

MODULAR HECKE ALGEBRAS AND THEIR HOPF SYMMETRY

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Dedicated to Pierre Cartier

ABSTRACT. We introduce and begin to analyse a class of algebras, associated to congruence subgroups, that extend both the algebra of modular forms of all levels and the ring of classical Hecke operators. At the intuitive level, these are algebras of ‘polynomial coordinates’ for the ‘transverse space’ of lattices modulo the action of the Hecke correspondences. Their underlying symmetry is shown to be encoded by the same Hopf algebra that controls the transverse geometry of codimension 1 foliations. Its action is shown to span the ‘holomorphic tangent space’ of the noncommutative space, and each of its three basic Hopf cyclic cocycles acquires a specific meaning. The Schwarzian 1-cocycle gives an inner derivation implemented by the level 1 Eisenstein series of weight 4. The Hopf cyclic 2-cocycle representing the transverse fundamental class provides a natural extension of the first Rankin–Cohen bracket to the modular Hecke algebras. Lastly, the Hopf cyclic version of the Godbillon–Vey cocycle gives rise to a 1-cocycle on $\mathrm{PSL}(2, \mathbb{Q})$ with values in Eisenstein series of weight 2, which, when coupled with the ‘period’ cocycle, yields a representative of the Euler class.

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INTRODUCTION

The aim of this paper is to introduce and begin to analyse a class of algebras, called modular Hecke algebras, which encode the two *a priori* unrelated structures on modular forms, namely the algebra structure given by the pointwise product on the one hand, and the action of the Hecke operators on the other. We associate to any congruence subgroup Γ of $\mathrm{SL}(2, \mathbb{Z})$ a crossed product algebra $\mathcal{A}(\Gamma)$, the *modular Hecke algebra* of level Γ , which is a direct extension of both the ring of classical Hecke operators and of the algebra $\mathcal{M}(\Gamma)$ of Γ -modular forms. With \mathcal{M}

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denoting the algebra of modular forms of arbitrary level, the elements of $\mathcal{A}(\Gamma)$ are maps with finite support

$$F: \Gamma \backslash \mathrm{GL}^+(2, \mathbb{Q}) \rightarrow \mathcal{M}, \quad \Gamma \alpha \mapsto F_\alpha \in \mathcal{M},$$

satisfying the covariance condition

$$F_{\alpha\gamma} = F_\alpha|_\gamma, \quad \text{for all } \alpha \in \mathrm{GL}^+(2, \mathbb{Q}), \gamma \in \Gamma$$

and their product is given by convolution. In the simplest case $\Gamma(1) = \mathrm{SL}(2, \mathbb{Z})$, the elements of $\mathcal{A}(\Gamma(1))$ are encoded by a finite number of modular forms $f_N \in \mathcal{M}(\Gamma_0(N))$ of arbitrary high level and the product operation is non-trivial. The algebra $\mathcal{A}(\Gamma)$ acts on $\mathcal{M}(\Gamma)$ extending the action of classical Hecke operators, and the ‘cuspidal’ elements form a two-sided ideal of $\mathcal{A}(\Gamma)$.

Our starting point is the basic observation that the Hopf algebra \mathcal{H}_1 , which was discovered (cf. [12]) in the analysis (cf. [11]) of codimension 1 foliations as underlying symmetry of the ‘transverse’ geometry, admits a natural action on the crossed product algebras $\mathcal{A}(\Gamma)$. This action is sufficiently non-trivial to span the ‘tangent space’ of the non-commutative space $Q(\Gamma)$ with coordinate ring $\mathcal{A}(\Gamma)$, and is thus a key ingredient in the understanding of the geometry of $Q(\Gamma)$. At an intuitive level these non-commutative spaces are obtained from the quotient of the space of lattices by the Hecke correspondences, while $\mathcal{A}(\Gamma)$ is the simplest algebra of polynomial coordinates on $Q(\Gamma)$.

To describe the action of \mathcal{H}_1 , we recall that, as an algebra, \mathcal{H}_1 coincides with the universal enveloping algebra of the Lie algebra with basis $\{X, Y, \delta_n; n \geq 1\}$ and brackets

$$[Y, X] = X, \quad [Y, \delta_n] = n\delta_n, \quad [X, \delta_n] = \delta_{n+1}, \quad [\delta_k, \delta_\ell] = 0, \quad n, k, \ell \geq 1,$$

while the coproduct which confers it the Hopf algebra structure is determined by the identities

$$\begin{aligned} \Delta Y &= Y \otimes 1 + 1 \otimes Y, \\ \Delta \delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1, \\ \Delta X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \end{aligned}$$

together with the property that $\Delta: \mathcal{H}_1 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ is an algebra homomorphism. The action of X on $\mathcal{A}(\Gamma)$ is given by a classical operator going back to Ramanujan ([25], [27]), which corrects the usual differentiation by the logarithmic derivative of the Dedekind eta function $\eta(z)$. The action of Y is given by the standard grading by the weight (the Euler operator) on modular forms. Finally, δ_1 and its higher ‘derivatives’ δ_n act by generalized cocycles on $\mathrm{GL}^+(2, \mathbb{Q})$ with values in modular forms.

The main result of the present paper is a rather complete understanding of the above action of \mathcal{H}_1 , viewed as the underlying symmetry of the noncommutative space $Q(\Gamma)$ obtained from the quotient of the space of lattices by the Hecke correspondences. The picture that emerges is that of a surprisingly close analogy between the action of Hecke operators on modular forms and the action of a discrete subgroup of $\mathrm{Diff}(S^1)$ on polynomial functions on the frame bundle of S^1 . The role of the angular variable $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ is assumed by the simplest Eichler integral,

namely the primitive $Z \in \mathbb{C}/\Lambda$ of the holomorphic differential form $\frac{2\pi i}{6}\eta^4 dz$ on the elliptic curve $X'(1) = \Gamma'(1)\backslash H^*$, where $\Gamma'(1)$ is the commutator subgroup of $\Gamma(1)$. The lattice $2\pi\mathbb{Z} \subset \mathbb{R}$ is replaced by the equilateral lattice $\Lambda \subset \mathbb{C}$ of periods of Z . The part of the circle

$$e^{i\theta} = \cos \theta + i \sin \theta, \quad x^2 + y^2 = 1$$

is thus played by the genus 1 curve $X'(1) \sim \mathbb{C}/\Lambda$ with Weierstrass parametrization and equation

$$(\wp_\Lambda(Z), \wp'_\Lambda(Z)) = \left(\sqrt[3]{j}, \frac{-2E_6}{\eta^{12}} \right), \quad y^2 = 4(x^3 - 1728)$$

using traditional notations. The diffeomorphisms $\phi(\theta)$ are replaced by Hecke transformations $Z|_0\gamma$, while the Jacobian $\phi'(\theta)$ becomes $J(\gamma) = \frac{dZ|_0\gamma}{dZ}$. Accordingly, the higher cocycles $\left(\frac{d}{d\theta}\right)^n \log \phi'(\theta)$ which allow us to define the action of δ_n take the form $\left(\frac{d}{dZ}\right)^n \log J(\gamma)$. Furthermore, the usual projective structure on S^1 , corresponding to $t = \tan(\frac{\theta}{2})$, which in terms of θ is given by the quadratic differential

$$\rho := \{t; \theta\} d\theta^2 = \frac{1}{2} d\theta^2,$$

has as its modular counterpart the rational projective structure on $X'(1)$ given by the quadratic differential

$$\varpi' = \frac{x dx^2}{2y^2} = \frac{x dx^2}{8(x^3 - 1728)}.$$

This Schwarzian relation between Z and ϖ' is similar to the classical relation between the modular invariant j and E_4 in the case of genus zero [22].

With the above dictionary in mind, one can proceed to transfer transverse geometry concepts and constructions to the setting of modular forms (cf. Theorem 12 below). The basic tool is the cyclic cohomology of Hopf algebras developed in ([12]). For \mathcal{H}_1 one gets three basic cyclic classes, δ'_2 , δ_1 and F which in the original action of \mathcal{H}_1 on the crossed product of functions on the frame bundle on S^1 by a discrete subgroup of $\text{Diff}(S^1)$ correspond respectively to

Schwarzian	$\delta'_2 := \delta_2 - \frac{1}{2}\delta_1^2,$
Godbillon–Vey Class	$\delta_1,$
Transverse Fundamental Class	$F := X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y.$

In the present paper we shall compute the above three classes in the action of \mathcal{H}_1 on the modular Hecke algebras.

We first show that, under the present action of \mathcal{H}_1 , the element δ'_2 is represented by an inner derivation implemented by the above mentioned Eisenstein series of weight 4 and level 1, denoted here by ω_4 . We then prove that no inner perturbation of the action by a 1-cocycle can annihilate the Schwarzian δ'_2 , and that the freedom one has in modifying the action of the Hopf algebra \mathcal{H}_1 by a 1-cocycle exactly changes the restriction of the action of X on modular forms and the value of ω_4 as

in the data used by Zagier [30] to define ‘canonical’ Rankin–Cohen algebras, where moreover the same ω_4 appears with a slightly different normalization.

We then show that the image under the canonical map from the Hopf cyclic cohomology of \mathcal{H}_1 to the Hochschild cohomology of the algebras $\mathcal{A}(\Gamma)$ of the Hopf 2-cocycle encoding the ‘transverse fundamental class’ gives the natural extension of the first Rankin–Cohen bracket for modular forms to $\mathcal{A}(\Gamma)$. Actually, in a sequel to this paper [15], we show that all the Rankin–Cohen brackets have natural extensions to modular Hecke algebras, which moreover are determined by a universal deformation formula based on \mathcal{H}_1 . The mere existence of the modular Hecke algebras is sufficient to prove in full generality the universal associativity property.

While the transverse fundamental class generates the even component of the periodic Hopf cyclic cohomology of \mathcal{H}_1 , the odd component is generated by the Hopf cyclic version of the Godbillon–Vey class, namely δ_1 . In the context of the present action of \mathcal{H}_1 , this class gives rise to a ‘transverse’ 1-cocycle on $\mathrm{GL}^+(2, \mathbb{Q})$ with values in Eisenstein series of weight 2, measuring the lack of $\mathrm{GL}^+(2, \mathbb{Q})$ -invariance of the connection ∇ corresponding to X . When coupled with the obvious extension of the classical period cocycle, it yields an analogue of the Bott–Thurston group 2-cocycle [7], that represents the Euler class $e \in H_d^2(\mathrm{SL}(2, \mathbb{R}), \mathbb{R})$ (cf. Theorem 16). By a theorem of Borel and Yang [5], the restriction of this class to $H^2(\mathrm{SL}(2, \mathbb{Q})^\delta, \mathbb{R})$ is nontrivial and generates $H^2(\mathrm{SL}(2, \mathbb{Q})^\delta, \mathbb{R})$.

Our last result (Theorem 17) gives an arithmetic representation for the rational Euler class $e \in H^2(\mathrm{SL}(2, \mathbb{Q})^\delta, \mathbb{Q})$ in terms of generalized Dedekind sums [16], [23]. The original Dedekind sums already arose in several different topological settings, as amply illustrated by Atiyah [2] and also by Kirby and Melvin [21].

In Section 1 we introduce the modular Hecke algebra $\mathcal{A}(\Gamma)$ for any congruence subgroup $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$. Its elements are form-valued Hecke operators; the classical Hecke algebra is the subalgebra of elements of degree 0 for the natural grading given by the weight.

Section 2 recalls the Hopf algebra \mathcal{H}_1 , in its original context of codimension 1 transverse geometry, and describes its most significant Hopf cyclic cocycles, that correspond to the Godbillon–Vey class, to the Schwarzian derivative, and to the transverse fundamental class.

In Section 3 we define and analyze in great detail the Hopf action of \mathcal{H}_1 on the crossed product of modular forms by Hecke operators. We show that the Schwarzian 1-cocycle gives an inner derivation. We relate the conjugates by inner perturbations of the Hopf action of \mathcal{H}_1 to the data used by Zagier [30] to define canonical Rankin–Cohen algebras. We also show that the Hopf cyclic 2-cocycle representing the transverse fundamental class provides a natural extension of the first Rankin–Cohen bracket to the crossed product algebras. The section concludes with a precise formulation of the geometric analogy with the case of $\mathrm{Diff}(S^1)$. In particular, we give a simple geometric explanation for the above formula for the quadratic differential ϖ .

In Section 4 we introduce the ‘transverse’ 1-cocycle on $\mathrm{GL}^+(2, \mathbb{Q})$ with values in Eisenstein series of weight 2, as a partial analogue of the Godbillon–Vey cocycle. We then complete the construction of its full analogue using the ‘period’ cocycle

and prove that it represents the Euler class of $\mathrm{SL}(2, \mathbb{R})$. Finally, we conclude by expressing the Euler class of $\mathrm{SL}(2, \mathbb{Q})$ in terms of generalized Dedekind sums.

Appendix A recalls the definition of cyclic cohomology for Hopf algebras, while Appendix B provides the details for the interpretation of the Godbillon–Vey class as a Hopf cyclic cohomological class.

1. MODULAR HECKE ALGEBRAS

Modular forms, regarded as lattice functions, may be viewed as natural coordinates on the lattice line bundle over the moduli space of isomorphism classes of elliptic curves over \mathbb{C} . We recall that every elliptic curve is isomorphic — as a complex manifold — to a 1-dimensional complex torus $E = \mathbb{C}/\Lambda$, where Λ is a lattice in \mathbb{C} ; furthermore, two lattices Λ and Λ' define isomorphic curves $E \simeq E'$ iff they are homothetic,

$$\Lambda' = \lambda\Lambda, \quad \text{for some } \lambda \in \mathbb{C}^\times.$$

Since one can always represent a lattice Λ in the form

$$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2, \quad \mathrm{Im} \frac{\omega_1}{\omega_2} > 0,$$

up to the obvious action of the group $\Gamma(1) = \mathrm{SL}(2, \mathbb{Z})$, it follows that the isomorphism classes of elliptic curves are parametrized by the quotient $\Gamma(1) \backslash H$, where $H = \{z \in \mathbb{C}, \mathrm{Im}(z) > 0\}$ is the upper half-plane. Thus, the set \mathcal{L} of all lattices in \mathbb{C} defines a line bundle

$$\mathbb{C}^\times \rightarrow \mathcal{L} \rightarrow \Gamma(1) \backslash H,$$

and the line bundle \mathcal{L}^{-2} is canonically isomorphic to the (complex) 1-jet bundle of $\Gamma(1) \backslash H$.

Any lattice function which is homogeneous of weight $2k$,

$$F(\lambda\Lambda) = \lambda^{-2k} F(\Lambda), \quad \lambda \in \mathbb{C}^\times,$$

is automatically of the form

$$F(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) = \omega_2^{-2k} f\left(\frac{\omega_1}{\omega_2}\right)$$

with the function f satisfying the modularity property

$$f|_{2k}\gamma = f, \quad \text{for all } \gamma \in \Gamma(1) = \mathrm{SL}(2, \mathbb{Z}), \quad (1.1)$$

where we used the standard ‘slash operator’ notation for the action of

$$G^+(\mathbb{R}) := \mathrm{GL}^+(2, \mathbb{R})$$

on functions on the upper half plane:

$$f|_k \alpha(z) = \det(\alpha)^{k/2} f(\alpha \cdot z) j(\alpha, z)^{-k}, \quad (1.2)$$

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+(\mathbb{R}), \quad \alpha \cdot z = \frac{az + b}{cz + d} \quad \text{and} \quad j(\alpha, z) = cz + d.$$

In particular, $f(z+1) = f(z)$, $z \in \mathbb{C}$, hence it induces a function on the punctured unit disc $f_\infty(q)$, $q = e^{2\pi iz}$. Such a function f is called a *modular form of weight $2k$*

if it is holomorphic on H and also at ∞ , that is f_∞ is holomorphic at $q = 0$; f is called a *cuspidal form* if in addition $f_\infty(0) = 0$. We denote by

$$\mathcal{M}(\Gamma(1)) := \bigoplus \mathcal{M}_{2k}(\Gamma(1)), \quad \text{resp.} \quad \mathcal{M}^0(\Gamma(1)) := \bigoplus \mathcal{M}_{2k}^0(\Gamma(1)),$$

the algebra of modular (resp. cusp) forms.

A richer and more interesting picture emerges when lattices are replaced by pairs (Λ, ϕ) , with Λ a lattice in \mathbb{C} , and $\phi: \frac{\mathbb{Q}\Lambda}{\Lambda} \xrightarrow{\sim} \frac{\mathbb{Q}^2}{\mathbb{Z}^2}$ an isomorphism which is ‘unimodular’ in a suitably defined sense, cf. [24, §17]. The set $\mathcal{L}_\mathbb{A}$ of all such pairs forms again a line bundle, only this time over a projective limit of Riemann surfaces,

$$\mathbb{C}^\times \rightarrow \mathcal{L}_\mathbb{A} \rightarrow H_\mathbb{A} := \varprojlim_N \Gamma(N) \backslash H,$$

where

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{Z}) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod{N} \right\}.$$

and the projective system is ordered by divisibility. The subscript \mathbb{A} is justified by the adelic interpretation of the fibration. Indeed, if one drops the ‘unimodularity’ condition on ϕ , then the resulting space of pairs can be canonically identified to the homogeneous space

$$\text{GL}(2, \mathbb{Q}) \backslash \text{GL}(2, \mathbb{A}). \quad (1.3)$$

The space $\mathcal{L}_\mathbb{A}$ is the closure of $\text{GL}(2, \mathbb{R})$ in this homogeneous space, and its structure of principal bundle comes from the natural inclusion of the multiplicative group \mathbb{C}^* in $\text{GL}(2, \mathbb{R})$. One can also describe $H_\mathbb{A}$ (cf. [24, §17]) as a double quotient of the form,

$$H_\mathbb{A} \simeq \text{GL}(2, \mathbb{Q}) \backslash \text{GL}(2, \mathbb{A})^0 / K_\infty Z_\infty, \quad (1.4)$$

where $K_\infty \simeq S^1$, $Z_\infty \simeq \text{GL}(1, \mathbb{R})$ viewed as diagonal matrices in $\text{GL}(2, \mathbb{R})$, and the subgroup $\text{GL}(2, \mathbb{A})^0$ is defined by requiring that the determinant belongs to $\mathbb{Q}^* \times \mathbb{R}^* \subset \text{GL}(1, \mathbb{A})$.

