

FEDOSOV QUANTIZATION IN ALGEBRAIC CONTEXT

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To B. L. Feigin on his 50th anniversary

ABSTRACT. We consider the problem of quantization of smooth symplectic varieties in the algebro-geometric setting. We show that, under appropriate cohomological assumptions, the Fedosov quantization procedure goes through with minimal changes. The assumptions are satisfied, for example, for affine and for projective varieties. We also give a classification of all possible quantizations.

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INTRODUCTION

Let V be a finite-dimensional vector space equipped with a non-degenerate 2-form. The algebra $S(V)$ of polynomial functions on V admits a well-known non-commutative one-parameter deformation $S(V)[[h]]$ called the Weyl algebra. The problem of deformation quantization consist in generalizing this construction to a deformation of the sheaf $\mathcal{A}(M)$ of functions on an arbitrary smooth symplectic manifold M . More precisely, one wants to know whether there exists a deformation with prescribed properties, and how many such deformations there are.

The reader will immediately notice that our language is ambiguous: “smooth manifold” can mean either a C^∞ -manifold, or a holomorphic manifold, or a smooth algebraic variety — over \mathbb{C} or over some other field, possibly of positive characteristic. This is intentional: the problem of deformation quantization makes perfect sense in all these situations.

When the problem was posed several decades ago, it soon became clear that the standard deformation theory methods take one only so far. General nonsense gives a series of obstruction classes lying in a certain group. However, this group is usually non-trivial. Thus to quantize a manifold, it is necessary to actually prove that the obstruction classes themselves vanish.

After a hiatus of several years, the problem was finally completely solved in the early 1980-ies independently by M. De Wilde–P. Lecomte and by B. Fedosov (see

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[DWL], [F] and a classic exposition of these results by P. Deligne in [D2]). The answer is that a quantization always exists, and that the space of all quantizations admits a simple description.

Both De Wilde–Lecomte and Fedosov worked with C^∞ -manifolds, by C^∞ methods. So did Deligne. When one looks at the proofs, though, one is tempted to think that the C^∞ context is not really essential — one only needs the vanishing of certain cohomology groups. This is implicit in [DWL] and [F], and less implicit in the gerb-theoretic version of the proof given in [D2]. However, Deligne does not state the necessary cohomology vanishing conditions either. Instead, he uses the softness of certain non-abelian group sheaves. Thus one cannot directly generalize either of the existing proofs to the holomorphic or algebraic setting — while there is a strong feeling that the results themselves should hold.

In the eight years which passed since the publication of [D2], the deformation quantization has been much better understood, and now there seems to be no doubt among experts as to what happens in the holomorphic and in the algebraic setting (at least in characteristic 0). Some proofs are actually published. In particular, R. Nest and B. Tsygan have given in [NT] a complete proof in the holomorphic case. They have also specified the cohomology vanishing condition which one needs to impose on the manifold in order for the argument to work.

However, it seems that the algebraic case still remains a folk knowledge, with no references in the literature. Thus a write-up of a purely algebraic proof would be useful. This is what the present paper is intended to be.

The results in the paper were discovered while trying to apply deformation quantization to a concrete algebro-geometric problem. The authors are definitely not experts in the field, and we lay no claim whatsoever to the novelty of our results. Moreover, even our approach is essentially the same as Fedosov’s, although retold in a more algebraic language. The main technical tool is the bundle of formal coordinate systems and the associated bundle of jets (“formal geometry” in the language of I. M. Gelfand). We are deeply grateful to B. Feigin who suggested this approach to us and more or less explained what to do.

We would like to mention also a recent paper [KV], by the second author jointly with M. Verbitsky, which contains certain results on purely commutative deformation of symplectic manifolds — more or less, a generalization of the unobstructedness theorem of F. Bogomolov [Bg]. The methods used there are different and somewhat simpler. However, the final result is completely parallel to what one has for quantizations. In particular, the required cohomology vanishing is precisely the same. In the latter part of the paper, we explain this similarity and show how to join quantizations and symplectic deformation into a single partially-commutative deformation of the manifold in question.

Finally — our proof only works in characteristic 0. What happens in characteristic $p > 0$? There are important reasons to study this question, and we believe that it is possible to prove some sort of a general statement. However, if one wants to apply our methods, one has to modify them in quite an essential way. We plan to return to this in future research.

Note added in proof. A short time after the first version of this paper was posted to the web, there appeared a preprint [Y] by A. Yekutieli devoted to a related problem. A. Yekutieli also read our paper and indicated some omissions which we now fix. Additionally, we would like to thank the referee for the detailed report and important suggestions, and the people at the Moscow Math Journal for their remarkable patience with our extensive last-minute rewrites.

