PICARD GROUPS IN POISSON GEOMETRY

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Dedicated to Pierre Cartier

ABSTRACT. We study isomorphism classes of symplectic dual pairs $P \leftarrow S \rightarrow T$, where $P$ is an integrable Poisson manifold, $S$ is symplectic, and the two maps are complete, surjective Poisson submersions with connected and simply-connected fibres. For fixed $P$, these Morita self-equivalences of $P$ form a group $\text{Pic}(P)$ under a natural “tensor product” operation. Variants of this construction are also studied, for rings (the origin of the notion of Picard group), Lie groupoids, and symplectic groupoids.

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1. Introduction

In his article on the occasion of the 40th anniversary of the IHES, Pierre Cartier [11] stressed the idea of “representations as points,” which was central to the work of Grothendieck, and which leads to categories in which morphisms map representations rather than “ordinary” points. In algebra, this idea underlies the notion of Morita equivalence [35], and the same term is now applied to objects which “encode” algebras, such as groupoids and Poisson manifolds. In all these settings, a morphism between objects $A$ and $B$ is an isomorphism class of “bimodules” of some sort, on which $A$ and $B$ have commuting actions from the left and right respectively. Morphisms are composed by an associative “tensor product” operation, and the resulting group of self-equivalences of an object is called its Picard group.

Morita equivalence for Poisson manifolds and symplectic groupoids was introduced by Xu [52], [53], based on notions of equivalence of topological groupoids by Haefliger [21], Muhly, Renault, and Williams [38], Moerdijk [32] and others, as well as on the idea of a symplectic dual pair [48]. The latter was itself an extension of Howe’s notion of linear dual pair [23], also introduced for the purpose of obtaining equivalences of representation categories. Xu’s ideas and their relation to
Rieffel’s [43] strong Morita equivalence of $C^*$-algebras have been explained and extended by Landsman in a series of publications [26], [27], [28], of which the present work may be seen as a further extension.

In this paper, we briefly review the work of Xu and Landsman, and then study several interesting examples, concentrating on their Picard groups. We also discuss briefly another notion of equivalence in Poisson geometry, gauge equivalence [46], which has been shown [9], [24] to be related to algebraic Morita equivalence via deformation quantization; we revisit and extend the results in [7] relating gauge equivalence to geometric Morita equivalence. Work of Rieffel and Schwarz [44] also suggests a relation between gauge equivalence and strong Morita equivalence of $C^*$-algebras; also see [47], in which Dirac structures play a key role.

In the final sections, we briefly treat some subjects which we have just begun to study. We investigate the Lie algebras of Picard groups in various categories. We also suggest a way of “enriching” the structure of the category of representations of a Poisson manifold so that representation equivalence implies Morita equivalence. Finally, we suggest a way of extending the notion of Morita equivalence for groupoids and Poisson manifolds. By enlarging the class of bimodules from manifolds to more general leaf spaces of foliations, we hope to eliminate some technical restrictions (integrability and regularity) which have limited previous work on geometric Morita equivalence. Here, the bimodules themselves are Morita equivalence classes!

The paper as a whole is somewhat like [50] in that we introduce a new concept in Poisson geometry, study some examples, and ask a lot of questions.

2. Bimodules as Generalized Morphisms of Algebras

Before passing to geometry, we will review the idea of generalized morphisms in the algebraic context where it first arose.

Let us consider the category whose objects are unital algebras over a fixed (commutative, unital) ring $k$ and arrows are algebra homomorphisms. Since homomorphisms act from the left, we adopt the convention that a homomorphism $\phi$ from $B$ to $A$ is generally written as an arrow $A \leftarrow B$. We will denote the automorphism group of a $k$-algebra $A$ by $\text{Aut}(A)$.

We can define another category with the same objects, a “generalized morphism” $A \leftarrow B$ being an $(A, B)$-bimodule $AXB$, i.e., a $k$-module which is a left $A$-module and right $B$-module on which the actions of the two algebras respect the $k$-module structure and commute. The “composition” of $AXB$ and $BYC$ is defined to be their tensor product over $B$, an $(A, C)$-bimodule. Any algebra homomorphism $A \leftarrow B$ can be regarded as such a generalized morphism, namely the $(A, B)$-bimodule $A\phi$, with $A$-action given by left multiplication and right $B$-action defined by $a \cdot b := a\phi(b)$.

A feature of the more general “morphisms” (and the reason for the quotes) is that, strictly speaking, they do not define a category. Rather, they are horizontal arrows in a bicategory, see, e.g., [5], [30]. For example, their composition is not
associative, but just associative up to a (natural) bimodule isomorphism:

\[(A \otimes_B Y) \otimes_C cZ \cong A \otimes_B (Y \otimes_C cZ).\]

Similarly, if we have homomorphisms \(A \xrightarrow{\phi} B\) and \(B \xrightarrow{\psi} C\), then the \((A, C)\)-bimodules \(A \otimes_B \phi B \phi A\) and \(A \otimes_B \psi B \psi A\) are naturally isomorphic, but not equal. We therefore introduce a category \(\text{Alg}\) in which the objects are unital algebras (the base ring \(k\) being fixed) and the morphisms \(A \leftrightarrow B\) are isomorphism classes of \((A, B)\)-bimodules. The map \(\phi \mapsto A_\phi\) induces a functor \(\phi \mapsto A_\phi\) from the usual category of algebras to \(\text{Alg}\). We will soon see that \(\phi\) is not faithful.

Invertible morphisms in \(\text{Alg}\) are known as Morita equivalences, and the group of automorphisms of \(A\) in \(\text{Alg}\) is called its Picard group, denoted by \(\text{Pic}(A)\). The functor \(\phi\) restricts to a group homomorphism from \(\text{Aut}(A)\) to \(\text{Pic}(A)\).

A simple computation shows that the automorphisms which become trivial in the Picard group are just the inner automorphisms; i.e., we have the following exact sequence of groups [3]:

\[1 \to \text{Inn}\text{Aut}(A) \to \text{Aut}(A) \overset{j}{\to} \text{Pic}(A).\]  

In particular, when \(A\) is commutative, \(\text{Aut}(A)\) sits inside \(\text{Pic}(A)\) as a subgroup via the map \(j\).

Now let \(\mathcal{Z}(A)\) denote the center of \(A\). There is a natural group homomorphism [3]

\[h: \text{Pic}(A) \to \text{Aut}(\mathcal{Z}(A))\]

which takes (the isomorphism class of) each invertible \((A, A)\)-bimodule \(X\) to \(h_X \in \text{Aut}(\mathcal{Z}(A))\), defined by the condition that \(h_X(z)x = zx\) for all \(z \in \mathcal{Z}(A)\) and \(x \in X\).

If we denote by \(\text{Pic}_{\mathcal{Z}(A)}(A)\) the subgroup of \(\text{Pic}(A)\) given by bimodules \(X\) satisfying \(zx = xz\) for all \(x \in X\) and \(z \in \mathcal{Z}(A)\), then we have the following exact sequence:

\[1 \to \text{Pic}_{\mathcal{Z}(A)}(A) \to \text{Pic}(A) \overset{h}{\to} \text{Aut}(\mathcal{Z}(A)).\]

When \(A\) is commutative, a simple computation shows that the map \(h\) is split by the map \(j\). In this case, one can write \(\text{Pic}(A)\) as a semi-direct product of \(\text{Aut}(A)\) and the group \(\text{Pic}_{\mathcal{Z}(A)}(A) := \text{Pic}_{\mathcal{Z}(A)}(A)\):

\[\text{Pic}(A) = \text{Aut}(A) \rtimes \text{Pic}_{\mathcal{Z}(A)}(A).\]

The action of \(\text{Aut}(A)\) on \(\text{Pic}_{\mathcal{Z}(A)}(A)\) is given by \(X \xrightarrow{\phi} \phi X\phi\), where the left and right \(A\)-module structures on \(\phi X\phi\) are given by \(a \cdot x := \phi(a)x\) and \(x \cdot a := x\phi(a)\). The group \(\text{Pic}_{\mathcal{Z}(A)}(A)\), consisting of bimodules for which the left and right module structures are the same, is often called the commutative Picard group of \(A\) (to avoid confusion with \(\text{Pic}(A)\), which makes sense even when \(A\) is noncommutative).

**Example 2.1.** Let \(M\) be a smooth manifold, and let \(A = C^\infty(M)\) be the \(C\)-algebra of complex-valued smooth functions on \(M\). In this case,

\[\text{Pic}(A) \cong \text{Pic}(M) \cong H^2(M, \mathbb{Z}),\]

where \(\text{Pic}(M)\) is the set of isomorphism classes of complex line bundles over \(M\), with group operation given by tensor product. The first isomorphism in (2.4) is a consequence of the Serre–Swan theorem (see, e.g., [3]), and the second is the Chern
class map. Since all the automorphisms of \( C^\infty(M) \) come from diffeomorphisms\(^1\) of \( M \), (2.3) becomes for this example

\[
1 \to H^2(M, \mathbb{Z}) \to \text{Pic}(C^\infty(M)) \xrightarrow{h} \text{Diff}(M),
\]

(2.5)

and we obtain the purely geometric description of \( \text{Pic}(C^\infty(M)) \) as the semidirect product \( \text{Diff}(M) \rtimes \text{Pic}(M) \), with action given by pull-back of line bundles.

In addition to the homomorphisms (2.2), one has for any invertible morphism between \( A \) and \( B \) in \( \text{Alg} \) an algebra isomorphism between \( \mathcal{Z}(A) \) and \( \mathcal{Z}(B) \). In particular, two unital commutative \( k \)-algebras are isomorphic in \( \text{Alg} \) (i.e., Morita equivalent) if and only if they are isomorphic in the usual sense [3]. Despite this, regarding commutative algebras as objects in \( \text{Alg} \) has the effect of enlarging the possible ways an algebra can be “isomorphic” to itself. As we saw in Example 2.1, for \( A = C^\infty(M) \) these “extra ways” (i.e., \( \text{Pic}_A(A) \)) have a geometric interpretation as the (isomorphism classes of) line bundles over \( M \).

As we will discuss in the next sections, groupoids, symplectic groupoids, and integrable Poisson manifolds can be similarly regarded as objects in more general categories, in which the morphisms are geometric “bimodules”. Inspired by Example 2.1, we shall investigate Picard groups in this setting, the contribution to them of the geometric automorphisms, and the analogues of the exact sequences (2.1) and (2.3).

3. Morita Equivalence and Picard Groups of Lie Groupoids

3.1. Generalized morphisms of Lie groupoids. Generalized morphisms of Lie groupoids were studied in detail by Mrčun [37], who called them \textit{Hilsum–Skandalis maps}, following [22] (see also [34] and references therein). We begin our discussion by fixing some notation and terminology.

Given a Lie groupoid \( \Gamma \) over a manifold \( P \), we denote its unit map by \( \varepsilon: P \to \Gamma \), the inversion by \( i: \Gamma \to \Gamma \) and the target (resp. source) map by \( t: \Gamma \to P \) (resp. \( s: \Gamma \to P \)). We will often identify an element \( x \) in \( P \) with its image \( \varepsilon(x) \in \Gamma \).