Modular (resp. cusp) forms f of weight k with respect to any congruence subgroup Γ are defined in a similar way as for $\Gamma(1)$, by requiring the holomorphicity (resp. cuspidality) of $f|_k \gamma$ at ∞ , with respect to any local parameter $q^{1/m}$, for every $\gamma \in \Gamma(1)$. One obtains, for each $N \geq 1$, graded algebras of *forms of level N*

$$\mathcal{M}(\Gamma(N)) := \bigoplus_{k \geq 1} \mathcal{M}_k(\Gamma(N)), \quad \text{resp.} \quad \mathcal{M}^0(\Gamma(N)) := \bigoplus_{k \geq 1} \mathcal{M}_k^0(\Gamma(N)),$$

which can be assembled together into algebras of modular (resp. cusp) forms of all levels:

$$\mathcal{M} := \varprojlim_N \mathcal{M}(\Gamma(N)), \quad \text{resp.} \quad \mathcal{M}^0 := \varprojlim_N \mathcal{M}^0(\Gamma(N)).$$

The group

$$G^+(\mathbb{Q}) := \text{GL}^+(2, \mathbb{Q})$$

acts ‘sideways’ on the tower defining the projective limit $H_\mathbb{A}$. Equivalently, one can view $G^+(\mathbb{Q})$ as diagonally embedded in $\text{GL}(2, \mathbb{A}_f)^0$, with the latter acting on $H_\mathbb{A}$

by right translations (cf. (1.4)). Using the fact that one can factor any $\alpha \in G^+(\mathbb{Q})$ in the form

$$\alpha = \gamma \cdot \beta, \quad \gamma \in \Gamma(1), \quad \beta \in B^+(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in G^+(\mathbb{Q}) \right\}, \quad (1.5)$$

it is easy to see that the action of $G^+(\mathbb{Q})$ preserves both holomorphicity and cuspidality. Thus, to this action of $G^+(\mathbb{Q})$ corresponds a first algebra of ‘noncommutative coordinates’ on the ‘transverse space’ $\mathcal{L}_{\mathbb{A}}/G^+(\mathbb{Q})$, namely the crossed-product algebra

$$\mathcal{A} \equiv \mathcal{A}_{G^+(\mathbb{Q})} := \mathcal{M} \rtimes G^+(\mathbb{Q}),$$

containing as an ideal the crossed product

$$\mathcal{A}^0 := \mathcal{M}^0 \rtimes G^+(\mathbb{Q}).$$

The elements of \mathcal{A} are finite sums of symbols of the form

$$\sum f U_{\gamma}^*, \quad \text{with } f \in \mathcal{M}, \gamma \in G^+(\mathbb{Q}),$$

and with the product given by the rule

$$f U_{\alpha}^* \cdot g U_{\beta}^* = (f \cdot g | \alpha) U_{\beta \alpha}^*, \quad (1.6)$$

where the ‘slash operation’ is relative to the weight of g as in (1.2).

On the other hand, to the right action of $\mathrm{GL}(2, \mathbb{A}_f)^0$ corresponds the inductive limit of the following *level N algebras*, $\mathcal{A}(N) := \mathcal{A}(\Gamma(N))$, obtained as crossed products of modular forms by Hecke operators at level N , which we proceed now to describe.

More generally, for any congruence subgroup Γ we shall construct an algebra $\mathcal{A}(\Gamma)$, which contains as subalgebras both the algebra of Γ -modular forms $\mathcal{M}(\Gamma)$ as well as the Hecke ring at level Γ (cf. [28, Chap. 3]), without being in general generated by these subalgebras.

Definition. Let Γ be a congruence subgroup. By a *Hecke operator form of level Γ* we mean a map

$$F: \Gamma \backslash G^+(\mathbb{Q}) \rightarrow \mathcal{M}, \quad \Gamma \alpha \mapsto F_{\alpha} \in \mathcal{M},$$

with *finite support* and satisfying the *covariance condition*

$$F_{\alpha \gamma} = F_{\alpha} | \gamma, \quad \text{for all } \alpha \in G^+(\mathbb{Q}), \gamma \in \Gamma. \quad (1.7)$$

A Hecke operator form F of level Γ is called *cuspidal* if

$$F_{\alpha} \in \mathcal{M}^0, \quad \text{for all } \alpha \in G^+(\mathbb{Q}). \quad (1.8)$$

Thus, by definition, F is determined by the (nonzero) values taken on a finite number of inequivalent representatives $\{\alpha_1, \dots, \alpha_r\}$ for double cosets in $\Gamma \backslash G^+(\mathbb{Q})/\Gamma$. It should be noted though that these values $F_{\alpha_i} \in \mathcal{M}$ are not required to be modular forms of level Γ . Maps from $\Gamma \backslash G^+(\mathbb{Q})/\Gamma$ to $\mathcal{M}(\Gamma)$ trivially fulfill equation (1.7), but they do not exhaust all its solutions. In fact, one has:

Lemma 1. *Let $f \in \mathcal{M}$ be any modular form, Γ a congruence subgroup. There exists a Hecke operator form F of level Γ such that $F_{\alpha} = f$ if and only if*

$$f | \gamma = f, \quad \text{for all } \gamma \in \Gamma \cap \alpha^{-1} \Gamma \alpha.$$

Proof. Assume $f = F_\alpha$. Then for any $\gamma \in \Gamma \cap \alpha^{-1}\Gamma\alpha$ one has $\alpha\gamma = \gamma'\alpha$, for some $\gamma' \in \Gamma$, and therefore $F_{\alpha\gamma} = F_{\gamma'\alpha} = F_\alpha$. On the other hand, by (1.7), $F_{\alpha\gamma} = F_\alpha|_\gamma$, which gives the required invariance property of f .

Conversely, define F to be 0 on the complement of the coset $\Gamma\alpha\Gamma$, while

$$F_{\gamma_1\alpha\gamma_2} = f|_{\gamma_2}, \quad \text{for all } \gamma_1, \gamma_2 \in \Gamma.$$

The definition is unambiguous, since if $\gamma'_1\alpha\gamma'_2 = \gamma_1\alpha\gamma_2$ then

$$\gamma'^{-1}_1\gamma_1\alpha = \alpha\gamma'_2\gamma_2^{-1}, \quad \text{with } \gamma'_2\gamma_2^{-1} \in \Gamma \cap \alpha^{-1}\Gamma\alpha,$$

so that $f|_{\gamma'_2} = f|_{\gamma'_2\gamma_2^{-1}}|_{\gamma_2} = f|_{\gamma_2}$. The condition (1.7) is automatically fulfilled. \square

As a simple illustration let us consider the case $\Gamma = \Gamma(1)$. The double cosets in $\Gamma(1)\backslash G^+(\mathbb{Q})/\Gamma(1)$ are represented by the elements $\alpha = r\beta_n$ where $r \in \text{GL}^+(1, \mathbb{Q})$ is the diagonal matrix $\begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$, $n \in \mathbb{N}$ and $\beta_n = \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}$. Then

$$\Gamma(1) \cap \alpha^{-1}\Gamma(1)\alpha = \Gamma_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) : c \equiv 0 \pmod{n} \right\}$$

and any $\Gamma_0(n)$ -modular form gives rise by the above recipe to a Hecke operator form of level $\Gamma(1)$. One can thus view a Hecke operator form of level $\Gamma(1)$ as a sequence with finite support of modular forms $f_{n,r} \in \mathcal{M}(\Gamma_0(n))$. Note that the action (1.2) on modular forms is trivial on the component $\text{GL}^+(1, \mathbb{R})$ of the center of $\text{GL}(2, \mathbb{R})$ and therefore the label r plays an unimportant role in the above labelling. We keep it however, in order to retain the ability to twist by characters of the center, for instance using the one-parameter group σ_z , $z \in \mathbb{C}$ of automorphisms of $\mathcal{A}(\Gamma)$ given by

$$(\sigma_z(F))_\alpha = \det(\alpha)^z F_\alpha.$$

These are automorphisms for the algebra structure defined below in Proposition 2. The weight of modular forms defines a natural grading on $\mathcal{A}(\Gamma)$, and one can see in the above example that the algebra $\mathcal{A}(\Gamma(1))$ of Hecke operator forms of level $\Gamma(1)$ is non-trivial in weight 2, while its subalgebra generated by modular forms of level 1 and standard Hecke operators has no element of weight 2.

Let us show that the space $\mathcal{A}(\Gamma)$ of Hecke operator forms of level Γ is indeed an algebra. We shall refer to it as the *modular Hecke algebra of level Γ* . This algebra acts canonically on the space $\mathcal{M}(\Gamma)$ of modular forms of level Γ , preserving the subspace $\mathcal{M}^0(\Gamma)$ of cusp forms.

Proposition 2. *Fix a congruence subgroup Γ .*

¹⁰. With $F^1, F^2 \in \mathcal{A}(\Gamma)$, the operation

$$(F^1 * F^2)_\alpha := \sum_{\beta \in \Gamma \backslash G^+(\mathbb{Q})} F^1_\beta \cdot F^2_{\alpha\beta^{-1}}|_\beta \quad (1.9)$$

turns the vector space $\mathcal{A}(\Gamma)$ of all Hecke operator forms of level Γ into an associative algebra.

2⁰. The equality

$$F * f := \sum_{\alpha \in \Gamma \backslash G^+(\mathbb{Q})} F_\alpha \cdot f|_\alpha \quad (1.10)$$

endows $\mathcal{M}(\Gamma)$ with the structure of an $\mathcal{A}(\Gamma)$ -module.

3⁰. The cuspidal Hecke operator forms of level Γ form an ideal $\mathcal{A}^0(\Gamma)$ of $\mathcal{A}(\Gamma)$; the subspace $\mathcal{M}^0(\Gamma)$ is a submodule of the $\mathcal{A}(\Gamma)$ -module $\mathcal{M}(\Gamma)$, and $\mathcal{A}^0(\Gamma)$ maps $\mathcal{M}(\Gamma)$ to $\mathcal{M}^0(\Gamma)$.

Proof. 1⁰. The right hand side of (1.9) is well-defined, since the expression $F_\beta^1 \cdot F_{\alpha\beta^{-1}}^2|\beta$ is invariant under the left multiplication of β by any $\gamma \in \Gamma$. Evidently, it defines a map with finite support from $\Gamma \backslash G^+(\mathbb{Q})$ to \mathcal{M} . Also, for any $\gamma \in \Gamma$,

$$\begin{aligned} (F^1 * F^2)_{\alpha\gamma} &= \sum_{\beta \in \Gamma \backslash G^+(\mathbb{Q})} F_\beta^1 \cdot F_{\alpha\gamma\beta^{-1}}^2|\beta \\ &= \sum_{\delta \in \Gamma \backslash G^+(\mathbb{Q})} F_\delta^1|\gamma \cdot F_{\alpha\delta^{-1}}^2|\delta|\gamma \\ &= (F^1 * F^2)_\alpha|\gamma, \end{aligned}$$

so that $F^1 * F^2 \in \mathcal{A}(\Gamma)$.

It is convenient to rewrite the product (1.9) in the form

$$(F^1 * F^2)_\alpha = \sum_{\alpha_2\alpha_1=\alpha} F_{\alpha_1}^1 \cdot F_{\alpha_2}^2|\alpha_1, \quad (1.11)$$

where the sum is taken over pairs $(\alpha_1, \alpha_2) \in G^+(\mathbb{Q})$ such that $\alpha_2\alpha_1 = \alpha$, modulo the equivalence relation

$$(\alpha_1, \alpha_2) \sim (\gamma\alpha_1, \alpha_2\gamma^{-1}), \quad \gamma \in \Gamma. \quad (1.12)$$

With this understood, the associativity of the product follows from the fact that, irrespective of the placement of putative parentheses,

$$(F^1 * F^2 * F^3)_\alpha = \sum_{\alpha_3\alpha_2\alpha_1=\alpha} F_{\alpha_1}^1 \cdot F_{\alpha_2}^2|\alpha_1 \cdot F_{\alpha_3}^3|\alpha_2\alpha_1,$$

where the triples $(\alpha_1, \alpha_2, \alpha_3) \in G^+(\mathbb{Q})$ with $\alpha_3\alpha_2\alpha_1 = \alpha$ are identified as above via the equivalence

$$(\alpha_1, \alpha_2, \alpha_3) \sim (\gamma\alpha_1, \delta\alpha_2\gamma^{-1}, \alpha_3\delta^{-1}), \quad \gamma, \delta \in \Gamma.$$

2⁰. To check that $F * f$ is Γ -modular, let $\gamma \in \Gamma$; then

$$(F * f)|_\gamma = \sum_{\alpha \in \Gamma \backslash G^+(\mathbb{Q})} F_\alpha|\gamma \cdot f|_\alpha\gamma = \sum_{\alpha \in \Gamma \backslash G^+(\mathbb{Q})} F_{\alpha\gamma} \cdot f|_\alpha\gamma = F * f.$$

The above proof of associativity also applies to show that we do have an action of $\mathcal{A}(\Gamma)$ on the vector space $\mathcal{M}(\Gamma)$.

Finally, 3⁰ follows from the very definitions, using the “Iwasawa decomposition” (1.5). \square

Let $\mathcal{H}(\Gamma)$ be the standard Hecke algebra of functions with finite support on double cosets in $\Gamma \backslash G^+(\mathbb{Q})/\Gamma$. It is canonically isomorphic to the subalgebra of weight 0 elements of $\mathcal{A}(\Gamma)$ by the isomorphism

$$j: \mathcal{H}(\Gamma) \rightarrow \mathcal{A}(\Gamma), \quad j(h)_\alpha := h(\alpha), \quad h \in \mathcal{H}(\Gamma).$$

The augmentation ϵ of the graded algebra $\mathcal{A}(\Gamma)$ gives a homomorphism $\epsilon: \mathcal{A}(\Gamma) \rightarrow \mathcal{H}(\Gamma)$ such that $\epsilon \circ j = \text{Id}$.

For $\Gamma = \Gamma(N)$ a Hecke operator form of level Γ is simply called “of level N ”. It gives rise for each multiple N' of N to a map

$$F': \Gamma(N') \backslash G^+(\mathbb{Q}) \rightarrow \mathcal{M}, \quad F' := F \circ p,$$

where p is the projection,

$$\Gamma(N') \backslash G^+(\mathbb{Q}) \rightarrow \Gamma(N) \backslash G^+(\mathbb{Q}).$$

The map F' has finite support and the covariance condition

$$F'_{\alpha\gamma} = F'_\alpha|_\gamma, \quad \text{for all } \alpha \in G^+(\mathbb{Q}), \gamma \in \Gamma(N'). \quad (1.13)$$

is still fulfilled. Let $r(N, N')$ be the order of $\Gamma(N') \backslash \Gamma(N)$, we then define a map $\rho_{N, N'}$ from $\mathcal{A}(N)$ to $\mathcal{A}(N')$ by

$$\rho(F) := \frac{1}{r(N, N')} F \circ p$$

for every Hecke operator form F of level N . This defines a homomorphism from $\mathcal{A}(N)$ to $\mathcal{A}(N')$ and allows to take the inductive limit,

$$\mathcal{A}(\infty) := \varinjlim_N \mathcal{A}(N),$$

which in turn can be interpreted as the crossed product of \mathcal{M} by $\text{GL}(2, \mathbb{A}_f)^0$.

2. THE HOPF ALGEBRA \mathcal{H}_1 AND ITS HOPF CYCLIC CLASSES

In this section we describe the Hopf algebra \mathcal{H}_1 and its Hopf cyclic cocycles corresponding respectively to the Godbillon–Vey class, to the Schwarzian derivative, and to the transverse fundamental class.

We begin by recalling the presentation of the Hopf algebra \mathcal{H}_1 (cf. [12]). As an algebra, it coincides with the universal enveloping algebra of the Lie algebra with basis $\{X, Y, \delta_n; n \geq 1\}$ and brackets

$$[Y, X] = X, [Y, \delta_n] = n\delta_n, \quad [X, \delta_n] = \delta_{n+1}, \quad [\delta_k, \delta_\ell] = 0, \quad n, k, \ell \geq 1. \quad (2.1)$$

As a Hopf algebra, the coproduct $\Delta: \mathcal{H}_1 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_1$ is determined by

$$\begin{aligned} \Delta Y &= Y \otimes 1 + 1 \otimes Y, \\ \Delta X &= X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y, \\ \Delta \delta_1 &= \delta_1 \otimes 1 + 1 \otimes \delta_1 \end{aligned} \quad (2.2)$$

and the multiplicativity property

$$\Delta(h^1 h^2) = \Delta h^1 \cdot \Delta h^2, \quad h^1, h^2 \in \mathcal{H}_1; \quad (2.3)$$

the antipode is determined by

$$S(Y) = -Y, \quad S(X) = -X + \delta_1 Y, \quad S(\delta_1) = -\delta_1 \quad (2.4)$$

and the anti-isomorphism property

$$S(h^1 h^2) = S(h^2) S(h^1), \quad h^1, h^2 \in \mathcal{H}_1; \quad (2.5)$$

finally, the counit is

$$\varepsilon(h) = \text{constant term of } h \in \mathcal{H}_1. \quad (2.6)$$

The role of \mathcal{H}_1 as symmetry in tranverse geometry comes from its natural action on crossed products ([12]). Given a one-dimensional manifold M^1 and a discrete subgroup $\Gamma \subset \text{Diff}^+(M^1)$, \mathcal{H}_1 acts on the crossed product algebra

$$\mathcal{A}_\Gamma = C_c^\infty(J_+^1(M^1)) \rtimes \Gamma,$$

by a Hopf action, where $J_+^1(M^1)$ is the oriented 1-jet bundle over M^1 . We use the coordinates in $J_+^1(M^1)$ given by the Taylor expansion,

$$j(s) = y + sy_1 + \cdots, \quad y_1 > 0,$$

and let diffeomorphisms act in the obvious functorial manner on the 1-jets,

$$\varphi(y, y_1) = (\varphi(y), \varphi'(y) \cdot y_1).$$

The action of \mathcal{H}_1 is then given as follows:

$$Y(fU_\varphi^*) = y_1 \frac{\partial f}{\partial y_1} U_\varphi^*, \quad X(fU_\varphi^*) = y_1 \frac{\partial f}{\partial y} U_\varphi^*, \quad (2.7)$$

$$\delta_n(fU_\varphi^*) = y_1^n \frac{d^n}{dy^n} \left(\log \frac{d\varphi}{dy} \right) fU_\varphi^*, \quad (2.8)$$

where we have identified $J_+^1(M^1) \simeq M^1 \times \mathbb{R}^+$ and denoted by (y, y_1) the coordinates on the latter.