1. STATEMENTS AND DEFINITIONS

1.1. Notation. Fix once and for all a base scheme S . Throughout the paper we will assume that S is a scheme of finite type over a fixed field k of characteristic 0. The most important case for us is $S = \operatorname{Spec} k$, a point. However, all the proofs work for non-trivial schemes just as well, and in some applications it is convenient to have the results available in a more general setting.

By an S -manifold X we will understand a scheme X/S of finite type and smooth over S —that is, we require that X is flat over S and the relative cotangent sheaf $\Omega_{X/S}^1$ is a locally free coherent sheaf. By the *dimension* of an S -manifold we will understand the relative dimension X/S , which coincides with the rank of the flat sheaf $\Omega_{X/S}^1$. For an S -manifold X , one defines the (relative) de Rham complex $\Omega_{X/S}^\bullet$ and its hypercohomology, known as the *de Rham cohomology groups* $H_{\mathrm{DR}}^\bullet(X/S)$. When $S = \operatorname{Spec} \mathbb{C}$ is the complex point, the de Rham cohomology groups are known to coincide with the topological cohomology groups $H^\bullet(X, \mathbb{C})$.

By a “vector bundle” we will understand a “locally free coherent sheaf”. For a vector bundle \mathcal{E} on X , we define a (relative) *flat connection* ∇ on \mathcal{E} as a differential operator $\nabla: \mathcal{E} \rightarrow \mathcal{E} \otimes \Omega^1(X/S)$ satisfying the usual compatibilities. Flat vector bundles on X form a tensor abelian category, with unit object \mathcal{O}_X (with the tautological connection). For every flat vector bundle \mathcal{E} , one defines the (relative) de Rham cohomology groups $H_{\mathrm{DR}}^\bullet(X, \mathcal{E})$. The de Rham cohomology of the unit bundle \mathcal{O}_X coincide with the de Rham cohomology groups $H_{\mathrm{DR}}^\bullet(X)$.

The coherent cohomology groups $H^\bullet(X, \mathcal{E})$ of a vector bundle \mathcal{E} on X can be interpreted as relative de Rham cohomology by means of relative jet bundles. To define those, let $\widehat{\Delta}$ be the completion of the fibered product $X \times_S X$ along the diagonal $\Delta \subset X \times_S X$, and let $\pi_1, \pi_2: \widehat{\Delta} \rightarrow X$ be the projections onto the first and the second factor. Then the jet bundle $J^\infty \mathcal{E}$ is given by

$$J^\infty \mathcal{E} = \pi_{1*} \pi_2^* \mathcal{E}.$$

The jet bundle carries a natural flat connection. The sheaf of its flat sections coincide with the sheaf \mathcal{E} , and the de Rham cohomology $H_{\mathrm{DR}}^\bullet(X, J^\infty \mathcal{E})$ is canonically isomorphic to $H^\bullet(X, \mathcal{E})$.

Note that a jet bundle $J^\infty \mathcal{E}$ is not finitely generated as a sheaf of \mathcal{O}_X -modules, thus not coherent. To be able to work with jet bundles, we have to complete the category of coherent sheaves on X by adding countable projective limits. The resulting category of pro-coherent sheaves is a tensor abelian category (although it no longer has good duality properties). For the details of the completion procedure, see [D1]. As an additional bonus for working with the completed category, we can interpret the de Rham cohomology groups $H_{\mathrm{DR}}^\bullet(X, \mathcal{E})$ of a flat vector bundle \mathcal{E} as

the Ext^\bullet -groups from \mathcal{O}_X to \mathcal{E} (in the usual category, this is not necessarily true even for $\mathcal{E} \cong \mathcal{O}_X$). For the proof, it suffices to consider the de Rham type resolution of \mathcal{E} by jet bundles $J^\infty \Omega_X^\bullet \otimes \mathcal{E}$.

To simplify notation, we will often drop S from the formulas and omit the word “relative” in the statements. The reader should always keep in mind that everything on X is understood relatively over S . Moreover, we will drop the prefix “pro” whenever there is no danger of confusion.

1.2. Assumptions. Let X be an S -manifold. All our results will be valid under the following assumption.

Definition 1.1. The manifold X is called *admissible* if the canonical map

$$H_{\text{DR}}^i(X) \rightarrow H^i(X, \mathcal{O}_X)$$

from the de Rham cohomology $H_{\text{DR}}^i(X)$ to the cohomology $H^i(X, \mathcal{O}_X)$ of the structure sheaf \mathcal{O}_X is surjective for $i = 1, 2$.

