PARABOLIC CHARACTER SHEAVES, I

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Dedicated to Pierre Cartier on the occasion of his 70th birthday

Abstract. We study a class of perverse sheaves on the variety of pairs $(P, gU_P)$ where $P$ runs through a conjugacy class of parabolics in a connected reductive group and $gU_P$ runs through $G/U_P$. This is a generalization of the theory of character sheaves.


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Introduction

0.1. Let $G$ be a connected reductive algebraic group over an algebraically closed field $k$. Let $Z$ be the variety (of the same dimension as $G$) consisting of all pairs $(P, gU_P)$ where $P$ runs through a fixed conjugacy class $\mathcal{P}$ of parabolics of $G$ and $gU_P \in G/U_P$ ($U_P$ is the unipotent radical of $P$). Now $G$ acts on $Z$ by $h: (P, gU_P) \mapsto (hPh^{-1}, hgh^{-1}U_{hPh^{-1}})$. In this paper we study a class of simple $G$-equivariant perverse sheaves on $Z$ which we call “parabolic character sheaves”. In the case where $\mathcal{P} = \{G\}$, these are the character sheaves on $Z = G$, in the sense of [L3]. In general, they have several common properties with the character sheaves: they are defined in arbitrary characteristic, and in the case where $k$ is an algebraic closure of a finite field $\mathbb{F}_q$ and $G, \mathcal{P}$ are defined over $\mathbb{F}_q$,

(a) the characteristic functions of the parabolic character sheaves that are defined over $\mathbb{F}_q$ form a basis of the space of $G(\mathbb{F}_q)$-invariant functions on $Z(\mathbb{F}_q)$

(under a mild assumption on the characteristic of $k$).

0.2. We now review the contents of this paper in more detail.

In Section 1 we describe a partition of a partial flag manifold of $G$ (defined over $\mathbb{F}_q$) into pieces that are $G(\mathbb{F}_q)$-stable, generalizing the partition [DL] of the full flag manifold. This partition has been considered by the author in 1977 (unpublished); the associated combinatorics has been developed by Bédard [B] (see Section 2). Although this partition is not needed in our theory of parabolic character sheaves,
it serves as motivation for the theory. In Section 3 we define a partition of \(Z\) into pieces which is governed by the same combinatorics as that in Section 2. In Section 4 we define the “parabolic character sheaves” on \(Z\) in two apparently different ways. The first one uses the partition of \(Z\) mentioned above and the usual notion of character sheaf on a connected component of a possibly disconnected reductive group. The second one imitates the definition of character sheaves in [L3]. One of our main results is that these two definitions coincide (Lemmas 4.13, 4.17). In Section 5, we define a map from the set of parabolic character sheaves on \(Z\) to the set of “tame” local systems on a maximal torus of \(G\) modulo the action of a certain subgroup of the Weyl group (this extends a known property of character sheaves).

Section 6 is Corollary 6.8. This has the following consequence. Assume that \(G\) and \(P\) (in Section 0.1) are defined over \(F\). Let \(\rho\) be the character of an irreducible representation of \(G(F)\) over \(\overline{\mathbb{Q}}_l\). By summing \(\rho\) over each fibre of \(G(F) \to (G/U)(F)\) for \(P \in \mathcal{P}(F)\), we obtain a \(G(F)\)-invariant function \(Z(F) \to \overline{\mathbb{Q}}_l\). We would like to express this function as a linear combination of a small number of elements in a fixed basis of the space of all \(G(F)\)-invariant functions on \(Z(F)\). Here “small” means “bounded by a number independent of \(q\”). The basis formed by the characteristic functions of the \(G(F)\)-orbits on \(Z(F)\) does not have this property. However, the basis 0.1(a) defined by the parabolic character sheaves does have the required property (under a mild assumption) as a consequence of Corollary 6.8. In Section 7 we prove property 0.1(a).

0.3. The theory of parabolic character sheaves on \(Z\) continues to make sense with only minor modifications when \(G\) and \(P\) (with \(G\) simply connected) are replaced by \(G(k((x)))\), \(x\) an indeterminate, and by a class of parahoric subgroups. (There is no known theory of character sheaves on \(G(k((x)))\) itself.) I believe that this theory is a necessary ingredient for establishing the (conjectural) geometric interpretation of affine Hecke algebras with unequal parameters proposed in [L8].

1. A Partition of a Partial Flag Manifold

1.1. For any affine connected algebraic group \(H\) let \(U_H\) be the unipotent radical of \(H\). Let \(G\) be as in Section 0.1. If \(P, Q\) are parabolics of \(G\) then so is \(PQ = (P \cap Q)U_P\) and we have

\[U(PQ) = U_P(P \cap U_Q).\]

1.2. Assume now that \(k\) is an algebraic closure of the finite field \(F\) and that we are given an \(F\)-rational structure on \(G\); let \(F: G \to G\) be the corresponding Frobenius map. Let \(\mathcal{P}\) be a conjugacy class of parabolics in \(G\). To any \(Q \in \mathcal{P}\) we attach a sequence of parabolics \(0Q \supset 1Q \supset 2Q \supset \ldots\) by

\[0Q = Q, \quad nQ = (n-1)^{-1}(Q)F^{n-1}(Q) \text{ for } n \geq 1.\]

We say that \(Q, Q' \in \mathcal{P}\) are equivalent if for any \(n \geq 0\) there exists an element of \(G\) that conjugates \(nQ\) to \(nQ'\) and \(F^n(Q)\) to \(F^n(Q')\). The equivalence classes form a partition of \(\mathcal{P}\) into finitely many \(G^F\)-stable subvarieties, or “pieces”. (In the
case where \( P \) is the variety of Borels, the pieces above are precisely the subvarieties \( X(u) \) (see [DL, 1.4]) of the flag manifold.) In the general case, using results in [DL], [L2] one can show that each piece has Euler characteristic in \( 1 + q \mathbb{Z} \). Hence

(a) the number of pieces in \( P \) is equal to the Euler characteristic of \( P \)

(at least for \( q \gg 0 \)). Based on this observation, around 1979, I asked R. Bédard (my Ph. D. student at the time) to find a combinatorial explanation for the equality (a). Bédard’s solution of this problem is reviewed in Section 2.

1.3. Let \( V \) be a vector space of dimension \( n \geq 2 \) over \( k \) (as in Section 1.2) with a fixed \( \mathbb{F}_q \)-rational structure and corresponding Frobenius map \( F: V \to V \). Then \( G = \text{GL}(V) \) is as in Section 1.2. We identify parabolics in \( G \) with partial flags in \( V \) in the usual way. Following [DL, 2.3] we partition the set of lines in \( V \) into locally closed subvarieties \( X_1, X_2, \ldots, X_n \) where \( X_j \) is the set of all lines \( L \) in \( V \) such that \( \dim(L + F(L) + F^2(L) + \cdots) = j \). The pieces \( X_j \) are special cases of the pieces in Section 1.2.

1.4. Let \( V \) be a vector space of dimension \( 2n \geq 4 \) over \( k \) (as in Section 1.2) with a given non-degenerate symplectic form \( \langle , \rangle : V \times V \to k \) and with a fixed \( \mathbb{F}_q \)-rational structure and corresponding Frobenius map \( F: V \to V \) such that \( \langle F(x), F(y) \rangle = \langle x, y \rangle \) for all \( x, y \in V \). Then the symplectic group \( G = \text{Sp}(V) \) is as in Section 1.2. We identify parabolics in \( G \) with partial isotropic flags in \( V \) in the usual way. Following [L1] we partition the set of lines in \( V \) into locally closed subvarieties \( X_1, X_2, \ldots, X_n, X'_1, \ldots, X'_J, X'_J \) where \( X_j \) is the set of all lines \( L \) in \( V \) such that \( L + F(L) + F^2(L) + \cdots \) is an isotropic subspace of dimension \( j \) and \( X'_j \) is the set of all lines \( L \) in \( V \) such that 

\[
\langle L, F(L) \rangle = \langle L, F^2(L) \rangle = \cdots = \langle L, F^{j-1}(L) \rangle = 0, \quad \langle L, F^j(L) \rangle \neq 0.
\]

The pieces \( X_j, X'_j \) are special cases of the pieces in Section 1.2.

2. Results of Bédard

2.1. Let \( W \) be a Coxeter group, let \( I \) be the set of simple reflections and let \( l: W \to \mathbb{N} \) be the length function. Let \( \epsilon: W \to W \) be a group isomorphism.

In this section we reformulate results of Bédard [B] in a more general setting. (In [B] it is assumed that \( W \) is finite and \( \epsilon(I) = I \); we do not assume this.)

If \( A, A' \) are subsets of a group and \( g \) is an element of that group we set \( gA = \text{Ad}(g)A = gAg^{-1}, N_{A'}(A) = \{ h \in A'; hA = A \} \).

For a subset \( X \) of \( W \) we write \( x = \min(X) \) if \( x \in X \) and \( l(x) < l(x') \) for all \( x' \in X \setminus \{x\} \). For \( J \subset I \) let \( W_J \) be the subgroup of \( W \) generated by \( J \). For \( J, K \subset I \) let

\[
KW = \{ w \in W; w = \min(W_Kw) \}, \quad W^J = \{ w \in W; w = \min(wW_J) \}, \quad KW^J = KW \cap W^J.
\]

We recall three known results.

(a) If \( x \in KW, u \in K' \cap \text{Ad}(x^{-1})K, u \in W_{K'} \), then \( xu \in KW \) and \( l(xu) = l(x) + l(u) \).
(b) If $x \in K W^{K'}$, $x' \in W_K x W_{K'}$, $x' \in K W$, then $x' = xu$ where $u \in W_{K'}$, $u \in K' \cap \text{Ad}(x^{-1})K W$.

(c) If $x \in W_K$, $x' \in W_K$, $K' \subset K$, then $x' \in W^{K'} \Leftrightarrow xx' \in W^{K'}$.

2.2. Let $J, J' \subset I$ be such that $\epsilon(J) = J'$. Let $T(J, \epsilon)$ be the set of all sequences $(J_n, w_n)_{n \geq 0}$ where $J_n \subset I$ and $w_n \in W$ are such that

(a) $J = J_0 \supset J_1 \supset J_2 \supset \ldots$,
(b) $J_n = J_{n-1} \cap \epsilon^{-1} \text{Ad}(w_{n-1})J_{n-1}$, for $n \geq 1$,
(c) $w_n \in \epsilon(J_n)W^{-J_n}$, for $n \geq 0$,
(d) $w_n \in W\epsilon(J_n)w_{n-1}W_{J_{n-1}}$, for $n \geq 1$.

2.3. In the setup of 2.2, let $S(J, \epsilon)$ be the set of all sequences $(J_n, J'_n, u_n)_{n \geq 0}$ where $J_n, J'_n$ are subsets of $I$ and $u_n$ are elements of $W$ such that

(a) $J = J_0 \supset J_1 \supset J_2 \supset \ldots$,
(b) $\epsilon(J_n) = \epsilon(J_{n-1}) \cap \text{Ad}(u_0u_1 \ldots u_{n-1})J_{n-1}$, for $n \geq 1$,
(c) $J'_n = J'_0' \supset J'_1' \supset J'_2' \supset \ldots$,
(d) $J'_0' = J'_0 \cap \text{Ad}(w_{n-1} \ldots w_0)^{-1} \epsilon(J_{n-1})$, for $n \geq 1$,
(e) $w_n \in W_{J_{n-1}}$, for $n \geq 1$,
(f) $u_n \in J'_n W^{-J_n}$, for $n \geq 0$.

From (b), (c) we see that

$$\epsilon(J_n) = \text{Ad}(u_0u_1 \ldots u_{n-1})J'_n,$$

for $n \geq 1$.

Proposition 2.4. There is a unique bijection $S(J, \epsilon) \xrightarrow{\sim} T(J, \epsilon)$ such that

$$(J_n, J'_n, u_n)_{n \geq 0} \mapsto (J_n, u_0u_1 \ldots u_{n-1})_{n \geq 0}.$$

Let $(J_n, w_n)_{n \geq 0} \in S(J, \epsilon)$. For $n \geq 1$ we set $J'_n = J_{n-1} \cap \text{Ad}^{-1}(w_{n-1}) \epsilon(J_{n-1})$. We set $J'_0 = J'_0, u_0 = w_0$. Then $u_0 \in J'_0 W_{J_0}$. Let $n \geq 1$. We can find $v \in W_{\epsilon(J_{n-1})}$ such that $v^{-1}w_n \in \epsilon(J_{n-1}) W$. We have

$$v^{-1}w_n \in W_{\epsilon(J_{n-1})}W_{J_{n-1}}, \quad w_{n-1} \in \epsilon(J_{n-1}) W^{-J_{n-1}}.$$

Hence, by 2.1(b) and 2.1(a) we have $v^{-1}w_n = w_{n-1}u_n$ with $u_n \in W_{J_{n-1}}, w_n \in J'_n W$, $l(w_{n-1}u_n) = l(w_{n-1}) + l(u_n)$. From 2.2(d) we deduce $vw_{n-1} \in W_{\epsilon(J_n)}w_{n-1}W_{J_{n-1}}$. From 2.2(b) we have $W_{\epsilon(J_n)}w_{n-1} \subset w_{n-1}W_{J_{n-1}}$, hence $vw_{n-1} \in w_{n-1}W_{J_{n-1}}$ and $v \in w_{n-1}W_{J_{n-1}}w_{n-1} \cap W_{\epsilon(J_n)} = W_{\epsilon(J_n)}$. From $v \in W_{\epsilon(J_n)}, w_n \in \epsilon(J_n) W$, $v^{-1}w_n \in \epsilon(J_n) W$, we deduce that $v = 1$. Hence $w_n = w_{n-1}u_n$. Let $v' \in W_{J_n}$. Since $w_n \in W^{J_n}$, we have

$$l(w_{n}v') = l(w_{n}) + l(v') = l(w_{n-1}u_n) + l(v') = l(w_{n-1}) + l(u_n) + l(v').$$

Since $w_{n-1} \in W^{J_{n-1}}$ and $w_{n}v' \in W_{J_{n-1}}$, we have $l(w_{n}v') = l(w_{n-1}u_n v') = l(w_{n-1}) + l(u_n v')$. Thus, $l(w_{n-1}) + l(u_n) + l(v') = l(w_{n-1}) + l(u_n v')$ hence $l(u_n) + l(v') = l(u_n v')$. We see that $u_n \in W^{J_n}$. Thus, $u_n \in J'_n W^{J_n}$ and $(J_n, J'_n, u_n)_{n \geq 0} \in S(J, \epsilon)$. Thus, we have a map $T(J, \epsilon) \to S(J, \epsilon)$. $(J_n, w_n)_{n \geq 0} \mapsto (J_n, J'_n, u_n)_{n \geq 0}$ where $u_0 = w_0, u_n = w_{n-1}w_n$ for $n \geq 1$. We now construct an inverse of this map. Let $(J_n, J'_n, u_n)_{n \geq 0} \in S(J, \epsilon)$. 

(a) For any $0 \leq k \leq n$ we have $u_ku_{k+1} \ldots u_n \in J^kW$ and $l(u_ku_{k+1} \ldots u_n) = l(u_k) + l(u_{k+1}) + \cdots + l(u_n)$.

We use descending induction on $k \in [0, n]$. For $k = n$, (a) is clear. Assume now that $k < n$ and that (a) is true when $k$ is replaced by $k + 1$. We have

$$u_k \in J^kW, \quad u_{k+1} \ldots u_n \in J^{k+1}W = J_k \cap \text{Ad}(u_{-1}^{-1})J'_kW,$$

and that (a) is true when $k$ is replaced by $k + 1$. We have

$$l(u_ku_{k+1} \ldots u_n) = l(u_k) + l(u_{k+1}) + \cdots + l(u_n).$$

Using (e), define

$$u_0u_1 \ldots u_n \in J^0W.$$

This follows from (b) and (e).

(b) For any $n \geq 0$ we have $u_0u_1 \ldots u_n \in J^0W$.

We use induction on $n$. For $n = 0$, (b) is clear. Assume now that $n > 0$ and that (b) is true when $n$ is replaced by $n - 1$. We have

$$u_0u_1 \ldots u_{n-1} \in W, \quad u_n \in W, \quad u_{n-1} \in W_{n-1}.$$

Using (2.1c) we deduce $u_0u_1 \ldots u_{n-1}u_n \in W$. This proves (b).