A (**left**) action of \( \Gamma \) on a manifold \( S \) consists of a map \( P \xleftarrow{s} S \) called the moment and a map from \( \{(g, x) \in \Gamma \times S : s(g) = J(x)\} \) to \( S \) satisfying the appropriate axioms (see for example [31]). Right actions are defined in a similar way (in which case we indicate the moment as \( S \xrightarrow{J} P \)). The action is **principal** with respect to a mapping \( p: S \to M \) if \( p \) is a surjective submersion and if \( \Gamma \) acts freely and transitively on each fibre of \( p \). (\( p: S \to M \) is sometimes referred to as a principal \( \Gamma \)-bundle.)

\(^1\)We do not know of any published source for this “folk theorem”. For connected \( M \), one usually starts by showing that any homomorphism from \( C^\infty(M) \) to \( \mathbb{C} \) is given by evaluation at a point. The complex case which interests us here can be reduced to the real case by a simple, elementary argument (shown to us by Bill Arveson). The case of real-valued functions is standard, though Maksim Maydanskiy has observed that the correspondence may fail if the set of components of \( M \) has cardinality greater than that of the continuum. On the other hand, Janez Mrčun has pointed out that one can prove the automorphism theorem directly, without any cardinality assumption. See Chapter IV of [25] for an extensive discussion of these issues.
If the groupoids $\Gamma_1$ over $P_1$ and $\Gamma_2$ over $P_2$ act on a manifold $S$ from the left and right, respectively, and the actions commute, we call $S$ a $(\Gamma_1, \Gamma_2)$-bibundle. Let $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$ be the moments. The bibundle induces a correspondence between the orbit spaces of $\Gamma_1$ and $\Gamma_2$ by the rule that orbits $O_1$ and $O_2$ are related if $J_1^{-1}(O_1)$ meets $J_2^{-1}(O_2)$.

Since the essential “ingredients” of a groupoid are its orbits and its isotropy groups, a bibundle should represent a generalized morphism $\Gamma_1 \leftarrow \Gamma_2$ only when the relation induced by $S$ is a map of orbit spaces $P_1/\Gamma_1 \leftarrow P_2/\Gamma_2$, and when each point $x \in S$ determines a homomorphism from the isotropy group of $J_2(x)$ in $\Gamma_2$ to that of $J_1(x)$ in $\Gamma_1$. Both of these conditions are met when the bibundle is left principal, i.e., when the left action of $\Gamma_1$ is principal with respect to the right moment map $J_2$. (Transitivity of the $\Gamma_1$ action on $J_2$ fibres makes the map on orbit spaces single-valued, while freeness gives the action on isotropy groups.) It follows from the assumption of left principality (see [28, Lemma 4.18]) that the $\Gamma_1$ action is proper and that the map between orbit spaces is smooth when the orbit spaces are manifolds.

It is sometimes useful to think of each element $x$ of $S$ as an arrow pointing from $J_2(x)$ to $J_1(x)$. The operations of $\Gamma_1$ and $\Gamma_2$ on $S$ may then be thought of as composition with the arrows in the groupoids. $S$ is left principal if the “source” map $J_2$ is a surjective submersion, and if any two arrows with the same source have a unique “quotient” in $\Gamma_1$. In other words, the bimodule is essentially the same thing as a (small) category whose set of objects is the disjoint union of $P_1$ and $P_2$, whose restriction to each $P_i$ is the groupoid $\Gamma_i$, and whose morphisms from $P_2$ to $P_1$ (there are none in the other direction) have a certain divisibility property.

A “tensor product” of bibundles [22] is then defined by “composition of arrows” as follows. Let $\Gamma_j$ be a groupoid over $P_j$, for $j = 1, 2, 3$. If $S$ is a $(\Gamma_1, \Gamma_2)$-bibundle with moments $P_1 \xrightarrow{J_1} S \xrightarrow{J_2} P_2$, and $S'$ is a $(\Gamma_2, \Gamma_3)$-bibundle with moments $P_2 \xrightarrow{J_2'} S' \xrightarrow{J_3'} P_3$, their product is the orbit space

$$S \star S' := (S \times_{P_2} S')/\Gamma_2,$$

where $S \times_{P_2} S' = \{(x, y) \in S \times S' : J_2(x) = J_2'(y)\}$ and the (right) $\Gamma_2$-action is given by

$$(x, y) \mapsto_{S} (xg, g^{-1}y).$$

Fortunately, the assumption that $S$ and $S'$ be left principal insures that $S \star S'$ is a smooth manifold, and that

$$\begin{array}{ccc}
\Gamma_1 & \xrightarrow{J_1} & S \star S' \xleftarrow{J_3'} \Gamma_3 \\
\xrightarrow{J_2} P_1 & & \xleftarrow{J_3} P_2
\end{array}$$

is a left principal $(\Gamma_1, \Gamma_3)$-bibundle.

Two $(\Gamma_1, \Gamma_2)$-bibundles $S_1$ and $S_2$ are isomorphic if there is a diffeomorphism between them which commutes with the groupoid actions (including their moments). It turns out that the product $\star$ is associative up to (natural) isomorphism, so that we
may define a category $LG$ in which the objects are Lie groupoids and the morphisms $\Gamma_1 \leftarrow \Gamma_2$ are isomorphism classes of left principal $(\Gamma_1, \Gamma_2)$-bibundles, which we call \textit{generalized morphisms} between groupoids. Following the terminology for algebras, we call two Lie groupoids $\Gamma_1$ and $\Gamma_2$ \textit{Morita equivalent} if they are isomorphic as objects in $LG$; this condition is equivalent to the existence of a $(\Gamma_1, \Gamma_2)$-bibundle which is left and right principal (i.e., \textit{biprincipal}). Such a bibundle is also called a \textit{Morita bibundle}.

\textbf{Example 3.1 (Gauge groupoids).} Let $\Gamma$ be a Lie groupoid over $P$, and let $E \xrightarrow{p} B$ be a right principal $\Gamma$-bundle. The \textit{gauge groupoid} associated with this principal bundle is the quotient of the groupoid $E \times E$ by the right diagonal action of $\Gamma$. More explicitly, it is a groupoid over $B$, with target (resp. source) map given by $[(x, y)] \mapsto p(x)$ (resp. $[(x, y)] \mapsto p(y)$), and multiplication defined by $[(x_1, y_1)][(x_2, y_2)] = [(x_1, y_2)]$, where we choose representatives so that $y_1 = x_2$ (which is always possible since $\Gamma$ acts transitively on $p$-fibres). We denote the gauge groupoid of $E$ by $G(E)$.

Let us assume that the moment $J$ for the $\Gamma$-action on $E$ is a surjective submersion. Note that $E$ carries a natural left $G(E)$-action, with moment $p: E \rightarrow B$, defined by $[(x, y)] \cdot z = x$, where $x$ is uniquely determined by the condition $y = z$. A simple computation shows that this action is principal with respect to $J: E \rightarrow P$, and it is easy to see that it commutes with the right $\Gamma$-action. As a result, 

\begin{equation} \begin{array}{c} G(E) \\ \downarrow p \downarrow E \downarrow \Gamma \\ B \downarrow \downarrow \downarrow \downarrow \downarrow J \\ \downarrow \Gamma \downarrow \downarrow P \end{array} \end{equation}

is a biprincipal $(G(E), \Gamma)$-bibundle, and $G(E)$ and $\Gamma$ are Morita equivalent.

\textbf{3.2. Picard groups of Lie groupoids.} Let us recall that a groupoid homomorphism is a functor, when groupoids are thought of as categories. Any groupoid homomorphism $\Gamma_1 \xrightarrow{\Phi} \Gamma_2$ can be seen as a generalized morphism as follows: we let $(\Gamma_1)_\Phi$ consist of pairs $(g, y) \in \Gamma_1 \times P_2$ such that $s(g) = \Phi(y)$; this space carries a left principal bibundle structure defined by a left $\Gamma_1$-action, with moment $(g, y) \mapsto \ell(g)$, by $g_1 \cdot (g, y) = (g_1g, y)$, and a right $\Gamma_2$-action, with moment $(g, y) \mapsto y$, by $(g, y) \cdot g_2 = (g\Phi(g_2), s(g_2))$. One can check that this correspondence of morphisms preserves their composition,

\begin{equation} (\Gamma_1)_\Phi \ast (\Gamma_2)_\Psi \cong (\Gamma_1)_\Phi \circ_\Psi, \end{equation}

and gives rise to a functor $j$ from the “conventional” category of groupoids and groupoid homomorphisms into $LG$. (See, for example, [34]. The case of étale groupoids is treated in [37], but the general case is essentially the same.)

Let $\Gamma$ be a Lie groupoid over $P$. The group of ordinary groupoid automorphisms of $\Gamma$ is denoted by $\text{Aut}(\Gamma)$, and we define the \textit{Picard group} of $\Gamma$, $\text{Pic}(\Gamma)$, as the group of automorphisms of $\Gamma$ regarded as an object in $LG$ (i.e., $\text{Pic}(\Gamma)$ is the group of isomorphism classes of biprincipal $(\Gamma, \Gamma)$-bibundles). The unit in $\text{Pic}(\Gamma)$ is the isomorphism class of $\Gamma$, regarded as a $(\Gamma, \Gamma)$-bibundle with respect to left and right multiplication.
As in the case of algebras, the functor $j$ restricts to a group homomorphism
\[ j: \text{Aut}(\Gamma) \to \text{Pic}(\Gamma), \quad \Phi \mapsto [(\Gamma)\Phi]. \] (3.3)

We now discuss the analogue of the exact sequence (2.1).

Recall that a submanifold $N$ of a groupoid $\Gamma$ over $P$ is called a bisection if the restrictions $l_N$ and $s_N$ of $t$ and $s$ are diffeomorphisms from $N$ to $P$. The bisections form a group $\text{Bis}(\Gamma)$ with the operations of setwise multiplication and inversion. A bisection $N$ induces diffeomorphisms (called left and right slidings by Albert and Dazord [1]) $l_N$ and $r_N$ of $\Gamma$ by $l_N(g) = ag$ where $a \in N$ is such that $s(a) = t(g)$, and $r_N(g) = gb$ where $b \in N$ is such that $s(g) = t(b)$. Left slidings commute with right slidings, and the composition of $l_N$ with $r_N^{-1}$ is an automorphism $\Phi_N$, which we call an inner automorphism of $\Gamma$. (Here $N^{-1} = i(N)$, where $i$ is the inversion map.) The group of inner automorphisms of $\Gamma$ is denoted by $\text{InnAut}(\Gamma)$.

**Proposition 3.2.** Let $\Gamma$ be a Lie groupoid over $P$. The following is an exact sequence:
\[ 1 \to \text{InnAut}(\Gamma) \to \text{Aut}(\Gamma) \xrightarrow{j} \text{Pic}(\Gamma). \] (3.4)

**Proof.** Let $\Phi \in \text{Aut}(\Gamma)$. Note that $(\Gamma)\Phi$ can be identified with the manifold $\Gamma$, carrying a left $\Gamma$-action by left multiplication and a right $\Gamma$-action, with moment $\Phi^{-1} \circ s: \Gamma \to P$, by $z \cdot g = z\Phi(g)$.