In the case when $S = \text{Spec } k$ is a point, examples of admissible manifolds are:

- (a) A projective manifold X — admissibility follows from the Hodge theory.
- (b) A smooth projective resolution $X \rightarrow Y$ of a singular affine variety Y such that the canonical bundle K_X is trivial — we have $H^i(X, \mathcal{O}_X) = 0$ for all $i \geq 1$ by the Grauert–Riemenschneider Vanishing Theorem.

For an S -manifold X , we will denote by $H_F^\bullet(X)$ the hypercohomology of the first piece $F^1 \Omega_X^\bullet$ of the de Rham complex Ω_X^\bullet with respect to the *filtration bête* — in other words, the third term in the natural cohomology long exact sequence

$$H_F^\bullet(X) \rightarrow H_{\text{DR}}^\bullet(X) \rightarrow H^\bullet(X, \mathcal{O}_X) \rightarrow \dots$$

associated to the map $H_{\text{DR}}^\bullet(X) \rightarrow H^\bullet(X, \mathcal{O}_X)$. If X is admissible, then the group $H_F^2(X)$ coincides with the kernel of the natural map $H_{\text{DR}}^2(X) \rightarrow H^2(X, \mathcal{O}_X)$.

1.3. Definitions. The prototype for quantization is the quantization of a formal polydisc. Let \mathcal{A} be the power series algebra

$$\mathcal{A} \cong k[[x_1, \dots, x_d, y_1, \dots, y_d]]$$

on $2d$ variables $x_1, \dots, x_d, y_1, \dots, y_d$. Roughly speaking, quantizing \mathcal{A} consist of passing to the so-called *Weyl algebra*.

Definition 1.2. The *formal Weyl algebra* (of fixed dimension $2d$) is the complete topological associative algebra

$$D = k[[x_1, \dots, x_d, y_1, \dots, y_d, h]]$$

topologically generated by elements $x_1, \dots, x_d, y_1, \dots, y_d, h$ subject to relations

$$\begin{aligned} [x_i, x_j] &= [y_i, y_j] = [x_i, h] = [y_j, h] = 0, \\ [x_i, y_j] &= \delta_{ij} h \end{aligned}$$

for all $0 < i, j \leq d$.

The formal Weyl algebra D is a flat algebra over the power series algebra $k[[h]]$. The subspace $hD \subset D$ is a two-sided ideal, and the quotient D/hD is isomorphic to the power series algebra \mathcal{A} .

The general definition of quantizations is as follows. Let X be an S -manifold with structure morphism $\pi: X \rightarrow S$, and denote by $\pi^{-1}\mathcal{O}_S$ the sheaf-theoretic pullback of the structure sheaf \mathcal{O}_S .

Definition 1.3. A *quantization* \mathcal{D} of an S -manifold X is a sheaf of associative flat $\pi^{-1}\mathcal{O}_S[[h]]$ -algebras on X complete in the h -adic topology and equipped with an isomorphism $\mathcal{D}/h\mathcal{D} \cong \mathcal{O}_X$.

We note that quantizations are compatible with base change. Namely, given an S -manifold X with quantization \mathcal{D} and a map $f: S' \rightarrow S$, we obtain a quantization $f^*\mathcal{D}$ of the S' -manifold $X \times_S S'$ by setting

$$f^*\mathcal{D} = f^*\mathcal{D} \hat{\otimes}_{f^*\mathcal{O}_S} \mathcal{O}_{S'}.$$