(c) For any $n \geq 0$ we have $u_0u_1 \ldots u_n \in J^0W$.

This follows from (b) and (e).

(d) For any $n \geq 0$ we have $u_0u_1 \ldots u_n \in J^0W$.

Proposition 2.5. $(J_n, J'_n, u_n)_{n \geq 0} \mapsto u_0u_1 \ldots u_m$ for $m \gg 0$ is a well defined bijection $\phi: S(J, \epsilon) \rightarrow J^0W$.

Let $(J_n, w_n)_{n \geq 0} \in T(J, \epsilon)$. There exists $n_0 \geq 1$ such that $J_{n-1} = J_n$ for $n \geq n_0$. For such $n$ we have

$$w_n \in \epsilon(J_n)W, \quad w_{n-1} \in \epsilon(J_n)W, \quad w_n \in W_{\epsilon(J_n)}w_{n-1}W_{J_n}.$$

hence $w_n = w_{n-1}$. Thus, if $(J_n, u_n)_{n \geq 0} \in S(J, \epsilon)$ then for $n \gg 0$ we have $u_n = 1$. Hence there is a well defined element $w \in W$ such that $w = u_0u_1 \ldots u_n$ for $n \gg 0$. By 2.4(d), we have $w \in J^0W$. Hence $\phi$ is a well defined map.

(a) For any $n \geq 0$ we have $u_0u_1 \ldots u_n = \min(W_{J_n}wW_{J_n})$.

By 2.4(d), we have $u_0u_1 \ldots u_n \in J^0W$. Hence it suffices to show that $w \in u_0u_1 \ldots u_nW_{J_n}$. Take $N > n$ such that $u_0u_1 \ldots u_N = w$. Then

$$w = (u_0 \ldots u_n)u_{n+1}u_{n+2} \ldots u_N.$$
and it suffices to show that $u_{n+1} u_{n+2} \ldots u_N \in W_{J_n}$. This follows from

$$u_{n+1} \in W_{J_n}, \quad u_{n+2} \in W_{J_{n+1}} \subset W_{J_n}, \quad \ldots, \quad u_N \in W_{J_{N-1}} \subset W_{J_n}.$$  

This proves (a).

We show that $\phi$ is a bijection. Assume that the images $w, \tilde{w}$ of

$$(J_n, J'_n, u_n)_{n \geq 0} \in S(J, \epsilon), \quad (\tilde{J}_n, \tilde{J}'_n, \tilde{u}_n)_{n \geq 0} \in S(J, \epsilon)$$

under $\phi$ satisfy $w = \tilde{w}$. We show by induction on $n \geq 0$ that

(b) $J'_k = \tilde{J}'_k$, $J_k = \tilde{J}_k$, $u_k = \tilde{u}_k$ for $k \in [0, n]$.

For $n = 0$ this holds since $J_0 = \tilde{J}_0 = J, J'_0 = \tilde{J}'_0 = J'$ and $u_0 = \tilde{u}_0 = \min(W_{J'_0} \cup W_{J_0})$ (see (a)). Assume now that $n > 0$ and that (b) holds when $n$ is replaced by $n - 1$.

From 2.3(b),(c) we deduce that $J_n = \tilde{J}_n, J'_n = \tilde{J}'_n$. From (a) we have

$$u_0 u_1 \ldots u_n = \min(W_{J'_n} \cup W_{J_n}) = \min(W_{J'_n} \cup W_{J_n}) = \tilde{u}_0 \tilde{u}_1 \ldots \tilde{u}_n = u_0 u_1 \ldots u_{n-1} \tilde{u}_n$$

hence $u_n = \tilde{u}_n$. Thus (b) holds. We see that $(J_n, J'_n, u_n)_{n \geq 0} = (\tilde{J}_n, \tilde{J}'_n, \tilde{u}_n)_{n \geq 0}$.

Thus, $\phi$ is injective.

We define an inverse to $\phi$. Let $w \in J' W$. We define by induction on $n \geq 0$ a sequence $(J_n, J'_n, u_n)_{n \geq 0}$ as follows. We set $J_0 = J, J'_0 = J', u_0 = \min(W_{J'_0} \cup W_{J_0})$.

Assume that $n > 0$ and that $J'_k, J_k, u_k$ are defined for $k \in [0, n-1]$. We define subsets $J_n, J'_n$ of $J_{n-1}$ by

$$\epsilon(J_n) = \epsilon(J_{n-1}) \cap \Ad(u_0 u_1 \ldots u_{n-1}) J_{n-1},$$

$$J'_n = J_{n-1} \cap \Ad(u_0 u_1 \ldots u_{n-1})^{-1} \epsilon(J_{n-1})$$

and we define $u_n$ by $u_0 u_1 \ldots u_n = \min(W_{J'_n} \cup W_{J_n})$. This completes the inductive definition. Using $\epsilon(J_{n-1}) \subset J'$, we see that

(c) $J'_n \subset \Ad(u_0 u_1 \ldots u_{n-1})^{-1} J'$ for $n \geq 1$.

We show that

(d) $u_n \in W_{J_{n-1}}$ for $n \geq 1$.

Now $u_0 u_1 \ldots u_n \in W_{J'_n} \cup W_{J_n} \subset W_{J'_n} \cup W_{J_n}$ and $u_0 u_1 \ldots u_{n-1} = \min(W_{J'_n} \cup W_{J_n})$.

Moreover, $u_0 u_1 \ldots u_n \in J'_W$. By 2.1(b) we have $u_0 u_1 \ldots u_n = (u_0 u_1 \ldots u_{n-1}) u$ with

$$u \in W_{J_{n-1}}, \quad u \in J_{n-1} \cap \Ad(u_0 u_1 \ldots u_{n-1})^{-1} J'_W.$$  

Thus $u = u_n$ and (d) follows. We show by induction on $n \geq 0$ that

(e) $u_n \in J'_n W_{J_n}$.

For $n = 0$ this is clear. Assume now that $n > 0$ and that (e) holds when $n$ is replaced by $n - 1$. By the argument in the proof of (d) we have $u_n \in J_{n-1} \cap \Ad(u_0 u_1 \ldots u_{n-1})^{-1} J'_W$. Since $J'_n \subset J_{n-1}$, we see from (c) that

$$J'_n \subset J_{n-1} \cap \Ad(u_0 u_1 \ldots u_{n-1})^{-1} J'.$$

Hence $u_n \in J'_n W$. Now

$$u_0 u_1 \ldots u_{n-1} \in W_{J_{n-1}}, \quad u_n \in W_{J_{n-1}}, \quad (u_0 u_1 \ldots u_{n-1}) u_n \in W_{J_n}, \quad J_n \subset J_{n-1}.$$
Using 2.1(c) we deduce that \( u_n \in W^{J_n} \). Combining this with \( u_n \in J_n^\delta W \) gives \( u_n \in J_n^\delta W^{J_n} \). Thus (e) is established. We see that \((J_n, J_n^\delta, u_n)_{n \geq 0} \in S(J, \epsilon)\). We show that

(f) if \( n \gg 0 \), then \( u_0 u_1 \ldots u_n = w \).

For any \( n \geq 0 \) we have \( u_0 u_1 \ldots u_n = \text{min}(W_{J'} w W_{J_n}) \). Also, \( w \in J' W \). Using 2.1(b) we have \( w = (u_0 u_1 \ldots u_n) u \) with

\[
    u \in W_{J_n} , \quad u \in J_n \cap \text{Ad}(u_0 u_1 \ldots u_n)^{-1} J' W.
\]

By (c), we have \( J_{n+1}^\delta \subset J_n \cap \text{Ad}(u_0 u_1 \ldots u_n)^{-1} J' \). Hence \( u \in J_{n+1}^\delta W \). Now \( J_{n+1}^\delta \subset J_n \). Assuming that \( n \gg 0 \), we have \( J_n = J_{n+1} \). From 2.3(f) we see that \( \# J_{n+1}^\delta = \# J_n \), hence \( \# J_n \). Hence the inclusion \( J_{n+1}^\delta \subset J_n \) must be an equality. We see that \( u \in J' W \). This, combined with \( u \in W_{J_n} \) implies \( u = 1 \) and proves (f). Thus, we have defined \( \psi : J' W \rightarrow S(J, \epsilon) \) such that \( \phi \psi = 1 \). Hence \( \phi \) is bijective.

2.6. Let \((J_n, J_n^\epsilon, u_n)_{n \geq 0} \in S(J, \epsilon)\) and let \( w \) be its image under \( \phi \). For \( n \gg 0 \) we have \( u_0 u_1 \ldots u_{n-1} = w \) hence \( \text{Ad}(w) J_n^\epsilon = \epsilon(J_n) \). By the proof of Proposition 2.5(f), for \( n \gg 0 \) we have also \( J_n = J_n^\epsilon = J_{n+1} = J_{n+1}^\epsilon = \ldots \) Thus there is a well defined subset \( J_\infty \) of \( J \) such that \( J_n = J_n^\epsilon = J_\infty \) for \( n \gg 0 \) and \( \text{Ad}(w) J_\infty = \epsilon(J_\infty) \).

2.7. Let \( G \) be as in Section 0.1. Let \( B \) be the variety of Borel subgroups of \( G \). Let \( W \) be the set of \( G \)-orbits on \( B \times B \) (\( G \) acts by conjugation on both factors). Then \( W \) is naturally a finite Coxeter group; let \( I \) be the set of simple reflections (the \( G \)-orbits of dimension \( \dim B + 1 \)). For \( B, B' \in B, w \in W \) we write \( \text{pos}(B, B') = w \) if the \( G \)-orbit of \((B, B')\) is \( w \). If \( P \) is a parabolic of \( G \), the set of all \( w \in W \) such that \( w = \text{pos}(B, B') \) for some \( B, B' \in B \in B \in P, B' \subset P \) is of the form \( W_P \) (as in Section 2.1) for a well defined subset \( J \subset I \). We then say that \( P \) has type \( J \). For \( J \subset I \), let \( \mathcal{P}_J \) be the set of all parabolics of type \( J \) of \( G \).

For \( P \in \mathcal{P}_J, Q \in \mathcal{P}_K \) there is a well defined element \( u = \text{pos}(P, Q) \in J W K \) such that \( \text{pos}(B, B') \geq u \) (standard partial order on \( W \)) for any \( B, B' \in B \in B \in P, B' \subset Q \) and \( \text{pos}(B_1, B_1') = u \) for some \( B_1, B_1' \in B \in B \in P, B_1' \subset Q \); we then have \( B_1 \subset P^Q, B_1' \subset Q^P \). We have

\[
    P^Q \in \mathcal{P}_{J \cap \text{Ad}(w) K}.
\]

Now \( (P, Q) \mapsto u \) defines a bijection between the set of \( G \)-orbits on \( \mathcal{P}_J \times \mathcal{P}_K \) and \( J W K \).

2.8. Let \( G, F \) be as in Section 1.2. Let \( W, I \) be as in Section 2.7. The bijection \( \delta : W \rightarrow W \) induced by \( F : G \rightarrow G \) satisfies \( \delta(I) = I \). Let \( J \subset I \). We show that the pieces of \( \mathcal{P}_J \) defined in Section 1.2 are naturally indexed by \( T(J, \delta) \).

Let \( Q \in \mathcal{P}_J \). To \( Q \) we associate a sequence \((J_n, w_n)_{n \geq 0} \) with \( J_n \subset I, w_n \in W \) and a sequence \((0, Q)_{n \geq 0} \) with \( 0 \in \mathcal{P}_J \). We set

\[
    0Q = Q, \quad J_0 = J, \quad w_0 = \text{pos}(F(0Q), 0Q).
\]
Assume that $n \geq 1$, that $mQ, J_m, w_m$ are already defined for $m < n$ and that $w_m = \text{pos}(F^{(m)}Q, mQ), mQ \in P^{J_m}$ for $m < n$. Let

$$J_n = J_{n-1} \cap \delta^{-1} \text{Ad}(w_{n-1})J_{n-1},$$

$$Q^{(n-1)}Q = (n-1)^{-1}Q^{(n-1)}Q \in P^{J_n},$$

$$w_n = \text{pos}(F^{(n)}Q, nQ) \in \delta(J_n)W^{J_n}.$$

This completes the inductive definition. From Lemma 3.2(c) (with $P, P', Z$ replaced by $n^{-1}Q, F(n^{-1}Q), nQ$) we see that $w_n \in w_{n-1}W^{J_{n-1}},$ for $n \geq 1$. Thus, $(J_n, w_n)_{n \geq 1} \in \mathcal{T}(J, \delta).$ Thus, $Q \mapsto (J_n, w_n)_{n \geq 0}$ is a map $\mathcal{P}_J \to \mathcal{T}(J, \delta)$. The fibre of this map at $t \in \mathcal{T}(J, \delta)$ is denoted by $\mathcal{P}_J^t$. Clearly, $(\mathcal{P}_J^t)_{t \in \mathcal{T}(J, \delta)}$ is a partition of $\mathcal{P}_J$ into locally closed subvarieties (the same as in Section 1.2). The $G^F$-action on $\mathcal{P}_J$ given by $h: Q \mapsto hQ$ preserves each of the pieces $\mathcal{P}_J^t$.

3. The Variety $Z_{J, \delta}$ and its Partition

3.1. Let $G$ be as in Section 0.1. Let $W, I$ be as in Section 2.7. Let $\hat{G}$ be a possibly disconnected reductive algebraic group over $k$ with identity component $G$ and let $G^1$ be a fixed connected component of $\hat{G}$. There is a unique isomorphism $\delta: W \rightleftarrows W$ such that $\delta(I) = I$ and

$$P \in \mathcal{P}_J, g \in G^1 \implies \delta P \in \mathcal{P}_{\delta(J)}.$$

Let $P \in \mathcal{P}_J, Q \in \mathcal{P}_K, u = \text{pos}(P, Q)$. We say that $P, Q$ are in good position if they have a common Levi or, equivalently, if $\text{Ad}(w^{-1})(J) = K$. In this case we have $PQ = P, PQ = Q$.

We fix $J \subset I$.

Lemma 3.2. Let $P \in \mathcal{P}_J, P' \in \mathcal{P}_J, J' \subset I$. Let $a = \text{pos}(P', P)$. Let $X = P'P, Y = P'P$ and let $Z$ be a parabolic subgroup of $P$. Let $b = \text{pos}(Y, Z)$.

(a) Let $Y'$ be a parabolic subgroup of $P$ of the same type as $Y$ such that $X, Y'$ are in good position and $\text{pos}(X, Y') = a$. Then $Y' = Y = P^X$.

(b) $X$ contains a Levi of $Y \cap Z$.

(c) $\text{pos}(X, Z) = ab$.

(d) $X(Y^Z) = XZ$.

We prove (a). Since $X, Y'$ are in good position, for any Borel $B'$ in $Y'$ there exists a Borel $B$ in $X$ such that $\text{pos}(B, B') = a$. Since $\text{pos}(X, P) = a$ and $B' \subset P$ we have $B' \subset P^X$. Since $Y'$ is the union of its Borels, we have $Y' \subset P^X$. Now $Y'$ is of type $J' \cap \text{Ad}(a^{-1})J'$, and $P^X$ is of type $J \cap \text{Ad}(a^{-1})(J' \cap \text{Ad}(a)J) = J \cap \text{Ad}(a^{-1})J'$. Thus $Y', P^X$ have the same type, hence $Y' = P^X$. Replacing in this argument $Y'$ by $Y$ we obtain $Y = P^X$.

We prove (b). Since $Z \subset P$ we have $U_P \subset Z$. Hence $Y \cap Z = P'P \cap Z = (P \cap P')U_P \cap Z = (P \cap P')U_Z$. Now $U_P \subset U_{Y \cap Z}$. Hence if $L$ is a Levi of $P \cap P' \cap Z$ then $L$ is a Levi of $Y \cap Z$. Now $L \subset P \cap P' \subset X$.