Suppose $f: (\Gamma)\Phi \to \Gamma$ is a $(\Gamma, \Gamma)$-bbundle isomorphism. Since $f$ commutes with moments, we must have $t(f(x)) = x$ and $s(f(x)) = \Phi^{-1}(x)$, for all $x \in P$. It then follows that the submanifold $N = f(P)$ is a bisection of $\Gamma$.

Now, if $z \in \Gamma$ and $x = s(z)$, then $f(z) = f(zx) = za$, where $a = f(x) \in N$ is uniquely determined by $t(a) = s(z)$. Thus $f = r_N$. On the other hand, since $f$ is a bisection isomorphism, we have $f(z\Phi(g)) = z\Phi(g)b$, for $b \in N$ satisfying $t(b) = s(\Phi(g))$, and also $f(z\Phi(g)) = f(z)g = zb'g$, for $b' \in N$ such that $t(b') = s(z) = t(\Phi(g))$. Note that $b'$ satisfies $s(b') = t(g)$, and this condition also determines it uniquely. So $\Phi(g)b = b'g$, which is equivalent to $\Phi = l_N \circ r_N^{-1} = \Phi_N$.

Conversely, if $\Phi_N$ is an inner automorphism of $\Gamma$, it is easy to check that the right sliding $r_N$ is a bisection isomorphism $(\Gamma)\Phi \to \Gamma$. $lacksquare$

The isotropy groups of points in the same orbit of $\Gamma$ are isomorphic to each other, but not in a canonical way. Their centers, however, can be canonically identified. Let $\text{ZBis}(\Gamma)$ be the subgroup of bisections with values in the centers of the isotropy subgroups of $\Gamma$ which are “constant on orbits”. This is the subgroup of $\text{Bis}(\Gamma)$ giving rise to trivial inner automorphisms; i.e., the following is an exact sequence:
\[ 1 \to \text{ZBis}(\Gamma) \to \text{Bis}(\Gamma) \to \text{InnAut}(\Gamma) \to 1. \] (3.5)

Let $\text{OutAut}(\Gamma) := \text{Aut}(\Gamma)/\text{InnAut}(\Gamma)$ denote the group of outer automorphisms of $\Gamma$. The next simple observation describes when $\text{Pic}(\Gamma)$ coincides with $\text{OutAut}(\Gamma)$.

**Lemma 3.3.** The exact sequence (3.4) extends to
\[ 1 \to \text{InnAut}(\Gamma) \to \text{Aut}(\Gamma) \xrightarrow{j} \text{Pic}(\Gamma) \to 1 \] (3.6)
(i.e., \( j \) is onto) if and only if for every biprincipal \((\Gamma, \Gamma)\)-bibundle \(S\), with moments \(P \xrightarrow{J_1} S \xrightarrow{J_2} P\), there exists a smooth map \(\sigma: P \to S\) with \(J_2 \circ \sigma = \id\) and \(J_1 \circ \sigma \in \Diff(P)\). In this case, \(\Pic(\Gamma) \cong \OutAut(\Gamma)\).

**Proof.** Let \(S = (\Gamma)_\Phi\), for \(\Phi \in \Aut(\Gamma)\). Then \(\sigma = \varepsilon \circ \Phi|_P\), where \(\varepsilon\) is the identity embedding, is a cross section for the moment \(J_2 = \Phi^{-1}\circ \varepsilon\), and \(J_1 \circ \sigma = t \circ \varepsilon \circ \Phi|_P = \Phi|_P \in \Diff(P)\).

Conversely, let \(S\) be a \((\Gamma, \Gamma)\)-Morita bibundle, with moments \(J_1\) and \(J_2\), and suppose \(\sigma: P \to S\) is a map satisfying \(J_2 \circ \sigma = \id\) and \(J_1 \circ \sigma \in \Diff(P)\). Let us define the map \(f: S \to \Gamma\) by \(f(z) = g\), where \(g\) is the unique element in \(\Gamma\) satisfying \(g\sigma(J_2(z)) = z\) in \(S\). It is clear that \(f\) preserves left \(\Gamma\)-actions, and it is a diffeomorphism as a result of the principality of the left \(\Gamma\)-action on \(S\) and the condition \(J_1 \circ \sigma \in \Diff(P)\). Note that the condition \(f(z \cdot g) = f(z)\Phi(g)\) defines an automorphism \(\Phi \in \Aut(\Gamma)\) so that \(S \cong (\Gamma)_\Phi\) as a bibundle. So \(j\) is onto. \(\square\)

Let us give some concrete examples.

**Example 3.4** (Lie groups). Let \(G\) be a Lie group, regarded as a Lie groupoid over a point. (\(G\) need not be connected; in particular, it may be discrete.) Let \(X\) be a biprincipal \((G, G)\)-bibundle, also called a \(G\)-bitorsor, defining a Morita self-equivalence of \(G\). A map \(\sigma\) from the base of \(G\) into \(X\) is just a choice of a point in \(X\). It immediately follows from Lemma 3.3 that there exists \(\phi \in \Aut(G)\) such that \(X\) is isomorphic, as a bibundle, to \(G_\phi\) (action on the left by left multiplication, and on the right by right multiplication composed with \(\phi\)). Hence the exact sequence (3.4) becomes

\[1 \to \InnAut(G) \to \Aut(G) \xrightarrow{j} \Pic(G) \to 1,\]

so \(\Pic(G) \cong \OutAut(G)\).

Since Morita equivalence is just isomorphism in the category \(\LG\), it is clear that Morita equivalent groupoids have isomorphic Picard groups. As a consequence of Examples 3.1 and 3.4, we obtain

**Corollary 3.5.** If \(\mathcal{G}(E)\) is the gauge groupoid of a principal \(\Gamma\)-bundle \(E\) (so that its moment \(J\) is a surjective submersion), then \(\Pic(\mathcal{G}(E)) \cong \Pic(\Gamma)\). In particular, if \(\Gamma = G\) is a Lie group, then \(\Pic(\mathcal{G}(E)) \cong \OutAut(G)\).

**Example 3.6** (Transitive groupoids). A transitive groupoid \(\Gamma\) over \(P\) is always isomorphic to a gauge groupoid of a Lie group principal bundle: for a fixed point \(x \in P\), we consider the set of arrows starting at \(x\), \(E_x := s^{-1}(x)\); if \(\Gamma_x\) denotes the isotropy subgroup of \(\Gamma\) at \(x\), a simple computation shows that \(E_x\) is a principal \(\Gamma_x\)-bundle, in such a way that \(\Gamma\) is isomorphic to the gauge groupoid \(\mathcal{G}(E_x)\). It follows from Corollary (3.5) that

\[\Pic(\Gamma) \cong \Pic(\mathcal{G}(E_x)) \cong \OutAut(\Gamma_x).\]

In particular, this shows that, although the isotropy groups of \(\Gamma\) at different points of \(P\) are isomorphic to one another only in a noncanonical way, their groups of outer isomorphisms can be canonically identified with each other.

The homomorphism \(j: \Aut(\Gamma) \to \OutAut(\Gamma_x)\) can be described directly as follows. An automorphism \(\Phi \in \Aut(\Gamma)\) induces an isomorphism \(\Phi_x: \Gamma_x \to \Gamma_{\Phi(x)}\),
Since $\Gamma$ is transitive, one can identify $\Gamma_x$ and $\Gamma_{\Phi(x)}$ by means of a choice of $g$ with $s(g) = x$, $t(g) = \Phi(x)$. This induces an automorphism

$\Phi^g_x : \Gamma_x \to \Gamma_x$, \quad $\Phi^g_x(z) = g^{-1}\Phi_x(z)g$,

and a simple computation shows that, if $h \in \Gamma$ also satisfies $s(h) = x$, $t(h) = \Phi(x)$, then $\Phi^g_x \circ (\Phi^h_x)^{-1} \in \text{InnAut}(\Gamma_x)$. So $\Phi$ determines an equivalence class $[\Phi_x] \in \text{OutAut}(\Gamma_x)$, which is the image of $\Phi$ under $j$. So, for a transitive groupoid, the exact sequence (3.4) becomes

$$1 \to \{\Phi \in \text{Aut}(\Gamma) : [\Phi_x] = \text{Id} \in \text{OutAut}(\Gamma_x)\} \to \text{Aut}(\Gamma) \to \text{OutAut}(\Gamma_x). \quad (3.7)$$

In this (indirect) way, we get a characterization of inner automorphisms of transitive groupoids as

$$\text{InnAut}(\Gamma) = \{\Phi \in \text{Aut}(\Gamma) : [\Phi_x] = \text{Id} \in \text{OutAut}(\Gamma_x)\}. \quad (3.8)$$

This identification can also be verified directly: if $\Phi_N$ is an inner automorphism associated with a bisection $N$ and we consider $(\Phi_N)_x : G(E)_x \to G(E)_{\Phi_N(x)}$, $x \in P$, then $(\Phi_N)^g_x$ is trivial if we choose $g \in N$; conversely, if $[\Phi_x] = 0$, we can reconstruct $N$ (uniquely up to $Z\text{Bis}(\Gamma)$, see (3.5)) as elements $g$ in $\Gamma$ so that $(\Phi_N)^g_x$ is trivial.

**Example 3.7** (Fundamental groupoids). Let $P$ be a connected manifold. The fundamental groupoid $\Pi(P)$, consisting of homotopy classes of paths in $P$ with fixed endpoints, is a transitive groupoid over $P$ whose isotropy group at $x \in P$ is the fundamental group $\pi_1(P, x)$. The corresponding principal bundle is the universal covering $\tilde{P} \to P$. Thus, the outer automorphism groups $\text{OutAut}(\pi_1(P, x))$ are all canonically isomorphic to $\text{Pic}(\Pi(P))$ and hence to one another.

We now discuss the groupoid analogue of (2.3). For a groupoid $\Gamma$ over $P$, the orbit space $Z(\Gamma) = P/\Gamma$ will play the role of its “center,” since the inner automorphisms act trivially on it. In this sense, a “commutative” groupoid is a groupoid with trivial orbits ($Z(\Gamma) = P$), and the groupoids with trivial center are the transitive ones.

In general, we regard the orbit space $Z(\Gamma)$ (the “center” of $\Gamma$) as a topological space, with the quotient topology. However, whenever $Z(\Gamma)$ is smooth and such that the quotient map is a submersion, we regard it as a smooth manifold. A $(\Gamma_1, \Gamma_2)$-Morita bibundle induces a homeomorphism (or diffeomorphism, in the smooth case) between $Z(\Gamma_1)$ and $Z(\Gamma_2)$, and therefore defines a homomorphism

$h : \text{Pic}(\Gamma) \to \text{Aut}(Z(\Gamma)). \quad (3.9)$

Here $\text{Aut}(Z(\Gamma))$ denotes the group of homeomorphisms (or diffeomorphisms, in the smooth case) of $Z(\Gamma)$ onto itself. The kernel of $h$ consists of those bibundles which induce the identity map on $Z(\Gamma)$; we denote this subgroup of $\text{Pic}(\Gamma)$ by $\text{Pic}_{Z(\Gamma)}(\Gamma)$. It seems appropriate to call this the *static Picard group* of $\Gamma$. The analogue of (2.3) is

$$1 \to \text{Pic}_{Z(\Gamma)}(\Gamma) \to \text{Pic}(\Gamma) \xrightarrow{h} \text{Aut}(Z(\Gamma)). \quad (3.10)$$

The next two examples discuss the Picard groups of Lie groupoids $\Gamma$ over $P$ with $Z(\Gamma) = P$ (i.e., $s = t$). These groupoids are just bundles of Lie groups, i.e., smooth
families of Lie groups parametrized by $P$. The description of the Picard groups of such groupoids follows from the study of their bitorsors by Moerdijk in [33].