A particular case of this construction allows one to define jet bundles for quantizations. Assume given a quantization \mathcal{D} of an S -manifold X . Consider the product $X \times_S X$ with the projections $p_1, p_2: X \times_S X \rightarrow X$. The second projection p_2 turns $X \times_S X$ into an X -manifold. Let $\mathcal{D}' = p_1^*\mathcal{D}$ be the quantization of the X -manifold $X \times_S X$ obtained by pullback with respect to the projection p_1 . Then \mathcal{D}' is a sheaf of $p_2^*\mathcal{O}_X$ -algebras on $X \times_S X$, and we have $\mathcal{D}'/h\mathcal{D}' \cong \mathcal{O}_{X \times_S X}$. The ideal $\mathcal{J}_\Delta \subset \mathcal{O}_{X \times_S X}$ of the diagonal $X \cong \Delta \subset X \times_S X$ lifts to a well-defined two-sided ideal $h\mathcal{D}' + \mathcal{J}_\Delta \subset \mathcal{D}'$. The completion $J^\infty\mathcal{D}$ of the sheaf of algebras \mathcal{D}' with respect to the sheaf of ideals $h\mathcal{D}' + \mathcal{J}_\Delta$ is supported on the diagonal, and it is naturally a sheaf of \mathcal{O}_X -algebras. Moreover, it is easy to see that $J^\infty\mathcal{D}$ is a pro-vector bundle on X . The fiber $J^\infty\mathcal{D}_x$ of the bundle $J^\infty\mathcal{D}$ at a closed point $x \in X$ is canonically isomorphic to the completion $\widehat{\mathcal{D}}_x$ of the stalk \mathcal{D}_x of the sheaf \mathcal{D} at the point $x \in X$ with respect to the topology generated by the ideal $h\mathcal{D}_x + \mathfrak{m}_x$, where $\mathfrak{m}_x \subset \mathcal{O}_x \cong \mathcal{D}_x/h\mathcal{D}_x$ is the maximal ideal in the local ring \mathcal{O}_x of germs of functions on X near $x \in X$.

Definition 1.4. The bundle $J^\infty\mathcal{D}$ is called the *jet bundle* of the quantization \mathcal{D} .

Quantizations are usually studied in connection with Poisson geometry (see e.g. [Kon]); we briefly recall this connection. Given a quantization \mathcal{D} on an S -manifold X , one considers the commutator in the non-commutative algebra \mathcal{D} and defines a skew-symmetric bracket operation

$$\{-, -\}: \mathcal{O}_X \otimes \mathcal{O}_X \rightarrow \mathcal{O}_X$$

by $\{a, b\} = \frac{1}{h}\tilde{a}\tilde{b} - \tilde{b}\tilde{a} \pmod{h^2}$ for any two local sections a, b of the sheaf \mathcal{O}_X lifted to local sections \tilde{a}, \tilde{b} of the sheaf \mathcal{D} . One checks easily that this bracket is well-defined and satisfies the axioms of a Poisson bracket, namely,

$$\{a, (b \cdot c)\} = \{a, b\} \cdot c + \{a, c\} \cdot b, \quad \{a, \{b, c\}\} + \{b, \{c, a\}\} + \{c, \{a, b\}\} = 0.$$

By definition, this means that X becomes a so-called *Poisson scheme* over S , and one says that \mathcal{D} is a quantization of the Poisson scheme X .

Since a Poisson bracket $\{-, -\}$ is a derivation with respect to both parameters, it is given by

$$\{a, b\} = da \wedge db \lrcorner \Theta$$

for some bivector field $\Theta \in \Lambda^2 \mathcal{T}_{X/S}$, where $\mathcal{T}_{X/S}$ is the relative tangent bundle of X over S . The bivector field Θ defines an \mathcal{O}_X -valued pairing on the relative cotangent bundle $\Omega^1(X/S)$.

In this paper, we will only be interested in Poisson brackets such that the associated pairing on $\Omega^1(X/S)$ is non-degenerate. Since the pairing is skew-symmetric, this in particular means that the dimension $\dim X/S$ must be even. Applying the non-degenerate pairing Θ , one identifies $\mathcal{T}_{X/S}$ and $\Omega^1(X/S)$, so that Θ induces a 2-form $\Omega \in \Omega^2(X/S)$. Conversely, given a non-degenerate 2-form $\Omega \in \Omega^2(X/S)$, one applies it to identify $\Omega^1(X/S)$ with $\mathcal{T}_{X/S}$ and obtains a non-degenerate bivector field $\Theta \in \Lambda^2 \mathcal{T}_{X/S}$. It is well-known that Θ defines a Poisson bracket if and only if Ω is a closed form. Thus giving a Poisson structure on an S -manifold X with non-degenerate pairing Θ is the same as giving a symplectic form $\Omega \in \Omega^2(X/S)$. Given a symplectic S -manifold X (of some even dimension $2d$), by a quantization of X we will understand a quantization of the S -manifold X such that the associated Poisson bracket on X coincides with the bracket induced by the symplectic form.

The definition of quantizations generalizes verbatim to the case of formal schemes; in particular, it applies to the formal polydisc $\mathrm{Spf} \mathcal{A}$ over a field k . Set-theoretically, $\mathrm{Spf} \mathcal{A}$ is a point, so that a quantization \mathcal{D} of $\mathrm{Spf} \mathcal{A}$ is an algebra over k . One example of such a quantization is the formal Weyl algebra D . Our approach to quantizations is based on the following standard fact (essentially, a version of the Darboux Theorem).