We prove (c). Let $\tilde{a} = \text{pos}(X, Z)$. We have $X \in \mathcal{P}_K$ with $K = J' \cap \text{Ad}(a)J$. We have $a \in J'W^J$ and $K \subset J'$ hence $a \in K^WJ$. From the definition of $\tilde{a}$ we have $\tilde{a} \in K^W$. Since $Z \subset P$, we have $\tilde{a} \in W_KaW_J$. Using 2.1(b) with $x, x'$
replaced by $a, \tilde{a}, v$, we see that $\tilde{a} = av$ with $v \in W_J$. Let $B, B' \in B$ be such that $B \subset X, B' \subset Z$, $\text{pos}(B, B') = \tilde{a}$. Since $a \in W^J$, we have $l(av) = l(a) + l(v)$ and there is a unique $B'' \in B$ such that $\text{pos}(B, B'') = a$, $\text{pos}(B', B'') = v$. Since $B' \subset P$ and $\text{pos}(B'', B) \in W_J$, we have $B'' \subset P$. Since $B \subset P', B'' \subset P$ and $\text{pos}(B, B'') = \text{pos}(P', P) = a$, we have $B'' \subset P^{P'} = Y$. Since $B'' \subset Y, B' \subset Z$, we have $v \geq b$. We can find $B_1, B_2 \in B$ such that $B_1 \subset Y, B_2 \subset Z, \text{pos}(B_1, B_2) = b$. Since $\text{pos}(P', P) = a$ and $B_1 \subset P^{P'}$, we can find $B_0 \in B$ such that $B_0 \subset P'$, $\text{pos}(B_0, B_1) = a$. Since $a \in W^J$ and $b \in W_J$, we have $\text{pos}(B_0, B_2) = ab$. We have $B_0 \subset P^{P'} = X, B_2 \subset Z$ hence $\text{pos}(B_0, B_2) \geq \text{pos}(X, Z)$, that is, $ab \geq \tilde{a} = av$. Thus, we have $v \geq b$ and $ab \geq av$. Since $a \in W^J$ and $b, v \in W_J$ we have $b = v$. Thus, $\tilde{a} = ab$.

We prove (d). Let $B$ be a Borel in $X^Z$. Using (c), we can find a Borel $B'$ in $Z$ such that $\text{pos}(B, B') = ab$. Let $B''$ be the unique Borel such that $\text{pos}(B, B'') = a$, $\text{pos}(B'', B') = b$. Now $B'' \subset P$ since $B' \subset P$ and $\text{pos}(B'', B') = b \in W_J$. Since $\text{pos}(X, P) = \text{pos}(B, B'') = a$, we have $B'' \subset P_X$ hence $B'' \subset Y$ (see (a)). Since $\text{pos}(X, Y, Z) = \text{pos}(B'', B') = b$, we have $B'' \subset Y^Z$. Since $B \subset C, B'' \subset Y^Z$ and $\text{pos}(B, B'') = a = \text{pos}(P', P)$ where $X \subset P', Y^Z \subset P$, we have $\text{pos}(B, B'') = a = \text{pos}(X, Y, Z)$. Hence $B \subset X(Y^Z)$. Thus any Borel in $X^Z$ is contained in $X(Y^Z)$. Since $X^Z$ is the union of its Borels, we have $X^Z \subset X(Y^Z)$. To show that $X^Z = X(Y^Z)$, it suffices to show that $X^Z, X(Y^Z)$ have the same type, or that $K \cap \text{Ad}(ab)K'' = K \cap \text{Ad}(a)K' \cap \text{Ad}(b)K''$ (where $X, Y, Z$ are of type $K, K', K''$) or that $\text{Ad}(a)K' = K$ which follows from $K = J' \cap \text{Ad}(a)J, K' = J \cap \text{Ad}(a^{-1})J'$. The lemma is proved.

3.3. For $(P, P') \in \mathcal{P}_J \times \mathcal{P}_{\delta(J)}$, $A(P, P') = \{g \in G^1; gP = P'\}$ is a single $P$-orbit (resp. $P'$-orbit) for right (resp. left) translation on $G$. Let $Z_{J, \delta}$ be the set of all triples $(P, P', g)$ where $P \in \mathcal{P}_J, P' \in \mathcal{P}_{\delta(J)}, g \in A(P, P')$. Let $Z_{J, \delta}$ be the set of all triples $(P, P', \gamma)$ where $P \in \mathcal{P}_J, P' \in \mathcal{P}_{\delta(J)}, \gamma \in U_{P'} \setminus A(P, P') = A(P, P')/U_P$.

Let $(P, P', g) \in Z_{J, \delta}$; let $z = \text{pos}(P, P)$. Let $J_1 = J \cap \delta^{-1} \text{Ad}(z)(J)$. We set $P^1 = g^{-1}P^gP \in \mathcal{P}_{J_1}, P^{P1} \in \mathcal{P}_{\delta(J)}$.

We write $(P^1, P^{P1}) = \alpha'(P, P', g)$. We have $(P^1, P^{P1}, g) \in Z_{J, \delta}$. We have $P^1 \subset P, P^{P1} \subset P'$ and, by Lemma 3.2(c) (with $Z = P^1$), we have $\text{pos}(P^{P1}, P^1) \in zW_J$.

Lemma 3.4. Let $g, g' \in A(P, P'), u' \in U_P$, $u \in U_P$ be such that $g' = u'gu$. Then

(a) $g' = u'_1gu_1$ where $u'_1 \in U_{P^{P1}}, u_1 \in U_P$;
(b) we have $\alpha'(P, P', g') = (P^1, P^{P1})$.

To prove (a), we may assume that $u' = 1$ (since $U_P g = gU_P$). Since $P^{P1} \subset P$, we have $U_P \subset U_{P^{P1}}$ hence $g' \in gU_{P^{P1}}$. This proves (a). To prove (b), we may assume that $u = 1$ (since $U_P g = gU_P$). Since $U_{P^{P1}} \subset P^{P1}$, we have $u' \in P^{P1}$ hence $g^{-1}(P^{P1}) = g^{-1}(u^{-1}(P^{P1})) = P^1$. The lemma is proved.
Lemma 3.5. Let \( g, g' \in A(P, P') \). Assume that \( \alpha'(P, P', g) = \alpha'(P, P', g') = (P_1, P_1') \) and that \( g' \in U_{P_1}g = gU_{P_1} \). There exist \( x \in U_P \cap P', w' \in U_{P'} \) such that \( g' = w'xg \).

We have \( g' = u'g \) where \( u' \in U_{P_0} \). We have \( u' = w'x \) where \( w' \in U_{P_1} \), \( x \in U_P \cap P' \). Hence \( g' = w'xg \).

Lemma 3.6. Let \( z \in \delta(J)W^J, J_1 = J \cap \delta^{-1}\text{Ad}(z)(J) \). Let 
\[
Z_{j, \delta} = \{(P, P', g) \in Z_{J, \delta}; \text{pos}(P', P) = z\},
\]
\[
Z_{j, \delta}^1 = \{(Q, Q', g) \in Z_{J, \delta}; \text{pos}(Q, Q') \in ZW_j\}.
\]
Define \( f: Z_{j, \delta} \rightarrow Z_{j, \delta}^1 \) by \( (P, P', g) \mapsto (P_1, P_1', g) \) where \( (P_1, P_1') = \alpha'(P, P', g) \). Then \( f \) is an isomorphism.

We show only that \( f \) is bijective. For \( (P, P', g) \in Z_{j, \delta}^1 \), \( (P, P', g) \in Z_{J, \delta} \) (resp. \( P \)) is the unique parabolic of type \( J \) (resp. \( \delta(J) \)) that contains \( P_1 \) (resp. \( P_1' \)). Hence \( f \) is injective. We show that \( f \) is surjective. Let \( (Q, Q', g) \in Z_{j, \delta}^1 \). Let \( P \) (resp. \( P' \)) be the unique parabolic of type \( J \) (resp. \( \delta(J) \)) that contains \( Q \) (resp. \( Q' \)). Clearly, \( \text{pos}(P', P) = z \) and \( gP = P' \). It suffices to show that \( P_{PP} = Q' \). We have \( \text{pos}(Q', Q) = zu \) where \( u \in W_J \). We can find \( B, B' \in B \) such that \( B \subset Q, B' \subset Q' \), \( \text{pos}(B', B) = z \). Since \( l(zu) = l(z) + l(u) \), we can find \( B'' \in B \) such that \( \text{pos}(B', B'') = z \), \( \text{pos}(B'', B) = u \). Since \( u \in W_J \) and \( B \subset P \), we have \( B'' \subset P \). Since \( B' \subset P', B'' \subset P \), \( \text{pos}(B', B'') = z \), we have \( B' \subset P_{PP} \). Since \( Q', P_{PP} \) are in \( \mathcal{P}_{\delta(J)} \) and both contain \( B' \), we have \( Q' = P_{PP} \). The lemma is proved.

3.7. We fix \( z \in \delta(J)W^J \). Let \( J_1 = J \cap \delta^{-1}\text{Ad}(z)(J) \). Let \( (Q, Q', \gamma_1) \in Z_{J, \delta} \) be such that \( \text{pos}(Q, Q) \in zW_J \). Let \( \mathcal{F} \) be the set of all \( (P, P', \gamma) \in Z_{J, \delta} \) such that \( \text{pos}(P', P) = z \), \( \gamma \subset \gamma_1 \) and \( \alpha'(P, P', g) = (Q, Q') \) for some \( g \in \gamma \) (see Lemma 3.4). (Note that \( P, P' \) are uniquely determined by \( Q, Q' \)) By Lemma 3.6 we have \( \mathcal{F} \neq \emptyset \).

Lemma 3.8. Let \( (P, P', \gamma) \in \mathcal{F} \). Let \( g \in \gamma \). Then \( v \mapsto (P, P', U_{P'v}gU_P) \) is a well defined, surjective map \( \kappa: U_P \cap P' \rightarrow \mathcal{F} \).

Let \( (P, P', \gamma') \in \mathcal{F} \). Let \( g' \in \gamma' \). By Lemma 3.5 there exist \( v \in U_P \cap P' \), \( w' \in U_{P'} \) such that \( g' = w'vg \). Then \( \gamma' = U_{P'v}gU_P \). The lemma is proved.

Lemma 3.9. In the setup of Lemma 3.8, the following two conditions for \( v, v' \) in \( U_P \cap P' \) are equivalent:
(i) \( \kappa(v) = \kappa(v') \);
(ii) \( v' = dv \) for some \( d \in U_P \cap U_{P'} \).

Assume that (i) holds. We have \( vgU_P = U_Pv'U_{P'}g \) hence \( v \in U_Pv'U_{P'} = U_{P'}v' \). Thus, \( v' = dv \) where \( d \in U_{P'} \) and we have automatically \( d \in U_P \cap U_{P'} \). Thus, (ii) holds. The converse is immediate.

3.10. We see that \( \kappa \) defines a bijection \( (U_P \cap P')/(U_P \cap P') \cong \mathcal{F} \). One can check that this is an isomorphism of algebraic varieties. Since \( U_P \cap P' \) is a connected unipotent group and \( U_P \cap U_{P'} \) is a connected closed subgroup of it, we see that
(a) \( \mathcal{F} \) is isomorphic to an affine space of dimension \( \dim((U_P \cap U_{P'})/(U_P \cap P')) \).
3.11. To any \((P, P', g) \in Z_{J, \delta}\) we associate a sequence \((J_n, w_n)_{n \geq 0}\) with \(J_n \subset I_n\), \(w_n \in W\) and a sequence \((P^n, P'^n, g)_{n \geq 0}\) with \((P^n, P'^n, g) \in Z_{J_n, \delta}\). We set
\[ P^0 = P, \quad P'^0 = P', \quad J_0 = J, \quad w_0 = \text{pos}(P^0, P'0). \]
Assume that \(n \geq 1\), that \(P^n, P'^n, J_n, w_n\) are already defined for \(m < n\) and that \(w_m = \text{pos}(P'^m, P^m)\) in \(Z_{J_m, \delta}\). Then \((P^n, P'^n) \in Z_{J_n, \delta}\) for \(m < n\). Let
\[ J_n = J_{n-1} \cap \delta^{-1}\text{Ad}(w_{n-1})(J_{n-1}), \]
\[ P^n = g^{-1}(P'^{n-1})^{P'n-1} g \in P_{J_n}, \quad P'^n = (P'^{n-1})^{P'n-1} \in P_{\delta(J_n)}, \]
\[ w_n = \text{pos}(P^n, P'^n) \in \delta(J_n)W_{J_n}. \]
This completes the inductive definition. From Lemma 3.2(c) (with \(P, P', Z\) replaced by \(P^n, P'^n, P_n\)) we see that \(w_n \in w_{n-1}W_{J_{n-1}}\) for \(n \geq 1\). Thus, \((J_n, w_n)_{n \geq 0} \in T(J, \delta)\). We write \((J_n, w_n)_{n \geq 0} = \beta(P, P', g)\). For \(t \in T(J, \delta)\) let
\[ tZ_{J, \delta} = \{(P, P', g) \in Z_{J, \delta}; \beta'(P, P', g) = t\}, \]
\[ t'Z_{J, \delta} = \{(P, P', \gamma) \in Z_{J, \delta}; \beta'(P, P', g) = t \text{ for some/any } g \in \gamma\}. \]
(The equivalence of “some/any” follows from Lemma 3.4.) Clearly, \((tZ_{J, \delta})_{t \in T(J, \delta)}\) is a partition of \(Z_{J, \delta}\) into locally closed subvarieties and \((t'Z_{J, \delta})_{t \in T(J, \delta)}\) is a partition of \(Z_{J, \delta}\) into locally closed subvarieties. The \(G\)-action on \(Z_{J, \delta}\) given by \(h: (P, P', g) \mapsto (hP, hP', hg)\) preserves each of the pieces \(tZ_{J, \delta}\). Similarly, the natural action of \(G\) on \(Z_{J, \delta}\) preserves each of the pieces \(t'Z_{J, \delta}\). Clearly, \((P, P', g) \mapsto (P^1, P'^1, g)\) is a morphism \(\vartheta': t'Z_{J, \delta} \to tZ_{J, \delta}\) where for \(t = (J_n, w_n)_{n \geq 0} \in T(J, \delta)\) we set \(t^1 = (J_n, w_n)_{n \geq 1} \in T(J, \delta)\); it induces a morphism
\[ \vartheta: t'Z_{J, \delta} \to tZ_{J, \delta}. \]

Lemma 3.12. (a) The morphism \(\vartheta': t'Z_{J, \delta} \to tZ_{J, \delta}\) is an isomorphism.

(b) The morphism \(\vartheta': t'Z_{J, \delta} \to tZ_{J, \delta}\) is an affine space bundle with fibres of dimension \(\text{dim}(P / P \cap P')\). For \(\vartheta: t'Z_{J, \delta} \to tZ_{J, \delta}\) is the same as \(t'Z_{J, \delta} \to tZ_{J, \delta}\).

(c) Consider the map \(\vartheta\) from the set of \(G\)-orbits on \(t'Z_{J, \delta}\) to the set of \(G\)-orbits on \(tZ_{J, \delta}\) induced by \(\vartheta\). Then \(\vartheta\) is a bijection.

Part (a) follows from Lemma 3.6. We prove (b). Now \(\text{dim}(P / P \cap P')\) in (b) is independent of the choice of \((P, P', \gamma)\); it depends only on \(\text{pos}(P, P')\) which is constant on \(tZ_{J, \delta}\). From 3.10(a) we see that each fibre of \(\vartheta\) is an affine space of the indicated dimension. The verification of local triviality is omitted.