**Example 3.8** (Bundles of contractible Lie groups). Let $\Gamma$ be a bundle of contractible Lie groups over $P$. Let $S$ be a $(\Gamma, \Gamma)$-Morita bibundle, with moments $J_1$ and $J_2$. The $J_2$-fibres are contractible (since each $J_2$-fibre is diffeomorphic to an $s$-fibre), so there exists a cross-section $\sigma: P \to S$, $J_2 \circ \sigma = \text{Id}$. Note that $J_1 \circ \sigma$ coincides with the diffeomorphism induced by $S$ on $Z(\Gamma) = P$. So, by Lemma 3.3, Pic($\Gamma$) $\cong$ OutAut($\Gamma$), which consists of automorphisms of the family of groups, modulo smooth sections acting by inner automorphisms. The static Picard group is thus

$$\text{Pic}_Z(\Gamma) \cong \{[\Phi] \in \text{OutAut}(\Gamma) : \Phi|_P = \text{Id}\}. \quad (3.11)$$

In particular, if $\Gamma$ is a bundle of contractible abelian Lie groups (e.g., a vector bundle with groupoid structure given by fibrewise addition), inner automorphisms are trivial, and so Pic($\Gamma$) $\cong$ Aut($\Gamma$).

**Example 3.9** (Bundles of abelian Lie groups). This example follows [33, Sec. 3.5].

Let $\Gamma$ be a bundle of abelian Lie groups. In this case, one can show that any $(\Gamma, \Gamma)$-Morita bibundle is completely determined by a pair $(S, \Phi)$, where $S$ is a principal $\Gamma$-bundle (also called a $\Gamma$-torsor) and $\Phi \in \text{Aut}(\Gamma)$. Moreover, there is a natural tensor product operation on $\Gamma$-torsors making their set of isomorphism classes into a group isomorphic to $\check{H}^1(P, \Gamma)$. The Picard group of $\Gamma$ can then be identified with the semi-direct product

$$\text{Pic}(\Gamma) \cong \text{Aut}(\Gamma) \ltimes \check{H}^1(P, \Gamma).$$

It also follows that

$$\text{Pic}_Z(\Gamma) \cong \text{Aut}_P(\Gamma) \ltimes \check{H}^1(P, \Gamma),$$

where $\text{Aut}_P(\Gamma) = \{\Phi \in \text{Aut}(\Gamma) : \Phi|_P = \text{Id}\}$.

4. **Symplectic Dual Pairs as Generalized Morphisms**

Let $S$ be a symplectic manifold. A pair of Poisson maps

$$\begin{array}{ccc}
S & \to & S \\
\downarrow J_1 & & \downarrow J_2 \\
P_1 & \to & P_2
\end{array}$$

is called a dual pair if the $J_1$- and $J_2$-fibres are the symplectic orthogonal of one another. A dual pair is called full if $J_1$ and $J_2$ are surjective submersions; it is called complete if the maps $J_1$ and $J_2$ are complete (recall that a Poisson map $J: Q \to P$ is complete if the hamiltonian vector field $X_J$ is complete whenever $X_f$ is complete, $f \in C^\infty(M)$).

We now discuss how dual pairs relate to generalized morphisms of symplectic groupoids and Poisson manifolds.

A symplectic groupoid [13, 49] is a Lie groupoid $\Gamma$ equipped with a symplectic structure $\omega$ for which the graph of the multiplication $m: \Gamma_2 \to \Gamma$ is lagrangian in $\Gamma \times \Gamma \times \Gamma$ (where $\Gamma$ is equipped with $-\omega$). This compatibility condition between
the groupoid structure and the symplectic form implies, in particular, that the identity embedding \( \varepsilon : P \to \Gamma \) is lagrangian, the inversion \( i : \Gamma \to \Gamma \) is an antisymplectomorphism and \( s \) and \( t \)-fibres are symplectically orthogonal to one another; furthermore, the identity section \( P \) inherits a Poisson structure \( \pi \), uniquely determined by the condition that \( t : \Gamma \to P \) (resp. \( s : \Gamma \to P \)) is a Poisson map (resp. anti-Poisson map).

Let \((S, \omega_S)\) be a symplectic manifold. A (left) action of \((\Gamma, \omega)\) on \(S\) is called **symplectic** if the graph of the action map \( \Gamma \times S \to S \) is lagrangian in \( \Gamma \times S \times S \). In this case, the moment \( J : S \to P \) is automatically a Poisson map \([31]\). Symplectic right actions are defined analogously, but their moments are anti-Poisson maps.

If \( \Gamma_1 \) and \( \Gamma_2 \) are symplectic groupoids, then a \((\Gamma_1, \Gamma_2)\)-bibundle \( S \) is a **symplectic bibundle** if both actions are symplectic. An isomorphism of symplectic bibundles is just an isomorphism preserving the symplectic structures. A **generalized morphism** \( \Gamma_1 \to \Gamma_2 \) is an isomorphism class of symplectic \((\Gamma_1, \Gamma_2)\)-bibundles which are left principal. It is proven in \([53]\) that the “tensor product” of symplectic bibundles (regarded just as bibundles, see Section 3.1) is automatically compatible with the symplectic structures, in such a way that the resulting bibundle is canonically symplectic. Thus, the tensor product operation induces a well-defined composition of generalized morphisms; we denote the resulting category by \(SG\).

We call two symplectic groupoids \( \Gamma_1 \) and \( \Gamma_2 \) **Morita equivalent** if they are isomorphic in \(SG\), or, equivalently, if there exists a biprincipal symplectic \((\Gamma_1, \Gamma_2)\)-bibundle \([27]\). One can check that, in this case, the moments \( P_1 J_1 \to S J_2 \to P_2 \) define a complete and full dual pair (here \( P_2 \) denotes the manifold \( P_2 \) with the opposite Poisson structure).

As mentioned earlier in this section, the identity section \( P \) of a symplectic groupoid \((\Gamma, \omega)\) naturally inherits a Poisson structure making \( t \) into a Poisson map (and \( s \) into an anti-Poisson map). A Poisson manifold \((P, \pi)\) is called **integrable** if there exists a symplectic groupoid over \( P \) for which the induced Poisson structure on \( P \) is \( \pi \). When \((P, \pi)\) is integrable, it has a canonically defined source-simply-connected\(^2\) symplectic groupoid \( \Gamma(P) \). Not every Poisson manifold is integrable, and the obstructions have been explicitly described in \([15]\) (we will return to this topic in Section 9.3). A Poisson structure \( \pi \) on \( P \) defines a Lie algebroid structure on \( T^*P \) (see, e.g., \([13]\)), and \( P \) is integrable if and only if \( T^*P \) is integrable as a Lie algebroid \([29]\) (see also \([15]\)).

Symplectic actions of symplectic groupoids and symplectic bibundles can be described purely in terms of the Poisson geometry of the identity sections. Recall that a **symplectic realization** of a Poisson manifold \( P \) is just a Poisson map from a symplectic manifold to \( P \). Any symplectic realization \( J : S \to P \) induces a canonical action of the Lie algebroid \( T^*P \) (induced by \( \pi \)) on \( S \) by assigning to each 1-form \( \alpha \) on \( P \) the vector field \( X \) on \( S \) defined by \( i_X \omega_S = J^*\alpha \). If \( P \) is integrable, this action extends to a symplectic action of \( \Gamma(P) \) when \( J \) is complete \([13], [15]\), in which case \( J \) is the moment of the action. Hence there is a natural correspondence between complete symplectic realizations of \( P \) and symplectic actions of \( \Gamma(P) \).

---

\(^2\)By “simply-connected,” we will always mean connected, with trivial fundamental group.
If $P_1$ and $P_2$ are integrable Poisson manifolds and

$$
\begin{array}{ccc}
S & \xrightarrow{J_1} & P_1 \\
\downarrow & & \downarrow \\
\downarrow & & \downarrow \\
P_2 & \xleftarrow{J_2} & \uparrow
\end{array}
$$

is a pair of complete symplectic realizations, then $S$ carries a left $\Gamma(P_1)$-action and a right $\Gamma(P_2)$-action. These actions commute if and only if they commute on the infinitesimal level, i.e.,

$$\{J_1^*C^\infty(P_1), J_2^*C^\infty(P_2)\} = 0. \tag{4.2}$$

In this case, we call (4.1) a $(P_1, P_2)$-symplectic bimodule. It then follows that there is a natural one-to-one correspondence between $(P_1, P_2)$-symplectic bimodules and $(\Gamma(P_1), \Gamma(P_2))$-symplectic bibundles. A generalized morphism of integrable Poisson manifolds $P_1$ and $P_2$ is an isomorphism class of $(P_1, P_2)$-symplectic bimodules which correspond to left principal $(\Gamma(P_1), \Gamma(P_2))$-symplectic bibundles.

Following the ideas in [52, Thm. 3.2], one obtains an alternative way to define generalized morphisms of Poisson manifolds just in terms of Poisson structures and moments, with no reference to symplectic groupoids: a symplectic bimodule $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} P_2$ is a generalized morphism if and only if $J_1$ and $J_2$ are complete Poisson maps, $J_1$ is a submersion, $J_2$ is a surjective submersion with simply-connected fibres, and the $J_1$- and $J_2$-fibres are the symplectic orthogonals of each other. The category whose objects are integrable Poisson manifolds and arrows are generalized morphisms is denoted by Poiss.

**Remark 4.1.** Any ordinary complete Poisson map $P_1 \xleftarrow{\phi} P_2$ defines a symplectic bimodule, namely $P_1 \xleftarrow{\phi} \Gamma(P_2) \xrightarrow{s_2} P_2$: we denote this symplectic bimodule by $\phi(\Gamma(P_2))$. It follows from our description of generalized morphisms of Poisson manifolds that such a symplectic bimodule is a generalized morphism if and only if $\phi$ is étale, which means that $\phi$ is a submersion between leaf spaces and a covering on each symplectic leaf.

If $\phi$ is étale, it lifts to a Lie algebroid morphism $T^*P_1 \xleftarrow{\Phi} T^*P_2$ and thus to a symplectic groupoid morphism $\Gamma(P_1) \xleftarrow{\Phi} \Gamma(P_2)$. Hence $\phi$ also defines a generalized morphism $\phi(\Gamma(P_2))_{\Phi}$, as discussed in the beginning of Section 3.2. We note that there is a natural $(P_1, P_2)$-bimodule map $\phi(\Gamma(P_2)) \rightarrow (\Gamma(P_1))_{\Phi}$ given by $g \mapsto (\Phi(g), s_2(g))$ which is étale; it becomes an isomorphism whenever $\phi$ is a Poisson diffeomorphism.