Lemma 1.5. *Let \mathcal{D} be any quantization of the formal polydisc $\mathrm{Spf} \mathcal{A}$ over a field k of characteristic 0 such that the associated Poisson pairing Θ is non-degenerate. Then \mathcal{D} is isomorphic to the formal Weyl algebra D . \square*

In particular, for any quantization \mathcal{D} of a smooth symplectic manifold X over a field of characteristic 0, the completion $\widehat{\mathcal{D}}_x$ of the stalk \mathcal{D}_x at some closed point $x \in X$ is a quantization of the formal neighborhood of x in X , which is isomorphic to the formal polydisc $\mathrm{Spf} \mathcal{A}$ over the residue field $k = k_x$ of the point $x \in X$. By Lemma 1.5, there exists a (non-canonical) isomorphism $\widehat{\mathcal{D}}_x \cong D$.

Remark 1.6. Assume that both the base $S = \mathrm{Spec} \mathcal{O}_S$ and a smooth S -manifold $X = \mathrm{Spec} \mathcal{O}_X$ are affine. Then for every quantization \mathcal{D} of the manifold X , the algebra $D_X = H^0(X, \mathcal{D})$ of global sections of the sheaf \mathcal{D} is a flat $\mathcal{O}_S[[h]]$ -algebra, complete in the h -adic topology and equipped with an isomorphism $D_X/hD_X \cong \mathcal{O}_X = H^0(X, \mathcal{O}_X)$ (the natural map $D_X \rightarrow \mathcal{O}_X$ is surjective because the sheaf \mathcal{O}_X has no cohomology). Conversely, there exist a non-commutative localization procedure called *Ore localization* which applies, in particular, to any one-parameter deformation of the algebra \mathcal{O}_X (see e.g. [Kap1, Section 2.1]) and gives a one-parameter deformation of the sheaf of algebras \mathcal{O}_X on X . The constructions are mutually inverse. Thus in the affine case, giving a quantization of $X = \mathrm{Spec} \mathcal{O}_X$ is equivalent to giving a one-parameter deformation of the \mathcal{O}_S -algebra \mathcal{O}_X (more

precisely, a flat $\mathcal{O}_S[[h]]$ -algebra D_X complete in the h -adic topology and equipped with an isomorphism $D_X/hD_X \cong \mathcal{O}_X$).

Remark 1.7. In general, a quantization \mathcal{D} of an S -manifold X does not have to be isomorphic to $\mathcal{O}_X[[h]]$ even as a sheaf of groups. However, this is true in the affine case over a point, $S = \operatorname{Spec} k$, k a field, $X = \operatorname{Spec} \mathcal{O}_X$. Indeed, deformations of the algebra \mathcal{O}_X in the class of associative k -algebras are controlled by the so-called *Hochschild cohomology groups* $HH_i^\bullet(\mathcal{O}_X)$, which are computed by means of the Hochschild cochain complex $C_i^\bullet(\mathcal{O}_X)$,

$$C_k^i(\mathcal{O}_X) = \operatorname{Hom}_k(\mathcal{O}_X^{\otimes i}, \mathcal{O}_X),$$

where the tensor product is taken in the category of k -modules. Inside $C_k^i(\mathcal{O}_X)$, one distinguishes a subcomplex

$$C_{\text{diff}}^i(\mathcal{O}_X) \subset C_k^i(\mathcal{O}_X)$$

of cochains given by polydifferential operators. For any quantization \mathcal{D} , the algebra D_X is isomorphic to $k[[h]]$ as a k -vector space; it is in the multiplication operation in D_X that the non-triviality of the quantization is contained (this multiplication is usually referred to as the *star-product*). The complex $C_{\text{diff}}^\bullet(\mathcal{O}_X)$ controls those deformations for which the star-product is given by a series with bidifferential operators as coefficients. However, an easy computation shows that the complexes $C_{\text{diff}}^\bullet(\mathcal{O}_X)$ and $C_k^\bullet(\mathcal{O}_X)$ are quasiisomorphic. Therefore any deformation D_X in fact can be given by a star-product whose Taylor coefficients in h are bidifferential operators. Since polydifferential operators are local, — that is, induced by sheaf maps $\mathcal{O}_X^{\otimes k} \rightarrow \mathcal{O}_X$, — after localization we can represent \mathcal{D} as the sheaf $\mathcal{O}_X[[h]]$ with some non-trivial star-product multiplication.