The volume form $\frac{dy \wedge dy_1}{y_1^2}$ on $J_+^1(M^1)$ is invariant under $\text{Diff}^+(M^1)$ and gives rise to the following trace $\tau: \mathcal{A}_\Gamma \rightarrow \mathbb{C}$,

$$\tau(fU_\varphi^*) = \begin{cases} \int_{J_+^1(M^1)} f(y, y_1) \frac{dy \wedge dy_1}{y_1^2} & \text{if } \varphi = 1, \\ 0 & \text{if } \varphi \neq 1. \end{cases} \quad (2.9)$$

This trace is ν -invariant with respect to the action $\mathcal{H}_1 \otimes \mathcal{A}_\Gamma \rightarrow \mathcal{A}_\Gamma$ and with the modular character $\nu \in \mathcal{H}_1^*$, determined by

$$\nu(Y) = 1, \quad \nu(X) = 0, \quad \nu(\delta_n) = 0;$$

the invariance property is given by the identity

$$\tau(h(a)) = \nu(h)\tau(a), \quad \text{for all } h \in \mathcal{H}_1. \quad (2.10)$$

The fact that

$$S^2 \neq \text{Id},$$

is automatically corrected by twisting with ν . Indeed, $\tilde{S} = \nu * S$ satisfies the involutive property

$$\tilde{S}^2 = \text{Id}. \quad (2.11)$$

One has

$$\tilde{S}(\delta_1) = -\delta_1, \quad \tilde{S}(Y) = -Y + 1, \quad \tilde{S}(X) = -X + \delta_1 Y. \quad (2.12)$$

Equation (2.11) shows that the pair $(\nu, 1)$ given by the character ν of \mathcal{H}_1 and the group-like element $1 \in \mathcal{H}_1$ is a modular pair in involution, which thus allows us to define the cyclic cohomology $HC_{\text{Hopf}}^*(\mathcal{H}_1)$ (cf. [12], [13]). We refer to Appendix A for the detailed definition of the Hopf cyclic cohomology.

The assignment

$$\chi_\tau(h^1 \otimes \dots \otimes h^n)(a^0, \dots, a^n) = \tau(a^0 h^1(a^1) \dots h^n(a^n)), \quad (2.13)$$

where $h^1, \dots, h^n \in \mathcal{H}_1$ and $a^0, a^1, \dots, a^n \in \mathcal{A}_\Gamma$, induces a characteristic homomorphism

$$\chi_\tau^*: HC_{\text{Hopf}}^*(\mathcal{H}_1) \rightarrow HC^*(\mathcal{A}_\Gamma).$$

In [12] we have constructed an isomorphism

$$\kappa_1^*: H^*(\mathfrak{a}_1, \mathbb{C}) \xrightarrow{\cong} PHC_{\text{Hopf}}^*(\mathcal{H}_1),$$

between the Gelfand–Fuchs cohomology of the Lie algebra of formal vector fields on \mathbb{R}^1 and the periodic cyclic cohomology of the Hopf algebra \mathcal{H}_1 .

Proposition 3. *The element $\delta_1 \in \mathcal{H}_1$ is a Hopf cyclic cocycle, which gives a nontrivial class*

$$[\delta_1] \in HC_{\text{Hopf}}^1(\mathcal{H}_1).$$

Moreover, $[\delta_1]$ is a generator for $PHC_{\text{Hopf}}^{\text{odd}}(\mathcal{H}_1)$ and corresponds to the Godbillon–Vey class in the isomorphism κ_1^* with the Gelfand–Fuchs cohomology.

Proof. Indeed, the fact that δ_1 is a 1-cocycle is easy to check:

$$b(\delta_1) = 1 \otimes \delta_1 - \Delta\delta_1 + \delta_1 \otimes 1 = 0,$$

while

$$\tau_1(\delta_1) = \tilde{S}(\delta_1) = S(\delta_1) = -\delta_1.$$

On the other hand, its image under the above characteristic map,

$$\chi_\tau^*([\delta_1]) \in HC^1(\mathcal{A}_\Gamma),$$

is precisely the anabelian 1-trace of [9] (cf. also [10, III.6.γ]), and the latter is known to give a nontrivial class on the transverse frame bundle to codimension 1 foliations. The remaining statement is proved in Appendix B (Proposition 18). \square

We shall now describe another Hopf cyclic 1-cocycle which corresponds to the Schwarzian derivative $\{y; x\}$, whose expression is

$$\{y; x\} := \frac{d^2}{dx^2} \left(\log \frac{dy}{dx} \right) - \frac{1}{2} \left(\frac{d}{dx} \left(\log \frac{dy}{dx} \right) \right)^2. \quad (2.14)$$

Proposition 4. *The element $\delta'_2 := \delta_2 - \frac{1}{2}\delta_1^2 \in \mathcal{H}_1$ is a Hopf cyclic cocycle, whose action on the crossed product algebra $\mathcal{A}_\Gamma = C_c^\infty(J_+^1(M^1)) \rtimes \Gamma$ is given by the Schwarzian derivative*

$$\delta'_2(fU_\varphi^*) = y_1^2 \{\varphi(y); y\} fU_\varphi^*$$

and whose class

$$[\delta'_2] \in HC_{\text{Hopf}}^1(\mathcal{H}_1)$$

is equal to $B(c)$, where c is the following Hochschild 2-cocycle,

$$c := \delta_1 \otimes X + \frac{1}{2}\delta_1^2 \otimes Y.$$

Proof. We shall give the detailed computation in order to illustrate the (b, B) bicomplex for Hopf cyclic cohomology. Let us compute $b(c)$. One has

$$\begin{aligned} b(\delta_1 \otimes X) &= 1 \otimes \delta_1 \otimes X - (\delta_1 \otimes 1 + 1 \otimes \delta_1) \otimes X \\ &\quad + \delta_1 \otimes (X \otimes 1 + 1 \otimes X + \delta_1 \otimes Y) - \delta_1 \otimes X \otimes 1 = \delta_1 \otimes \delta_1 \otimes Y. \end{aligned} \quad (2.15)$$

Also

$$\begin{aligned} b(\delta_1^2 \otimes Y) &= 1 \otimes \delta_1^2 \otimes Y - (\delta_1^2 \otimes 1 + 2\delta_1 \otimes \delta_1 + 1 \otimes \delta_1^2) \otimes Y \\ &\quad + \delta_1^2 \otimes (Y \otimes 1 + 1 \otimes Y) - \delta_1^2 \otimes Y \otimes 1 = -2\delta_1 \otimes \delta_1 \otimes Y. \end{aligned} \quad (2.16)$$

This shows that

$$b(c) = 0,$$

so that c is a Hochschild cocycle.

Let us now compute $B(c)$. First, we recall that

$$B_0(h^1 \otimes h^2) = \tilde{S}(h^1)h^2.$$

Since $\tilde{S}(\delta_1) = -\delta_1$, one then has

$$B_0(c) = -\delta_1 X + \frac{1}{2}\delta_1^2 Y.$$

Since $\tilde{S}(Y) = -Y + 1$ and $\tilde{S}(X) = -X + \delta_1 Y$, it follows that

$$\begin{aligned} \tilde{S}(B_0 c) &= \tilde{S}(X)\delta_1 + \frac{1}{2}\tilde{S}(Y)\delta_1^2 \\ &= (-X + \delta_1 Y)\delta_1 + \frac{1}{2}(-Y + 1)\delta_1^2 \\ &= -X\delta_1 + \delta_1^2 Y + \delta_1^2 - \frac{1}{2}(\delta_1^2 Y + \delta_1^2) \\ &= -X\delta_1 + \frac{1}{2}\delta_1^2 Y + \frac{1}{2}\delta_1^2. \end{aligned} \quad (2.17)$$

Therefore,

$$B(c) = B_0 c - \tilde{S}(B_0 c) = -\delta_1 X + \frac{1}{2}\delta_1^2 Y - \left(-X\delta_1 + \frac{1}{2}\delta_1^2 Y + \frac{1}{2}\delta_1^2\right) = \delta'_2. \quad (2.18)$$

which shows that the class of δ'_2 is trivial in the periodic cyclic cohomology $PHC_{\text{Hopf}}^*(\mathcal{H}_1)$. \square

We conclude this section by noting that while the class of the unit constant $[1] \in HC_{\text{Hopf}}^0(\mathcal{H}_1)$ is trivial in the periodic theory (since $B(Y) = 1$), a generator of $PHC_{\text{Hopf}}^{\text{even}}(\mathcal{H}_1)$ is the class of the cyclic 2-cocycle

$$F := X \otimes Y - Y \otimes X - \delta_1 Y \otimes Y, \quad (2.19)$$

which in the foliation context represents the ‘transverse fundamental class’.

3. HOPF SYMMETRY OF MODULAR HECKE ALGEBRAS

We shall define and analyze in this section the natural action of the Hopf algebra \mathcal{H}_1 on the algebras $\mathcal{A}_{G^+(\mathbb{Q})}$ and $\mathcal{A}(\Gamma)$, for any congruence subgroup Γ . We shall then formulate in precise terms the geometric analogy with the action of \mathcal{H}_1 on the crossed products of the polynomial functions on the frame bundle of S^1 by discrete subgroups of $\text{Diff}(S^1)$.

In order to define the action of the generator X , we shall use the most natural derivation of the algebra of modular forms, which is in fact a classical one, originating in the work of Ramanujan (cf. [25], [27]). It is the operator of degree 2 on the algebra of modular forms given by

$$X := \frac{1}{2\pi i} \frac{d}{dz} - \frac{1}{12\pi i} \frac{d}{dz} (\log \Delta) \cdot Y = \frac{1}{2\pi i} \frac{d}{dz} - \frac{1}{2\pi i} \frac{d}{dz} (\log \eta^4) \cdot Y, \quad (3.1)$$

where

$$\Delta(z) = (2\pi)^{12} \eta^{24}(z) = (2\pi)^{12} q \prod_{n=1}^{\infty} (1 - q^n)^{24}, \quad q = e^{2\pi i z} \quad (3.2)$$

is the discriminant cusp form of weight 12, expressed in terms of the Dedekind η -function, while Y stands for the grading operator

$$Y(f) = \frac{k}{2} \cdot f, \quad \text{for all } f \in \mathcal{M}_k. \quad (3.3)$$

The derivation X provides the analogue of a connection, given by means of its horizontal vector field, while the operator Y serves as the analogue of the fundamental vector field. Indeed, one has

$$[Y, X] = X.$$

As a matter of fact, if we interpret the modular forms of weight k as sections of the $\frac{k}{2}$ power of the line bundle of 1-forms, via the identification

$$f \mapsto f \cdot (2\pi i dz)^{\frac{k}{2}}, \quad \text{for all } f \in \mathcal{M}_k, \quad (3.4)$$

X effectively defines a connection ∇ , uniquely determined by the property

$$\nabla(\eta^{2k} dz^{\frac{k}{2}}) = 0, \quad k \in 2\mathbb{N}; \quad (3.5)$$

explicitly,

$$\nabla(f \cdot (2\pi i dz)^{\frac{k}{2}}) = X(f) \cdot (2\pi i dz)^{\frac{k}{2}+1}, \quad \text{for all } f \in \mathcal{M}_k. \quad (3.6)$$

With this understood, we can now state the following:

Lemma 5. *For any $\gamma \in G^+(\mathbb{Q})$ and $f \in \mathcal{M}_k$, one has*

$$(X(f|_k \gamma^{-1}))|_{k+2\gamma} = X(f) - \mu_\gamma \cdot Y(f),$$

where

$$\mu_\gamma(z) = \frac{1}{12\pi i} \frac{d}{dz} \log \frac{\Delta|\gamma}{\Delta} = \frac{1}{2\pi i} \frac{d}{dz} \log \frac{\eta^4|\gamma}{\eta^4}. \quad (3.7)$$

Proof. This follows from (3.6), with μ_γ accounting for the lack of invariance of the section $\eta^4 dz$. \square

Remark 1. Alternatively, one can proceed by direct calculation and use (3.1) to obtain, for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+(\mathbb{Q})$,

$$\mu_\gamma(z) = \frac{1}{12\pi i} \left(\det(\gamma)(cz + d)^{-2} \frac{\Delta'}{\Delta} \left(\frac{az + b}{cz + d} \right) - \frac{\Delta'}{\Delta}(z) - \frac{12c}{cz + d} \right).$$

The right hand side is the explicit form of that in (3.7). To interpret it, we recall that (cf., e. g., [26, III, §2])

$$\frac{\Delta'(z)}{\Delta(z)} = -\frac{6}{\pi i} G_2^*(z), \quad (3.8)$$

where

$$G_2^*(z) = 2\zeta(2) + 2 \sum_{m \geq 1} \sum_{n \in \mathbb{Z}} \frac{1}{(mz + n)^2} = \frac{\pi^2}{3} - 8\pi^2 \sum_{m, n \geq 1} m e^{2\pi i m n z} \quad (3.9)$$

is the holomorphic Eisenstein series of weight 2. The latter fails to be modular, and satisfies the transformation formula

$$G_2^*|_\alpha(z) = G_2^*(z) - \frac{2\pi i c}{cz + d}, \quad \text{for all } \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1). \quad (3.10)$$

Thus, $\mu_\alpha \equiv 0$ when $\alpha \in \text{SL}(2, \mathbb{Z})$, while for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G^+(\mathbb{Q})$

$$\mu_\gamma(z) = \frac{1}{2\pi^2} \left(G_2^*|_\gamma(z) - G_2^*(z) + \frac{2\pi i c}{cz + d} \right) \quad (3.11)$$

measures the failure of the gauge transformation γ of preserving the ‘connection’.

By its very definition (3.7), μ takes values in the space of weight 2 modular forms \mathcal{M}_2 . It is not difficult to see that in fact the range of μ is contained in the canonical complement $\mathcal{E}_2 \subset \mathcal{M}_2$ to the subspace of cuspidal modular forms \mathcal{M}_2^0 , generated by Eisenstein series. Although we shall later prove a more precise result, let us record for now the following:

Lemma 6. *For any $\gamma \in G^+(\mathbb{Q})$, μ_γ is an Eisenstein series of weight 2.*

Proof. From (3.7) it follows that for any $\gamma_1, \gamma_2 \in G^+(\mathbb{Q})$

$$\mu_{\gamma_1 \cdot \gamma_2} = \mu_{\gamma_1}|_{\gamma_2} + \mu_{\gamma_2}; \quad (3.12)$$

in particular,

$$\mu_{\gamma_1 \cdot \gamma_2} = \mu_{\gamma_2}, \quad \text{for all } \gamma_1 \in \Gamma(1). \quad (3.13)$$

Using the decomposition

$$G^+(\mathbb{Q}) = \Gamma(1) \cdot T^+(\mathbb{Q}) \cdot \Gamma(1), \quad \text{where } T^+(\mathbb{Q}) = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \in G^+(\mathbb{Q}) \right\},$$

one can write any $\gamma \in G^+(\mathbb{Q})$ as a product

$$\gamma = \alpha_1 \cdot \delta \cdot \alpha_2, \quad \alpha_1, \alpha_2 \in \Gamma(1) \quad \text{and} \quad \delta \in T^+(\mathbb{Q});$$

when γ has integer entries, so does δ . Applying (3.12) and the fact that μ vanishes on $\Gamma(1)$ one obtains:

$$\mu_\gamma = \mu_{\alpha_1}|\delta \cdot \alpha_2 + \mu_{\delta \cdot \alpha_2} = \mu_\delta|\alpha_2 + \mu_{\alpha_2} = \mu_\delta|\alpha_2. \quad (3.14)$$

Thus, the range of μ is spanned by elements of elements of the form

$$\mu_\delta|\alpha = \frac{1}{2\pi^2} \cdot (G_2^*|\delta - G_2^*)|\alpha, \quad \delta \in T^+(\mathbb{Q}) \quad \text{and} \quad \alpha \in \Gamma(1). \quad (3.15)$$

The fact that the range of μ consists of weight 2 Eisenstein series follows now from the identity, cf. [26, VII, §3.5],

$$NG_2^*(Nz) - G_2^*(z) = N^{-1} \sum_{k=1}^{N-1} \wp_{(0, \frac{k}{N})}(z), \quad N \geq 2, \quad (3.16)$$

where $\wp_{\mathbf{a}}(z)$ is the \mathbf{a} -division value of the Weierstrass \wp -function and the collection of functions

$$\left\{ \wp_{\mathbf{a}}; \mathbf{a} \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z} \right)^2 \setminus \mathbf{0} \right\}$$

generates the space of weight 2 Eisenstein series of level N . \square

We now analyze the action of the Hopf algebra \mathcal{H}_1 on the algebra $\mathcal{A}_{G^+(\mathbb{Q})}$, the corresponding results for $\mathcal{A}(\Gamma)$ for any congruence subgroup Γ will be discussed afterwards.