We call two integrable Poisson manifolds $P_1$ and $P_2$ Morita equivalent [52] if their source-simply-connected symplectic groupoids are Morita equivalent; this coincides with the notion of isomorphism in Poiss [27]. Equivalently, we can define Morita equivalence of Poisson manifolds in Poisson geometrical terms [52, Thm. 3.2]: $P_1$ and $P_2$ are Morita equivalent if there is a symplectic manifold $S$ with complete Poisson and anti-Poisson maps $J_1: S \rightarrow P_1$ and $J_2: S \rightarrow P_2$, so that $P_1 \xleftarrow{J_1} S \xrightarrow{J_2} \overline{P}_2$ is a complete full dual pair for which $J_1$- and $J_2$-fibres are simply connected.
In this case, the diagram

\[
\begin{array}{ccc}
  S & \xleftarrow{J_1} & J_2 \\
  P_1 & & P_2
\end{array}
\]

is called a \textit{Morita bimodule}.

Morita equivalent Poisson manifolds share many properties. For example, they have isomorphic first Poisson cohomology groups \[19\], homeomorphic spaces of symplectic leaves \[7, 15\] and isomorphic transverse geometry at corresponding symplectic leaves \[48\]. For a symplectic manifold, its fundamental group is a complete Morita invariant \[52\]; in particular, simply-connected symplectic manifolds are Morita equivalent to a point. We will discuss further examples of complete invariants of Morita equivalence in Section 8.

**Remark 4.2.** For later use in Section 8, we note that symplectic leaves that correspond to each other in a Morita equivalence are themselves Morita equivalent \[15\]. When the symplectic leaves are open, one may simply restrict the equivalence. (In fact, one may also restrict a Morita equivalence to any open subset which is a union of symplectic leaves.) In general, consider symplectic leaves \(L_i\) in \(P_i\), \(i = 1, 2\), and \(N = J_1^{-1}(L_1) = J_2^{-1}(L_2)\) (the pull-back foliations by \(J_1\) and \(J_2\) coincide); then \(J_i\) maps \(N\) onto \(L_i\), and the map has simply-connected fibers. So we have an induced isomorphism of fundamental groups \(\pi_1(L_1) \cong \pi_1(N) \cong \pi_1(L_2)\).

5. **Picard Groups of Symplectic Groupoids and Poisson Manifolds**

Let \(\Gamma\) be a symplectic groupoid. We denote its group of symplectic groupoid automorphisms by \(\text{Aut}(\Gamma)\), and its group of automorphisms in \(\text{SG}\) by \(\text{Pic}(\Gamma)\); similarly, if \(P\) is an integrable Poisson manifold, the group of Poisson diffeomorphisms \(f: P \to P\) is denoted by \(\text{Aut}(P)\), whereas the group of automorphisms of \(P\) in \(\text{Poiss}\) is denoted by \(\text{Pic}(P)\). Note that, if \(P\) is integrable, then we have a natural identification

\[
\text{Pic}(P) = \text{Pic}(\Gamma(P)),
\]

where \(\Gamma(P)\) is the canonical source-simply-connected symplectic groupoid integrating \(P\).

As in the case of Lie groupoids (3.3), we have a group homomorphism

\[
j: \text{Aut}(\Gamma) \to \text{Pic}(\Gamma), \quad \Phi \mapsto [(\Gamma)_\Phi],
\]

which induces, for Poisson manifolds, a group homomorphism

\[
j: \text{Aut}(P) \to \text{Pic}(P), \quad \phi \mapsto [(\Gamma(P))_\phi],
\]

where \((\Gamma(P))_\phi\) is the symplectic bimodule

\[
\begin{array}{ccc}
  \Gamma(P) & \xleftarrow{t} & \phi^{-1}_s \\
  P & & P
\end{array}
\]
On a symplectic groupoid $\Gamma$, the group of lagrangian bisections (i.e., bisections which are lagrangian submanifolds), $\text{LBis}(\Gamma)$, plays a special role: if $L$ is a lagrangian bisection, then the inner automorphism $\Phi_L$ is a symplectomorphism of $\Gamma$, and $\phi_L = (\Phi_L)|_P: P \to P$ preserves the induced Poisson structure on $P$ [31]. We refer to the subgroup

$$\text{InnAut}(\Gamma) := \{\Phi_L: L \in \text{LBis}(\Gamma(P))\} \subseteq \text{Aut}(\Gamma)$$

as the subgroup of inner automorphisms of $\Gamma$; similarly, the subgroup

$$\text{InnAut}(P) := \{\phi_L = (\Phi_L)|_P: L \in \text{LBis}(\Gamma(P))\} \subseteq \text{Aut}(P)$$

is called the subgroup of inner Poisson automorphisms of $P$. We denote the quotients $\text{OutAut}(\Gamma)$ by $\text{Out}(\Gamma)$ and $\text{Out}(P)$ by $\text{Out}(P)$. For example, when the modular class of a Poisson manifold is nonzero, the flow of any modular vector field represents a nontrivial one-parameter group of outer automorphisms.

The following result follows from Proposition 3.2.

**Proposition 5.1.** Let $\Gamma$ be a symplectic groupoid. Then the following is an exact sequence:

$$1 \to \text{InnAut}(\Gamma) \to \text{Aut}(\Gamma) \xrightarrow{j} \text{Pic}(\Gamma). \quad (5.4)$$

Similarly, if $P$ is a Poisson manifold, then the sequence below is exact:

$$1 \to \text{InnAut}(P) \to \text{Aut}(P) \xrightarrow{j} \text{Pic}(P). \quad (5.5)$$

In particular, the outer automorphisms $\text{OutAut}(\Gamma)$ (resp. $\text{OutAut}(P)$) sit in $\text{Pic}(\Gamma)$ (resp. $\text{Pic}(P)$) as a subgroup.

To describe the inner automorphisms in terms of the groupoid, we introduce the subgroup $\text{IsoLBis}(\Gamma(P))$ of $\text{LBis}(\Gamma(P))$ consisting of bisections with values in the isotropy subgroupoid of $\Gamma(P)$, and its subgroup $\text{ZLBis}(\Gamma(P))$ consisting of lagrangian bisections with values in the centers of the isotropy subgroups which are “constant on orbits.” We have the exact sequence corresponding to (3.5):

$$1 \to \text{IsoLBis}(\Gamma(P)) \to \text{LBis}(\Gamma(P)) \to \text{InnAut}(P) \to 1 \quad (5.6)$$

as well as a sequence which describes the kernel in (5.4):

$$1 \to \text{ZLBis}(\Gamma(P)) \to \text{LBis}(\Gamma(P)) \to \text{InnAut}(\Gamma(P)) \to 1. \quad (5.7)$$

If $\Gamma$ is a symplectic groupoid, then $Z(\Gamma)$ is the space of symplectic leaves of the Poisson structure on its identity section. So, for an integrable Poisson manifold $P$, we define $Z(P)$ as the leaf space $P/F$, where $F$ is the symplectic foliation of $P$. As in the case of Lie groupoids, we generally consider this leaf space as a topological space, but we regard it as a smooth manifold whenever $F$ is simple. In this way, $\text{Aut}(Z(P))$ denotes the group of homeomorphisms (or diffeomorphisms, in the smooth case) of $Z(P)$ onto itself. It follows from the discussion for Lie groupoids (see also [7], [15]) that a $(P_1, P_2)$-Morita bimodule induces a homeomorphism (or diffeomorphism, in the smooth case) between $Z(P_1)$ and $Z(P_2)$. As a result, we get a group homomorphism

$$h: \text{Pic}(P) \to \text{Aut}(Z(P)). \quad (5.8)$$
The kernel of $h$ consists of those $(P, P)$-Morita bimodules which induce the identity map on $\mathcal{Z}(P)$; we denote this subgroup of Pic$(P)$ by Pic$_Z(P)$. We then obtain an analogue of (2.3) and (3.10):

$$1 \to \text{Pic}_Z(P) \to \text{Pic}(P) \xrightarrow{h} \text{Aut}(\mathcal{Z}(P)).$$ (5.9)

As with groupoids, we have two extreme situations: the “commutative” case, where $\mathcal{Z}(P) = P$, occurs for the zero Poisson structure, while the “most noncommutative” case, where $\mathcal{Z}(P)$ is a point, corresponds to symplectic manifolds. We will discuss both situations in the next section.

**Remark 5.2 (Variation lattices).** The regular part $P_r$ of a Poisson manifold $P$ is the open dense subset consisting of the regular leaves, i.e., those where the transverse structure is a zero Poisson structure. Any Morita self-equivalence of $P$ restricts to one on $P_r$, and $\mathcal{Z}(P_r)$ is open and dense in $\mathcal{Z}(P)$. Thus, for any $X \in \text{Pic}(P)$, $h(X)$ is determined by the restriction to $\mathcal{Z}(P_r)$. Suppose, now, that the symplectic leaf foliation of $P_r$ is a fibration with simply connected fibres, so that $\mathcal{Z}(P_r)$ is a manifold. The proof of Theorem 5.3 of [52] shows that the restriction of $h(X)$ to $\mathcal{Z}(P_r)$ preserves the variation lattice of Dazord [17], which is a collection of closed 1-forms attached to integer second homology classes in the fibres, measuring the variation of the integrals of the fibre symplectic structures over these classes. This lattice of closed forms sometimes provides $\mathcal{Z}(P_r)$ with an additional structure which limits how $h(\text{Pic}(P))$ may act on $\mathcal{Z}(P)$. For example, see Section 6.3. (Rigidity of $\mathcal{Z}(P_r)$ may also come from the modular vector field, as in Section 8.)

### 6. Examples of Picard Groups

#### 6.1. Symplectic Poisson structures.

We start with a “trivial” example. Any discrete group $G$ is a symplectic groupoid over a point. Its symplectic Morita bibundles are discrete as well, so Pic$(G)$ for this symplectic groupoid is isomorphic to OutAut$(G)$, just as in Example 3.4.

Next, suppose that $P = S$ is a connected symplectic manifold. In this case, $\mathcal{Z}(S)$ is just a point and the exact sequence (5.9) is trivial. The source-simply-connected symplectic groupoid $\Gamma(S)$ is the fundamental groupoid $\Pi(S)$ (with symplectic structure pulled back from $S \times S$), which is transitive since $S$ is connected. We then have from Example 3.7:

**Proposition 6.1.** Let $S$ be a connected symplectic manifold. Then $\Gamma(S)$ is Morita equivalent as a symplectic groupoid to $\pi_1(S, x)$ for each $x \in S$, and hence $\text{Pic}(S) \cong \text{OutAut}(\pi_1(S, x))$.

**Proof.** It suffices to notice that the Morita bibundle coming from (3.1),

$$\begin{array}{ccc}
\Gamma(S) & \xrightarrow{pr} & S \\
\downarrow & & \downarrow \pi_1(S, x) \\
S & \xrightarrow{\{x\}} & \{x\}
\end{array}$$ (6.1)
is in fact a symplectic Morita bibundle, which is clear since the moments are symplectic realizations.