We prove (c). Since \(\vartheta\) is surjective (see (b)) and \(G\)-equivariant, \(\vartheta\) is well defined and surjective. We show that \(\vartheta\) is injective. Let \((P, P', \gamma), (P', P''', \gamma)\) be two triples in \(t'Z_{J, \delta}\) whose images under \(\vartheta\) are in the same \(G\)-orbit; we must show that these two triples are in the same \(G\)-orbit. Since \(\vartheta\) is \(G\)-equivariant, we may assume that \(\vartheta(P, P', \gamma) = \vartheta(P', P'', \gamma) = (Q, Q', \gamma_1) \in tZ_{J_1, \delta}\). Define \(F\) in terms of \((Q, Q', \gamma_1)\) as in Section 3.7. Then \((P, P', \gamma) \in F, (P', P'', \gamma) \in F\). Since \(P, \hat{P}\) are parabolics of the same type containing \(Q\) we have \(P = \hat{P}\). Since \(P', P''\) are parabolics of the same type containing \(Q'\) we have \(P' = \hat{P}'\). Let \(g \in \gamma\). By Lemma 3.8, we have \(\hat{\gamma} = U_P v g U_P\) for some \(v \in U_P \cap P'\). We have also \(\hat{\gamma} = P (\text{since } v \in P)\),
Our notation for perverse sheaves follows \(\text{[4.1.]}\). Let \(\gamma = U_P v g U_P = v U_P g U_P = v U_P g U_P v^{-1} = v \gamma v^{-1}\) (since \(v\) normalizes \(U_P\) and \(v \in U_P\)). Thus, \((\tilde{P}, \tilde{P}^\prime, \tilde{\gamma})\) is obtained by the action of \(v \in G\) on \((P, P', \gamma)\), hence \((\tilde{P}, \tilde{P}^\prime, \tilde{\gamma})\) is in the \(G\)-orbit of \((P, P', \gamma)\). The lemma is proved.

**Lemma 3.13.** Let \(t = (J_n, w_n)_{n \geq 0} \in \mathcal{T}(J, \delta)\). Then \(\mathcal{Z}_{J, \delta}\) is an iterated affine space bundle over a fibre bundle over \(P_{J_n}\) with fibres isomorphic to \(P/U_P\) where \(P \in P_{J_n}\) \((n \gg 0)\). In particular, \(\mathcal{Z}_{J, \delta} \neq \emptyset\).

Assume first that \(t\) is such that \(J_n = J\) for all \(n\) and \(w_n = w\) for all \(n\) (here \(w \in W\)). In this case, \(\mathcal{Z}_{J, \delta}\) is the set of all \((P, P', \gamma)\) where \(P \in P_J, P' \in P_{\mathfrak{A}(J)}, \text{pos}(P', P) = w, P', P\) are in good position, \(\gamma \in U_P \setminus A(P, P')/U_P\). (The associated sequence \(P^n, P'^n\) is in this case \(P^n = P, P'^n = P\)). Thus, in this case, \(\mathcal{Z}_{J, \delta}\) is a locally trivial fibration over \(P_J\) with fibres isomorphic to \(P/U_P\) for \(P \in P_J\) so the lemma holds.

We now consider a general \(t\). For any \(r \geq 0\) let \(t_r = (J_n, w_n)_{n \geq r} \in \mathcal{T}(J_r, \delta)\).

By Lemma 3.12(b) we have a sequence of affine space bundles

\[(a) \quad t_r \mathcal{Z}_{J, \delta} \to t_r t_r \mathcal{Z}_{J, \delta} \to t_r t_r t_r \mathcal{Z}_{J, \delta} \to \ldots\]

where for \(r \gg 0\), \(t_r \mathcal{Z}_{J, \delta}\) is as in the first part of the proof. The lemma follows.

**3.14.** In the setup of Lemma 3.13, the maps in 3.13(a) induces bijections on the sets of \(G\)-orbits (see Lemma 3.12(c)). Thus we obtain a canonical bijection between the set of \(G\)-orbits on \(t_r \mathcal{Z}_{J, \delta}\) and the set of \(G\)-orbits on \(t_r \mathcal{Z}_{J, \delta}\) where \(r\) is chosen large enough so that \(J_r = J_{r+1} = \ldots\), and \(w_r = w_{r+1} = \ldots = w\). This last set of orbits is canonically the set of \((P \cap P')\)-orbits on \(U_P \setminus A(P, P')/U_P\) where \(P \in P_J, P' \in P_{\mathfrak{A}(J, J)}\) and \(\text{pos}(P', P) = w\) (good position). Let \(L_t\) be a common Levi of \(P, P'\). Then \(C_t = \{g \in G^1; g L_t = L_t, \text{pos}(\tilde{g} P, \tilde{P}) = w\}\) is a connected component of \(N_G(L_t)\). Under the obvious bijection \(C_t \sim U_P \setminus A(P, P')/U_P\), the conjugation action of \(L_t\) on \(C_t\) corresponds to the conjugation action of \(P \cap P'\) on \(U_P \setminus A(P, P')/U_P\). Thus we obtain a canonical bijection between the set of \(G\)-orbits on \(t_r \mathcal{Z}_{J, \delta}\) and the set of \(L_t\)-conjugacy classes in \(C_t\) (a connected component of an algebraic group with identity component \(L_t\)). Putting together these bijections we obtain a bijection

\[G \setminus Z_{J, \delta} \leftrightarrow \bigcup_{t \in \mathcal{T}(J, \delta)} L_t \setminus C_t\]

where \(G \setminus Z_{J, \delta}\) is the set of \(G\)-orbits on \(Z_{J, \delta}\) and \(L_t \setminus C_t\) is the set of \(L_t\)-orbits on \(C_t\) (for the conjugation action).

4. **Parabolic Character Sheaves on \(Z_{J, \delta}\)**

4.1. Our notation for perverse sheaves follows \([BBD]\). For an algebraic variety \(X\) over \(k\) we write \(D(X)\) instead of \(D^b(X, \mathbb{Q}_l)\); \(l\) is a fixed prime number invertible in \(k\). If \(f: X \to Y\) is a smooth morphism with connected fibres of dimension \(d\), and
Let \( K \) be a perverse sheaf on \( Y \), we set \( f(K) = f'(K)[d] \), a perverse sheaf on \( X \). If \( K \in \mathcal{D}(X) \) and \( A \) is a simple perverse sheaf on \( X \) we write \( A \upharpoonright K \) instead of \( \text{“} A \text{”} \) is a composition factor of \( p^iH^i(K) \) for some \( i \in \mathbb{Z}^n \).

We preserve the setup of Section 3.1. Let \( J \subset I \). Let \( B^* \) be a Borel of \( G \) and let \( T \) be a maximal torus of \( B^* \). Let \( N^1 = \{ n \in G^1 \mid nTn^{-1} = T \} \). The map \( N^1 \to W, n \mapsto \text{pos}(B^*, nB^*) \) induces a bijection \( T\backslash N^1 \to \tilde{W} \); let \( N^1_\delta \) be the \( T \)-coset in \( N^1 \) corresponding to \( x \in W \). We choose \( \hat{x} \in N^1_\delta \). For \( x \in W \) we define a morphism of algebraic varieties \( \alpha_x : U_{B^*}N^1_\delta U_{B^*} \to N^1_x \) by \( \alpha_x(um) = n \) for \( u, u' \in U_{B^*}, n \in N^1_\delta \). Consider the diagram

\[
N^1_x \xrightarrow{\phi} \hat{Y}_x \xrightarrow{\rho} Y_x \xrightarrow{\pi} \mathbb{Z}_{J, \delta}
\]

where

\[
Y_x = \{(B, B', g) \in \mathbb{Z}_{\mathcal{S}, \delta} \mid \text{pos}(B, B') = x\},
\]

\[
\hat{Y}_x = \{(hU_{B^*}, g) \in G/U_{B^*} \times G^1 \mid h^{-1}gh \in U_{B^*}N^1_\delta U_{B^*}\},
\]

\[
\rho(hU_{B^*}, g) = (hB^*, \gamma^gB^*, g), \quad \phi(hU_{B^*}, g) = \alpha_x(h^{-1}gh),
\]

\[
\pi(B, B', g) = (P, P', gU_P), \quad B \subset P, \quad B' \subset P'.
\]

Let \( \mathcal{S}(T) \) be the set of isomorphism classes of \( \hat{Q}_T \)-local systems \( \mathcal{L} \) of rank 1 on \( T \) such that \( \mathcal{L}^\otimes m \cong \hat{Q}_L \) for some integer \( m \geq 1 \) invertible in \( k \). For \( \mathcal{L} \in \mathcal{S}(T) \) let \( W^1_{\mathcal{L}} = \{ x \in W \mid \text{Ad}(\hat{x})^*\mathcal{L} \cong \mathcal{L} \} \). (We have \( \text{Ad}(\hat{x}) : T \to T \).) Let \( \mathcal{L} \in \mathcal{S}(T), x \in W^1_{\mathcal{L}} \). The inverse image \( \mathcal{L}_x \) of \( \mathcal{L} \) under \( N^1_\delta \to T, n \mapsto \hat{x}^{-1}n \) is a \( T \)-equivariant local system on \( N^1_\delta \) for the conjugation action of \( T \) on \( N^1_\delta \). Now \( T \) acts on \( \hat{Y}_x \) by \( t : (x, g) \mapsto (xt^{-1}, g) \) and on \( Y_x \), trivially. The \( T \)-actions are compatible with \( \phi, \rho, \pi \). Since \( \mathcal{L}_x \) is \( T \)-equivariant, \( \phi^*\mathcal{L}_x \) is a \( T \)-equivariant local system on \( \hat{Y}_x \); it is of the form \( \rho^*\hat{\mathcal{L}} \) for a well defined local system \( \hat{\mathcal{L}} \) on \( Y_x \), since \( \rho \) is a principal \( T \)-bundle. (We will often denote the restriction of \( \hat{\mathcal{L}} \) to various subvarieties of \( Y_x \) again by \( \hat{\mathcal{L}} \).) We set

\[
K^\mathcal{L}_x = \pi_1\hat{\mathcal{L}} \in \mathcal{D}(\mathbb{Z}_{J, \delta}).
\]

Let \( \hat{\mathcal{L}} \) be the simple perverse sheaf on \( \mathbb{Z}_{\mathcal{S}, \delta} \) whose support is the closure of \( Y_x \) and whose restriction to \( Y_x \) is a shift of \( \hat{\mathcal{L}} \). Let \( \hat{K}^\mathcal{L}_x = \hat{\pi}_1\hat{\mathcal{L}} \) where \( \hat{\pi} : \mathbb{Z}_{\mathcal{S}, \delta} \to \mathbb{Z}_{J, \delta} \) is given by the same formula as \( \pi \).

4.2. Let \( x = (x_1, x_2, \ldots, x_r) \) be a sequence in \( W \), let \( x = x_1x_2\ldots x_r \) and let

\[
Y_x = \{(B_0, B_1, \ldots, B_r, g) \in B \times B \times \ldots \times B \times G^1 \mid \text{pos}(B_{i-1}, B_i) = x_i, \ i \in [1, r], \ B_r = gB_0\}.
\]

Define \( \pi_x : Y_x \to \mathbb{Z}_{J, \delta} \) by \( (B_0, B_1, \ldots, B_r, g) \mapsto (P, P', gU_P) \) where \( P \in \mathcal{P}_J, \ P' \in \mathcal{P}_{\mathcal{S}(J)} \) are given by \( B_0 \subset P, \ B_r \subset P' \). Let

\[
\hat{Y}_x = \{(h_0U_{B^*}, h_1B^*, \ldots, h_rB^*, g) \in G/U_{B^*} \times G/B^* \times \ldots \times G/B^* \times G^1 \mid h^{-1}_i h_i \in B^* x_i B^* \text{ for } i \in [1, r], \ h^{-1}_r g h_0 \in \mathcal{N}_{G^1}(B^*)\}.
\]
Define $\rho_\mathbf{x}: \hat{Y}_\mathbf{x} \to Y_{\mathbf{x}}$, $\phi_\mathbf{x}: \hat{Y}_\mathbf{x} \to N^1_\mathbf{x}$ by

$$
\rho_\mathbf{x}(h_0 B^r, h_1 B^r, \ldots, h_r B^r, g) = (h_0 B^r, h_1 B^r, \ldots, h_r B^r, g),
$$

$$
\phi_\mathbf{x}(h_0 B^r, h_1 B^r, \ldots, h_r B^r, g) = n_1 n_2 \ldots n_r n
$$

where $n_i \in N_G(T)$ is given by $h_{i-1}^{-1} h_i \in U_{B^r} n_i U_{B^r}$ and $n \in N^1 \cap N_G(T^+)$ is given by $h_r^{-1} g h_0 \in U_{B^r} n$. Now $\phi_\mathbf{x}$ is $T$-equivariant where $T$ acts on $\hat{Y}_\mathbf{x}$ by

$$
t: (h_0 B^r, h_1 B^r, \ldots, h_r B^r, g) \mapsto (h_0 t^{-1} U_{B^r}, h_1 B^r, \ldots, h_r B^r, g).
$$

Hence, if $L \in S(T)$ and $x \in W^1_L$ then $\phi_\mathbf{x}^* (L_\mathbf{x})$ is $T$-equivariant (see Section 4.1); it is of the form $p_\mathbf{x}^* \hat{L}$ for a well defined local system $\hat{L}$ on $Y_{\mathbf{x}}$, since $\rho_\mathbf{x}$ is a principal $T$-bundle. We set

$$
K^\xi_\mathbf{x} = (\pi_\mathbf{x})_! \hat{L} \in D(Z_{I, \delta}).
$$

In the case where $\mathbf{x}$ reduces to a single element $x$, we have clearly $K^\xi_\mathbf{x} = K^\xi_x$. In general, $Y_\mathbf{x}$ is smooth and connected. An equivalent statement is that

$$
\{(h_0, h_1, \ldots, h_r, g) \in G^{r+1} \times G^1; h_{i-1}^{-1} h_i \in B^r x_i B^r \ (i \in [1, r]), \ h_r^{-1} g h_0 B^r = B^r h_r^{-1} g h_0\}
$$

is smooth and connected. By the substitution $n = h_r^{-1} g h_0$, $h_{i-1}^{-1} h_i = y_i$, $i \in [1, r]$, we are reduced to the statement that

$$
\{(h_0, y_1, y_2, \ldots, y_r, n) \in G^{r+1} \times G^1; n B^r = B^r, y_i \in B^r x_i B^r\}
$$

is smooth and connected, which is clear.

4.3. Let $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ be a sequence in $I \cup \{1\}$ and let $L \in S(T)$ be such that $x_1 x_2 \ldots x_r \in W^1_L$. Let

$$
\hat{Y}_\mathbf{x} = \{(B_0, B_1, \ldots, B_r, g) \in B \times B \times \ldots \times B \times G^1; \ \text{pos}(B_{i-1}, B_i) \in \{x_i, 1\}, i \in [1, r], B_r = g B_0\}.
$$

Let $Y'_\mathbf{x}$ (resp. $Y''_\mathbf{x}$) be the set of all $(B_0, B_1, \ldots, B_r, \gamma)$ where $(B_0, B_1, \ldots, B_r) \in B^{r+1}$, $\gamma \in G^1/U_P$ (with $P \in \mathcal{P}_J$ given by $B_0 \subset P$) such that $(B_0, B_1, \ldots, B_r, g)$ is in $\hat{Y}_\mathbf{x}$ (resp. in $Y_\mathbf{x}$) for some/any $g \in \gamma$. Now $Y''_\mathbf{x}$ is an open dense smooth subset of $Y'_\mathbf{x}$. The local system $\hat{L}$ on $Y_\mathbf{x}$ is the inverse image under $Y_\mathbf{x} \to Y''_\mathbf{x}$ (an affine space bundle) of a local system on $Y'_\mathbf{x}$ denoted again by $\hat{L}$. Let $IC(Y''_\mathbf{x}, \hat{L})$ be the corresponding intersection cohomology complex on $Y''_\mathbf{x}$. Define $\pi_\mathbf{x}: Y''_\mathbf{x} \to Z_{I, \delta}$ by $(B_0, B_1, \ldots, B_r, \gamma) \mapsto (P, P', \gamma)$ where $P \in \mathcal{P}_J$, $P' \in \mathcal{P}_{J(J)}$, are given by $B_0 \subset P$, $B_r \subset P'$. We set

$$
K^\xi_\mathbf{x} = (\pi_\mathbf{x})_! IC(Y'_\mathbf{x}, \hat{L}) \in D(Z_{I, \delta}).
$$

Since $\pi$ is proper, we may apply the decomposition theorem [BBD] and we see that $K^\xi_\mathbf{x}$ is a semisimple complex on $Z_{I, \delta}$. 
Proposition 4.4. Let $\mathcal{L} \in \mathcal{S}(T)$ and let $A$ be a simple perverse sheaf on $Z_{J,\delta}$. The following conditions on $A$ are equivalent:

(i) $A \cong K_x^L$ for some $x \in W_L^1$.
(ii) $A \cong K_x^L$ for some sequence $x = (x_1, x_2, \ldots, x_r)$ in $W$ with $x_1x_2\ldots x_r \in W_L^1$.
(iii) $A \cong K_x^L$ for some sequence $x = (x_1, x_2, \ldots, x_r)$ in $I \cup \{1\}$ with $x_1x_2\ldots x_r \in W_L^1$.
(iv) $A \cong K_x^L$ for some $x \in W_L^1$.