Proposition 7. *There is a unique Hopf action of \mathcal{H}_1 on $\mathcal{A}_{G^+(\mathbb{Q})}$, determined by*

$$X(fU_\gamma^*) = X(f)U_\gamma^*, \quad Y(fU_\gamma^*) = Y(f)U_\gamma^*, \quad \text{and} \quad \delta_1(fU_\gamma^*) = \mu_\gamma \cdot fU_\gamma^*.$$

Proof. Indeed, one has by construction

$$[Y, X] = X, \quad [Y, \delta_1] = \delta_1.$$

The action of δ_n is uniquely determined by induction using $[X, \delta_n] = \delta_{n+1}$, which gives

$$\delta_n(fU_\gamma^*) = X^{n-1}(\mu_\gamma) \cdot fU_\gamma^*. \quad (3.17)$$

The relations $[\delta_k, \delta_n] = 0$ follow easily. Since as an algebra \mathcal{H}_1 is the envelopping algebra of the Lie algebra with presentation

$$[Y, X] = X, \quad [Y, \delta_n] = n\delta_n, \quad [X, \delta_n] = \delta_{n+1}, \quad [\delta_k, \delta_n] = 0,$$

one gets the required action of \mathcal{H}_1 . Let us check its compatibility with the coproduct. For any $a^1, a^2 \in \mathcal{A}_{G^+(\mathbb{Q})}$ one checks that,

$$\begin{aligned} Y(a^1 a^2) &= Y(a^1)a^2 + a^1 Y(a^2), \\ X(a^1 a^2) &= X(a^1)a^2 + a^1 X(a^2) + \delta_1(a^1)Y(a^2), \\ \delta_1(a^1 a^2) &= \delta_1(a^1)a^2 + a^1 \delta_1(a^2), \end{aligned}$$

which is enough to prove the required compatibility since X, Y, δ_1 generate \mathcal{H}_1 as an algebra. \square

We shall now show that, after a change of coordinates, the action of \mathcal{H}_1 on $\mathcal{A}_{G^+(\mathbb{Q})}$ in the new coordinates becomes strikingly similar to the action of \mathcal{H}_1 on the crossed product algebra $\mathcal{A}_\Gamma = C_c^\infty(J_+^1(M^1)) \rtimes \Gamma$ (cf. Section 1). To this end, let us introduce the primitive of the differential form $\frac{2\pi i}{6}\eta^4 dz$, which is, in classical terminology, an integral of the 1st kind for $\Gamma(6)$. Thus,

$$Z(z) := \frac{2\pi i}{6} \int_{i\infty}^z \eta^4 dz, \quad (3.18)$$

and

$$dZ := \frac{1}{6}\eta^4 \frac{dq}{q} = \frac{2\pi i}{6}\eta^4 dz. \quad (3.19)$$

Let χ be the unique character of $\Gamma(1)$ such that

$$\chi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right) = e^{\frac{2\pi i}{6}}. \quad (3.20)$$

One has by construction

$$dZ|_0\gamma = \chi(\gamma) dZ \quad \text{for all } \gamma \in \Gamma(1).$$

The kernel of χ is the commutator subgroup $\Gamma'(1)$ which is a free group on the two generators,

$$\gamma_1 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}. \quad (3.21)$$

For $\gamma \in \Gamma'(1)$ one has $dZ|_0\gamma = dZ$ and $Z|_0\gamma - Z = L(\gamma)$, where L is the unique homomorphism from the free group $\Gamma'(1)$ to the additive group \mathbb{C} such that,

$$L(\gamma_1) = L_0 e^{\frac{2\pi i}{3}}, \quad L(\gamma_2) = L_0 e^{\frac{2\pi i}{6}} \quad (3.22)$$

where $2L_0 \sim 1.402182\dots$ is one third of the real half period of the elliptic integral $\int \frac{dx}{\sqrt{x^3+1}}$. This is easily seen by evaluating $Z|_0\gamma_k(i\infty)$. It follows in particular that Z gives an isomorphism of the elliptic curve $\Gamma'(1)\backslash H^*$ with the quotient \mathbb{C}/Λ where Λ is the cubic lattice which is the range of L . Extending L to $\Gamma(1)$ we obtain the following,

$$Z|_0\gamma = \chi(\gamma)Z + L(\gamma) \quad \text{for all } \gamma \in \Gamma(1). \quad (3.23)$$

We shall now show that the variable $Z \in \mathbb{C}/\Lambda$ plays exactly the same role as the variable $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ of the circle case.

Using formula (3.6), one has

$$X(f) \cdot (2\pi i dz)^2 = d\left(\frac{2\pi i f dz}{dZ}\right) dZ, \quad \text{for all } f \in \mathcal{M}_2. \quad (3.24)$$

Since

$$2\pi i \mu_\gamma dz = d \log \frac{\eta^4|_\gamma}{\eta^4} = \frac{d}{dZ} \left(\log \frac{d(Z|_0\gamma)}{dZ} \right) dZ,$$

it follows from (3.24) that

$$X(\mu_\gamma) \cdot (2\pi i dz)^2 = \left(\frac{d}{dZ}\right)^2 \left(\log \frac{d(Z|_0\gamma)}{dZ} \right) (dZ)^2. \quad (3.25)$$

Iterating the above calculation one obtains for the higher horizontal derivatives of μ_γ the expression

$$X^{n-1}(\mu_\gamma) \cdot (2\pi i dz)^n = \left(\frac{d}{dZ}\right)^n \left(\log \frac{d(Z|_0\gamma)}{dZ}\right) (dZ)^n, \quad \text{for all } n \in \mathbb{N} \quad (3.26)$$

so that we can state the following exact analogue of (2.8).

Proposition 8. *The generalized cocycles $\delta_n \in \mathcal{H}_1$ act on $\mathcal{A}_{G^+(\mathbb{Q})}$ by the formulae:*

$$\delta_n(fU_\gamma^*) = \left(\frac{d}{dZ}\right)^n \left(\log \frac{d(Z|_0\gamma)}{dZ}\right) (dZ)^n fU_\gamma^*. \quad (3.27)$$

In the above statement, we have used the identification (3.4) between modular forms and higher differentials, so that the modular form f of weight k stands here for the higher differential $f \cdot (2\pi i dz)^{\frac{k}{2}}$.

In the context of codimension 1 foliations discussed in Section 2, we showed (cf. Proposition 4) that the 1-cocycle

$$\delta'_2 = \delta_2 - \frac{1}{2}\delta_1^2 \in \mathcal{H}_1,$$

corresponds to the Schwarzian derivative. Here, using Proposition 7, we obtain:

Corollary 9. *The equality*

$$\delta'_2(fU_\gamma^*) = \left(X(\mu_\gamma) - \frac{1}{2}\mu_\gamma^2\right) \cdot fU_\gamma^* \quad (3.28)$$

defines a derivation of the algebra $\mathcal{A}_{G^+(\mathbb{Q})}$.

Proof. Indeed, one easily checks that the action of δ'_2 on $\mathcal{A}_{G^+(\mathbb{Q})}$ is given by the stated formula. \square

We shall next analyze the above derivation, it is given by construction by the 1-cocycle on $G^+(\mathbb{Q})$ with values in \mathcal{M}_4 ,

$$\sigma_\gamma = X(\mu_\gamma) - \frac{1}{2}\mu_\gamma^2. \quad (3.29)$$

We recall that the classical Schwarzian, for which we use the standard notation $\{y; x\}$, is given by the expression

$$\{y; x\} := \frac{d^2}{dx^2} \left(\log \frac{dy}{dx}\right) - \frac{1}{2} \left(\frac{d}{dx} \left(\log \frac{dy}{dx}\right)\right)^2. \quad (3.30)$$

The Schwarzian vanishes precisely on the Möbius transformations

$$z \mapsto \gamma \cdot z = \frac{az + b}{cz + d}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{R}), \quad (3.31)$$

which in particular act on S^1 viewed as $\mathbb{P}^1(\mathbb{R})$. In the present context of modular forms we shall show that the Schwarzian is an inner derivation of $\mathcal{A}_{G^+(\mathbb{Q})}$.

The Schwarzian version of the chain rule can be expressed as the identity

$$\{y; x\} = \left(\frac{dz}{dx}\right)^2 (\{y; z\} - \{x; z\}), \quad (3.32)$$

where $y = y(x)$ and $x = x(z)$. Using (3.31), it follows that if

$$Y(z) = (y|_0\gamma)(z) = y\left(\frac{az+b}{cz+d}\right),$$

then

$$\{Y; z\} = \{y; z\}|_4\gamma. \quad (3.33)$$

Now, in view of (3.25), one has

$$\left(X(\mu_\gamma) - \frac{1}{2}\mu_\gamma^2\right) \cdot (2\pi i dz)^2 = \{Z|_0\gamma; Z\}(dZ)^2. \quad (3.34)$$

By the above chain rule,

$$\{Z|_0\gamma; Z\} = \left(\frac{dz}{dZ}\right)^2 (\{Z|_0\gamma; z\} - \{Z; z\}), \quad (3.35)$$

while by (3.33),

$$\{Z|_0\gamma; z\} = \{Z; z\}|_4\gamma. \quad (3.36)$$

Thus, from the identities (3.34)–(3.36) one obtains for the “Schwarzian” 1-cocycle σ defined by (3.29) the formula

$$\sigma_\gamma = (2\pi i)^{-2}(\{Z; z\}|_4\gamma - \{Z; z\}). \quad (3.37)$$

This shows that σ is the coboundary of the weight 4 modular form

$$\omega_4 = (2\pi i)^{-2}\{Z; z\}. \quad (3.38)$$

A direct computation, using the definitions (3.30) and (3.1) together with (3.19) gives

$$\omega_4 = X(g_2^*) + \frac{1}{2}(g_2^*)^2, \quad (3.39)$$

where

$$g_2^* := \frac{1}{2\pi i} \frac{d}{dz}(\log \eta^4) = -\frac{2}{(2\pi i)^2} G_2^*. \quad (3.40)$$

In turn this implies (cf., e. g., [30]) that ω_4 is a multiple of the classical Eisenstein series G_4 , namely

$$\omega_4 = -\frac{E_4}{72} = -\frac{10}{(2\pi)^4} G_4, \quad (3.41)$$

where

$$E_4(q) := 1 + 240 \sum_{n=1}^{\infty} n^3 \frac{q^n}{1-q^n}. \quad (3.42)$$

We can thus conclude with the following statement.

Proposition 10. *The derivation δ'_2 of the algebra $\mathcal{A}_{G^+(\mathbb{Q})}$ is inner and implemented by the weight 4 modular form $\omega_4 = -\frac{E_4}{72}$,*

$$\delta'_2(a) = [a, \omega_4], \quad \text{for all } a \in \mathcal{A}_{G^+(\mathbb{Q})}.$$

The obvious question then, is whether one can perturb the action of \mathcal{H}_1 on \mathcal{A} by a 1-cocycle in such a way that, for the perturbed action, $\delta'_2 = 0$. The new action of \mathcal{H}_1 on \mathcal{A} would then fulfill, more generally,

$$\delta_n = \frac{(n-1)!}{2^{n-1}} \delta_1^n$$

and therefore would come from a much smaller Hopf algebra.

We shall show however that such a perturbation does not exist, and we shall actually relate the class of the action of \mathcal{H}_1 modulo such perturbations to the data that Zagier [30] introduced to define canonical Rankin–Cohen algebras.

To fix the notation, let us recall a few simple generalities on cocycle perturbations of Hopf actions (see, e. g., [4]).

Let \mathcal{A} be an algebra and \mathcal{H} a Hopf algebra. We let $\mathcal{L}(\mathcal{H}, \mathcal{A})$ be the convolution algebra of linear maps from \mathcal{H} to \mathcal{A} . The product of two elements of $\mathcal{L}(\mathcal{H}, \mathcal{A})$ is given by

$$uv(h) = \sum u(h_{(1)})v(h_{(2)}), \quad \text{for all } h \in \mathcal{H},$$

using the standard short-hand notation, where one denotes the coproduct by

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)}, \quad \text{for all } h \in \mathcal{H}.$$

Let us also assume that the Hopf algebra \mathcal{H} is acting, by a Hopf action, on \mathcal{A} . A 1-cocycle is then an invertible element of the convolution algebra of linear maps from \mathcal{H} to \mathcal{A} , $u \in \mathcal{L}(\mathcal{H}, \mathcal{A})$, such that

$$u(hh') = \sum u(h_{(1)})h_{(2)}(u(h')), \quad \text{for all } h \in \mathcal{H}. \quad (3.43)$$

Note that in view of the above equation, in the case of our Hopf algebra, we only need to compute u on the generators X, Y, δ_1 of \mathcal{H}_1 in order to determine it uniquely.

The perturbed action of \mathcal{H} on \mathcal{A} is given by,

$$\tilde{h}(a) := \sum u(h_{(1)})h_{(2)}(a)u^{-1}(h_{(3)})$$

We look for a cocycle u such that $\tilde{\delta}'_2 = 0$ for the perturbed action, or equivalently

$$u(\delta'_2) = \omega_4.$$

Since the action of \mathcal{H}_1 under consideration actually commutes with the natural coaction of $G^+(\mathbb{Q})$ on $\mathcal{A}_{G^+(\mathbb{Q})}$, it is natural to only consider cocycles u which preserve this property. It then follows that the values taken by such a 1-cocycle u on generators must belong to the subalgebra $\mathcal{M} \subset \mathcal{A}_{G^+(\mathbb{Q})}$ and have to be of the following form:

$$u(X) = t \in \mathcal{M}_2, \quad u(Y) = \lambda \in \mathbb{C}, \quad u(\delta_1) = m \in \mathcal{M}_2.$$

Proposition 11. *Let $t \in \mathcal{M}_2$, $\lambda \in \mathbb{C}$, $m \in \mathcal{M}_2$.*

¹⁰. *There exists a unique 1-cocycle $u \in \mathcal{L}(\mathcal{H}_1, \mathcal{A}_{G^+(\mathbb{Q})})$ such that*

$$u(X) = t, \quad u(Y) = \lambda, \quad u(\delta_1) = m.$$

2⁰. The conjugate under u of the action of \mathcal{H}_1 is given on generators as follows:

$$\begin{aligned}\tilde{Y} &= Y, & \tilde{X}(a) &= X(a) + [(t - \lambda m), a] - \lambda \delta_1(a) + mY(a), \\ \tilde{\delta}_1(a) &= \delta_1(a) + [m, a], & a &\in \mathcal{A}_{G^+(\mathbb{Q})}.\end{aligned}$$

3⁰. The conjugate under u of δ'_2 is given by the operator

$$\tilde{\delta}'_2(a) = \left[X(m) + \frac{m^2}{2} - \omega_4, a \right], \quad a \in \mathcal{A}_{G^+(\mathbb{Q})},$$

and there is no choice of u for which $\tilde{\delta}'_2 = 0$.

Proof. 1⁰. The uniqueness is obvious but one needs to check the existence. Any element of \mathcal{H}_1 can be uniquely written as a linear combination of monomials of the form $P(\delta_1, \delta_2, \dots, \delta_k)X^n Y^l$. Let $m^{(k)}$ be defined by induction by $m^{(1)} = m$ and

$$m^{(k+1)} = X(m^{(k)}) + mY(m^{(k)}). \quad (3.44)$$

Similarly, let $t^{(k)}$ be defined by induction by $t^{(1)} = m$ and

$$t^{(k+1)} = X(t^{(k)}) + mY(t^{(k)}) + t \cdot t^{(k)}. \quad (3.45)$$

Let us then define u by the equality,

$$u(P(\delta_1, \delta_2, \dots, \delta_k)X^n Y^l) := \lambda^l P(m^{(1)}, m^{(2)}, \dots, m^{(k)})t^{(n)}. \quad (3.46)$$

One checks directly that u is a 1-cocycle and that u^{-1} is given by

$$u^{-1}(P(\delta_1, \delta_2, \dots, \delta_k)X^n Y^l) := (-\lambda)^l P(n^{(1)}, n^{(2)}, \dots, n^{(k)})s^{(n)} \quad (3.47)$$

where $n^{(k)} := -X^k(m)$ and $s^{(n)}$ is defined by induction using $s^{(1)} := -t + \lambda m$ and

$$s^{(n+1)} := X(s^{(n)}) + s^{(n)}(-t + \lambda m). \quad (3.48)$$

2⁰. One has as above $u^{-1}(X) = -t + \lambda m$. Since

$$\Delta^{(2)}(X) = X \otimes 1 \otimes 1 + 1 \otimes X \otimes 1 + 1 \otimes 1 \otimes X + \delta_1 \otimes Y \otimes 1 + \delta_1 \otimes 1 \otimes Y + 1 \otimes \delta_1 \otimes Y$$

it follows that \tilde{X} is given by

$$\tilde{X}(a) = X(a) + [(t - \lambda m), a] - \lambda \delta_1(a) + mY(a).$$

The computation of \tilde{Y} and $\tilde{\delta}_1$ is straightforward.

3⁰. Using the above formula (3.46) gives

$$u(\delta'_2) = X(m) + \frac{m^2}{2}. \quad (3.49)$$

We are thus reduced to showing that one cannot find a modular form (of arbitrary level) which solves the equation

$$X(m) + \frac{m^2}{2} = \omega_4. \quad (3.50)$$

To do that let us use the “quasimodular” ([31]) solution to the above equation, given by g_2^* (3.40). Note that not only g_2^* fulfills (3.49) but also one has

$$X(f) = \frac{1}{2\pi i} \frac{d}{dz} f - g_2^* f,$$

for any modular form of weight 2. Thus,

$$\frac{1}{2\pi i} \frac{d}{dz} g_2^* - \frac{1}{2} (g_2^*)^2 = \omega_4$$

so that one gets a minus sign and not a plus sign when trading X for $q \frac{d}{dq}$.