As a result of (3.8) in Example 3.6, we have

**Corollary 6.2.** If $S$ is a connected symplectic manifold, then $\text{InnAut}(S)$ has the following description:

$$\text{InnAut}(S) = \{ \phi \in \text{Sympl}(S) : [\phi_*] = \text{Id} \in \text{OutAut}(\pi_1(S, x)) \}.$$ 

So for a connected symplectic manifold, the exact sequence (5.5) becomes

$$1 \to \{ \phi \in \text{Sympl}(S) : [\phi_*] = \text{Id} \in \text{OutAut}(\pi_1(S, x)) \} \to \text{Sympl}(S) \to \text{Pic}(S). \quad (6.2)$$

**Remark 6.3.** The isomorphism $\eta : \text{Pic}(S) \to \text{OutAut}(\pi_1(S, x))$ obtained in Proposition 6.1 can also be seen as a consequence of the classification of complete Poisson maps with symplectic targets by their holonomy [10]. Recall [10] that any complete symplectic realization $M \to S$ is a fibration with a natural flat connection. A typical fiber, $F$, carries a natural symplectic structure, preserved under the holonomy action of $\pi_1(S)$. For a choice of base point $x \in S$, the realization $M \to S$ is isomorphic to $(\tilde{S} \times F)/\pi_1(S)$, and hence it is completely determined, up to isomorphism, by the holonomy action.

As a consequence, any Morita bimodule $S \xleftarrow{J_1} M \xrightarrow{J_2} S$ is isomorphic to one of the form

$$\begin{array}{ccc}
\tilde{S} \times F & \xrightarrow{pr} & \tilde{S} \\
\downarrow{q_2} & & \downarrow{pr} \\
S & \to & S
\end{array}$$

The map $q_2$, restricted to $F$, is a covering of $S$, and, since $F$ is simply connected, there exists a symplectomorphism $\tilde{q}_2 : F \to \tilde{S}$ making the following diagram commute:

$$\begin{array}{ccc}
F & \xrightarrow{\tilde{q}_2} & \tilde{S} \\
\downarrow{q_2} & & \downarrow{pr} \\
S & \to & S
\end{array}$$

The action of $\pi_1(S)$ on $F$ and the map $\tilde{q}_2$ induce an action of $\pi_1(S)$ on $\tilde{S}$ by deck transformations, and this action defines an automorphism $\phi$ of $\pi_1(S)$. One can check that

$$\eta([M]) = [\phi] \in \text{OutAut}(\pi_1(S)).$$

**Remark 6.4.** Gompf [20] has shown that every finitely presented group is the fundamental group of a compact symplectic 4-manifold. (Without the compactness and dimension restriction, it is much easier to realize these groups as fundamental groups of cotangent bundles.) This shows that the subcategories of $\text{Poiss}$ consisting of groups and of symplectic manifolds are essentially the same as far as their objects are concerned. Comparing them on the level of morphisms means deciding
which outer automorphisms of the fundamental group are realizable by symplectic diffeomorphisms. (For closed surfaces, Nielsen’s theorem on mapping classes [40] and Moser’s theorem on volume elements [36] together imply that all outer automorphisms are realizable by symplectic or antisymplectic diffeomorphisms.)

6.2. The zero Poisson structure. Let \((P, \pi)\) be a Poisson manifold with \(\pi = 0\). In this case, \(Z(P) = P\), \(\text{Aut}(P) = \text{Diff}(P) = \text{Aut}(P)\), and the inner Poisson automorphisms are trivial. The exact sequence (5.5) becomes

\[
1 \to \text{Aut}(P) \xrightarrow{j} \text{Pic}(P). \tag{6.3}
\]

For (5.9), we get

\[
1 \to \text{Pic}_Z(P) \to \text{Pic}(P) \xrightarrow{h} \text{Aut}(P), \tag{6.4}
\]

and, as in the case of commutative algebras, \(h \circ j = \text{Id}\).

Let \(\text{pr}: T^*P \to P\) be the natural projection, and let \(\omega\) denote the canonical symplectic form on \(T^*P\). The following lemma is closely related to [39, Thm. 4.1] and follows from [51, Prop. 4.7.1].

**Lemma 6.5.** Let \((P, \pi)\) be a Poisson manifold with \(\pi = 0\). Let \(\omega'\) be a symplectic form on \(T^*P\) for which the \(\text{pr}\)-fibres and the zero section \(P \hookrightarrow T^*P\) are lagrangian submanifolds and \(\text{pr}\) is a complete Poisson map. Then there exists a symplectomorphism \(f: (T^*P, \omega) \to (T^*P, \omega')\) so that \(f(P) \equiv P\) and \(f \circ \text{pr} = \text{pr} \circ f\).

We now give an explicit description of \(\text{Pic}_{Z(P)}(P)\) and \(\text{Pic}(P)\).

**Proposition 6.6.** Let \((P, \pi)\) be a Poisson manifold with \(\pi = 0\). Then

(i) \(\text{Pic}_{Z(P)}(P) \cong H^2(P, \mathbb{R})\);

(ii) \(\text{Pic}(P) \cong \text{Diff}(P) \ltimes H^2(P, \mathbb{R})\), where the semi-direct product is with respect to the natural action of diffeomorphisms on cohomology by pull-back.

**Proof.** The canonical source-simply-connected symplectic groupoid of \(P\) is \(T^*P\), equipped with its canonical symplectic form \(\omega\) and groupoid structure given by fibrewise addition.

It is easy to check that, for any closed 2-form \(\alpha \in \Omega^2(P)\), \((T^*P, \omega + \text{pr}^*\alpha)\) is a \((P, P)\)-Morita bimodule defining an element in \(\text{Pic}_{Z(P)}(P)\). Let us consider the map

\[
\tau: \{ \alpha \in \Omega^2(P), \ \alpha \text{ closed} \} \to \text{Pic}_{Z(P)}(P), \quad \alpha \mapsto [(T^*P, \omega + \text{pr}^*\alpha)]. \tag{6.5}
\]

**Claim 1:** The map \(\tau\) is onto.

In order to prove Claim 1, note that, by Example 3.8 (see (3.11)), any \((P, P)\)-Morita bimodule \(S\) inducing the identity map on \(P\) must be of the form \((T^*P, \nu)\), where \(\nu\) is a symplectic form for which the \(\text{pr}\)-fibres are lagrangian and \(\text{pr}\) is complete. Consider the closed 2-form \(\beta = \iota^*\nu \in \Omega^2(P)\), where \(\iota: P \to T^*P\) is the zero-section embedding. Then \(\omega' = \omega - \text{pr}^*\beta\) satisfies the conditions of Lemma 6.5, so there is a symplectic bimodule isomorphism \(f: (T^*P, \omega) \to (T^*P, \omega')\). But \(f^*\omega' = \omega\) implies that

\[
f^*\omega - f^*\text{pr}^*\beta = f^*\omega - \text{pr}^*f^*\beta = \omega.
\]

So \(f^*\omega = \omega + \text{pr}^*(f^*\beta)\), and therefore \([S] = \tau(\alpha)\), where \(\alpha = f^*\beta\).
Claim 2: Ker(τ) = {α ∈ Ω^2(P), α exact}.

If α ∈ Ker(τ), then there exists a diffeomorphism \( f: T^*P \to T^*P \) with \( f^*(ω + pr^*α) = ω \). Since ω is exact, so is α. On the other hand, if \( α = dθ \), for \( θ ∈ Ω^1(P) \), then the map \( f: T^*P \to T^*P, (x, ξ) \mapsto (x, ξ - θ) \) satisfies \( f^*(ω + pr^*dθ) = ω \) and defines a symplectic bimodule isomorphism. So \( α ∈ Ker(τ) \).

It is not hard to check that τ is a group homomorphism. So we have an exact sequence of groups

\[
1 \to \{α ∈ Ω^2(P), α \text{ exact}\} \to \{α ∈ Ω^2(P), α \text{ closed}\} \overset{ε}{\to} Pic_{Z}(P) \to 1,
\]

so \( Pic_{Z}(P) \cong H^2(P, ℝ) \). This proves (i).

Since the maps \( j \) and \( h \) in (6.3) and (6.4) satisfy \( h \circ j = Id \), we have an identification

\[
\text{Diff}(P) \times H^2(P, ℝ) \to Pic(P), \quad (φ, α) ↦ \{[(T^*P)φ], ω + pr^*α\},
\]

under which the tensor product on \( Pic(P) \) becomes

\[
((φ, α), (ψ, β)) ↦ (φ \circ ψ, α + (φ^{-1})^*β).
\]

This proves (ii). \( □ \)

Remark 6.7. As Dmitry Roytenberg has suggested to us, one can pass directly from an element in \( Pic_{Z}(P) \) to a 1-dimensional extension of the tangent bundle Lie algebroid \( TP \). Such extensions are classified by elements of \( H^2(P, ℝ) \); a vector bundle splitting of the extension has a “curvature” whose cohomology class determines the extension up to isomorphism.

Let \( S \) be a \((P, P)\)-Morita bimodule with maps \( J_1 = J_2 = J \). The fibres of \( J \) are the leaves of a lagrangian foliation, hence they carry flat affine structures. For each \( x \in P \), the affine functions on \( J^{-1}(x) \) form a vector space which is a fibre of the vector bundle \( E \) over \( P \) whose sections are the fibrewise affine functions on \( S \). Given any such section, its fibre derivative may be identified with a vector field on \( P \); the resulting bundle map from \( E \) to \( TP \) is the anchor map for a Lie algebroid structure in which the bracket is the Poisson bracket of \( S \), restricted to fibrewise affine functions. Splittings of this Lie algebroid extension correspond to cross sections of \( J \); this correspondence may be used to show that the characteristic class of the Lie algebroid is the same as that attached to the bimodule in Proposition 6.6.

The Picard group of \( P \) is the same as that of the source-simply-connected symplectic groupoid \( Γ(P) = T^*P \). However, there are other symplectic groupoids over \( P \), for which the calculation of the Picard group is more complicated.

We have already seen that, when \( P \) is a point, dropping the source-connectivity assumption leads to the class of discrete groups as symplectic groupoids. In general, the symplectic groupoids over \( P \) for which the (components of the) source fibres have trivial fundamental groups consist of action groupoids for the actions of discrete groups on \( T^*P \) which preserve the foliation by the fibres. These actions may be described as “affinizations” of the standard cotangent lifts of actions on \( P \).

On the other hand, we may keep source-connectivity but allow source fibres to have nontrivial fundamental groups. In this case, the groupoid is obtained by dividing the cotangent bundle \( T^*P \) by a “lattice” \( Λ \) which is a bundle of discrete subgroups with local bases consisting of \( k \)-tuples of closed 1-forms for some
$k \leq \dim(P)$. Such lattices arise naturally in the theory of action-angle variables as analyzed by Duistermaat [18]. The lattice describes what Duistermaat refers to as the monodromy of a lagrangian fibration. Once the monodromy (i.e., the groupoid) has been fixed, then the integrable systems with this monodromy are precisely the symplectic bibundles for this groupoid; their isomorphism classes are the Picard group, with respect to the “tensor product” operation.