In the case where $G^1 = G$, the proof follows word by word that in [L3, 2.11–2.16], [L4, 12.7]. The general case can be treated in a similar way.

4.5. Let $C_{J,\delta}^L$ be the set of (isomorphism classes) of simple perverse sheaves on $Z_{J,\delta}$ which satisfy the equivalent conditions 4.4(i)–(iv) with respect to $\mathcal{L}$. The simple perverse sheaves on $Z_{J,\delta}$ which belong to $C_{J,\delta}^L$ for some $\mathcal{L} \in \mathcal{S}(T)$ are called parabolic character sheaves; they (or their isomorphism classes) form a set $C_{J,\delta}$. In particular, the notion of parabolic character sheaf on $Z_{J,\delta} = G^1$ is well defined; we thus recover the definition of character sheaves in [L3], [L6].

We describe the set $C_{\varnothing,\delta}$. For $x$, $\mathcal{L}$ as in Section 4.1, the simple perverse sheaf $\hat{\mathcal{L}}_x$ on $Z_{\varnothing,\delta}$ (see Section 4.1) is a shift of the inverse image under the obvious map $Z_{\varnothing,\delta} \to Z_{\varnothing,\delta}$ of a well defined simple perverse sheaf on $Z_{\varnothing,\delta}$. These simple perverse sheaves on $Z_{\varnothing,\delta}$ constitute $C_{\varnothing,\delta}$.

4.6. Let $t = (J_n, w_n)_{n \geq 0} \in T(J, \delta)$. For $r \gg 0$ we have $J_r = J_{r+1} = \ldots$, and $w_r = w_{r+1} = \cdots = w$. For such $r$ we define a class $C_{t,\delta}^L$ of simple perverse sheaves on $t^*Z_{J,\delta}$. Let $P \in \mathcal{P}_{J_r}$, $P' \in \mathcal{P}_{\delta(J_r)}$ and $\operatorname{pos}(P', P) = w$ (good position). Let $L$ be a common Levi of $P'$, $P$. Then $C = \{g \in G^1; \; gL = L, \; gP = P'\}$ is a connected component of $N_G^L(L)$. Let $X$ be a character sheaf on $C$ (the definition in Section 4.5 is applicable since $C$ is a connected component of an algebraic group with identity component $L$). Now $X$ is $L$-equivariant for the conjugation action of $L$ hence also $P \cap P'$ equivariant where $P \cap P'$ acts via its quotient $(P \cap P')/U_{P\cap P'} = L$. Hence there is a well defined simple perverse sheaf $X'$ on $G \times_{P \cap P'} C$ (here $P \cap P'$ acts on $G$ by right translation) whose inverse image under $G \times C \to G \times P \cap P'$ is a shift of the inverse image of $X$ under $pr_2: G \times C \to C$. We may regard $X'$ as a simple perverse sheaf on $t^*Z_{J,\delta}$ via the isomorphism

$$G \times_{P \cap P'} C \xrightarrow{\sim} t^*Z_{J,\delta}, \quad (g, c) \mapsto (gP, \; gU_{P'}, \; cU_Pg^{-1}).$$

Now let $\vartheta: t^*Z_{J,\delta} \to t^*Z_{J,\delta}$ be a composition of maps in Lemma 3.13(a); thus $\vartheta$ is smooth with connected fibres. Then $\tilde{X} = \vartheta(X')$ is a simple perverse sheaf on $t^*Z_{J,\delta}$.

Let $t^*Z_{J,\delta}$ be the class of simple perverse sheaves on $Z_{J,\delta}$ whose support is the closure in $Z_{J,\delta}$ of $\operatorname{supp} \tilde{X}$ and whose restriction to $t^*Z_{J,\delta}$ is $\tilde{X}$.

Let $C_{t,\delta}^L$ be the class of simple perverse sheaves on $t^*Z_{J,\delta}$ consisting of all $\tilde{X}$ as above. (It is independent of the choice of $r$.) Let $C_{J,\delta}^L$ be the class of simple perverse sheaves on $Z_{J,\delta}$ consisting of all $\tilde{X}$ as above. The set of isomorphism classes of objects in $C_{J,\delta}^L$ is in bijection with the set of pairs $(t, X)$ where $t \in T(J, \delta)$ and $X$ is a character sheaf on $C$ (as above).
4.7. We fix $b \in W$. Let $V$ be a locally closed subvariety of $Y_b$ (see Section 4.1). For $x, z \in W$ let

$$X_{x,z} = \{(\tilde{B}, B, \tilde{B}', B', g) \in B^4 \times G^4; (\tilde{B}, B, g) \in V, \; \; \tilde{B} = B', \; \; \text{pos}(\tilde{B}, B) = x, \; \; \text{pos}(B, \tilde{B}') = z, \; \; \text{pos}(B', B') = \delta(x)\}.$$

Define $\kappa: X_{x,z} \to V$ by $\kappa(\tilde{B}, B, \tilde{B}', B', g) = (\tilde{B}, B, g)$. Let $\mathcal{L} \in \mathcal{S}(T)$ be such that $z\delta(x) \in W^1_1$. The inverse image of the local system $\tilde{\mathcal{L}}$ on $Y_{x,\delta(x)}$ (see Section 4.2) under

$$X_{x,z} \to Y_{z,\delta(x)}, \quad (\tilde{B}, B, \tilde{B}', B', g) \mapsto (\tilde{B}, B, \tilde{B}', B', g)$$

is denoted again by $\tilde{\mathcal{L}}$.

**Lemma 4.8.** Let $m: V \to Z$ be a morphism of varieties. Let $A$ be a simple perverse sheaf on $Z$ such that $A \dashv (m/_{Z})_{!}\tilde{\mathcal{L}}$. Then there exists $\mathcal{L}_1 \in \mathcal{S}(T)$ such that $b \in W^1_1$ and such that $A \dashv m_{!}\mathcal{L}_1$. Here $\mathcal{L}_1$ is the local system on $Y_b$ (or its restriction to $V$) defined as in Section 4.1 for $b, \mathcal{L}_1$ instead of $x, \mathcal{L}$.

For simplicity we assume that $\mathcal{L} = \tilde{\mathcal{Q}}$. We argue by induction on $l(z)$. If $l(z) = 0$ then $x = b$, $\kappa$ is an isomorphism and the result is obvious. Assume now that $l(z) > 0$. We can find $s \in W$ such that $l(s) = 1, l(z) > l(sz)$.

Assume first that $l(xs) = l(x) + 1$. Consider the isomorphism

$$\iota: X_{x,z} \xrightarrow{\sim} X_{xs,sz}, \quad (\tilde{B}, B, \tilde{B}', B', g) \mapsto (\tilde{B}, B_1, \tilde{B}', B'_1, g)$$

where $B_1 \in B$ is given by

$$(a) \quad \text{pos}(B, B_1) = s, \quad \text{pos}(B_1, \tilde{B}') = sz$$

and $B'_1 = \gamma B_1$. We have $\kappa = \kappa'$ where $\kappa': X_{xs,sz} \to V$ is given by

$$\kappa'(\tilde{B}, B_1, \tilde{B}', B'_1, g) = (\tilde{B}, B', g).$$

Now $\nu_{\tilde{\mathcal{Q}}} = \tilde{\mathcal{Q}}$ and $A \dashv (m\kappa')_{!}\nu_{\tilde{\mathcal{Q}}}$. Using the induction hypothesis for $xs, sz$ instead of $x, z$, we get the conclusion of the lemma.

Assume next that $l(xs) = l(x) - 1$. Then we have a partition $X_{x,z} = X'_{x,z} \sqcup X''_{x,z}$ where $X'_{x,z}$ is the open subset defined by $\text{pos}(\tilde{B}, B_1) = x$ (and $B_1$ is given by (a)) and $X''_{x,z}$ is the closed subset defined by $\text{pos}(\tilde{B}, B_1) = xs$ (and $B_1$ is given by (a)).

Let $j' = \kappa|x'_{x,z}, j'' = \kappa'|x''_{x,z}$. By general principles, either

(b) $A \dashv (mj')_{!}\tilde{\mathcal{Q}}$, or

(c) $A \dashv (mj'')_{!}\tilde{\mathcal{Q}}$.

Assume that (c) holds. We have $j'' = \kappa'' \iota''$ where $\kappa'': X_{xs,sz} \to V$ is given by $\kappa''(\tilde{B}, B_1, \tilde{B}', B'_1, g) = (\tilde{B}, B', g)$ and

$$\iota'': X''_{x,z} \to X_{xs,sz}, \quad (\tilde{B}, B, \tilde{B}', B', g) \mapsto (\tilde{B}, B_1, \tilde{B}', B'_1, g)$$

(where $B_1, B'_1$ are as in (a)) is a line bundle; also $\iota''_{!}\tilde{\mathcal{Q}} = \tilde{\mathcal{Q}}$ (up to shift). Hence $A \dashv (m\kappa'')_{!}\tilde{\mathcal{Q}}$. Using the induction hypothesis for $xs, sz$ instead of $x, z$, we get the conclusion of the lemma.
Assume now that (b) holds. We have \( j' = \kappa' \ell' \) where \( \kappa': X_{x,sz} \to V \) is given by \( \kappa'(B, B_1, B', B'_1, g) = (\bar{B}, B', g) \) and

\[
\ell': X'_{x,z} \to X_{x,sz}, \quad (\bar{B}, B, \bar{B}', B', g) \mapsto (\bar{B}, B_1, B', B'_1, g)
\]

(with \( B_1, B'_1 \) as in (a)) makes \( X'_{x,z} \) into the complement of the zero section of a line bundle over \( X_{x,sz} \). Hence we have an exact triangle consisting of \( \ell_! \tilde{Q}_1, \tilde{Q}_2 \) and a shift of \( \tilde{Q}_2 \). Hence \( A \cong (mk')_! \tilde{Q}_2 \). Using the induction hypothesis for \( x, sz \) instead of \( x, z \), we get the conclusion of the lemma. The lemma is proved.

4.9. Let \( t = (J_n, w_n)_{n \geq 0} \in T(J, \delta) \). Let \( Y_t \) be the set of all \((B, B', g) \in Z_{\mathcal{Z}, \delta}\) such that, if \( P \in \mathcal{P}_J, P' \in \mathcal{P}_{\delta(J)} \) are given by \( B \subset P, B' \subset P' \), then \((P, P', g) \in t \mathcal{Z}_{J,\delta}\). We have morphisms

\[
\xi_t: Y_t \to t^* \mathcal{Z}_{J,\delta}, \quad (B, B', g) \mapsto (P, P', g),
\]

\[
\xi'_t: Y_t \to t^* \mathcal{Z}_{J,\delta}, \quad (B, B', g) \mapsto (P, P', gU_P).
\]

For \( a \in W \) let \( Y_{t,a} = \{(B, B', g) \in Y_t; \text{pos}(B, B') = a \} \). Let \( \xi_{t,a} = \xi_t|_{Y_{t,a}}, \xi'_{t,a} = \xi'_t|_{Y_{t,a}} \).

**Lemma 4.10.** Let \( A' \) be a simple perverse sheaf on \( t^* \mathcal{Z}_{J,\delta} \). Let \( \mathcal{L} \in \mathcal{S}(T) \) be such that \( a \in W \). Assume that \( A' \cong (\xi'_{t,a})_! \tilde{\mathcal{L}} \) (notation of Section 4.1.) Then there exist \( \mathcal{L}_1 \in \mathcal{S}(T), b \in W \subset \mathcal{L}_1 \) such that \( A' \cong \vartheta^*(\xi_{t,b})_! \mathcal{L}_1 \) \((\vartheta^*: t^* \mathcal{Z}_{J,\delta} \to t^* \mathcal{Z}_{J,\delta}\) as in Lemma 3.12(b)).

It suffices to prove the following variant of the lemma.

\((*)\) Let \( A \) be a simple perverse sheaf on \( t^* \mathcal{Z}_{J,\delta} \) such that \( A \cong (\xi_{t,a})_! \tilde{\mathcal{L}} \). Then there exist \( \mathcal{L}_1 \in \mathcal{S}(T), b \in W \subset \mathcal{L}_1 \) such that \( A \cong \vartheta^*(\xi_{t,b})_! \mathcal{L}_1 \) \((\vartheta^*: t^* \mathcal{Z}_{J,\delta} \to t^* \mathcal{Z}_{J,\delta}\) as in Lemma 3.12(a)).

Define \( f: Y_{t,a} \to Y_{t,b} \) by \( f(B, B', g) = (g^{-1}(\tilde{P}B'), \tilde{P}B') \) where \( \xi_{t,a}(B, B', g) = (P, P', g) \) and \( \tilde{P}' \) is the restriction of \( L \). We have a partition \( Y_{t,a} = \bigsqcup_{b \in W} Y_{t,a,b} \). Setting \( Y_{t,a,b} = f^{-1}(Y_{t,b}) \) we get a partition \( Y_{t,a} = \bigsqcup_{b \in W} Y_{t,a,b} \) into locally closed subvarieties. Let \( \xi_{t,a,b}: Y_{t,a,b} \to t^* \mathcal{Z}_{J,\delta} \) be the restriction of \( \xi_{t,a} \). If \( A \) is as in (*) then, by general principles, \( A \cong (\xi_{t,a,b})_! \tilde{\mathcal{L}} \) for some \( b \in W \). We have \( \vartheta'^*(\xi_{t,a,b})_! \tilde{\mathcal{L}} \). Hence, if \( f_b: Y_{t,a,b} \to Y_{t,b} \) is the restriction of \( f \), we have \( \vartheta'^*(\xi_{t,a,b})_! = \xi_{t,b,f_b} \) hence \( \vartheta'^*(\xi_{t,a,b})_! = \vartheta'^{-1} \xi_{t,b,f_b} \). Thus, \( A \cong (\vartheta'^{-1})_!(\xi_{t,b,f_b})_! \tilde{\mathcal{L}} \) and

\[
\vartheta'^*(A) \cong (\xi_{t,b,f_b})_! \tilde{\mathcal{L}}.
\]

We can write uniquely \( a = a_2a_1 \) where \( a_1 \in W \), \( a_2 \in W \). We show that for \( (B, B', g) \in Y_{t,b} \), we have

\[(b) \quad f_b^{-1}(B, B', g) = \{(B, B', g); B, B' \in B, \delta B = B', \text{pos}(B, B') = a_2, \text{pos}(\bar{B}, B) = \delta^{-1}(a_1)\}.
\]

Assume first that \( (B, B', g) \in f_b^{-1}(B, B', g) \). Define \( P \in \mathcal{P}_J, P' \in \mathcal{P}_{\delta(J)} \) by \( B \subset P, B' \subset P' \). Set \( \bar{P}' = P \cdot P' \). We know that \( \text{pos}(B, B') = a, \bar{B}' = \bar{P}' \). We
have \( \text{pos}(P^B, B') \in W_{\delta}(J) \) (both \( B' \), \( P^B \) are contained in \( P' \)) and \( \text{pos}(B, P^B) = \text{pos}(B, P') \in W_{\delta}(J) \). We have automatically \( \text{pos}(P^B, B') = a_1 \), \( \text{pos}(B, P^B) = a_2 \).

It also follows that \( \text{pos}(y^{-1}(P^B), B) = \delta^{-1}(a_1) \). We show that \( P^B = \tilde{P}^B \). We have \( P' \cap B = P' \cap P \cap B \subset P \cap B \). Also \( U_{P'} \subset U_{P'} \) (since \( \tilde{P}' \subset P' \)). Hence \((P' \cap B)U_{P'} \subset (\tilde{P}' \cap B)U_{P'} \), that is \( \tilde{P}^B \subset P^B \); since \( \tilde{P}^B \), \( P^B \) are Borels, we have \( P^B = \tilde{P}^B = B' \). We see that \( \text{pos}(B, \tilde{B}') = a_2 \), \( \text{pos}(y^{-1}(\tilde{B}'), B) = \delta^{-1}(a_1) \).