Assuming now that f is a solution of (3.50), one gets

$$\frac{1}{2\pi i} \frac{d}{dz} f - g_2^* f + \frac{1}{2} f^2 = \frac{1}{2\pi i} \frac{d}{dz} g_2^* - \frac{1}{2} (g_2^*)^2.$$

In turn, this implies

$$\frac{1}{2\pi i} \frac{d}{dz} (f - g_2^*) + \frac{1}{2} (f - g_2^*)^2 = 0.$$

But one can trivially integrate this non-linear equation. Indeed, the substitution $y = \frac{1}{f - g_2^*}$ gives

$$\frac{1}{2\pi i} \frac{d}{dz} y = \frac{1}{2}, \quad \text{so that} \quad f - g_2^* = \frac{1}{\pi i z + c};$$

the latter fails to be periodic in z except when it vanishes. \square

Remark 2. Note that the freedom we have in modifying the action of the Hopf algebra \mathcal{H}_1 by a 1-cocycle exactly modifies the restriction of the action of X on modular forms and the value of ω_4 as in the data used by Zagier [30] to define “canonical” Rankin–Cohen algebras. More precisely the derivation ∂ and the element Φ of degree 4 used by Zagier in [30, Proposition 1] correspond to the restriction of X to \mathcal{M} and to $\omega_4 = 2\Phi$. The gauge transformation associated in *loc. cit.* to an element ϕ of degree 2 corresponds to the value $m = -\frac{\phi}{2}$ for our cocycle.

A first step in the understanding of the relationship between actions of the Hopf algebra \mathcal{H}_1 such that the derivation δ'_2 is inner and ‘canonical’ Rankin–Cohen algebras consists in statement 3⁰ of the following theorem, which we shall formulate for the modular Hecke algebra $\mathcal{A}(\Gamma)$ associated to an arbitrary congruence subgroup Γ of $\mathrm{SL}(2, \mathbb{Z})$.

We shall in fact show that all the above results also apply to the algebras $\mathcal{A}(\Gamma)$. In view of the adelic interpretation of $\mathcal{A}(\Gamma)$ as a reduced algebra of the crossed product of \mathcal{M} by $\mathrm{GL}(2, \mathbb{A}_f)^0$, the proofs will be similar, provided one checks that the reducing projection e_Γ is invariant under the action of \mathcal{H}_1 . It is more instructive however to proceed directly and concretely describe the action of \mathcal{H}_1 on $\mathcal{A}(\Gamma)$, as follows below.

The action of the generators of \mathcal{H}_1 on $\mathcal{A}(\Gamma)$ is given by the following operators:

$$\begin{aligned} Y(F)_\alpha &:= Y(F_\alpha), \\ X(F)_\alpha &:= X(F_\alpha), \\ \delta_n(F)_\alpha &:= \mu_{n,\alpha} \cdot F_\alpha, \end{aligned} \tag{3.51}$$

for all $F \in \mathcal{A}(\Gamma)$, $\alpha \in G^+(\mathbb{Q})$, where

$$\mu_{n,\alpha} := X^{n-1}(\mu_\alpha), \quad \text{for all } n \in \mathbb{N}. \tag{3.52}$$

The right hand side of each of the formulae in (3.51) gives an element of $\mathcal{A}(\Gamma)$. This is true for the second because μ_γ vanishes for $\gamma \in \Gamma \subset \Gamma(1)$, and it follows for the third since, cf. (3.13),

$$\mu_{n,\alpha\gamma} = \mu_{n,\alpha}, \quad \gamma \in \Gamma \subset \Gamma(1).$$

The algebra $\mathcal{A}(\Gamma)$ is graded by the weight of modular forms, i.e., by the operator Y . The weight 0 subalgebra given by $Y = 0$, is the standard Hecke algebra associated to Γ .

Theorem 12. *Let Γ be any congruence subgroup.*

1⁰. *The formulae (3.51) define a Hopf action of the Hopf algebra \mathcal{H}_1 on the algebra $\mathcal{A}(\Gamma)$.*

2⁰. *The Schwarzian derivation $\delta'_2 = \delta_2 - \frac{1}{2}\delta_1^2$ is inner and is implemented by $\omega_4 \in \mathcal{A}(\Gamma)$.*

3⁰. *The image of the generator (2.19) of the tranverse fundamental class $[F] \in HC_{\text{Hopf}}^2(\mathcal{H}_1)$ under the canonical map from the Hopf cyclic cohomology of \mathcal{H}_1 to the Hochschild cohomology of $\mathcal{A}(\Gamma)$,*

$$\chi(F)(a_1, a_2) := X(a_1)Y(a_2) - Y(a_1)X(a_2) - \delta_1(Y(a_1))Y(a_2),$$

gives the natural extension of the first Rankin–Cohen bracket $\{\cdot, \cdot\}_1$ to the algebra $\mathcal{A}(\Gamma)$.

4⁰. *The cuspidal ideal $\mathcal{A}^0(\Gamma)$ is globally invariant under the action of \mathcal{H}_1 .*

Proof. The verification of the first two statements is identical to that already given for $\mathcal{A}_{G^+(\mathbb{Q})}$. The third statement follows from 1⁰ and the cocycle property of F , together with the formula

$$\{a_1, a_2\}_1 = X(a_1)Y(a_2) - Y(a_1)X(a_2)$$

for the Rankin–Cohen bracket of modular forms. The fourth statement is true because X preserves the cuspidal ideal \mathcal{M}^0 . \square

Note that, exactly as before, inner perturbations of the action of \mathcal{H}_1 by a cocycle of the form

$$u(X) = t \in \mathcal{M}_2, \quad u(Y) = \lambda \in \mathbb{C}, \quad u(\delta_1) = m \in \mathcal{M}_2,$$

yield a new action of \mathcal{H}_1 with

$$\tilde{Y} = Y, \quad \tilde{X} = X + \text{ad}(t - \lambda m) - \lambda \delta_1 + mY, \quad \tilde{\delta}_1 = \delta_1 + \text{ad}(m),$$

and with the new Schwarzian $\tilde{\delta}'_2$ inner, implemented by $\omega_4 - X(m) - \frac{m^2}{2}$.

In the simplest case $\Gamma = \Gamma(1)$ one can witness the non-triviality of the action of \mathcal{H}_1 on $\mathcal{A}(\Gamma) = \mathcal{A}(1)$ by computing the higher derivatives $\delta_k(T_n)$ of the standard Hecke operators T_n .

We conclude this section by describing in precise terms the analogy between the action of \mathcal{H}_1 on crossed products of polynomial functions on the frame bundle of S^1 by discrete subgroups of $\text{Diff}(S^1)$ (cf. [12]) and its action on modular Hecke algebras.

Since η^4 has level 6, the natural domain of definition of the differential form defined by equation (3.19) is the modular elliptic curve

$$E := X(6) \cong X_0(36),$$

where the isomorphism comes from the fact that $\Gamma(6)$ is conjugate to $\Gamma_0(36)$ by the matrix $\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \in G^+(\mathbb{Q})$. The operator X of equation (3.1) is simply giving the canonical flat connection on the canonical line bundle of E . Proposition 8 then shows that the action of \mathcal{H}_1 on \mathcal{A} is governed by exactly the same rules as its action on crossed products for the case of $\text{Diff}(S^1)$ (cf. [12]). By Theorem 12 the same holds for the action of \mathcal{H}_1 on the modular Hecke algebras $\mathcal{A}(\Gamma)$, where the diffeomorphisms are replaced by Hecke correspondences. Thus, both ‘differential’ operators X and Y have clear geometric meaning, while the δ_n , $n \in \mathbb{N}$, are uniquely determined by the Hopf action rules.

Pursuing the above analogy we shall now describe the projective structure on the curve E which is responsible for the fact that the derivation δ'_2 of the modular Hecke algebras (Theorem 12) is inner. The simplest projective structure on S^1 is given by the familiar identification $S^1 \cong \mathbb{P}^1(\mathbb{R})$. The relation between the angular variable $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and the variable $t \in \mathbb{R}$ of the rational parametrization defining the projective structure is $t = \tan(\frac{\theta}{2})$, which shows that, when expressed in terms of θ , this projective structure on S^1 is given by the quadratic differential

$$\rho := \{t; \theta\} d\theta^2 = \frac{1}{2} dt^2.$$

In our case the elliptic curve E plays the role of the circle S^1 , Z plays the role of the angular variable θ and the projective structure is defined by the variable z , and hence given by the quadratic differential

$$\varpi := \{z; Z\} dZ^2 = -\{Z; z\} dz^2. \quad (3.53)$$

Now the elliptic curve E has Weierstrass equation

$$y^2 = x^3 + 1 \quad (3.54)$$

and in these terms,

$$dZ = \frac{dx}{2y}. \quad (3.55)$$

We need to express the above quadratic differential in terms of its Weierstrass coordinates x and y . Since by equation (3.39)

$$\{Z; z\} = (2\pi i)^2 \omega_4 = -\frac{(2\pi i)^2}{72} E_4,$$

what is needed is to express the ratio $\frac{E_4}{\eta^8}$ as a rational function of x and y . The function x is the only solution of the differential equation

$$\left(\frac{dx}{dZ}\right)^2 = 4(x^3 + 1) \quad (3.56)$$

having an expansion near $q = 0$ of the form

$$x = q^{-\frac{1}{3}} \left(1 + \sum_{n=1}^{\infty} a_n q^n \right). \quad (3.57)$$

It is given by the ratio

$$x = \frac{\mu}{\eta^8},$$

where μ is the $\Gamma_0(6)$ -modular form of weight 4

$$\mu = \frac{1}{5} \left(g_4 - 4g_4 \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} - 9g_4 \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} + 36g_4 \begin{vmatrix} 6 & 0 \\ 0 & 1 \end{vmatrix} - 36S_{\text{new}} \right)$$

with $g_4 := \frac{1}{240}E_4$ and

$$S_{\text{new}}(q) = q - 2q^2 - 3q^3 + 4q^4 + 6q^5 + 6q^6 - 16q^7 + \dots$$

denoting the new form of weight 4 and level $\Gamma_0(6)$.

The transformation $\alpha = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(6)$ defines an involution on our curve E , with respect to which $\eta^4|\alpha = -\eta^4$. Thus, with $2y := \frac{dx}{dZ}$, one has

$$x|\alpha = x \quad \text{and} \quad y|\alpha = -y.$$

On the other hand,

$$E_4|\alpha = E_4,$$

which shows that $\frac{E_4}{\eta^8}$ is a rational function of the above x . Explicitly, this rational function $R(x)$ can be computed as

$$\frac{E_4}{\eta^8} = R(x) = \frac{(x^3 + 4)(x^9 + 228x^6 + 48x^3 + 64)}{x^2(x^3 - 8)^2(x^3 + 1)}, \quad (3.58)$$

where the poles correspond precisely to the 12 cusps of $E = X(6)$.

From the equation (3.19), one has

$$\varpi = \frac{(2\pi i)^2}{72} E_4(dz)^2 = \frac{1}{2} \frac{E_4}{\eta^8} (dZ)^2$$

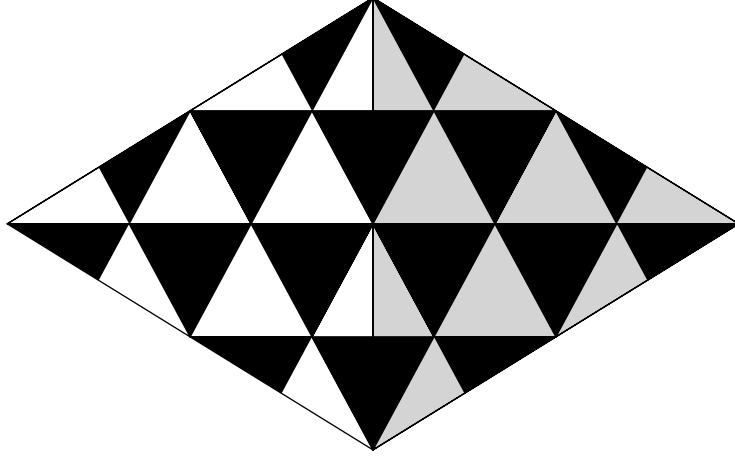
and using equation (3.56) in the form

$$4(dZ)^2 = \frac{(dx)^2}{x^3 + 1}$$

we finally obtain

$$\varpi = \frac{(x^3 + 4)(x^9 + 228x^6 + 48x^3 + 64)}{8(x(x^3 - 8)(x^3 + 1))^2} dx^2. \quad (3.59)$$

The curve E is the simplest elliptic curve in that it admits a triangulation by two equilateral triangles (the two big triangles in the picture). It is well-known, cf. [6, Theorem I.5.2], that the existence of an equilateral triangulation defining the conformal structure characterizes curves defined over \mathbb{Q} among compact connected Riemann surfaces. The group $\text{PSL}(2, \mathbb{Z}/6)$ acts by automorphisms of E and its commutator subgroup $C \cong (\mathbb{Z}/3) \cdot (\mathbb{Z}/2)^2$ preserves the 1-form $\eta^4 dz$ and hence acts by translations, and can thus be identified with a subgroup of the abelian group E .

FIGURE 1. The elliptic curve E and the subgroup C

The corresponding elements of E are the 12 cusps. They provide the vertices of a triangulation of E by 24 equilateral triangles which is invariant by the group C and is displayed in the above picture. The quotient E/C is the elliptic curve $\Gamma'(1)\backslash H^*$, i. e., the quotient \mathbb{C}/Λ which is again triangulated by two equilateral triangles. One then gets the following geometric expression for the rational fraction $R(x)$ of (3.58),

$$R(x) = \sum_{\tau \in C} x^\tau \quad (3.60)$$

where x^τ is the transform of x by the translation τ of the elliptic curve. Of course $\eta^4 dz$ does make sense on the quotient $E/C = E'$ and the formula for the projective structure looks simpler if we work directly there; it becomes

$$\varpi' = \frac{x dx^2}{2y^2} = \frac{x dx^2}{8(x^3 - 1728)}. \quad (3.61)$$

In order to recast the above developments in a completely geometric perspective one would still need to be able to characterize the Hecke correspondences $\sigma(\alpha)$ over E associated to $\alpha \in \Gamma \backslash G^+(\mathbb{Q})/\Gamma$ (cf. [28, 7.2.3]) in geometric terms. Letting \mathcal{L} be the ample line bundle associated to the divisor of cusps on E , the triple

$$(E, \sigma(\alpha), \mathcal{L})$$

is then strikingly similar to the geometric data associated to quadratic algebras (see [1] for instance) with the important nuance that instead of the graph of a translation on the elliptic curve we are dealing in our context with the irreducible curve representing a Hecke correspondence.

4. THE TRANSVERSE COCYCLE AND THE EULER CLASS

In this section we shall concentrate on the analogue of the Godbillon–Vey cocycle obtained from the cyclic cocycle $[\delta_1] \in HC_{\text{Hopf}}^1(\mathcal{H}_1)$ (cf. Proposition 3) in the above

action of \mathcal{H}_1 on the crossed product algebras. We need for that a substitute for the invariant trace τ which was used in the context of codimension 1 foliations. We first consider the projection map (already used in the *residue* definition in [8])

$$P: \mathcal{M} \rightarrow \mathcal{M}_2,$$

which projects onto the component of weight 2, and promote it to a projection denoted by the same letter $P: \mathcal{A}_{G+(\mathbb{Q})} \rightarrow \mathcal{M}_2$, by setting

$$P(fU_\gamma^*) = \begin{cases} P(f) & \text{if } \gamma = 1, \\ 0 & \text{if } \gamma \neq 1. \end{cases}$$

The next result expresses its covariance under the action of \mathcal{H}_1 .

Lemma 13. *For any $a, b \in \mathcal{A}_{G+(\mathbb{Q})}$ and any $h \in \mathcal{H}_1$, one has*

$$P(h(a) \cdot b) = P(a \cdot \tilde{S}(h)(b)), \quad (4.1)$$

where $\tilde{S} = \nu * S$ is the twisted antipode.

Proof. One can assume that a, b are of the form $a = fU_\gamma^*$, $b = gU_{\gamma^{-1}}^*$ and that h is a monomial in the generators of \mathcal{H}_1 . We endow \mathcal{H}_1 with its natural grading so that $\deg(Y)=0$, $\deg(X)=1$, $\deg(\delta_n)=n$. Since $P(f)=0$ unless the weight of f is 2, both sides of (4.1) vanish unless

$$\text{weight}(f) + \text{weight}(g) + 2\deg(h) = 2.$$

If $\deg(h) \neq 0$, both f and g are of weight 0 and hence are constants. Then both sides of (4.1) vanish unless the monomial is of the form $Y^m\delta_1$. In that case, we have $h(a) = \mu_\gamma U_\gamma^*$ and $h(a) \cdot b = \mu_\gamma$. Since

$$\tilde{S}(Y) = -Y + 1 \quad \text{and} \quad \tilde{S}(\delta_1) = -\delta_1,$$

we get

$$\tilde{S}(Y^m\delta_1) = -\delta_1(1-Y)^m \quad \text{and} \quad \tilde{S}(h)(b) = -\mu_{\gamma^{-1}}U_{\gamma^{-1}}^*.$$

Then

$$a \cdot \tilde{S}(h)(b) = -\mu_{\gamma^{-1}}|_{2\gamma}$$

and thus the cocycle property of μ gives (4.1) for the case considered.