1.4. Statements. We can now formulate our main result.

Theorem 1.8. *Let X be an admissible S -manifold of dimension $2d$ equipped with a closed non-degenerate relative form $\Omega \in H^0(X, \Omega_{X/S}^2)$. Denote by $[\Omega] \in H_{\text{DR}}^2(X/S)$ the cohomology class of the symplectic form. Let $Q(X, \Omega)$ be the set of isomorphism classes of quantizations of X compatible with the form Ω .*

Then there exists a natural injective map

$$\operatorname{Per}: Q(X, \Omega) \hookrightarrow H_{\text{DR}}^2(X/S)[[h]],$$

called the non-commutative period map. Moreover, for every quantization $q \in Q(X, \Omega)$, the power series $f = \operatorname{Per}(q) \in H_{\text{DR}}^2(X)[[h]]$ has constant term $[\Omega]$. Finally, any splitting $P: H_{\text{DR}}^2(X/S) \rightarrow H_F^2(X/S)$ of the canonical embedding $H_F^2(X/S) \rightarrow H_{\text{DR}}^2(X/S)$ induces an isomorphism

$$P \circ \operatorname{Per}: Q(X, \Omega) \xrightarrow{\sim} P([\Omega]) + hH_F^2(X/S)[[h]] \subset H_F^2(X/S)[[h]]$$

between $Q(X, \Omega)$ and the set of all power series in h with coefficients in $H_F^2(X/S)$ and constant term $P([\Omega])$.

In particular, quantizations always exist (provided the manifold in question is admissible). Moreover, one can define a preferred quantization:

Definition 1.9. A quantization $\mathcal{D} \in Q(X, \Omega)$ of an admissible symplectic S -manifold is called *canonical* if its period $\text{Per}(\mathcal{D}) \in H_{\text{DR}}^2(X/S)[[h]]$ is the constant power series $[\Omega]$.

The period map itself is completely canonical. However, the parametrization of quantizations by formal power series with coefficients in $H_F^2(X/S)$ does depend on the splitting $F: H_{\text{DR}}^2(X/S) \rightarrow H_F^2(X/S)$. Sometimes there is a canonical choice of this splitting—for instance, when X is projective over \mathbb{C} , such a splitting is provided by Hodge theory. The canonical quantization enjoys several nice properties, but we should warn the reader that it *does not* have to exist, even for an admissible manifold—unless $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$, so that the period map is surjective.

In the basic case when $S = \text{Spec } k$ is a point, Theorem 1.8 is completely parallel to what one has for symplectic deformations—in other words, for commutative deformations of the pair $\langle X, \Omega \rangle$. The commutative version of the period map was introduced in [KV], and it is very simple: it sends a deformation to the associated family of cohomology classes $[\Omega]_h \in H_{\text{DR}}^2(X)$ of the corresponding symplectic forms. This motivates our terminology. Unfortunately, we do not have a similar interpretation of the non-commutative period map.

Our definition of the period map is also quite simple, in fact, it takes one paragraph—and the full proof of Theorem 1.8 takes only two pages. Both are contained in Section 4. But both the definition and the proof require some preliminary machinery. All the facts we need are essentially standard, but there are no suitable references in the literature. Thus we have to devote Section 2 and Section 3 to these preliminaries. So as not to overwhelm the reader with technicalities, some proofs are postponed till Section 5 (which only depends on Section 2). Section 6 contains some extensions of our results to other frameworks. In particular, we consider the equivariant version of Theorem 1.8. We also clarify the relation between quantizations and the universal symplectic deformation constructed in [KV] by showing that both can be incorporated into a single multi-parameter partially non-commutative deformation. It is here that the general relative setting of Theorem 1.8 plays a crucial role. Finally, Section 7 is taken up with some concluding remarks—we try to place our results in the general context and compare them with existing alternative approaches to deformation quantization.

2. PRELIMINARIES ON HARISH-CHANDRA TORSORS

2.1. Harish-Chandra pairs. The following definition was first introduced most probably by A. Beilinson and J. Bernstein, [BB].

Definition 2.1. A *Harish-Chandra pair* $\langle G, \mathfrak{h} \rangle$ over the field k is a pair of a connected affine algebraic group G over k , a Lie algebra \mathfrak{h} over k equipped with a G -action, and an embedding $\mathfrak{g} \rightarrow \mathfrak{h}$ of the Lie algebra \mathfrak{g} of the group G into the Lie algebra \mathfrak{h} such that the adjoint action of \mathfrak{g} on \mathfrak{h} is the differential of the given G -action.

A *module* V over a Harish-Chandra pair $\langle G, \mathfrak{h} \rangle$ is a representation V of the Lie algebra \mathfrak{h} whose restriction to $\mathfrak{g} \subset \mathfrak{h}$ is integrated to an algebraic representation of the group G .