Thus, \((B, B', g)\) belongs to the right hand side of (b).

Conversely, assume that \((B, B', g)\) belongs to the right hand side of (b). Since \( l(a_2a_1) = l(a_2) + l(a_1) \) we have \( \text{pos}(B, B') = a_2a_1 = a \). Define \( P \in \mathcal{P}_J \), \( P' \in \mathcal{P}_{\delta}(J) \) by \( B \subset P, B' \subset P' \). Set \( \tilde{P}' = P^P \). As in the earlier part of the proof we see that \( \text{pos}(P^B, B') \in W_{\delta}(J) \), \( \text{pos}(B, P^B) \in W_{\delta}(J) \) and \( P^B = \tilde{P}^B \). This forces \( P^B = \tilde{B}' \) hence \( \tilde{B}' = \tilde{P}^B \). Thus, \((B, B', g) \in f_b^{-1}(\tilde{B}, \tilde{B}', g)\), proving (b).

We see that we may identify \( Y_{t,a,b} \) with \( X_{t,\delta^{-1}(a_1),a_2} \) as defined in Section 4.7 relative to \( V = Y_{t,b} \). Moreover, \( J, \mathcal{P}_J \) may be identified with \( \kappa \) in Section 4.7.

Now \( \vartheta(A) \) (a simple perverse sheaf on \( tZ_{J,\delta} \)) satisfies the hypothesis of Lemma 4.8 with \( Z = tZ_{J,\delta} \) and \( m = \xi_{t,b} \) (see (a)). As in Lemma 4.8, there exists \( \mathcal{L}_1 \in \mathcal{S}(T) \) such that \( b \in W_{\mathcal{L}_1} \) and such that \( \vartheta(A) \triangleright (\xi_{t,b})_{\mathcal{L}_1} \). Since \( \vartheta' \) is an isomorphism, we see that \( A \triangleright \vartheta'(\xi_{t,b})_{\mathcal{L}_1} \). Since (*) is proved. The lemma is proved.

**Lemma 4.11.** Let \( \mathcal{L} \in \mathcal{S}(T) \) and let \( a \in W^1_{\mathcal{L}} \). Let \( t = (J_n, w_n)_{n \geq 0} \in \mathcal{T}(J, \delta) \). Let \( \xi_{t,a} : Y_{t,a} \to \mathcal{Z}_{J,\delta} \) be as in Section 4.9. Then any composition factor of \( \bigoplus_i P^{'H'_i}((\xi'_{t,a})_{\mathcal{L}}) \) belongs to \( C'_{t,\delta} \).

More generally we show that the lemma holds whenever \( J, \mathcal{P}_J \) are replaced by \( J_n, \mathcal{P}_{J_n} \), \( n \geq 0 \). First we show:

(a) if the result is true for \( n = 1 \), then it is true for \( n = 0 \).

Let \( A' \) be a composition factor of \( \bigoplus_i P^{'H'_i}((\xi'_{t,a})_{\mathcal{L}}) \). Let \( b, \mathcal{L}_1 \) be as in Lemma 4.10. Then \( A' \triangleright \vartheta'((\xi'_{t,b})_{\mathcal{L}_1}) \) hence \( A' \triangleright \vartheta\bigoplus_i P^{'H'_i}((\xi'_{t,b})_{\mathcal{L}_1}) \). Since \( \vartheta \) is an affine space bundle, there exists a composition factor \( A'' \) of \( \bigoplus_i P^{'H'_i}((\xi'_{t,b})_{\mathcal{L}_1}) \) such that \( A' = \tilde{\vartheta}A'' \). By our hypothesis we have \( A'' \in C'_{t,\delta} \). From the definitions we have \( \tilde{\vartheta}A'' \in C'_{t,\delta} \). Thus, (a) holds.

Similarly, if the result holds for some \( n \geq 1 \), then it holds for \( n - 1 \). (The proof is the same as for \( n = 1 \).) In this way we see that it suffices to prove the result for \( n \gg 0 \). Thus, we may assume that \( J_0 = J_1 = \cdots = J \) and \( w_0 = w_1 = \cdots = w \). We can write uniquely \( a = a_2a_1 \), where \( a_1 \in W_{\delta}(J) \), \( a_2 \in \mathcal{P}_{\delta}(J) \). We have \( w_0 \in \delta(J)W \) and \( a \in W_jw^{-1}W_{\delta}(J) = w^{-1}W_{\delta}(J) \). Thus, \( a_2 = w^{-1} \).

Let \( P \in \mathcal{P}_J \), \( P' \in \mathcal{P}_{\delta}(J) \) be such that \( \text{pos}(P', P) = w \). Let \( L \) be a common Levi of \( P', P \). We may assume that \( T \subset L \). Let \( C = \{ c \in G^1; c^L = L, cP = P \} \). Let \( Y' \) be the set of all \( (\beta, \beta', c) \) where \( \beta, \beta' \) are Borels of \( L \) such that \( \text{pos}(\beta, \beta') = a_1 \) (position relative to \( L \) with Weyl group \( W_{\delta}(J) \)) and \( c \in C \) satisfies \( c^L \beta = \beta' \).

Then \( P \cap P' \) acts on \( Y' \times U_{P'} \) by \( p : (\beta, \beta', c, w) \mapsto (\{ \beta, l\beta', l^c, p \} u) \) where \( l \in L \).
$p \in lU_{P \cap P'}$. We have a commutative diagram
\[ G \times_{P \cap P'} (Y' \times U_{P'}) \sim Y_{t,a} \]
\[ G \times_{P \cap P'} C \sim \zeta_{t,a} \]
where the upper horizontal map is $(g, \beta, \beta', c, u) \mapsto (gB, gB', g(cu))$ with $B' = \beta'U_{P'}$, $B = \beta U_{P'}$, and the lower horizontal map is $(g, \beta, \beta', c, u) \mapsto (g, c)$. This commutative diagram shows that any composition factor of $\bigoplus_i \chi_i^{\mathcal{P}(H)}(|\mathcal{L}_{t,a}\mathcal{L})$ is of the form $X'$ where $X$ is a character sheaf on $C$ (notation of Section 4.6); hence it is in $C_{t,\delta}$. The lemma is proved.

**Lemma 4.12.** For $A \in C_{J,\delta}$, $t \in T(J, \delta)$, we set $\tau A = A|_{Z_{J,\delta}}$. Then any composition factor of $\bigoplus_i \chi_i^{\mathcal{P}(H)}(\tau A)$ belongs to $C_{t,\delta}$.

We can find $\mathcal{L} \in \mathcal{S}(T)$ and $x = (x_1, x_2, \ldots, x_r)$ as in Section 4.3 such that $x_1x_2\ldots x_r \in W_{L}$ and $A \in \mathcal{K}_L$ (see Sections 4.3, 4.5). Since $\mathcal{K}_L$ is semisimple, we have $K_L \cong A[m] \oplus K'$ for some $K' \in \mathcal{D}(Z_{J,\delta})$ and some $m \in \mathbb{Z}$. Hence $K_L|_{Z_{J,\delta}} \cong A[m] \oplus K_1'$ for some $K_1' \in \mathcal{D}(Z_{J,\delta})$. It suffices to show that, if $A'$ is a composition factor of $\bigoplus_i \chi_i^{\mathcal{P}(H)}(\mathcal{K}_L|_{Z_{J,\delta}})$, then $A' \in C_{t,\delta}$. As in [L3, 2.11–2.16] we see that there exists $\mathcal{L} \in \mathcal{S}(T)$, $x \in W_L$ such that $A' = K_L|_{Z_{J,\delta}}$. Using Lemma 4.11 we have $A' \in C_{t,\delta}$. The lemma is proved.

**Lemma 4.13.** If $A \in C_{J,\delta}$, then $A \in C_{t,\delta}$.

Since $Z_{J,\delta} = \bigcup_{t \in T(J, \delta)} Z_{t,\delta}$, we can find $t \in T(J, \delta)$ such that $\text{supp}(A) \cap Z_{t,\delta}$ is open dense in $\text{supp}(A)$. Then $\tau A = A|_{Z_{t,\delta}}$ is a simple perverse sheaf on $Z_{J,\delta}$ and $\tau A \in C_{t,\delta}$ (Lemma 4.12). Now $A, \tau A$ are related just as $X, \bar{X}$ are related in Section 4.6. Hence $A \in C_{t,\delta}$. The lemma is proved.

**Lemma 4.14.** Let $t = (J_n, w_n)_{n \geq 0} \in T(J, \delta)$. Define $d_n \in W^{\delta(J)}$ by $a^{-1} = w_n$ for $n \gg 0$. Let $b \in W_{\delta(J_n)}$ (see Section 2.6). Let $(B, B', g) \in Y_{ab}$. Define $P \in \mathcal{P}, P' \in \mathcal{P}_{\delta(J)}$ by $B \subset P, B' \subset P'$.

(a) We have $(P, P', g) \in I_{Z_{J,\delta}}$.
(b) If $b = 1, (B, B', g) \mapsto (P, P', g)$ is a surjective map $Y_{ab} \to I_{Z_{J,\delta}}$.

We prove (a). Recall that $w_n = \min(W_{\delta(J)}a^{-1}W_{J_n}) = \min(W_{\delta(J)}b^{-1}a^{-1}W_{J_n})$ for $n \geq 0$. In particular, $\text{pos}(P', P) = \min(W_{\delta(J)}b^{-1}a^{-1}W_{J_n}) = w_0$. Define $P^n, P'^n$ in terms of $(P, P', g)$ in Section 3.1.1. We have $P^1 = P^1 \in \mathcal{P}_{\delta(J_1)}$, $P^1 = P_{J_1}$. As in the proof of Lemma 4.10 we have $ab = a_2a_1, \text{pos}((P^1)^B, B') = a_1 \in W_{\delta(J_1)}$, $a_2 \in W^{\delta(J)}$. Since $a \in W^{\delta(J)}$, we have $a_1 = b \in W_{\delta(J_n)}$. Thus, $\text{pos}((P^1)^B, B') \in W_{\delta(J)}$ hence $B' \subset P^1$ and $B \subset P^1$. We have
\[ \text{pos}(P^1, P^1) = \min(W_{\delta(J_1)}|_{\text{pos}(B', B)}W_{J_1}) = \min(W_{\delta(J_1)}b^{-1}a^{-1}W_{J_1}) = w_1. \]

By the same argument for $B, B', P^1, P^1, g, t_1$ instead of $B, B', P, P', g, t$, we see that $\text{pos}(P^2, P^2) = w_2$ and $B \subset P^2, B' \subset P^2$. (We have $a^{-1} \in \delta(J_1)W$...
since $\delta(J_1) \subset \delta(J)$.) Continuing in this way, we find $\text{pos}(P'^n, P^n) = w_n$ and $B \subset P^n$, $B' \subset P^n$ for all $n \geq 0$. In particular, $(P, P', g) \in \mathcal{t}Z_{J, \delta}$. This proves (a).

We prove (b). Assume that $(P, P', g) \in \mathcal{t}Z_{J, \delta}$. Define $P'^n, P^n$ in terms of $(P, P', g)$ as in Section 3.11. By assumption, if $n \gg 0$ we have $\text{pos}(P'^n, P^n) = a^{-1}$ (good position). Hence

$$\text{pr}_1: \{ (B, B') \in B \times B; B' \subset P^n, B \subset P^n, \text{pos}(B, B') = a \} \rightarrow \{ B \in B; B \subset P^n \}$$

is an isomorphism with inverse $B \mapsto (B, (P'^n)_B)$. The condition that $(B, B')$ in the domain of $\text{pr}_1$ satisfies $B' = gB$ is that $B$ is fixed by the map $B \mapsto g^{-1}(\{P'^n\}_B)$ of the flag manifold of $P^n$ into itself. This map has at least one fixed point. Hence there exist Borels $B \subset P^n$, $B' \subset P^n$ such that $B' = gB$ and $\text{pos}(B, B') = a$. We then have $B \subset P$, $B' \subset P'$, proving (b).

**Lemma 4.15.** Let $t = (J_n, w_n)_{n \geq 0} \in \mathcal{T}(J, \delta)$. Define $a_2 \in W^{\delta(J)}$ by $w_n = a_2^{-1}$ for $n \gg 0$. Let $a_1 \in W_{\delta(J_\infty)}$ and let $a = a_2a_1$, $a' = \delta^{-1}(a_1)a_2$. Let $L' \in \mathcal{S}(T)$ be such that $a' \in W^1_{L'}$. Let $A'$ be a simple perverse sheaf on $\mathcal{t}Z_{J_1, \delta}$ such that $A' = (\xi_{t_1, a'})^*L'$. There exists $L \in \mathcal{S}(T)$ such that $a \in W^1_L$ and $\vartheta A'$ (a simple perverse sheaf on $\mathcal{t}Z_{J, \delta}$) satisfies $\vartheta A' = (\xi_{t_0, a})^*L$.

It suffices to prove the following variant of the lemma.

(*) Let $t = (J_n, w_n)_{n \geq 0} \in \mathcal{T}(J, \delta)$. Define $a_2 \in W^{\delta(J)}$ by $w_n = a_2^{-1}$ for $n \gg 0$. Let $a_1 \in W_{\delta(J_\infty)}$ and let $a = a_2a_1$, $a' = \delta^{-1}(a_1)a_2$. Let $L' \in \mathcal{S}(T)$ be such that $a' \in W^1_{L'}$. Let $A$ be a simple perverse sheaf on $\mathcal{t}Z_{J_1, \delta}$ such that $A = (\xi_{t_1, a'})^*L'$. There exists $L \in \mathcal{S}(T)$ such that $a \in W^1_L$ and $\vartheta^* A$ (a simple perverse sheaf on $\mathcal{t}Z_{J, \delta}$) satisfies $\vartheta^* A = (\xi_{t_0, a})^*L$.

Define $f: Y_{t, a} \rightarrow Y_{t_1}$ as in the proof of Lemma 4.10. Assume that $f(B, B', g) = (B, \bar{B}', g)$. As in the proof of Lemma 4.10 we have

$$\text{pos}(ar{B}, B) = \delta^{-1}(a_1), \quad \text{pos}(B, \bar{B'}) = a_2, \quad \text{pos}(\bar{B'}, B') = a_1.$$  

By Section 2.6 we have $\text{Ad}(a_2^{-1})J_\infty = \delta(J_\infty)$ hence $a_2^{-1}\delta^{-1}(a_1)a_2 \in W_{\delta(J_\infty)}$ and $l(a_2^{-1}\delta^{-1}(a_1)a_2) = l(\delta^{-1}(a_1))$; since $a_2 \in W^{\delta(J)} \subset W^{\delta(J_\infty)}$, we have

$$l(\delta^{-1}(a_1)a_2) = l(a_2(a_2^{-1}\delta^{-1}(a_1)a_2)) = l(a_2) + l(a_2^{-1}\delta^{-1}(a_1)a_2) = l(a_2) + l(\delta^{-1}(a_1)).$$

Hence pos$(\bar{B}, \bar{B'}) = \delta^{-1}(a_1)a_2 = a'$. We see that $f$ defines a map $f': Y_{t, a} \rightarrow Y_{t_1, a'}$. Define $f''': Y_{t, a'} \rightarrow Y_{t, a}$ by $f'''(\bar{B}, \bar{B'}, g) = (B, B', g)$ with $B \in B$ given by pos$(\bar{B}, B) = \delta^{-1}(a_1)$, pos$(B, \bar{B'}) = a_2$ and $B' = gB$. (From pos$(\bar{B}, B') = a_2$, pos$(\bar{B'}, B') = a_1$ and $l(a_2a_1) = l(a_2) + l(a_1)$ we deduce that pos$(B, B') = a_2a_1$.) From the proof of Lemma 4.10 we see that $f'''$ is an inverse to $f'$. From the definitions we have a commutative diagram

$$
\begin{array}{ccc}
Y_{t, a} & \xrightarrow{f} & Y_{t, a'} \\
\downarrow{\xi_{t, a}} & & \downarrow{\xi_{t_1, a'}} \\
\mathcal{t}Z_{J, \delta} & \xrightarrow{\vartheta'} & \mathcal{t}Z_{J_1, \delta}
\end{array}
$$

This proves (*) since $f'$ and $\vartheta'$ are isomorphisms. The lemma is proved.
Lemma 4.16. Let \( t = (J_n, w_n)_{n \geq 0} \in T(J, \delta) \), \( C, X, X', \tilde{X} \) be as in Section 4.6. For any \( n \geq 0 \) define a simple perverse sheaf \( X'_n \) on \( t; Z_j, \delta \) by \( X'_n = \tilde{\nu}(X_{n+1}) \) where \( \tilde{\nu}; Z_{j+1, \delta} \to t; Z_j, \delta \) is as in Lemma 3.13(a) for \( n \geq 0 \) and \( X'_n = X' \) for \( n = 0 \). Define \( a_2 \in W^{\delta}(J) \) by \( a_2^{-1} = w_m \) for \( m \gg 0 \). For any \( n \geq 0 \) there exists \( \mathcal{L}_n \in \mathcal{S}(T) \) and \( b'_n \in W^{\delta}(J) \) (see Section 2.6) such that \( a_2 b'_n \in W^{\delta}_n \) and \( X'_n \sto (\xi, a_2 b'_n); \tilde{\nu} \).