If $\deg(h) = 0$, then h is of the form $h = Y^m$. Let then $f \in \mathcal{M}_k$. Both sides of (4.1) vanish unless $g \in \mathcal{M}_{2-k}$. Since $\tilde{S}(Y) = -Y + 1$, it follows that

$$P(Y^m(a)b) = \left(\frac{k}{2}\right)^m ab,$$

while

$$P(a \cdot \tilde{S}(Y^m)(b)) = \left(1 - \frac{2-k}{2}\right)^m ab,$$

which gives the required equality. \square

At this point, inspired by the construction of the Godbillon–Vey cocycle, we set

$$\text{GV}(a, b) = P(a \cdot \delta_1(b)), \quad \text{for all } a, b \in \mathcal{A}_{G^+(\mathbb{Q})}.$$

In the absence of an invariant trace, the construction of a formal analogue of the Godbillon–Vey cocycle will be completed by means of coupling with the analogue of the classical period cocycle. Due to the specificity of the situation, the entire construction can be conveniently phrased in terms of group cohomology rather than cyclic cohomology. Thus, GV reduces to the group 1-cocycle $\mathbf{E} \in Z^1(G^+(\mathbb{Q}), \mathcal{M}_2)$,

$$\mathbf{E}(\gamma) := \text{GV}(U_{\gamma^{-1}}^*, U_{\gamma}^*) = \mu_{\gamma}|_2 \gamma^{-1}, \quad \text{for all } \gamma \in G^+(\mathbb{Q}), \quad (4.2)$$

which we call the *transverse cocycle*. This cocycle $[\mathbf{E}] \in H^1(G^+(\mathbb{Q}), \mathcal{M}_2)$ is non-trivial. We can afford to omit the easy and direct proof, since the subsequent results will yield a finer understanding.

In order to provide a conceptual framework for the next proposition we shall now describe in details the identification of the 1-jet bundle $J^1(H_{\mathbb{A}})$ with the line bundle $\mathcal{L}_{\mathbb{A}}^{-2}$ where $\mathcal{L}_{\mathbb{A}}$ is the lattice line bundle.

A modular form of weight $2k$ is a differential form $f(z) dz^k$ on the complex curve $H_{\mathbb{A}}$, and hence can be viewed directly as a polynomial function on the (complex) 1-jet bundle $J^1(H_{\mathbb{A}})$. The careful description of the isomorphism $J^1(H_{\mathbb{A}}) \sim \mathcal{L}_{\mathbb{A}}^{-2}$ does give a very useful description of the invariant connections on these line bundles. We use the canonical identification of the Poincaré disk with the space of translation invariant conformal structures on \mathbb{C} , by means of Beltrami differentials. Thus, with μ in the Poincaré disk, the conformal structure on \mathbb{C}_{μ} is defined by declaring that $dz + \mu d\bar{z}$ is a 1-form of type (1,0). The 1-forms $\{d\mu, dz + \mu d\bar{z}\}$ are then a basis of forms of type (1,0) on the total space of the pairs (z, μ) . Viewing a fixed lattice Λ as a lattice in each of the complex lines \mathbb{C}_{μ} , $|\mu| < 1$, defines a map from lattices to 1-parameter families of lattices. Equivalently, this amounts to associate to a given lattice Λ the following germ of map from \mathbb{C} to lattices $\Lambda(\mu)$:

$$\Lambda(\mu) := \left\{ \omega + \frac{\mu}{\text{cov}(\Lambda)} \bar{\omega}, \quad \omega \in \Lambda \right\},$$

where $\text{cov}(\Lambda)$ is $-2i$ times the covolume of Λ and μ is a complex number such that

$$|\mu| < |\text{cov}(\Lambda)|.$$

The \mathbb{Q} structure of $\Lambda(\mu)$ is trivially obtained from that of Λ . The point of the normalization by the covolume is that one has

$$(a\Lambda)(\mu) = a\Lambda\left(\frac{\mu}{a^2}\right), \quad \text{for all } a \in \mathbb{C}^{\times},$$

which is exactly the correct homogeneity to identify the -2 power of the bundle of lattices with the 1-jet bundle.

For $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$, one has

$$\Lambda(\mu) = \mathbb{Z} \left(\omega_1 + \frac{\mu}{\text{cov}(\Lambda)} \bar{\omega}_1 \right) + \mathbb{Z} \left(\omega_2 + \frac{\mu}{\text{cov}(\Lambda)} \bar{\omega}_2 \right)$$

and

$$\text{cov}(\Lambda) = \omega_2 \bar{\omega}_1 - \omega_1 \bar{\omega}_2.$$

The inhomogeneous coordinate $z = \omega_1/\omega_2$ of $\Lambda(\mu)$ is thus given by

$$z(\mu) = \frac{\omega_1(\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1) - \mu\bar{\omega}_1}{\omega_2(\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1) - \mu\bar{\omega}_2},$$

The Taylor expansion of $z(\mu)$ at $\mu = 0$ gives,

$$z(\mu) = \frac{\omega_1}{\omega_2} + \frac{\mu}{\omega_2^2} + \frac{\mu^2\bar{\omega}_2}{\omega_2^3(\omega_1\bar{\omega}_2 - \omega_2\bar{\omega}_1)} + \dots$$

where the terms of degree two and higher are not holomorphic in the lattice Λ . The 1-jet however is holomorphic in Λ which gives the required canonical identification

$$J^1(H_{\mathbb{A}}) \sim \mathcal{L}_{\mathbb{A}}^{-2}.$$

The above construction yields a canonical connection on these line bundles. In terms of homogeneous lattice functions F of weight $2k$, the connection is given by definition as follows:

$$\nabla_{\mathbb{R}}(F)(\Lambda) := \frac{1}{2\pi i} \frac{d}{d\mu}(F(\Lambda(\mu)))|_{\mu=0}.$$

Note that $\nabla_{\mathbb{R}}(F)$ is homogeneous of weight $2k+2$ but is not in general a holomorphic function of Λ , even when F is holomorphic. In the inhomogeneous coordinate $z = \omega_1/\omega_2$ one gets,

$$\nabla_{\mathbb{R}} = \frac{1}{2\pi i} \frac{d}{dz} + \frac{2k}{2\pi i(z - \bar{z})},$$

which one recognizes as a standard connection in the theory of modular forms ([31]). It is the canonical connection on this holomorphic line bundle, associated to its natural invariant Hermitian metric. This connection has non-zero curvature, given by the area element. By its very construction, the connection $\nabla_{\mathbb{R}}$ is invariant under $G^+(\mathbb{Q})$ and its relation to the flat connection ∇ is given by

$$X = \nabla_{\mathbb{R}} - 2\phi_0 Y. \quad (4.3)$$

Here $\phi_0 = \frac{1}{4\pi^2} G_0$, with G_0 denoting the modular — but nonholomorphic — weight 2 Eisenstein series

$$G_0(z) = G_0(z, 0),$$

obtained by taking the value at $s = 0$ of the analytic continuation of the series

$$\begin{aligned} G_0(z, s) &= \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} (mz + n)^{-2} |mz + n|^{-s} \\ &= 2\zeta(2+s) + 2 \sum_{m \geq 1} \sum_{n \in \mathbb{Z}} (mz + n)^{-2} |mz + n|^{-s}, \quad \text{Re } s > 0. \end{aligned}$$

It is related to G_2^* by the identity

$$G_2^*(z) = G_0(z) + \frac{2\pi i}{z - \bar{z}}, \quad (4.4)$$

which makes the check of (4.3) straightforward. The $G^+(\mathbb{Q})$ -invariance of $\nabla_{\mathbb{R}}$ then entails the following result.

Proposition 14. *For any $\gamma \in G^+(\mathbb{Q})$ one has*

$$\mu_{\gamma} = 2(\phi_0|\gamma - \phi_0). \quad (4.5)$$

In order to obtain a better description of the right hand side of (4.5) we rely on Hecke's construction in [20, §2] of canonical generators for the \mathbb{Q} -vector space $\mathcal{E}_2(\mathbb{Q})$ of rational Eisenstein series. An Eisenstein series $E \in \mathcal{E}_2(\mathbb{Q})$ if the constant term of the q -expansion of E at each cusp is a rational number.

We shall closely follow the exposition in [29, §2.4]. For each $\mathbf{a} = (a_1, a_2) \in (\mathbb{Q}/\mathbb{Z})^2$ let

$$G_{\mathbf{a}}(z, s) := \sum_{\substack{\mathbf{m} \in \mathbb{Q}^2 \setminus \mathbf{0}, \\ \mathbf{m} \equiv \mathbf{a} \pmod{\mathbb{Z}}} (m_1 z + m_2)^{-2} |m_1 z + m_2|^{-s}, \quad \operatorname{Re} s > 0;$$

for fixed $z \in H$, $G_{\mathbf{a}}(z, s)$ may be analytically continued to a meromorphic function in the complex s -plane which is holomorphic at $s = 0$, so that one can define

$$G_{\mathbf{a}}(z) := G_{\mathbf{a}}(z, 0).$$

One then has

$$G_{\mathbf{a}}|\gamma = G_{\mathbf{a} \cdot \gamma}, \quad \text{for all } \gamma \in \Gamma(1),$$

which shows that $G_{\mathbf{a}}(z)$ behaves like a weight 2 modular form of some level N . It fails to be holomorphic, but only in a controlled way; more precisely, the function

$$z \mapsto G_{\mathbf{a}}(z) + \frac{2\pi i}{z - \bar{z}}$$

is holomorphic on H . Moreover, for $\mathbf{a} = (a_1, a_2) \in (\mathbb{Q}/\mathbb{Z})^2 \setminus \mathbf{0}$,

$$\wp_{\mathbf{a}}(z) = G_{\mathbf{a}}(z) - G_{\mathbf{0}}(z)$$

is the \mathbf{a} -division value of the Weierstrass \wp -function and the collection of functions

$$\left\{ \wp_{\mathbf{a}}; \mathbf{a} \in \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z} \right)^2 \setminus \mathbf{0} \right\}$$

generates the space of weight 2 Eisenstein series of level N .

In order to obtain a basis for $\mathcal{E}_2(\mathbb{Q})$, one considers for each $\mathbf{x} = (x_1, x_2) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2$ the additive character $\chi_{\mathbf{x}}: (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2 \rightarrow \mathbb{C}^\times$ defined by

$$\chi_{\mathbf{x}}\left(\frac{\mathbf{a}}{N}\right) := e^{2\pi i(a_2 x_1 - a_1 x_2)}$$

and one forms the 'twisted' Eisenstein series

$$\phi_{\mathbf{x}}(z) := (2\pi N)^{-2} \sum_{\mathbf{a} \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2} \chi_{\mathbf{x}}(\mathbf{a}) \cdot G_{\mathbf{a}}(z). \quad (4.6)$$

The definition is independent of N , and if $\mathbf{x} = (x_1, x_2) \in (\frac{1}{N}\mathbb{Z}/\mathbb{Z})^2 \setminus \mathbf{0}$ then $\phi_{\mathbf{x}}$ is a weight 2 Eisenstein series of level N . The function $\phi_{\mathbf{x}}$ is also known in the literature as the logarithmic derivative of the generalized Dedekind function $\eta_{\mathbf{x}}$,

$$\phi_{\mathbf{x}} = \frac{1}{2\pi i} \cdot \frac{d}{dz} \log \eta_{\mathbf{x}}.$$

To account for the special case when $\mathbf{x} = \mathbf{0}$, one adjoins the non-holomorphic but fully modular function $\phi_{\mathbf{0}}$ given as above by

$$2\pi i \cdot \phi_{\mathbf{0}}(z) - \frac{1}{z - \bar{z}} = 2 \cdot \frac{d}{dz} \log \eta. \quad (4.7)$$

All the linear relations among the functions $\phi_{\mathbf{x}}$, $\mathbf{x} \in (\mathbb{Q}/\mathbb{Z})^2$ are encoded in the *distribution property*

$$\phi_{\mathbf{x}} = \sum_{\mathbf{y} \cdot \tilde{\gamma} = \mathbf{x}} \phi_{\mathbf{y}} | \gamma, \quad (4.8)$$

where

$$\tilde{\gamma} = \det \gamma \cdot \gamma^{-1}.$$

This allows to equip the extended Eisenstein space

$$\mathcal{E}_2^*(\mathbb{Q}) = \mathcal{E}_2(\mathbb{Q}) \oplus \mathbb{Q} \cdot \phi_0,$$

with a linear $\mathrm{PGL}^+(2, \mathbb{Q})$ -action, as follows. Denoting

$$\mathcal{S} := (\mathbb{Q}/\mathbb{Z})^2, \quad \text{resp.} \quad \mathcal{S}' := \mathcal{S} \setminus \mathbf{0}$$

and identifying in the obvious way

$$\mathrm{PGL}^+(2, \mathbb{Q}) \cong M_2^+(\mathbb{Z}) / \{\text{scalars}\},$$

where $M_2^+(\mathbb{Z})$ stands for the set of integral 2×2 -matrices of determinant > 0 , one defines the action of $\gamma \in M_2^+(\mathbb{Z})$ by:

$$\mathbf{x} | \gamma := \sum_{\mathbf{y} \cdot \tilde{\gamma} = \mathbf{x}} \mathbf{y} \in \mathbb{Q}[\mathcal{S}]. \quad (4.9)$$

With this definition one has

$$\phi_{\mathbf{x}} | \gamma = \phi_{\mathbf{x} | \gamma}, \quad \gamma \in M_2^+(\mathbb{Z}). \quad (4.10)$$

Modulo the subspace of *distribution relations*

$$\mathcal{R} := \mathbb{Q} - \text{span of } \left\{ \mathbf{x} - \mathbf{x} | \begin{pmatrix} n & 0 \\ 0 & n \end{pmatrix}; \mathbf{x} \in \mathcal{S}, n \in \mathbb{Z} \setminus 0 \right\},$$

the assignment $\mathbf{x} \in \mathcal{S} \mapsto \phi_{\mathbf{x}}$ induces an isomorphism of $\mathrm{PGL}_2^+(2, \mathbb{Q})$ -modules

$$\mathbb{Q}[\mathcal{S}] / \mathcal{R} \cong \mathcal{E}_2^*(\mathbb{Q}). \quad (4.11)$$

We are now in a position to state the following result, which supersedes Lemma 6.

Proposition 15. *For any $\gamma \in M_2^+(\mathbb{Z})$ one has*

$$\mu_{\gamma} = 2(\phi_0 | \gamma - \phi_0) = 2 \left(\sum_{\mathbf{y} \cdot \tilde{\gamma} = \mathbf{0}} \phi_{\mathbf{y}} - \phi_0 \right). \quad (4.12)$$

Proof. This follows from Proposition 14 and the equalities (4.9), (4.10). \square

The role of the invariant 1-cocycle needed to complete the construction of the analogue of the Godbillon–Vey class will be played by the extension to $G^+(\mathbb{Q})$ of the well-known *period cocycle* for modular forms. We start by recalling its definition.

Fix $z_0 \in H$ and, for each $\gamma \in G^+(\mathbb{Q})$, define a linear functional $\Psi_{z_0}(\gamma) \equiv \Psi(\gamma)$ on Ω^1 by setting

$$\Psi(\gamma)(\omega) \equiv \langle \omega, \Psi(\gamma) \rangle := \int_{z_0}^{\gamma \cdot z_0} \omega, \quad \text{for all } \omega \in \Omega^1.$$

One easily checks the 1-cocycle property

$$\Psi(\gamma_1 \gamma_2)(\omega) = \Psi(\gamma_2)(\gamma_1^* \omega) + \Psi(\gamma_1)(\omega), \quad \text{for all } \gamma_1, \gamma_2 \in G^+(\mathbb{Q}),$$

as well as the fact that the cohomology class $[\Psi] \in H^1(G^+(\mathbb{Q}), (\Omega^1)^*)$ is independent of the choice of $z_0 \in H$.

The cup product of the two 1-cocycles gives a 2-dimensional cohomology class $[\mathbf{E} \cup \Psi] \in H^2(G^+(\mathbb{Q}), \Omega^1 \otimes (\Omega^1)^*)$. We shall be interested in the contraction of this class, $[\text{Tr}(\mathbf{E} \cup \Psi)] \in H^2(G^+(\mathbb{Q}), \mathbb{C})$, obtained by forming the ‘trace’ 2-cocycle

$$\tau(\gamma_1, \gamma_2) \equiv \text{Tr}(\mathbf{E} \cup \Psi)(\gamma_1, \gamma_2) := \langle \mathbf{E}(\gamma_1), \gamma_1 \cdot \Psi(\gamma_2) \rangle, \quad \gamma_1, \gamma_2 \in G^+(\mathbb{Q}).$$

Explicitly, with $\psi_\gamma := \frac{\Delta|\gamma|}{\Delta}$,

$$\begin{aligned} \tau(\gamma_1, \gamma_2) &= \langle \gamma_1^{-1} \cdot \mathbf{E}(\gamma_1), \Psi(\gamma_2) \rangle = \langle \gamma_1^* (\mathbf{E}(\gamma_1)), \Psi(\gamma_2) \rangle \\ &= \frac{1}{12\pi i} \left\langle \frac{d\psi_{\gamma_1}}{\psi_{\gamma_1}}, \Psi(\gamma_2) \right\rangle = \frac{1}{12\pi i} \int_{z_0}^{\gamma_2 \cdot z_0} \frac{d\psi_{\gamma_1}}{\psi_{\gamma_1}}, \end{aligned}$$

that is

$$\tau(\gamma_1, \gamma_2) = \frac{1}{12\pi i} \cdot (\log \psi_{\gamma_1}(\gamma_2 \cdot z_0) - \log \psi_{\gamma_1}(z_0)), \quad (4.13)$$

where the determination of the logarithm is unimportant.