Just as when working with jet bundles, in applications it is important to allow groups which are not finite-dimensional, or, more precisely, to allow G to be the projective limit of affine algebraic groups. To extend Definition 2.1 to this case, we make the following modifications. The Lie algebra \mathfrak{g} is a topological vector space equipped with a “compact” topology—namely, it is a projective limit of finite-dimensional vector spaces. Note that topological vector spaces of this type form an abelian category (the one dual to the category of usual vector spaces). The Lie algebra \mathfrak{h} is also a projective limit of finite-dimensional vector spaces and moreover, $\mathfrak{g} \subset \mathfrak{h}$ is of finite codimension (in other words, \mathfrak{g} is closed in \mathfrak{h}). The group G will always be an affine group scheme and a projective limit of affine algebraic groups of finite type over k .

A module V over a Harish-Chandra pair will also be a projective limit of finite-dimensional vector spaces, and we will assume that both G and \mathfrak{h} act in a way compatible with this topology. In the case of the group scheme $G = \operatorname{Spec} A$, this means that the G -action on V is given by a coaction $V^* \rightarrow V^* \otimes_k A$ of the Hopf algebra A on the (discrete, although infinite-dimensional) vector space V^* topologically dual to V . Modules defined in this way also form an abelian category. This category comes equipped with a symmetric tensor product (defined in the obvious way). The unit object for this product is the one-dimensional trivial representation k .

As usual both for groups and for Lie algebras, given a $\langle G, \mathfrak{h} \rangle$ -module V , by the *cohomology groups* $H^\bullet(\langle G, \mathfrak{h} \rangle, V)$ of the module V we will understand the Ext-groups $\operatorname{Ext}^\bullet(k, V)$ (taken in the category of topological Harish-Chandra modules).

Remark 2.2. There is a more general notion of a Harish-Chandra pair (see [BFM]), where the Lie algebra \mathfrak{h} is allowed to be a so-called *Tate topological vector space*. In this paper, we do not need it.

2.2. Torsors. Let X be an S -manifold. To keep things precise, we will say that given a group scheme G , by a G -torsor over X we will understand a scheme Y faithfully flat over X and equipped with an action map $G \times Y \rightarrow Y$ which commutes with the projection to X and induces an isomorphism $G \times Y \rightarrow Y \times_X Y$. (In our applications, all torsors will be locally trivial in Zariski topology.) Assume given a Harish-Chandra pair $\langle G, \mathfrak{h} \rangle$. For any G -torsor \mathcal{M} over X we have the Lie algebra bundles $\mathfrak{g}_{\mathcal{M}}$ and $\mathfrak{h}_{\mathcal{M}}$ on X associated to the G -modules \mathfrak{g} and \mathfrak{h} . The map $\mathfrak{g} \rightarrow \mathfrak{h}$ induces a map $\mathfrak{g}_{\mathcal{M}} \rightarrow \mathfrak{h}_{\mathcal{M}}$. Moreover, since we work in characteristic 0, the scheme G is smooth, so that the faithfully flat projection $\rho: \mathcal{M} \rightarrow X$ is also smooth. Therefore we have a G -equivariant short exact sequence

$$0 \rightarrow \mathcal{T}_{\mathcal{M}/X} \rightarrow \mathcal{T}_{\mathcal{M}/S} \rightarrow \rho^* \mathcal{T}_{X/S} \rightarrow 0$$

of relative tangent bundles, which by descent gives the so-called *Atiyah extension*

$$0 \rightarrow \mathfrak{g}_{\mathcal{M}} \xrightarrow{\iota_{\mathcal{M}}} \mathcal{E}_{\mathcal{M}} \rightarrow \mathcal{T}_{X/S} \rightarrow 0 \quad (2.1)$$

of bundles on X . Recall that a G -invariant connection¹ on the principal G -bundle \mathcal{M} is by definition given by a bundle map $\theta_{\mathcal{M}}: \mathcal{E}_{\mathcal{M}} \rightarrow \mathfrak{g}_{\mathcal{M}}$ which splits the extension (2.1)—in other words, the composition $\theta_{\mathcal{M}} \circ \iota_{\mathcal{M}}: \mathfrak{g}_{\mathcal{M}} \rightarrow \mathfrak{g}_{\mathcal{M}}$ is the identity map. Equivalently, one can specify the corresponding G -invariant \mathfrak{g} -valued 1-form

¹As noted in Section 1.1, connections are understood relatively over S .