Analysis of this Picard group begins as follows [18], [55]. When the monodromy is "trivial," i.e., $\Lambda$ has a global basis of sections, the next invariant of a torus fibration is its Chern class, which is an element of $H^2(P)$ with values in the sections of $\Lambda$. Finally, when the Chern class is trivial, there is a lagrangian class in $H^2(P, \mathbb{R})$ whose vanishing is the condition for the fibration to admit a lagrangian cross section. In general, these invariants are mixed in a complicated way (for instance, the Chern class lies in the cohomology of a sheaf determined by the monodromy). The starting point for the description of the Picard group in terms of these invariants is Example 3.9 combined with an extension of Lemma 6.5 to more general lagrangian foliations. We will leave this discussion to a separate paper.

6.3. Lie–Poisson structures on duals of Lie algebras of compact groups.
For any Lie group $G$, the cotangent bundle $T^*G$ is a symplectic groupoid over the dual Lie algebra $g^*$; it is source-simply-connected just when $G$ is (connected and) simply-connected. What is the Picard group of $T^*G$?

When $G$ is a torus, we are in the situation described after Proposition 6.6. In the remainder of this section, we will discuss the case where $G$ is compact and simply-connected, so that we are also studying the Picard group of $g^*$. Here, the regular part $g^*_r$ is an open dense subset, in which the leaves are all simply connected, and the leaf space $\mathcal{Z}(g^*_r)$ may be identified with the interior of a Weyl chamber $W$. $g^*_r$ itself is a (trivial) bundle of symplectic manifolds over $\mathcal{Z}(g^*_r)$ whose fibres are flag manifolds $G/T$, with symplectic structure depending on the base point in $W$. By Remark 5.2 above, for any $X \in \text{Pic}(g^*)$ the restriction to $\mathcal{Z}(g^*_r)$ of its action $h(X)$ on the leaf space must preserve the variation lattice. On the other hand, this variation lattice is well known to be the weight lattice, once one identifies the tangent spaces to $W$ with the vector space $\mathfrak{t}^*$. Since the forms making up the variation lattice are constant with respect to the natural affine structure of the Weyl chamber, $h(X)$ must be (the restriction of) a linear map. Linear maps of this type, preserving the Weyl chamber and the weight lattice, arise from permutations of the fundamental weights. It follows from Xu’s work [52] that all these permutations arise from the Picard group of $g^*_r$. On the other hand, for the full $g^*$, one may restrict a Picard group element to the fibre of $T^*G$ over the zero element of $g^*$, obtaining an outer automorphism of $G$. These correspond to automorphisms of the Dynkin diagram, which are generally quite special compared to the permutations of the fundamental weights. Taking into account the simple-connectivity of the symplectic leaves, we believe that this homomorphism from $\text{Pic}(g^*)$ to $\text{OutAut}(G)$ may in fact be an isomorphism.

7. Gauge Versus Morita Equivalence

We recall here the notion of a gauge transformation of a Poisson structure associated with a closed 2-form [46].
Let \((P, \pi)\) be a Poisson manifold, and let \(B \in \Omega^2(P)\) be a closed 2-form. We identify \(\pi\) (resp. \(B\)) with the bundle map \(\tilde{\pi} : T^*P \to TP\), \(\tilde{\pi}(\alpha)(\beta) = \pi(\beta, \alpha)\) (resp. \(\tilde{B} : TP \to T^*P\), \(\tilde{B}(u)(v) = B(u, v)\)). If the bundle endomorphism
\[
1 + \tilde{B}\tilde{\pi} : T^*P \to T^*P
\]
is invertible, we define the gauge transformation \(\tau_B\) of \(\pi\) associated with \(B\) by
\[
\tilde{\tau}_B(\pi) = \tilde{\pi}(1 + \tilde{B}\tilde{\pi})^{-1}.
\]
(7.2)
When \(\pi\) is nondegenerate, (7.2) just says that \(\tilde{\tau}_B(\pi)^{-1} = \tilde{\pi}^{-1} + B\); in particular, any two symplectic structures on a given manifold are gauge equivalent. In general, a gauge transformation \(\tau_B\) produces a Poisson structure whose leaf decomposition is the same as before, but the symplectic structures along the leaves differ by the pullbacks of \(B\).\(^3\) We call two Poisson structures \(\pi_1\) and \(\pi_2\) on \(P\) gauge equivalent if there exists a closed 2-form \(B\) with \(\tau_B(\pi_1) = \pi_2\). More generally, two Poisson manifolds \((P_1, \pi_1)\) and \((P_2, \pi_2)\) are called gauge equivalent up to Poisson diffeomorphism if there exists a Poisson diffeomorphism \(f : (P_1, \pi_1) \to (P_2, \tau_B(\pi_2))\) for some closed 2-form \(B \in \Omega^2(P_2)\).

The close relationship between gauge and Morita equivalences is shown by the following result [7].

**Theorem 7.1.** If two integrable Poisson manifolds \((P_1, \pi_1)\) and \((P_2, \pi_2)\) are gauge equivalent up to Poisson diffeomorphism, then they are Morita equivalent. The converse does not hold in general.

The proof of the theorem follows from the description of the effect of gauge transformations on symplectic groupoids. Let \((P, \pi)\) be an integrable Poisson manifold, and let \(\Gamma(P)\) be its source-simply-connected symplectic groupoid. Since \(\pi\) and any gauge equivalent Poisson structure \(\tau_B(\pi)\) have isomorphic Lie algebroids, \(\tau_B(\pi)\) is integrable, and its (source-simply-connected) symplectic groupoid can be identified, as a Lie groupoid, with \(\Gamma(P)\). The source and target maps are unchanged, but the original symplectic structure \(\omega\) on \(\Gamma(P)\) is changed to \(\tau_B(\omega) := \omega + t^*B - s^*B\). (The additional terms form a coboundary in the “bar-de Rham” double complex attached to a groupoid, as discussed, for instance, in [4]. The meaning of this fact is not very clear to us, but see [6, Ex. 6.6] and Section 9.1.)

In other words, we have the correspondences:

\[
\begin{array}{ccc}
(Gamma(P), \omega) & \longrightarrow & (Gamma(P), \tau_B(\omega)) \\
\downarrow & & \downarrow \\
(P, \pi) & \longrightarrow & (P, \tau_B(\pi))
\end{array}
\]

(7.3)

\(^3\)Since \(1 + \tilde{B}\tilde{\pi}\) might not be invertible, we are not quite dealing here with a group action, but with a groupoid, obtained by restricting to the Poisson structures the groupoid associated with the action of the additive group of closed 2-forms on the space of Dirac structures [14] on \(P\) (see [6]).
A Morita bimodule for $\pi$ and $\tau_B(\pi)$ is obtained by means of a “half” twist:

\[
\begin{array}{c}
\Gamma(P, \omega') \\
\downarrow t \quad \downarrow s \\
(P, \pi) & (P, \tau_B(\pi))
\end{array}
\]

where $\omega' = \omega - s^*B$.

To see that Morita equivalence does not imply gauge equivalence, even up to Poisson diffeomorphism, we note that two symplectic manifolds are gauge equivalent up to Poisson diffeomorphism if and only if they are symplectomorphic, whereas Morita equivalence just amounts to isomorphism of fundamental groups. Example 5.2 in [7] shows that even Morita equivalent Poisson structures on the same manifold can fail to be gauge equivalent up to Poisson diffeomorphism. On the other hand, we will present in Section 8 a class of Poisson structures on surfaces for which the notions of gauge and Morita equivalence do coincide.

**Remark 7.2.** As we mentioned above, the gauge transformation construction extends to an action of the closed 2-forms on $P$ on the set of Dirac structures on $P$. If $(P, \pi)$ is an integrable Poisson manifold with symplectic groupoid $(\Gamma(P), \omega)$ such that $\tau_B(\pi)$ fails to be a Poisson structure, then the 2-form $\omega + t^*B - s^*B$ becomes degenerate, and the diagram (7.3) suggests that Dirac structures should be “integrated” to groupoids carrying multiplicative pre-symplectic forms, generalizing the correspondence between integrable Poisson manifolds and symplectic groupoids. The precise relationship between Dirac structures and “pre-symplectic groupoids” is developed in [6], where the situation of Dirac structures “twisted” by closed 3-forms [46] is also investigated. Following [54], one may use the correspondence

\[
\text{(twisted) Dirac structures} \leftrightarrow \text{(twisted) pre-symplectic groupoids}
\]

to give a groupoid interpretation of quasi-hamiltonian actions and their group-valued momentum maps [2].

**Remark 7.3.** The results in [44], on Morita equivalence of quantum tori, and [9], [24], on Morita equivalence of formal deformation quantizations, suggest that one should single out for special attention the gauge transformations by closed 2-forms which belong to integer cohomology classes. It should be interesting to investigate how the geometric properties of the symplectic bimodules given by Theorem 7.1 in this case might relate to the Morita equivalence on the quantum level.

### 8. Topologically Stable Structures on Surfaces

This section has been written in collaboration with Olga Radko.

Let $\Sigma$ be a compact connected oriented surface equipped with a Poisson structure $\pi$ which has at most linear degeneracies. We call such a structure **topologically stable**, since the topology of its zero set is preserved under small perturbations of $\pi$. This zero set consists of $n$ smooth disjoint, closed curves on $\Sigma$, for some $n \geq 0$. Each of them carries a natural orientation given by any modular vector field for $\pi$. We denote by $Z(\Sigma, \pi)$ the zero set, considered as an oriented 1-manifold.
We call two topologically stable surfaces \((\Sigma, \pi)\) and \((\Sigma', \pi')\) \textit{topologically equivalent} if there is an orientation-preserving diffeomorphism \(\varphi : \Sigma \to \Sigma'\) such that \(\varphi(Z(\Sigma, \pi)) = Z(\Sigma', \pi')\). The associated equivalence class is denoted by \([Z(\Sigma, \pi)]\).

Each topologically stable structure \((\Sigma, \pi)\) with \(n\) zero curves has \((n + 1)\) numerical invariants: \(n\) modular periods (periods of a modular vector field around the zero curves) and a regularized volume invariant (generalizing the Liouville volume in the symplectic case). Together with the topological equivalence class, these invariants completely classify topologically stable structures up to orientation-preserving Poisson isomorphisms [12].

A topological equivalence class \([Z(\Sigma, \pi)]\) can be encoded by an oriented labeled graph \(G(\Sigma, \pi)\), with a vertex for each 2-dimensional leaf of the structure, two vertices being connected by an edge for each boundary zero curve which they share. Each edge is oriented so that it points toward the vertex for which the Poisson structure is positive with respect to the orientation of \(\Sigma\). We label each vertex by the genus of the corresponding leaf. (Note that this genus, together with the number of edges at the vertex, completely determines the topology of the leaf.) If we also assign to each edge of \(G(\Sigma, \pi)\) the modular period of \((\Sigma, \pi)\) around the corresponding zero curve, we obtain a more elaborately labeled graph which we denote by \(G_T(\Sigma, \pi)\).