At this juncture we recall that, as a special case of the main result in [5], the natural map

$$H_d^*(\text{SL}(2, \mathbb{R}), \mathbb{R}) \rightarrow H^*(\text{SL}(2, \mathbb{Q}), \mathbb{R})$$

obtained as the composition of the natural homomorphism

$$H_d^*(\text{SL}(2, \mathbb{R}), \mathbb{R}) \rightarrow H^*(\text{SL}(2, \mathbb{R})^\delta, \mathbb{R})$$

with the restriction map

$$H^*(\text{SL}(2, \mathbb{R})^\delta, \mathbb{R}) \rightarrow H^*(\text{SL}(2, \mathbb{Q}), \mathbb{R})$$

is an isomorphism. Therefore,

$$H^2(\text{SL}(2, \mathbb{Q}), \mathbb{R}) = \mathbb{R} \cdot e, \quad (4.14)$$

where $e \in H^2(\text{SL}(2, \mathbb{Q}), \mathbb{R})$ stands for the Euler class. Noting that the expression (4.13) actually makes sense for any $\gamma_1, \gamma_2 \in \text{GL}^+(2, \mathbb{R})$, we can view τ as a 2-cocycle on $\text{SL}(2, \mathbb{R})^\delta$.

Theorem 16. *The 2-cocycle $\text{Re } \tau \in Z^2(\text{SL}(2, \mathbb{R})^\delta, \mathbb{R})$ represents the Euler class $e \in H^2(\text{SL}(2, \mathbb{R})^\delta, \mathbb{R})$.*

Proof. By construction, one has

$$\text{Re } \tau(\gamma_1, \gamma_2) = \int_{z_0}^{\gamma_2 \cdot z_0} \text{Re}(\mu_{\gamma_1}(z) dz), \quad \text{for all } \gamma_1, \gamma_2 \in G^+(\mathbb{R}) := \text{GL}^+(2, \mathbb{R}),$$

Using Proposition 14 we may write

$$\text{Re } \tau(\gamma_1, \gamma_2) = \int_{z_0}^{\gamma_2 \cdot z_0} 2 \text{Re}((\phi_0 | \gamma_1 - \phi_0)(z) dz) = \int_{z_0}^{\gamma_2 \cdot z_0} (\gamma_1^*(\omega_0) - \omega_0), \quad (4.15)$$

where

$$\omega_0(z) := 2 \text{Re}(\phi_0(z) dz) = \frac{1}{2\pi^2} \text{Re}(G_0(z) dz).$$

Because G_0 is not holomorphic, ω_0 is not a closed form. Explicitly, by (4.4) one has

$$\omega_0(z) = \frac{1}{2\pi^2} \operatorname{Re}(G_2^*(z) dz) - \frac{1}{2\pi y} dx, \quad z = x + iy,$$

with G_2^* holomorphic, therefore

$$d\omega_0(z) = -\frac{1}{2\pi} \frac{dx \wedge dy}{y^2}. \quad (4.16)$$

Integrating along geodesic segments in the upper half-plane, we can rewrite (4.15) as follows:

$$\begin{aligned} \operatorname{Re} \tau(\gamma_1, \gamma_2) &= \int_{[z_0, \gamma_2 \cdot z_0]} \gamma_1^*(\omega_0) - \int_{[z_0, \gamma_2 \cdot z_0]} \omega_0 \\ &= \int_{[\gamma_1 \cdot z_0, \gamma_1 \gamma_2 \cdot z_0]} \omega_0 - \int_{[z_0, \gamma_2 \cdot z_0]} \omega_0. \end{aligned}$$

By subtracting the coboundary

$$\int_{[z_0, \gamma_1 \gamma_2 \cdot z_0]} \omega_0 - \int_{[z_0, \gamma_1 \cdot z_0]} \omega_0 - \int_{[z_0, \gamma_2 \cdot z_0]} \omega_0$$

we see that $-\operatorname{Re} \tau$ is cohomologous to the area cocycle (comp. Appendix B, Remark 4)

$$\begin{aligned} A(\gamma_1, \gamma_2) &:= - \int_{[\gamma_1 \cdot z_0, \gamma_1 \gamma_2 \cdot z_0]} \omega_0 + \int_{[z_0, \gamma_1 \gamma_2 \cdot z_0]} \omega_0 - \int_{[z_0, \gamma_1 \cdot z_0]} \omega_0 \\ &= - \int_{\Delta(z_0, \gamma_1 \cdot z_0, \gamma_1 \gamma_2 \cdot z_0)} d\omega_0 = \frac{1}{2\pi} \operatorname{Area}(\Delta(z_0, \gamma_1 \cdot z_0, \gamma_1 \gamma_2 \cdot z_0)). \quad \square \end{aligned}$$

Remark 3. One can also show that the imaginary part $\operatorname{Im} \tau$ is a coboundary.

Indeed, by (4.13) one has, for any $\gamma_1, \gamma_2 \in \operatorname{PSL}(2, \mathbb{R})$,

$$\begin{aligned} 12\pi i \tau(\gamma_1, \gamma_2) &= \log \frac{\Delta|\gamma_1}{\Delta}(\gamma_2 \cdot z_0) - \log \frac{\Delta|\gamma_1}{\Delta}(z_0) \\ &= \log \Delta|\gamma_1(\gamma_2 \cdot z_0) - \log \Delta|\gamma_1(z_0) - (\log \Delta(\gamma_2 \cdot z_0) - \log \Delta(z_0)) \end{aligned} \quad (4.17)$$

after the cancellation of an additive constant which depends only on γ_1 .

Since

$$\log \Delta|\gamma(z_0) = \log \Delta(\gamma \cdot z_0) - 6 \log j^2(\gamma, z_0) + 2\pi i k(\gamma),$$

for some $k(\gamma) \in \mathbb{Z}$ and with the choice of the principal branch for $\mathbb{C} \setminus [0, \infty)$ of the logarithm of the square of the automorphy factor, the succession of identities in (4.17) can be continued with

$$\begin{aligned} 12\pi i \tau(\gamma_1, \gamma_2) &= \log \Delta(\gamma_1 \gamma_2 \cdot z_0) - \log \Delta(\gamma_1 \cdot z_0) - \log \Delta(\gamma_2 \cdot z_0) \\ &\quad + \log \Delta(z_0) - 6 (\log j^2(\gamma_1, \gamma_2 \cdot z_0) - \log j^2(\gamma_1, z_0)). \end{aligned} \quad (4.18)$$

The equality

$$\log j^2(\gamma_1 \gamma_2, z_0) = \log j^2(\gamma_1, \gamma_2 \cdot z_0) + \log j^2(\gamma_2, z_0) + 2\pi i c(\gamma_1, \gamma_2), \quad (4.19)$$

determines a cocycle $c \in Z^2(\mathrm{PSL}(2, \mathbb{R}), \mathbb{Z})$, which is precisely the cocycle discussed in [3, §B-2]. Inserting (4.19) into (4.18) one obtains

$$\begin{aligned} 12\pi i \tau(\gamma_1, \gamma_2) &= \log \Delta(z_0) + \log \Delta(\gamma_1 \gamma_2 \cdot z_0) - \log \Delta(\gamma_1 \cdot z_0) - \log \Delta(\gamma_2 \cdot z_0) \\ &\quad - 6(\log j^2(\gamma_1 \gamma_2 \cdot z_0) - \log j^2(\gamma_2, z_0) - \log j^2(\gamma_1, z_0)) + 12\pi i c(\gamma_1, \gamma_2). \end{aligned} \quad (4.20)$$

This shows that the cocycles τ and c differ by a coboundary, which proves our claim. At the same time, in conjunction with the above theorem, it gives an alternate proof to the statement (cf. [3, Lemma 2.1]) that the cocycle c represents the Euler class.

We shall now refine the above construction to produce a remarkable rational representative for the Euler class $e \in H^2(\mathrm{SL}(2, \mathbb{Q}), \mathbb{Q})$. In the process, we shall make extensive use of the results in [29, Chap. 2]. To begin with, we replace the 1-cocycle $\Psi = \Psi_{z_0}$ by a cohomologous 1-cocycle—introduced by Stevens in [29, Def. 2.3.1]—which is independent of the base point $z_0 \in H$. The new cocycle, $\Pi \in Z^1(G^+(\mathbb{Q}), (\mathcal{M}_2)^*)$, is given by the formula

$$\Pi(\gamma)(\omega) := \int_{z_0}^{\gamma \cdot z_0} f(z) dz - z_0 \cdot \mathbf{a}_0(f|\gamma - f) + \int_{z_0}^{i\infty} (\widetilde{f|\gamma} - \widetilde{f})(z) dz,$$

where $f \in \mathcal{M}_2(\Gamma(N))$ for some level N ,

$$f(z) = \sum_{n=0}^{\infty} \mathbf{a}_n e^{\frac{2\pi i n z}{N}}$$

represents its Fourier expansion at ∞ , and

$$\widetilde{f}(z) := f(z) - \mathbf{a}_0(f);$$

note that $\mathbf{a}_0(f)$, and therefore \widetilde{f} too, are independent of the level.

By coupling \mathbf{E} with Π instead of Ψ , one gets a 2-cocycle equivalent to the restriction of τ to $G^+(\mathbb{Q})$,

$$\theta(\gamma_1, \gamma_2) := \langle \mathbf{E}(\gamma_1), \gamma_1 \cdot \Pi(\gamma_2) \rangle, \quad \gamma_1, \gamma_2 \in G^+(\mathbb{Q});$$

explicitly,

$$\begin{aligned} \theta(\gamma_1, \gamma_2) &= \int_{z_0}^{\gamma_2 \cdot z_0} \mu_{\gamma_1}(z) dz - z_0 \cdot \mathbf{a}_0(\mu_{\gamma_1}|\gamma_2 - \mu_{\gamma_1}) \\ &\quad + \int_{z_0}^{i\infty} (\widetilde{\mu_{\gamma_1}|\gamma_2} - \widetilde{\mu_{\gamma_1}})(z) dz. \end{aligned} \quad (4.21)$$

Furthermore, as follows from the preceding proof, cf. (4.20), retaining only the real part of θ gives a cohomologous cocycle

$$\rho(\gamma_1, \gamma_2) := \mathrm{Re} \theta(\gamma_1, \gamma_2) = \mathrm{Re} \langle \mathbf{E}(\gamma_1), \gamma_1 \cdot \Pi(\gamma_2) \rangle, \quad \gamma_1, \gamma_2 \in G^+(\mathbb{Q}). \quad (4.22)$$

A priori, $\rho \in Z^2(G^+(\mathbb{Q}), \mathbb{R})$, and we want to show that it actually takes rational values, which moreover can be explicitly computed. To this end, we now proceed to unfold the expression of the cocycle $\rho(\gamma_1, \gamma_2) = \mathrm{Re} \theta(\gamma_1, \gamma_2)$, defined by (4.22), for

$$\gamma_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in M_2^+(\mathbb{Z}). \quad (4.23)$$

The expression of $\rho(\gamma_1, \gamma_2)$ is simpler when

$$\gamma_2 \in B^+(\mathbb{Z}), \quad \text{i. e.,} \quad c_2 = 0.$$

Indeed, in that case if we let $z_0 \rightarrow i\infty$ then $\gamma_2 \cdot z_0 \rightarrow i\infty$ too, and the limit can be computed as in [29, §2.3], giving

$$\theta(\gamma_1, \gamma_2) = \frac{b_2}{d_2} \mathbf{a}_0(\mu_{\gamma_1}).$$

Thus, by (4.12)

$$\rho(\gamma_1, \gamma_2) = \frac{b_2}{d_2} \cdot 2\mathbf{a}_0(\phi_{\mathbf{0}}|\gamma_1 - \phi_{\mathbf{0}}).$$

Since according to [29, Prop. 2.5.1 (ii)],

$$\mathbf{a}_0(\phi_{\mathbf{x}}) = \frac{1}{2} \mathbf{B}_2(x_1), \quad (4.24)$$

from (4.9), (4.10) it follows that, when $c_2 = 0$,

$$\rho(\gamma_1, \gamma_2) = \frac{b_2}{d_2} \left(\sum_{y \cdot \gamma_1 = \mathbf{0}} \mathbf{B}_2(y_1) - \frac{1}{6} \right). \quad (4.25)$$

Here $\mathbf{B}_2: \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}$ is the periodized function corresponding to the Bernoulli polynomial $B_2(X) = X^2 - X + \frac{1}{6}$ so that $\mathbf{B}_2(x) := B_2(x - [x])$ where $[x]$ is the integer part of x .

The complementary case, when

$$\gamma_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in M_2^+(\mathbb{Z}) \quad \text{with} \quad c_2 > 0,$$

can be dealt with by means of Stevens' *Dedekind symbol* map [29, Def. 2.5.2] $\mathbf{S}_E: \mathbb{P}^1(\mathbb{Q}) \rightarrow \mathbb{Q}$ associated to an Eisenstein series $E \in \mathcal{E}_2(\mathbb{Q})$. According to [29, Prop 2.5.4], the Dedekind symbol of a basis element $\phi_{\mathbf{x}}$ is given by the following generalized Dedekind sum, introduced by Meyers in [23]: for $m, n \in \mathbb{Z}$ with $(m, n) = 1$ and $n > 0$,

$$\mathbf{S}_{\phi_{\mathbf{x}}} \left(\frac{m}{n} \right) = \sum_{j=0}^{n-1} \mathbf{B}_1 \left(\frac{x_1 + j}{n} \right) \mathbf{B}_1 \left(\frac{m(x_1 + j)}{n} + x_2 \right), \quad (4.26)$$

where $\mathbf{B}_1: \mathbb{Q}/\mathbb{Z} \rightarrow \mathbb{Q}$ denotes the periodized function corresponding to the Bernoulli polynomial $B_1(X) = X - \frac{1}{2}$. Relying again on the results in [29, §2.5], in particular on formula (2.5.3), one obtains:

$$\rho(\gamma_1, \gamma_2) = \frac{a_2}{c_2} \mathbf{a}_0(\mu_{\gamma_1}) + \frac{d_2}{c_2} \mathbf{a}_0(\mu_{\gamma_1}|\gamma_2) - \mathbf{S}_{\mu_{\gamma_1}} \left(\frac{a'_2}{c'_2} \right),$$

where $\frac{a'_2}{c'_2}, (a'_2, c'_2) = 1$, is the fraction $\frac{a_2}{c_2}$ in lowest terms form. Using once more (4.12), as well as (4.24) and (4.26), one finally obtains, for the case $c_2 > 0$,

$$\begin{aligned} \rho(\gamma_1, \gamma_2) &= \frac{a_2}{c_2} \left(\sum_{\mathbf{y} \cdot \tilde{\gamma}_1 = \mathbf{0}} B_2(y_1) - \frac{1}{6} \right) \\ &\quad + \frac{d_2}{c_2} \left(\sum_{\mathbf{z} \cdot \tilde{\gamma}_2 \tilde{\gamma}_1 = \mathbf{0}} B_2(z_1) - \sum_{\mathbf{y} \cdot \tilde{\gamma}_2 = \mathbf{0}} B_2(y_1) \right) \\ &\quad - 2 \sum_{\mathbf{y} \cdot \tilde{\gamma}_1 = \mathbf{0}} \sum_{j=0}^{c'_2-1} B_1 \left(\frac{y_1 + j}{c'_2} \right) B_1 \left(\frac{a'_2(y_1 + j)}{c'_2} + y_2 \right) \\ &\quad + 2 \sum_{j=0}^{c'_2-1} B_1 \left(\frac{j}{c'_2} \right) B_1 \left(\frac{a'_2 j}{c'_2} \right). \end{aligned} \quad (4.27)$$

In conclusion, we have proved:

Theorem 17. *The 2-cocycle $\rho \in Z^2(\mathrm{PSL}(2, \mathbb{Q}), \mathbb{Q})$ given by the formulae (4.25) and (4.27) represents the Euler class $e \in H^2(\mathrm{PSL}(2, \mathbb{Q}), \mathbb{Q})$.*

APPENDIX A. HOPF CYCLIC COHOMOLOGY

We shall recall in this appendix the definition of cyclic cohomology for Hopf algebras endowed with a modular pair in involution (ν, σ) , in the special case $\sigma = 1$ (cf. [12]. We refer to [13] for the general case).

Let \mathcal{H} be a Hopf algebra, ν a character of \mathcal{H} such that the twisted antipode $\tilde{S} = \nu * S$ satisfies the involutive property

$$\tilde{S}^2 = \mathrm{Id}.$$

The cyclic cohomology groups $HC_{\mathrm{Hopf}}^*(\mathcal{H})$ are defined by means of the cyclic module associated to the Hopf algebra \mathcal{H} as follows. For each $n \in \mathbb{N}$, $C^n(\mathcal{H}) = \mathcal{H}^{\otimes n}$; the face operators $\partial_i: C^{n-1}(\mathcal{H}) \rightarrow C^n(\mathcal{H})$, $0 \leq i \leq n$, are

$$\begin{aligned} \partial_0(h^1 \otimes \dots \otimes h^{n-1}) &= 1 \otimes h^1 \otimes \dots \otimes h^{n-1}, \\ \partial_j(h^1 \otimes \dots \otimes h^{n-1}) &= h^1 \otimes \dots \otimes \Delta h^j \otimes \dots \otimes h^{n-1}, \quad 1 \leq j \leq n-1, \\ \partial_n(h^1 \otimes \dots \otimes h^{n-1}) &= h^1 \otimes \dots \otimes h^{n-1} \otimes 1; \end{aligned}$$

the degeneracy operators $\sigma_i: C^{n+1}(\mathcal{H}) \rightarrow C^n(\mathcal{H})$, $0 \leq i \leq n$, are

$$\sigma_i(h^1 \otimes \dots \otimes h^{n+1}) = h^1 \otimes \dots \otimes \varepsilon(h^{i+1}) \otimes \dots \otimes h^{n+1};$$

finally, the cyclic operator $\tau_n: C^n(\mathcal{H}) \rightarrow C^n(\mathcal{H})$ is given by

$$\tau_n(h^1 \otimes \dots \otimes h^n) = (\Delta^{n-1} \tilde{S}(h^1)) \cdot h^2 \otimes \dots \otimes h^n \otimes 1.$$

The Hopf cyclic cohomology is computed from the normalized bicomplex $(CC^{*,*}(\mathcal{H}), b, B)$, where:

$$\begin{aligned} CC^{p,q}(\mathcal{H}) &= \bar{C}^{q-p}(\mathcal{H}), \quad q \geq p, \\ CC^{p,q}(\mathcal{H}) &= 0, \quad q < p; \end{aligned}$$

with

$$\bar{C}^n(\mathcal{H}) = \bigcap \text{Ker } \sigma_i, \quad \text{for all } n \geq 1, \quad \bar{C}^0(\mathcal{H}) = \mathbb{C}.$$

The operator

$$b: \bar{C}^{n-1}(\mathcal{H}) \rightarrow \bar{C}^n(\mathcal{H}), \quad b = \sum_{i=0}^n (-1)^i \partial_i$$

has the expression

$$\begin{aligned} b(h^1 \otimes \dots \otimes h^{n-1}) &= 1 \otimes h^1 \otimes \dots \otimes h^{n-1} \\ &\quad + \sum_{j=1}^{n-1} (-1)^j \sum_{(h_j)} h^1 \otimes \dots \otimes h_{(1)}^j \otimes h_{(2)}^j \otimes \dots \otimes h^{n-1} \\ &\quad + (-1)^n h^1 \otimes \dots \otimes h^{n-1} \otimes 1, \end{aligned}$$

while for $n = 0$, $b(\mathbb{C}) = 0$.