Assume that the result holds for \( n = 1 \); by Lemma 4.15, it holds for \( n = 0 \). (In Lemma 4.15 we have \( a' = \delta^{-1}(a_1) a_2 = a_2(a_2^{-1} \delta^{-1}(a_1) a_2) \) where \( a_2^{-1} \delta^{-1}(a_1) a_2 \in W^{\delta}(J) \) by Section 2.6.) The same argument shows that, if the result holds for some \( n \geq 1 \) then it also holds for \( n - 1 \). In this way it suffices to show that the result holds for \( n \gg 0 \). Replacing \( t, n \) by \( t_n, 0 \), we may assume that \( J_0 = J_1 = \cdots = J, w_0 = w_1 = \cdots = a_2^{-1} \) and \( n = 0 \). In this case the result follows using a commutative diagram as in Lemma 4.11. The lemma is proved.

Lemma 4.17. If \( A \in C_{J, \delta} \), then \( A \in C_{J, \delta} \).

Let \( t, \tilde{X}, X \) be as in the proof of Lemma 4.16. We may assume that \( A = \tilde{X} \). By Lemma 4.16 we have \( \tilde{X} \sto (\xi, a_2 b') \tilde{\nu} \) where \( a_2 \) is as in Lemma 4.16 and \( b' \in W^{\delta}(J) \) for some \( \mathcal{L} \in \mathcal{S}(T) \) with \( a_2 b' \in W^{\delta}_n \). By Lemma 4.14, \( \pi_2: Y_{a_2 b'} \to Z_{J, \delta} \) factors through a map \( \pi': Y_{a_2 b'} \to Z_{J, \delta} \) and \( \tilde{X} \sto \pi \tilde{\nu} \). Thus there exists a simple perverse sheaf on \( Z_{J, \delta} \) whose support is the closure in \( Z_{J, \delta} \) of \( \text{supp} \tilde{X} \), whose restriction to \( t; Z_{J, \delta} \) is \( \tilde{X} \) and which is a composition factor of \( \bigoplus \pi H^i(\pi \tilde{\nu}) \); this is necessarily \( \tilde{X} \). We see that \( \tilde{X} \in C_{J, \delta} \). The lemma is proved.

5. Central Character

5.1. We preserve the setup of Section 3.1. Let \( J \subset I \). The following result is similar to [L4, 11.3].

Lemma 5.2. Let \( \mathcal{L}, \mathcal{L}' \in \mathcal{S}(T), x \in W^{\delta}_n \), \( x' \in W^{\delta}_n \). Then \( x \in W^{\delta}_{n-1} \). Assume that \( \mathcal{L}' \not\cong \mathrm{Ad}(w) \mathcal{L} \) for any \( w \in W_J \). Then \( H^i(Z_{J, \delta}, K^{\xi^{-1}} \otimes K^{\xi_\ell}) = 0 \) for all \( i \in \mathbb{Z} \).

An equivalent statement is \( \bigoplus_i H^i(Y_x \times Z_{J, \delta} \mathcal{L}^{\xi^{-1}} \otimes \mathcal{L}') = 0 \) (notation of Section 4.1). Here the fibre product is taken with respect to the maps \( Y_x \to Z_{J, \delta} \to Y_{x'} \) as in Section 4.1. This fibre product is the set of all \( (B_1, B_2, B_3, B_4, g, g') \in \mathcal{B}^4 \times G^4 \times G^4 \) such that \( gB_1 = B_2, gB_3 = B_4, \) pos\( (B_1, B_2) = x, \) pos\( (B_1, B_3) = x' \), \( B_1, B_3 \) are contained in the same parabolic \( P \) of type \( J \) and \( g^{-1}g' \in U_P \). We partition this set into locally closed pieces \( Z_w(w \in W_J) \); here \( Z_w \) is defined by the condition \( \text{pos}(B_1, B_3) = w \). The restriction of \( \mathcal{L}^{\xi^{-1}} \otimes \mathcal{L}' \) to a subvariety of the fibre product is denoted in the same way. It suffices to show that \( H^i(Z_w, \mathcal{L}^{\xi^{-1}} \otimes \mathcal{L}') = 0 \) for all \( w \in W_J \). For fixed \( w \in W_J \), we have an obvious map \( \zeta : Z_w \to Z_w \) to

\[ (B_1, B_2, B_3, B_4) \in \mathcal{B}^4; \text{pos}(B_1, B_2) = x, \text{pos}(B_3, B_4) = x', \text{pos}(B_1, B_3) = w, \text{pos}(B_2, B_4) = \delta(w) \].
Using the Leray spectral sequence of $\zeta$ we see that it suffices to show that $H^i_c$ of any fibre of $\zeta$ with coefficient in the local system above is 0. Let

$$\Psi = \zeta^{-1}(g_1 B^*, g_2 B^*, g_3 B^*, g_4 B^*)$$

where $g_1, g_2, g_3, g_4 \in G$. We can find $g_0 \in G$ such that $g_0 g_1 B^* = g_2 B^*, g_0 g_2 B^* = g_3 B^*$. We can assume that $g_0 g_1 = g_2, g_0 g_3 = g_4$. A point in $\Psi$ is completely determined by its $(g, g')$-component. Thus we may identify $\Psi$ with the set of all $(g, g') \in G \times G$ such that $g_0 g_i B^* = g_0 g_j B^* = g_0 g_k B^*$ and $g^{-1} g' \in U_P$ where $P \in \mathcal{P}_J$ contains $g_1 B^*$. Thus we may identify

$$\Psi = \{(g, u) \in G \times U_P; g_0^{-1} g \in g_1 B^* \cap g_3 B^*\}$$

where $P$ is as above. Here $g_1, g_3 \in G$ are fixed such that $\text{pos}(B^*, g_1^{-1} g_3 B^*) = w$. Define $\tau: \Psi \to T$ by $g_0^{-1} g_0 g_i \in \tau(g, u) U_B$, (an affine space bundle). One checks that the local system $\mathcal{L}^{-1} \otimes \mathcal{L}'$ on $\Psi$ is $\tau^*(\mathcal{L}^{-1} \otimes \text{Ad}(w^{-1})^* \mathcal{L}')$. It then suffices to show that $H_c^i(T, \mathcal{L}^{-1} \otimes \text{Ad}(w^{-1})^* \mathcal{L}') = 0$ for all $i$. This follows from the fact that $\mathcal{L}^{-1} \otimes \text{Ad}(w^{-1})^* \mathcal{L}'$ is $\mathcal{S}(T)$ is not isomorphic to $\hat{Q}_i$. The lemma is proved.

5.3. From Lemma 5.2 we deduce as in [L4, p. 268] that there is a well defined map $C_{J, \delta} \to \{W_J\text{-orbits in } \mathcal{S}(T)\}$, $A \mapsto (W_J\text{-orbit of } \mathcal{L})$ where $A \in C_{J, \delta}^J$.

6. The Functors $f_j^\prime, e_j^\prime$

6.1. We preserve the setup of Section 3.1. Let $J \subset J' \subset I$. Let $Z_{J, J', \delta}$ be the set of all triples $(P, P', g U_Q)$ where $(P, P') \in \mathcal{P}_J \times \mathcal{P}_J'$, $Q \in \mathcal{P}_P$ is given by $P \subset Q$ and $g U_Q \in G / U_Q$ is such that $g P = P'$ (the last condition is well defined since $U_Q \subset U_P$). Consider the diagram

$$Z_{J, J', \delta} \leftarrow Z_{J', J, \delta} \xrightarrow{d} Z_{J', J'}$$

where $c(P, P', g U_Q) = (P, P', g U_P)$ and $d(P, P', g U_Q) = (Q, Q', g U_Q)$ (with $Q$ as above and $Q' = g Q$). Define

$$f_j^\prime: D(Z_{J, J', \delta}) \to D(Z_{J', J', \delta}), \quad e_j^\prime: D(Z_{J', J, \delta}) \to D(Z_{J, J', \delta})$$

by $f_j^\prime(A) = d e^* A, e_j^\prime(A') = c d^* A'$.

Lemma 6.2. For $J \subset J' \subset J'' \subset I$ we have $f_j^{\prime', f_j^{\prime''}} = f_j^{\prime''}$, $e_j^{\prime', e_j^{\prime''}} = e_j^{\prime''}$.

We have a diagram

$$Z_{J, J', \delta} \xleftarrow{c} Z_{J, J', J''} \xrightarrow{d} Z_{J', J', \delta} \xrightarrow{c'} Z_{J', J', J''} \xrightarrow{d'} Z_{J'', J'', \delta}$$

where $c, d$ are as in Section 6.1 and $c', d'$ are analogous to $c, d$. We have a cartesian diagram

$$Z_{J, J', J''} \xrightarrow{d'} Z_{J', J'', \delta} \xleftarrow{c'} Z_{J', J', \delta} \xrightarrow{d} Z_{J, J', \delta}$$
where $c''$, $d''$ are the obvious maps. Using the change of basis theorem we have
\[
\begin{align*}
f_{J''} f_J^\prime &= d'' c'' d e^* = d'' d'' c'' c^* = (d' d'')(ce'')(c e') = f_J^\prime, \\
e_J^\prime c_J^G &= c_1 d'' c d' = c c_1' d'' d = (ce'')(d' d'') = e_J^\prime.
\end{align*}
\]

The lemma is proved.

6.3. For $J \subset I$, let $D_0(Z_{J,\delta})$ be the full subcategory of $D(Z_{J,\delta})$ whose objects are the $A \in D(Z_{J,\delta})$ such that any composition factor of $\bigoplus \phi_i p_i^J(A)$ belongs to $C_{J,\delta}$.

**Lemma 6.4.** For $J \subset J' \subset I$ and $A \in D_0(Z_{I,\delta})$, we have $f_J^A(A) \in D_0(Z_{J',\delta})$.

We may assume that $A \in C_{J,\delta}$. Then there exists $A_0 \in C_{\mathfrak{g},\delta}$ such that $A \oplus f_J^A(A_0)$. By the decomposition theorem [BBD], $f_J^A(A_0)$ is a semisimple complex, hence some shift of $A$ is a direct summand of $f_J^A(A_0)$. Hence $f_J^A(A)$ is a direct summand of $f_J^A(A_0) = f_J^A(A_0)$. Using the definitions and the decomposition theorem, we see that $f_J^A(A_0)$ is a direct sum of shifts of objects in $C_{J,\delta}$. In particular, $f_J^A(A_0) \in D_0(Z_{J,\delta})$. Since $f_J^A(A)$ is a direct summand of $f_J^A(A_0)$, we must have $f_J^A(A) \in D_0(Z_{J',\delta})$. The lemma is proved.

**Lemma 6.5.** Let $J \subset I$. If $A \in D_0(Z_{J,\delta})$, then $e_J^A(A) \in D_0(Z_{J,\delta})$.

We can assume that $A = K_{\mathfrak{g},\delta}$ where $L$, $x$ are as in Section 4.1. Using the known relationship between $K_{\mathfrak{g},\delta}^\mathfrak{z}$, $K_{\mathfrak{g},\delta}^\mathfrak{w}$ (compare [L4, 12.7]) we may assume that $A = K_{\mathfrak{g},\delta}^\mathfrak{z}$. For simplicity we assume also that $L = Q_l$. Thus, $A = K_{\mathfrak{g},\delta}^\mathfrak{w}$ where $x \in W$. Let
\[
Z = \{(B, B', B_1, B_1', g) \in B^4 \times G^1; \, ^gB = B', \, ^gB_1 = B_1', \quad pos(B, B') = x, \, pos(B, B_1) \in W_J\},
\]

Define $\phi: Z \rightarrow Z_{\mathfrak{g},\delta}$ by $\phi(B, B', B_1, B_1', g) = (B_1, B_1', gU_{B_1})$. It suffices to show that $\phi \mathcal{Q}_l \in D_0(Z_{\mathfrak{g},\delta})$. For any $z \in W_J$ let
\[
\begin{align*}
Z^z &= \{(B, B', B_1, B_1', g) \in Z; \, pos(B, B_1) = z\}.
\end{align*}
\]
Now $(Z^z)_{z \in W_J}$ is a partition of $Z$ into locally closed subvarieties. Let $\phi^z: Z^z \rightarrow Z_{\mathfrak{g},\delta}$ be the restriction of $\phi$. It suffices to show that $\phi^z \mathcal{Q}_l \in D_0(Z_{\mathfrak{g},\delta})$. Consider the partition $Z_{\mathfrak{g},\delta} = \bigsqcup_{w \in W} Z_{\mathfrak{g},\delta}^w$ where
\[
Z_{\mathfrak{g},\delta}^w = \{(B_1, B_1', gU_{B_1}) \in Z_{\mathfrak{g},\delta}; \, pos(B_1, B_1') = w\}.
\]

It suffices to show that for any $w$, the restriction of $\phi^z \mathcal{Q}_l$ to $Z_{\mathfrak{g},\delta}^w$ has cohomology sheaves which are local systems with all composition factors isomorphic to $Q_l$. Let us fix $(B_1, B_1') \in B \times B$ such that $pos(B_1, B_1') = w$. It suffices to show that the restriction of $\phi^z \mathcal{Q}_l$ to $\{(B_1, B_1', gU_{B_1}); \, g \in G^1; \, ^gB_1 = B_1'\}$ has cohomology sheaves which are local systems with all composition factors isomorphic to $Q_l$. Let $T_1$ be a maximal torus of $B_1 \cap B_1'$. We can find $\zeta \in G^1$, $h \in G$ such that
\[
\begin{align*}
^\zeta B_1 &= B_1', \quad ^\zeta T_1 = T_1, \quad ^h T_1 = T_1, \quad pos(hB_1, B_1) = z.
\end{align*}
\]
Let
\[
\begin{align*}
Z' &= \{(B, u, t) \in B \times U_{B_1} \times T_1; \, pos(B, \, ^{\zeta u}B) = x, \, pos(B, B_1) = z\}.
\end{align*}
\]
Consider the projection \( pr'_3 : Z' \to T_1 \). We must show that \((pr'_3)_* \tilde{Q}_l \in \mathcal{D}(T_1)\) has cohomology sheaves which are local systems with all composition factors isomorphic to \( \tilde{Q}_l \). Let
\[
Z'' = \{(v, u, t) \in U_{B_1} \times U_{B_1} \times T_1 ; \ pos^{(v^h B_1, \zeta^h B_1)} = x\}.
\]
Define \( Z'' \to Z' \) by \((v, u, t) \mapsto (v^h B_1, u, t)\) (an affine space bundle). Consider the projection \( pr''_3 : Z'' \to T_1 \). It suffices to show that \((pr''_3)_* \tilde{Q}_l \in \mathcal{D}(T_1)\) has cohomology sheaves which are local systems that are direct sums of copies of \( \tilde{Q}_l \). We make the change of variable \((v, u, t) \mapsto (v', u', t)\) where \( u' = twt^{-1} \). Since \( th \in hB_1 \), \( Z'' \)
becomes \( \tilde{Z} \times T_1 \) where
\[
\tilde{Z} = \{(v, u') \in U_{B_1} \times U_{B_1} ; \ pos^{(v^h B_1, \zeta^h B_1)} = x\}
\]
and \( pr'_3 \) becomes the second projection \( \tilde{Z} \times T_1 \to T_1 \). The desired conclusion follows. The case where \( L \not\cong \tilde{Q}_l \) is treated similarly (compare with \([L7, 2.2]\)).