8.1. Morita equivalence of topologically stable structures. Topologically stable structures (TSS) on surfaces form a class of Poisson structures for which Morita equivalence is the same as the gauge equivalence up to a diffeomorphism. The first step toward establishing this is:

**Theorem 8.1.** Two TSS \((\Sigma_1, \pi_1)\) and \((\Sigma_2, \pi_2)\) are Morita equivalent iff there is an isomorphism of labeled graphs \(\Theta_T(\Sigma_1, \pi_1) \simeq \Theta_T(\Sigma_2, \pi_2)\).

**Proof.** Assume that \((\Sigma_1, \pi_1)\) and \((\Sigma_2, \pi_2)\) are Morita equivalent. As was shown in [7] for the case of the sphere, the fact that Morita equivalence implies homeomorphism of the leaf spaces means that \(\Theta(\Sigma_1, \pi_1) \simeq \Theta(\Sigma_2, \pi_2)\). Since the restriction of the Morita equivalence bimodule to each leaf is a Morita equivalence (see Remark 4.2), it follows that the genera of the associated leaves are the same. Finally, since modular periods are invariants of Morita equivalence of topologically stable structures [7], \(\Theta_T(\Sigma_1, \pi_1) \simeq \Theta_T(\Sigma_2, \pi_2)\) as labeled graphs.

Conversely, if \(\Theta_T(\Sigma_1, \pi_1) \simeq \Theta_T(\Sigma_2, \pi_2)\), there is a diffeomorphism \(\alpha : \Sigma_1 \to \Sigma_2\) which maps \(Z(\Sigma_1, \pi_1)\) to \(Z(\Sigma_2, \pi_2)\), since the graph \(\Theta(\Sigma, \pi)\) completely encodes the topology of the decomposition of \(\Sigma\) into its 2-dimensional symplectic leaves. On \(\Sigma = \Sigma_1\), let \(\pi = \pi_1, \pi' = \alpha^{-1}_*(\pi_2)\). Since \(\Theta_T(\Sigma_1, \pi_1) \simeq \Theta_T(\Sigma_2, \pi_2)\), we know that, for any zero curve \(\gamma \in Z(\pi) = Z(\pi')\), the modular periods for the two structures are the same. By Theorem 6.2 of [7], \(\pi\) and \(\pi'\) are gauge equivalent up to Poisson diffeomorphism, and so are \((\Sigma_1, \pi_1)\) and \((\Sigma_2, \pi_2)\); hence they are Morita equivalent by Theorem 7.1. \(\square\)

The fact that gauge equivalence up to a diffeomorphism implies Morita equivalence and the second part of the proof above imply:

**Corollary 8.2.** For TSS the notions of Morita and gauge equivalence up to a diffeomorphism coincide.
8.2. Picard groups of topologically stable structures. In this section, we merely raise the problem of computing the Picard group of a TSS \((\Sigma, \pi)\). It appears from the arguments above that the ingredients of \(\text{Pic}(\Sigma, \pi)\) should be:

1. the automorphism group of the labeled graph \(G_T(\Sigma, \pi)\);
2. the torus which is the product of the groups of rotations of the zero curves;
3. the outer automorphism groups of the fundamental groups of the 2-dimensional leaves.

How these ingredients are combined should be described by an algebraic/combinatorial object which is a further refinement of \(G_T(\Sigma, \pi)\), and which encodes the inclusions of the fundamental groups of the zero curves into those of the adjacent 2-dimensional leaves.

9. Further Questions

We hope that this paper represents the beginning of an interesting line of research. With that in mind, we conclude with discussion of some issues which remain at least partly unresolved.

9.1. The Lie algebra of the Picard group. In all the examples above, the Picard group consists of a discrete part associated with fundamental groups, and a continuous part associated with diffeomorphisms of leaf spaces. It seems useful, therefore, to think of Picard groups as (possibly infinite-dimensional) Lie groups, and to study the infinitesimal objects which should play the role of their Lie algebras.

It should be possible to define Picard Lie algebras \(\text{pic}(A)\), \(\text{pic}(\Gamma)\), and \(\text{pic}(P)\) for algebras, (Lie, symplectic) groupoids, and Poisson manifolds. Their elements should be isomorphism classes of infinitesimal deformations of the identity bimodule (or bibundle), with the underlying object held fixed. (This is to be contrasted with the analysis in [8] of deformations of bimodules as the underlying algebra is deformed.) In the case of algebras, if one fixes the underlying \(k\)-module of the bimodule to be \(A\) itself, \(\text{pic}(A)\) has a description in terms of Hochschild cocycles and Gerstenhaber brackets. For symplectic groupoids, one may fix the underlying manifold of the bibundle and deform the symplectic structure along with the bibundle structure. The former deformations should be related to the groupoid double complex mentioned after Theorem 7.1 and produce 2-cohomology classes on the base, while the latter should lead to the “diffeomorphism” part of \(\text{pic}(\Gamma)\).

Revisiting the exact sequences in Section 5 and passing to Lie algebras, we obtain first of all from (5.5) and (5.6) the sequence

\[
0 \to \text{Pic}(P) / H_{\pi}^2(P) \xrightarrow{d_{\pi}} \mathcal{X}_{\pi}(P) \to \text{pic}(P),
\]

so that \(H_{\pi}^1(P)\) sits inside \(\text{pic}(P)\) as a subalgebra. (Here \(\mathcal{X}_{\pi}(P)\) denotes the space of Poisson vector fields, \(H_{\pi}^r(P)\) are Poisson cohomology spaces and \(d_{\pi}\) is the Poisson differential, see, e.g., [10].) Similarly, (5.9) becomes

\[
0 \to \text{Pic}_{\mathcal{Z}(P)} \to \text{pic}(P) \to \mathcal{X}(\mathcal{Z}(P)).
\]
Here, \( \mathcal{X}(Z(P)) \) denotes the space of vector fields on \( Z(P) \), restricted when appropriate to those whose flows preserve an additional structure such as a variation lattice.

### 9.2. Representation equivalence vs. Morita equivalence

Although, for algebras, representation equivalence implies Morita equivalence, this is not the case in other categories, where an object may not serve as its own identity bimodule. Landsman [27] discusses this phenomenon, including the case of \( C^* \)-algebras, where the notion of representation must be suitably chosen so that representation equivalence does imply Morita equivalence [43].

For Poisson manifolds, Xu gave an explicit counterexample in [52]. One simply takes a Poisson manifold and multiplies the Poisson structure by 2. This clearly does not change the category of representations, but examples may be given in which the Morita equivalence class does change, since the variation lattice on the leaf space is multiplied by 2. (See Remark 5.2.) In this setting, it seems that, rather than changing the notion of representation, we should give a refined structure to the category of representations. For instance, in Xu’s examples, one may distinguish the two representation categories by considering the Lie algebras of the automorphism groups as (codimension 1 quotients of) Poisson algebras. An interesting approach would be to define a notion of representation for Poisson manifolds which is refined enough to detect the difference between the two examples, but we will reserve this discussion for a future work.

### 9.3. Nonintegrable Poisson manifolds

If we try to build a category in which the Morita equivalences are isomorphisms, and which contains all Poisson manifolds and complete Poisson maps, two problems immediately arise. First of all, if we define morphisms to be (symplectic) bimodules, then only the integrable Poisson manifolds admit identity morphisms. Second, although we can associate a bimodule to any complete Poisson map between integrable Poisson manifolds, these bimodules are generally not generalized morphisms (see Remark 4.1), so that it becomes difficult to define compositions. Both of these problems occur because certain natural constructions lead to leaf spaces of foliations, which may not be manifolds.

It seems that the difficulties just described can be circumvented if one admits as symplectic bimodules not only symplectic manifolds, but also the leaf spaces of transversely symplectic foliations. The technical aspects of this approach are yet to be worked out, requiring an appropriate category of “leaf spaces,” so we merely sketch some ideas here.

The problem of identity morphisms is handled as follows. Cattaneo and Felder [12] constructed, for every Poisson manifold \( P \), a groupoid \( \Gamma(P) \) over \( P \) which they identify as the “phase space” for a Poisson sigma model associated with \( P \). More concretely, \( \Gamma(P) \) is the quotient of a Banach manifold of paths in \( T^* P \) by a foliation of finite codimension (twice the dimension of \( P \)). Crainic and Fernandes [16] extended this construction to the case where the Poisson manifold \( P \) are replaced by an arbitrary manifold \( M \) and a Lie algebroid \( A \) over \( M \) (the Lie algebroid in the Poisson case being the cotangent bundle \( T^* P \)), obtaining a canonical groupoid \( \Gamma(A) \) over \( M \) which they call the “Weinstein groupoid” of \( A \). The ideas of this
construction have appeared independently in [45]. By abuse of notation, we will
denote $\Gamma(T^*P)$ for a Poisson manifold $P$ by $\Gamma(P)$.

Since the elements of $\Gamma(A)$ are homotopy classes of paths (in a sense adapted to
the presence of $A$) in $M$, we propose, following [45], to call $\Gamma(A)$ the \textit{fundamental
groupoid} of $(M, A)$ or, when $A$ is the cotangent Lie algebroid $T^*P$ of a Poisson
manifold $P$, the \textit{Poisson fundamental groupoid} of $P$. Notice that $\Gamma(A)$ is the usual
fundamental groupoid of $M$ just when $A$ is isomorphic to $TM$ (i.e., when $P$ is
symplectic in the Poisson case).

Although $\Gamma(A)$ it is generally not a manifold, it is always the leaf space of a
foliation. This foliation originally lives in the Banach manifold of $A$-paths (see [16]),
but it is finite-codimensional, so if one prefers to work with finite dimensional spaces,
one may identify the leaf space with the the orbit space of an \’{e}tale groupoid obtained
by restricting the holonomy groupoid of the foliation to a complete transversal.
Crainic and Fernandes gave explicit criteria in [16] for $\Gamma(A)$ to be an ordinary

Leaf spaces may still be treated for many purposes as if it they were manifolds.
The best way to do this seems to define a \textit{leafspace structure} on $X$ as a Morita
equivalence class of \’{e}tale groupoids whose orbit spaces are identified with $X$. Mappings
between leafspaces are then equivalence classes of regular bimodules over these
groupoids, and a symplectic structure on $X$ is given by groupoid-invariant symplec-
tic structures on the manifolds of objects of \’{e}tale groupoids realizing $X$.

All this analysis is perhaps best carried out in the language of “differentiable stacks” [41].

We may now define a category in which the objects are Poisson manifolds, and in
which a morphism $P_2 \leftrightarrow P_1$ is an isomorphism class of left principal bibundles of the
form (4.1) in which $S$ is a symplectic leafspace. An isomorphism in this category will
still be called a Morita equivalence. Fortunately, it follows from the local triviality of
principal bundles [34] that the total space of a Morita equivalence between \textit{integrable}
Poisson manifolds must be a smooth manifold, so that the manifolds are already
Morita equivalent in the usual sense.

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\footnote{Note that the term “symplectic leaf spaces” has quite a different meaning.}
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