The B -operator $B: \bar{C}^{n+1}(\mathcal{H}) \rightarrow \bar{C}^n(\mathcal{H})$ is defined by the formula

$$B = A \circ B_0, \quad n \geq 0,$$

where $B_0: \bar{C}^{n+1}(\mathcal{H}) \rightarrow \bar{C}^n(\mathcal{H})$ is the (extra degeneracy) operator

$$\begin{aligned} B_0(h^1 \otimes \dots \otimes h^{n+1}) &= (\Delta^{n-1} \tilde{S}(h^1)) \cdot h^2 \otimes \dots \otimes h^{n+1} \\ &= \sum_{(h^1)} S(h_{(n)}^1) h^2 \otimes \dots \otimes S(h_{(2)}^1) h^n \otimes \tilde{S}(h_{(1)}^1) h^{n+1}, \\ B_0(h) &= \nu(h), \quad h \in \mathcal{H}, \end{aligned}$$

and

$$A = 1 + (-1)^n \tau_n + \dots + (-1)^{n^2} \tau_n^n.$$

The groups $HC_{\text{Hopf}}^n(\mathcal{H})$ are computed from the first quadrant total complex $(TC^*(\mathcal{H}), b + B)$,

$$TC^n(\mathcal{H}) = \sum_{p=0}^n CC^{p, n-p}(\mathcal{H}),$$

while the $(\mathbb{Z}/2$ -graded) periodic groups $PHC_{\text{Hopf}}^*(\mathcal{H})$ are computed from the full total complex $(PTC^*(\mathcal{H}), b + B)$,

$$PTC^n(\mathcal{H}) = \sum_p CC^{p, n-p}(\mathcal{H}).$$

APPENDIX B. GODBILLON–VEY AS HOPF CYCLIC CLASS

After recalling its original definition, we shall give in this appendix the promised detailed account of the interpretation of the Godbillon–Vey as Hopf cyclic class for \mathcal{H}_1 .

Let V be a closed, smooth manifold, foliated by a transversely oriented codimension 1 foliation \mathcal{F} . Then $T\mathcal{F} = \text{Ker } \omega \subset TV$, for some $\omega \in \Omega^1(V)$ such that $\omega \wedge d\omega = 0$. Equivalently, $d\omega = \omega \wedge \alpha$ for some $\alpha \in \Omega^1(V)$, which implies $d\alpha \wedge \omega = 0$.

In turn, the latter ensures that $d\alpha = \omega \wedge \beta$, $\beta \in \Omega^1(V)$. Thus, $\alpha \wedge d\alpha \in \Omega^3(V)$ is closed. Its de Rham cohomology class,

$$\text{GV}(V, \mathcal{F}) = [\alpha \wedge d\alpha] \in H^3(V, \mathbb{R}),$$

is independent of the choices of ω and α and represents the original definition of the Godbillon–Vey class.

The Godbillon–Vey class acquires a universal status when viewed as a characteristic class (cf. [18]) associated to the Gelfand–Fuchs cohomology [17] of the Lie algebra $\mathfrak{a}_1 = \mathbb{R}[[x]]\partial_x$ of formal vector fields on \mathbb{R} . The cohomology $H^*(\mathfrak{a}_1, \mathbb{R})$ is finite dimensional and the only nontrivial groups are:

$$H^0(\mathfrak{a}_1, \mathbb{R}) = \mathbb{R} \cdot 1 \quad \text{and} \quad H^3(\mathfrak{a}_1, \mathbb{R}) = \mathbb{R} \cdot gv,$$

where

$$gv(p_1\partial_x, p_2\partial_x, p_3\partial_x) = \begin{vmatrix} p_1(0) & p_2(0) & p_3(0) \\ p'_1(0) & p'_2(0) & p'_3(0) \\ p''_1(0) & p''_2(0) & p''_3(0) \end{vmatrix},$$

that is, with the obvious notation,

$$gv = \theta^0 \wedge \theta^1 \wedge \theta^2.$$

Given any oriented 1-dimensional manifold M^1 , the Lie algebra cocycle gv can be converted into a 3-form on the jet bundle (of orientation preserving jets)

$$J_+^\infty(M^1) = \varprojlim_n J_+^n(M^1),$$

invariant under the pseudogroup $\mathcal{G}^+(M^1)$ of all orientation preserving local diffeomorphisms of M^1 . Indeed, sending the formal vector field

$$p = j_0^\infty \left(\frac{dh_t}{dt} \Big|_{t=0} \right) \in \mathfrak{a}_1,$$

where $\{h_t\}$ is a 1-parameter family of local diffeomorphisms of \mathbb{R} preserving the origin, to the $\mathcal{G}^+(M^1)$ -invariant vector field

$$j_0^\infty \left(\frac{d(f \circ h_t)}{dt} \Big|_{t=0} \right) \in T_{j_0^\infty(f)} J_+^\infty(M^1)$$

gives a natural identification of the Lie algebra complex of \mathfrak{a}_1 with the invariant forms on the jet bundle,

$$\theta \in C^*(\mathfrak{a}_1) \mapsto \theta \in \Omega^*(J_+^\infty(M^1))^{\mathcal{G}^+(M^1)}.$$

In local coordinates on $J_+^\infty(M^1)$, given by the coefficients of the Taylor expansion at 0,

$$f(s) = y + sy_1 + s^2y_2 + \cdots, \quad y_1 > 0,$$

one has

$$\begin{aligned} dy &= y_1\theta^0, \\ dy_1 &= y_1\theta^1 + 2y_2\theta^0, \\ dy_2 &= y_1\theta^2 + 2y_2\theta^1 + 3y_3\theta^0, \end{aligned}$$

therefore

$$gv = \frac{1}{y_1^3} dy \wedge dy_1 \wedge dy_2 \in \Omega^3(J_+^\infty(M^1))^{\mathcal{G}^+(M^1)}.$$

Given a codimension 1 foliation (V, \mathcal{F}) as above, one can find an open covering $\{U_i\}$ of V and submersions $f_i: U_i \rightarrow T_i \subset \mathbb{R}$, whose fibers are plaques of \mathcal{F} , such that $f_i = g_{ij} \circ f_j$ on $U_i \cap U_j$, with $g_{ij} \in \mathcal{G}^+(M^1)$ a 1-cocycle. Then $M^1 = \bigcup_i T_i \times \{i\}$ is a complete transversal. Let $J^\infty(\mathcal{F})$ denote the bundle over V whose fiber at $x \in U_i$ consists of the ∞ -jets of local submersions of the form $\varphi \circ f_i$ with $\varphi \in \mathcal{G}^+(M^1)$. Using the $\mathcal{G}^+(M^1)$ -invariance of $gv \in \Omega^3(J^\infty(M^1))$ one can pull it back to a closed form $gv(\mathcal{F}) \in \Omega^3(J^2(\mathcal{F}))$. Its de Rham class $[gv(\mathcal{F})] \in H^3(J^2(\mathcal{F}), \mathbb{R})$, when viewed as a class in $H^3(V, \mathbb{R})$ (the fibers of $J^2\mathcal{F}$ being contractible), is precisely the Godbillon–Vey class $GV(V, \mathcal{F})$.

In [12], [13] we proved in full generality that the Hopf cyclic cohomology of the Hopf algebra of transverse differential operators on an n -dimensional manifold is canonically isomorphic to the Gelfand–Fuchs cohomology of the Lie algebra of formal vector fields on \mathbb{R}^n , via an isomorphism which can be explicitly realized at the cochain level in terms of a fixed torsion-free connection. In particular, one has a canonical isomorphism

$$\kappa_1^*: H^*(\mathfrak{a}_1, \mathbb{C}) \xrightarrow{\sim} PHC_{\text{Hopf}}^*(\mathcal{H}).$$

Proposition 18. *The canonical cochain map associated to the trivial connection on $J_+^1(\mathbb{R})$ sends the universal Godbillon–Vey cocycle gv to the Hopf cyclic cocycle δ_1 , implementing the identity*

$$\kappa_1^*([gv]) = [\delta_1].$$

Proof. To begin with recall that as a form on the jet bundle $J_+^2(\mathbb{R})$

$$gv = \frac{1}{y_1^3} dy \wedge dy_1 \wedge dy_2.$$

According to the two-step definition in [12], [14] of the isomorphism κ_1^* , one first turns the Lie algebra cocycle $gv \in C^2(\mathfrak{a}_1, \mathbb{R})$ into a group 1-cocycle $C_{1,0}(gv)$ on $\mathcal{G} = \text{Diff}^+(\mathbb{R})$ with values in currents on $J_+^1(\mathbb{R})$, and then one takes its image in the cyclic bicomplex under the canonical map Φ . The resulting cyclic cocycle

$$(\Phi(C_{1,0}(gv)))(f^0 U_{\varphi_0}^*, f^1 U_{\varphi_1}^*)$$

is automatically supported at the identity, i.e., it is nonzero only when $\varphi_1 \varphi_0 = 1$. Moreover, it is of the form

$$(\Phi(C_{1,0}(gv)))(f^0 U_{\varphi}^*, f^1 U_{\varphi^{-1}}^*) = -\langle C_{1,0}(gv)(1, \varphi), f^0 \cdot f^1 \circ \varphi \rangle.$$

By definition,

$$\langle C_{1,0}(gv)(1, \varphi), f \rangle = \int_{\Delta^1 \times J_+^1(\mathbb{R})} f \tilde{\sigma}(1, \varphi)^*(gv)$$

where Δ^1 is the 1-simplex and

$$\tilde{\sigma}(1, \varphi): \Delta^1 \times J_+^1(\mathbb{R}) \rightarrow J_+^\infty(\mathbb{R})$$

has the expression

$$\tilde{\sigma}(1, \varphi)(t, y, y_1) = \sigma_{(1-t)\nabla_0 + t\nabla_0^\varphi}(y, y_1),$$

whose meaning we now proceed to explain.

First, ∇_0 stands for the trivial linear connection on \mathbb{R} , given by the connection form on $J_+^1(\mathbb{R})$

$$\omega_0 = y_1^{-1} dy_1,$$

while ∇_0^φ denotes its transform under the prolongation

$$\varphi(y, y_1) = (\varphi(y), \varphi'(y) \cdot y_1),$$

of the diffeomorphism $\varphi \in \mathcal{G}$; the latter corresponds to the connection form

$$\varphi^*(\omega_0) = \frac{1}{\varphi'(y)y_1}(\varphi'(y) dy_1 + \varphi''(y) \cdot y_1 dy) = y_1^{-1} dy_1 + \frac{d}{dy}(\log \varphi'(y)) dy.$$

Furthermore, for any linear connection ∇ on \mathbb{R} , σ_∇ denotes the jet

$$\sigma_\nabla(y, y_1) = j_0^\infty(Y(s))$$

of the local diffeomorphism

$$s \mapsto Y(s) := \exp_y^\nabla \left(s y_1 \frac{d}{dy} \right), \quad s \in \mathbb{R}.$$

Now $Y(s)$ satisfies the geodesics ODE

$$\begin{cases} \ddot{Y}(s) + \Gamma_{11}^1(Y(s)) \cdot \dot{Y}(s)^2 = 0, \\ Y(0) = y, \\ \dot{Y}(0) = y_1. \end{cases}$$

Since we only need the 2-jet of the exponential map, suffices to retain that

$$Y(0) = y, \quad \dot{Y}(0) = y_1 \quad \text{and} \quad \ddot{Y}(0) = -\Gamma_{11}^1(y)y_1^2.$$

Thus,

$$\sigma_\nabla(y, y_1) = y + y_1 s - \Gamma_{11}^1(y)y_1^2 s^2 + \text{higher order terms}.$$

In our case $\nabla = (1-t)\nabla_0 + t\nabla_0^\varphi$, which gives

$$\Gamma_{11}^1(t, y) = t \frac{d}{dy}(\log \varphi'(y)),$$

and therefore

$$\tilde{\sigma}(1, \varphi)(t, y, y_1) = y + y_1 s - t \frac{d}{dy}(\log \varphi'(y))y_1^2 s^2 + \text{higher order terms}.$$

It follows that

$$\tilde{\sigma}(1, \varphi)^*(dy) = dy, \quad \tilde{\sigma}(1, \varphi)^*(dy_1) = dy_1$$

and

$$\begin{aligned} \tilde{\sigma}(1, \varphi)^*(dy_2) = & - \left(\frac{d}{dy}(\log \varphi'(y)) dt + t \frac{d}{dy} \left(\frac{d}{dy}(\log \varphi'(y)) \right) dy \right) y_1^2 \\ & - 2t \frac{d}{dy}(\log \varphi'(y)) y_1 dy_1. \end{aligned}$$

Hence on $\Delta^1 \times J_+^1(\mathbb{R})$,

$$\tilde{\sigma}(1, \varphi)^*(gv) = -\frac{1}{y_1} \frac{d}{dy}(\log \varphi'(y)) dt \wedge dy \wedge dy_1.$$

Going back to the definition of the group cochain, one gets

$$\begin{aligned} \langle C_{1,0}(gv)(1, \varphi), f \rangle &= - \int_{J_+^1(\mathbb{R})} f(y, y_1) \cdot \int_0^1 dt \cdot \frac{1}{y_1} \frac{d}{dy}(\log \varphi'(y)) dy \wedge dy_1 \\ &= - \int_{J_+^1(\mathbb{R})} f(y, y_1) \left(y_1 \frac{d}{dy}(\log \varphi'(y)) \right) \frac{dy \wedge dy_1}{y_1^2}, \end{aligned}$$

which finally gives

$$\begin{aligned} (\Phi(C_{1,0}(gv)))(f^0 U_\varphi^*, f^1 U_{\varphi^{-1}}^*) &= \int_{J_+^1(\mathbb{R})} f^0 \cdot f^1 \circ \varphi \cdot \left(y_1 \frac{d}{dy}(\log \varphi'(y)) \right) \frac{dy \wedge dy_1}{y_1^2} \\ &= \tau(f^0 U_\varphi^* \cdot \delta_1(f^1 U_{\varphi^{-1}}^*)) = \chi_\tau(\delta_1)(f^0 U_\varphi^*, f^1 U_{\varphi^{-1}}^*). \quad \square \end{aligned}$$

Remark 4. Since

$$\begin{aligned} C_{1,0}(gv)(1, \varphi) &= - \left(y_1 \frac{d}{dy}(\log \varphi'(y)) \right) \cdot \frac{dy \wedge dy_1}{y_1^2} \\ &= - \frac{d}{dy}(\log \varphi'(y)) \frac{dy \wedge dy_1}{y_1} = d \left(\frac{d}{dy}(\log \varphi'(y)) \log y_1 dy \right), \end{aligned}$$

it follows (as in [19, pp. 41–42]) that, in the (non-homogeneous) (d, δ) -bicomplex

$$\{\Gamma^{*,*}(\text{Diff}^+(S^1), J_+^1(S^1)), d, \delta\}$$

computing the $\text{Diff}^+(S^1)$ -equivariant cohomology of $J_+^1(S^1)$,

$$\mu(\varphi) := C_{1,0}(gv)(1, \varphi)$$

is cohomologous to the δ -boundary of

$$\nu(\varphi) = \frac{d}{dy}(\log \varphi'(y)) \log y_1 dy,$$

that is with

$$\delta \nu(\varphi, \psi) = d(\log \varphi' \circ \psi) \cdot \log \psi'.$$

The latter is precisely Thurston's formula for the Godbillon–Vey class. By integration along the fiber one gets the Bott–Thurston cocycle (see [7]) on $\text{Diff}(S^1)$

$$\text{BT}(\varphi, \psi) = \int_{S^1} \log \psi' \cdot d \log(\varphi' \circ \psi).$$

In turn, its restriction to $\text{PSL}(2, \mathbb{R})$, acting by fractional linear transformation on the projective real line $\mathbb{P}^1(\mathbb{R})$, gives a multiple of the *area cocycle* representing the Euler class (cf. [7, Appendix by R. Brooks]):

$$A(\gamma_1, \gamma_2) = \text{Area}(\Delta(i, \gamma_2 \cdot i, \gamma_2 \gamma_1 \cdot i)), \quad \gamma_1, \gamma_2 \in \text{PSL}(2, \mathbb{R}),$$

where $\text{Area}(\Delta(z_1, z_2, z_3))$ denotes the hyperbolic area of the geodesic triangle in the upper half-plane with vertices z_1, z_2, z_3 .

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