided infinite sequences $x = \{x_i\}_{i \in \mathbb{Z}}$ with the following symmetry condition: for each sequence $x$, there exists a $k = k(x) \geq 0$ such that $x_{-i - k} = x_{i+k+r}$, $i = 1, 2, \ldots$, and the multiset $\{x_{-1}, \ldots, x_{-N}\}$ is contained in the multiset $\{x_1, \ldots, x_{N+r}\}$. If $r = 0$, then $X_r$ is the set $X_0$ of eventually symmetric sequences defined above. Consider the two actions of the symmetric group on this space defined above and a $\sigma$-finite measure $\mu'$ on $X_r$ which is an extension of the measure $\mu_0'$ defined on the set of sequences $X_{0,r} \equiv \{x: x_{-i} = x_{i+r}, \ i = 1, 2, \ldots\}$ (shifted symmetric sequences) as above, i.e., as the Bernoulli measure with factor $\nu$. The extension $\mu'$ of the measure $\mu_0'$ to the whole space is defined by invariance under the actions of the symmetric groups.

We take the set $X_{0,r}$ as a fundamental domain for the action of the group of permutations of the negative indices and its subset $X'_{0,r} \subset X_{0,r}$ consisting of sequences $\{x_i\} \subset X_{0,r}$ such that $x_{-i} = x_i$ for $|i| \leq r$, $i \neq 0$, as a fundamental domain for the action of the group of permutations of the positive indices. This condition guarantees that the intersection of the orbit and the fundamental domain consists of a single point.

Let a measure $\nu$ on a finite or countable alphabet $A$ be determined by a vector $\alpha_1, \alpha_2, \ldots$, where $1 > \alpha \geq \alpha_2 \geq \cdots \geq 0$ and $\sum_k \alpha_k = 1$. Then, assuming that the measure of the first fundamental domain is equal to one, we find that the measure of the second fundamental domain is

$$\left(\sum_i \alpha_i^2\right)^r.$$ 

This is precisely the value of the coupling constant for the factor representation of the group $S_\infty$ acting on the negative indices. Thus, for a positive $r$, the coupling constant is not equal to one.

In this example, the commutant of the factor representation of the group $S_\infty$ acting on the negative indices is generated by the representation of the group $S_\infty$ acting on the positive indices and a finite or countable (depending on the alphabet $A$) number of projections to pairwise orthogonal subspaces of the form $H_a$, $a = (a_1, \ldots, a_r)$, which consist of functions from the space $L^2(X_r)$ that vanish outside the cylinder set of sequences with first $r$ symbols equal to $a_1, \ldots, a_r$. It
is easy to see that these projections also lie in the commutant. In the decomposition of the space $L^2(X_r)$ into the direct sum of the subspaces $H_{\alpha}$ over all $r$-tuples from the alphabet $A$, the weight of each subspace is equal to the product $\alpha_1 \cdots \alpha_r$ (probability of the tuple), and in each subspace, the value of the coupling constant is equal to the same product, whence we obtain again the desired value of the coupling constant in the whole space. The example admits wide generalizations.

In conclusion, note that the values of the coupling constants of factor representations of groups and algebras that arise in natural constructions are closely related to the $K$-functors of the corresponding algebras. For modules over the rotation algebras $A_\theta$, this fact was mentioned in a paper by A. Connes [1].

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