**Lemma 6.6.** Let \( J \subset I \). If \( A \in \mathcal{D}(Z_{J, \delta}) \), then some shift of \( A \) is a direct summand of \( f_J^* c_J^1(A) \).

The argument in this proof is inspired by one in \([G, 8.5.1]\). Let
\[
\mathcal{V} = \{(Q, hU_Q) ; \ Q \in \mathcal{P}_J, hU_Q \in Q/U_Q\}.
\]
Define \( m : Z_{J, \delta} \times \mathcal{P}_J, \mathcal{V} \to Z_{J, \delta} \) by \( m((P, hU_P), (P', P', gU_P)) = (P, P', gU_P) \). For \( X \in \mathcal{D}(Z_{J, \delta}), C \in \mathcal{D}(\mathcal{V}) \) we set \( X \circ C = m(X \boxtimes C) \in \mathcal{D}(Z_{J, \delta}) \).

Let \( \mathcal{V}_0 = \{(Q, hU_Q) \in \mathcal{V} ; \ h \in U_Q\} \) and let \( j : \mathcal{V}_0 \to \mathcal{V} \) be the inclusion. Let \( \tilde{\mathcal{V}} \) be the set of all pairs \((B, hU_Q)\) where \( B \in B \) and \( hU_Q \in U_{B}/U_Q \) (with \( Q \in \mathcal{P}_J \) given by \( B \subset Q \)). Define \( \pi : \tilde{\mathcal{V}} \to \mathcal{V} \) by \( \pi(B, hU_Q) = (Q, hU_Q) \). Clearly, the lemma is a consequence of (a), (b), (c) below.

(a) \( A \circ (j_! \tilde{Q}_l) = A \),

(b) \( A \circ (\pi_! \tilde{Q}_l) = f_J^* c_J^1(A) \),

(c) \( j_! \tilde{Q}_l[n] \) is a direct summand of \( \pi_! \tilde{Q}_l \) for some \( n \).

Now (a) is obvious. We prove (b). We have \( f_J^* c_J^1(A) = d_c c_0 d^*(A) \) where
\[
Z_{J, \delta} \xleftarrow{\text{c}} Z_{J, \delta} \xrightarrow{\text{d}} Z_{J, \delta}
\]
are defined by
\[
c(B, B', gU_P) = (B, B', gU_B), \quad d(B, B', gU_P) = (P, P', gU_P)
\]
(with \( P \in \mathcal{P}_J, P' \in \mathcal{P}_{B \subset P} \)). Let \( Z' \) be the set of all quadruples \((B, B', gU_P, g'U_P)\) where \((B, B') \in B \times B, P \in \mathcal{P}_J\) is given by \( B \subset P \) and \( gU_P, g'U_P \in G'/U_P \) are such that \( gB = B' \) and \( gU_B = g'U_B \). Let \( b, b' \) be the projections \( pr_{123} : Z' \to Z_{J, \delta}, pr_{124} : Z' \to Z_{J, \delta} \). Define \( a, a' : Z' \to Z_{J, \delta} \) by
\[
a(B, B', gU_P, g'U_P) = (P, P', gU_P), \quad a'(B, B', gU_P, g'U_P) = (P, P', g'U_P),
\]
where \( P, P' \) are as above. Then \( a \circ db, a' \circ db' \). By the change of basis theorem we have \( c^*_c = db^* a^* \) hence
\[
f_J^* c_J^1(A) = d_b b^* d^*(A) = (db')_!(db)^*(A) = a'_! a^*(A).
\]
Let $Z''$ be the set of all triples $(B, gU_p, hU_p)$ where $B \in B$, $P \in \mathcal{P}_I$ is given by $B \subset P$, $gU_p \in G^1/U_p$ and $h \in U_B$. Define $\tilde{a}, \tilde{a}' : Z'' \to Z_{I,\delta}$ by

$$\tilde{a}(B, gU_p, hU_p) = (P, \varphi, gU_p), \quad \tilde{a}'(B, gU_p, hU_p) = (P, \varphi, ghU_p)$$

with $P$ as above. From the definitions we have $A \circ (\pi_0\tilde{Q}_I) = \tilde{a}'^{*}(A)$. The isomorphism

$$\iota : Z'' \isom Z', \quad (B, gU_p, hU_p) \mapsto (B, \varphi^B, gU_p, gU_p)$$

satisfies $a = \tilde{a}, a' = \tilde{a}'$. Thus, $\tilde{a}'^{*}(A) = a'_1 \iota^{*} a^{*}(A) = a^{*}(A)$ and (b) is proved.

We prove (c). Let $Q \in \mathcal{P}_J$. Let

$$\tilde{Q}/U_Q = \{(B, hU_Q) : B \in B, B \subset Q, hU_Q \in U_B/U_Q\}.$$

Let $\pi' : \tilde{Q}/U_Q \to Q/U_Q$ be the second projection. It is known that $\pi'$ is a semismall map onto its image (the set of unipotent elements in $Q/U_Q$) and that $\pi'(\tilde{Q})$ contains as a direct summand a shift of the skyscraper sheaf of $Q/U_Q$ at the unit element of $Q/U_Q$. Using $G$-equivariance, we see that a shift of $j_!\tilde{Q}$ is a direct summand of $\pi_0\tilde{Q}_I$. This proves (c). The lemma is proved.

**Proposition 6.7.** (a) Let $J \subset I$. Let $A \in \mathcal{D}(Z_{I,\delta})$. We have $A \in \mathcal{D}_0(Z_{I,\delta})$ if and only if $e''_0(A) \in \mathcal{D}_0(Z_{S,\delta})$.

(b) Let $J \subset J' \subset I$. If $A' \in \mathcal{D}_0(Z_{J',\delta})$ then $e''_0(A') \in \mathcal{D}_0(Z_{J,\delta})$.

We prove (a). If $A \in \mathcal{D}_0(Z_{J,\delta})$ then $e''_0(A) \in \mathcal{D}_0(Z_{S,\delta})$ by Lemma 6.5. Conversely, assume that $e''_0(A) \in \mathcal{D}_0(Z_{S,\delta})$. Using Lemma 6.4, we have $f''_0 e''_0(A) \in \mathcal{D}_0(Z_{J,\delta})$. Using this and Lemma 6.6, we see that $A \in \mathcal{D}_0(Z_{J,\delta})$.

We prove (b). Applying (a) to $A = e''_0(A')$ we see that it suffices to show that $e''_0 e''_0(A') \in \mathcal{D}_0(Z_{S,\delta})$ or equivalently (see Lemma 6.2) that $e''_0(A') \in \mathcal{D}_0(Z_{S,\delta})$. But this follows from Lemma 6.5.

**Corollary 6.8.** Let $J \subset I$. Let $A$ be a character sheaf on $G^1$. Then $e''_0(A) \in \mathcal{D}_0(Z_{J,\delta})$.

7. Characteristic Functions

7.1. We reserve the setup of Section 3.1. Assume that $k, F_q$ are as in Section 1.2 and that we are given an $F_q$-rational structure on $G$ with Frobenius map $F : G \to G$ such that $G^1$ is defined over $F_q$. Now $F$ induces on the Weyl group $W$ an automorphism denoted by $F : W \to W$; it commutes with $\delta : W \to W$ and it carries $I$ onto itself. Let $J \subset I$ be such that $F(J) = J$. Then $\mathcal{P}_I, \mathcal{P}_{I(K)}$ are defined over $F_q$ and $Z_{J,\delta}$ is defined over $F_q$ with corresponding Frobenius map $F : Z_{J,\delta} \to Z_{J,\delta}$ given by $F(P, P', gU_p) = (F(P), F(P'), F(g)U_{f(P)})$. The natural $G$-action on $Z_{J,\delta}$ (see Section 3.11) restricts to a $G(F_q)$-action on $Z_{I,\delta}(F_q)$. Let $E_{J,\delta}$ be the vector space of all functions $Z_{J,\delta}(F_q) \to F_q$ that are constant on the orbits of $G(F_q)$.

7.2. If $X$ is an algebraic variety over $k$ with a fixed $F_q$-structure and with Frobenius map $F : X \to X$ and if we are given $K \in \mathcal{D}(X)$ together with an isomorphism $\phi : F^*K \to K$, we denote by $\chi_{K,\phi} : X(F_q) \to Q$ the corresponding characteristic function; for $x \in X(F_q)$, $\chi_{K,\phi}(x)$ is the alternating sum over $i \in \mathbb{Z}$ of the trace of the map induced by $\phi$ on the stalk at $x$ of the $i$-th cohomology sheaf of $K$. 


7.3. In the setting of Section 7.1, let $C_{t,δ}^F = \{ A \in \mathcal{C}_{t, δ}; F^* A \cong A \}$. For any $A \in C_{t, δ}^F$, we choose an isomorphism $φ_A: F^* A \to A$. (It is unique up to multiplication by an element in $\mathbb{Q}_{t}^*.$) Then $\chi_{A, φ_A}: Z_{t, δ}^F \to \mathbb{Q}_{t}$ is constant on the orbits of $G(\mathbb{F}_q)$ since $A$ is $G$-equivariant.

Define $F: T(J, δ) \to T(J, δ)$ by $F((J_n, w_n)_{n \geq 0}) = (F(J_n), F(w_n)_{n > 0})$. Clearly, $F: Z_{t, δ} \to Z_{t, δ}$ carries $tZ_{t, δ}$ onto $tF(t)Z_{t, δ}$. In particular, $tZ_{t, δ}$ is $F$-stable if and only if $t \in T(J, δ)^F$ (that is, $F(t) = t$). For any $t \in T(J, δ)^F$, let $\mathcal{E}_{t, δ}$ be the vector space of all functions $\bar{F}_{t, δ}(\mathbb{F}_q) \to \mathbb{Q}_{t}$ that are constant on the orbits of $G(\mathbb{F}_q)$. We may identify $\mathcal{E}_{t, δ}$ with a subspace $\mathcal{E}_{t, δ}$ of $\bar{E}_{t, δ}$ by associating to $f \in \mathcal{E}_{t, δ}$ the function $\tilde{f} \in \mathcal{E}_{t, δ}$ whose restriction to $\mathcal{E}_{t, δ}$ is $f$ and which is $0$ on $Z_{t, δ}(\mathbb{F}_q)$.

(a) $\mathcal{E}_{t, δ} = \bigoplus_{t \in T(J, δ)^F} \mathcal{E}_{t, δ}$.

For $t \in T(J, δ)^F$ let $C_{t, δ}^F = \{ A' \in C_{t, δ}^F; F^* A' \cong A' \}$. For any $A' \in C_{t, δ}^F$, we choose an isomorphism $\phi_{A'}: F^* A' \to A'$. (It is unique up to multiplication by an element in $\mathbb{Q}_{t}^*.$) Then $\chi_{A', φ_{A'}}: Z_{t, δ}^{A'}^F \to \mathbb{Q}_t$ is constant on the orbits of $G(\mathbb{F}_q)$. Thus $\chi_{A', φ_{A'}} \otimes \mathcal{E}_{t, δ}$ and $\tilde{\chi}_{A', φ_{A'}} \in \mathcal{E}_{t, δ} \subseteq \mathcal{E}_{t, δ}$.

Lemma 7.4. Let $A \in C_{t, δ}^F$. Let $d = \dim \text{supp}(A)$. For any $t \in T(J, δ)$ let $tA = A|_{Z_{t, δ}}$. Define $t^0 \in T(J, δ)$, $A^0 \in C_{t, δ}^F$ by $t^0 A = A^0$, $\dim \text{supp} A^0 = d$ (see Lemma 4.13). We have $t^0 \in T(J, δ)^F$, $A^0 \in C_{t^0, δ}^F$ and $\chi_{A, φ_A} = \lambda \tilde{\chi}_{A^0, φ_{A^0}} + \sum_{t \in T(J, δ)^F} \sum_{A' \in C_{t, δ}^F} \lambda_{t, A'} \tilde{\chi}_{A', φ_{A'}}$.

where $\lambda_{t, A'} \in \mathbb{Q}_t$, $\lambda \in \mathbb{Q}^*_t$.

For any $t \in T(J, δ)^F$, $φ_A$ induces an isomorphism $F^*(tA) \to tA$ denoted again by $φ_A$ and we have $\chi_{A, φ_A} = \sum_{t \in T(J, δ)^F} \tilde{\chi}_{A, φ_A}$.

For $t = t^0$ we have $\tilde{\chi}_{A, φ_A} = \lambda \tilde{\chi}_{A^0, φ_{A^0}}$ where $\lambda \in \mathbb{Q}^*_t$. It remains to show that, for any $t \in T(J, δ)^F - \{ t^0 \}$, $\tilde{\chi}_{A, φ_A}$ is a $\mathbb{Q}_t$-linear combination of functions $\chi_{A', φ_{A'}}$ where $A' \in C_{t, δ}^F$, $\dim \text{supp}(A') < d$. Clearly,

$$\chi_{tA, φ_A} = \sum_{t} (-1)^i \chi_{p^H(tA), φ_A}$$

where the isomorphism $F^*(p^H(tA)) \to p^H(tA)$ induced by $φ_A$ is denoted again by $φ_A$. It then suffices to show that $\chi_{p^H(tA), φ_A}$ is a $\mathbb{Q}_t$-linear combination of functions $\chi_{A', φ_{A'}}$ where $A' \in C_{t, δ}^F$, $\dim \text{supp}(A') < d$.

From Lemma 4.13 we see that $\dim \text{supp} tA < d$. Hence $\dim \text{supp} p^H(tA) < d$ and any composition factor $A'$ of $p^H(tA)$ has support of dimension $< d$. By Lemma 4.12, any such $A'$ is in $C_{t, δ}^F$ and (a) follows. The lemma is proved.
7.5. We consider the following statement.

(*) The functions $\chi_{A,\phi_A}$ where $A$ runs over $C_{I,\delta}^F$ form a $\bar{\mathbb{Q}}_l$-basis of $\mathcal{E}_{I,\delta}$.

In the case where $J = I$, so that $Z_{I,\delta} = G^1$, and assuming that the characteristic of $k$ is not too small, statement (*) appears without proof in [L6, 5.2.1] and with proof (when $G^1 = G$) in [L5, 25.2]. In the remainder of this section we assume that (*) holds when $J = I$ (for $G$ and for groups of smaller dimension).

We show that (*) holds. By Lemma 7.4, the functions $\chi_{A,\phi_{A'}}$ are related to the functions $\tilde{\chi}_{A',\phi_{A'}}$ (with $t \in T(J, \delta)^F$, $A' \in C_{I,\delta}^F$) by an upper triangular matrix with invertible entries on the diagonal. Hence it suffices to show that the functions $\tilde{\chi}_{A',\phi_{A'}}$ (with $t \in T(J, \delta)^F$, $A' \in C_{I,\delta}^F$) form a $\bar{\mathbb{Q}}_l$-basis of $\mathcal{E}_{I,\delta}$. Using 7.3(a), we see that it suffices to show that for any $t \in T(J, \delta)^F$, the functions $\chi_{A',\phi_{A'}}$ form a $\bar{\mathbb{Q}}_l$-basis of $\mathcal{E}_{I,\delta}$. Using 3.14 and the definition of $C_{I,\delta}$ (see Section 4.6), we see that it suffices to prove the statement in the previous sentence when $Z_{I,\delta}$ is replaced by $Z_{I,\delta}$ (notation of Section 3.14) for $r$ large.

We may assume that $P, P^r, L_t, C_t$ (as in Section 3.14) are defined over $\mathbb{F}_q$. We are reduced to the statement that the characteristic functions of the character sheaves on $C_t$ (relative to the $\mathbb{F}_q$-structure) form a $\bar{\mathbb{Q}}_l$-basis for the space of $L_t(\mathbb{F}_q)$-invariant functions $C_t(\mathbb{F}_q) \to \bar{\mathbb{Q}}_l$, which is part of our assumption.

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