$\rho^*\theta_{\mathcal{M}} \in H^0(\mathcal{M}, \Omega_{\mathcal{M}}^1 \otimes \mathfrak{g})$, where $\rho: \mathcal{M} \rightarrow X$ is the projection. A connection $\theta_{\mathcal{M}}$ is *flat* if the corresponding one-form $\Omega = \rho^*\theta_{\mathcal{M}}$ satisfies the Maurer–Cartan equation $2d\Omega + \Omega \wedge \Omega = 0$. Generalizing this, by an \mathfrak{h} -valued connection on \mathcal{M} we will understand a bundle map $\theta_{\mathcal{M}}: \mathcal{E}_{\mathcal{M}} \rightarrow \mathfrak{h}_{\mathcal{M}}$ such that the composition $\theta_{\mathcal{M}} \circ \iota_{\mathcal{M}}: \mathfrak{g}_{\mathcal{M}} \rightarrow \mathfrak{h}_{\mathcal{M}}$ is the given embedding. Again, an \mathfrak{h} -valued connection $\theta_{\mathcal{M}}$ is called *flat* if the corresponding \mathfrak{h} -valued 1-form $\Omega = \rho^*\theta_{\mathcal{M}}$ satisfies $2d\Omega + \Omega \wedge \Omega = 0$.

Definition 2.3. By a *Harish-Chandra $\langle G, \mathfrak{h} \rangle$ -torsor* \mathcal{M} over the S -manifold X we will understand a pair $\langle \mathcal{M}, \theta_{\mathcal{M}} \rangle$ of a G -torsor \mathcal{M} over X and a flat \mathfrak{h} -valued connection $\theta_{\mathcal{M}}: \mathcal{E}_{\mathcal{M}} \rightarrow \mathfrak{h}_{\mathcal{M}}$ on \mathcal{M} .

The notion of a Harish-Chandra torsor has the usual functorialities. In particular, if we have a map of Harish-Chandra pairs $f: \langle G, \mathfrak{h} \rangle \rightarrow \langle G_1, \mathfrak{h}_1 \rangle$ and a $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} , then we canonically obtain the induced $\langle G_1, \mathfrak{h}_1 \rangle$ -torsor $\mathcal{M}_1 = f_*\mathcal{M} = \mathcal{M} \times^G G_1$. For a tautological Harish-Chandra pair $\langle G, \mathfrak{g} \rangle$, a $\langle G, \mathfrak{g} \rangle$ -torsor over X is the same as a principal G -bundle equipped with a G -invariant flat connection.

The set of isomorphism classes of all $\langle G, \mathfrak{h} \rangle$ -torsors over an S -manifold X will be denoted by $H^1(X, \langle G, \mathfrak{h} \rangle)$. The torsors themselves form a category. This category is a groupoid, which we will denote by $\mathcal{H}^1(X, \langle G, \mathfrak{h} \rangle)$.

An important special class of Harish-Chandra torsors is the following one.

Definition 2.4. A $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} over an S -manifold X is called *transitive* if the connection map $\theta_{\mathcal{M}}: \mathcal{E}_{\mathcal{M}} \rightarrow \mathfrak{h}_{\mathcal{M}}$ is an isomorphism.

Given a transitive $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} , one can invert the connection map and obtain a bundle map $\theta_{\mathcal{M}}^{-1}: \mathfrak{h}_{\mathcal{M}} \rightarrow \mathcal{E}_{\mathcal{M}}$, or, equivalently, a G -equivariant map $\mathfrak{h} \otimes \mathcal{O}_{\mathcal{M}} \rightarrow \mathcal{T}_{\mathcal{M}/S}$ of vector bundles on \mathcal{M} . The latter is in turn equivalent to giving a G -equivariant map $\mathfrak{h} \rightarrow H^0(\mathcal{M}, \mathcal{T}_{\mathcal{M}/S})$, and it is easy to check that this is a Lie algebra map if and only if the connection $\theta_{\mathcal{M}}$ is flat. Thus for a transitive $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} , the whole Harish-Chandra pair $\langle G, \mathfrak{h} \rangle$ acts on the scheme \mathcal{M} —that is, the Lie algebra \mathfrak{h} acts by derivations of the structure sheaf, and the restriction of this action to $\mathfrak{g} \subset \mathfrak{h}$ is the differential of the G -action. The \mathfrak{h} -action is transitive, which explains our terminology. Conversely, a G -torsor \mathcal{M} equipped with a compatible transitive action of \mathfrak{h} gives rise to a transitive $\langle G, \mathfrak{h} \rangle$ -torsor in the sense of Definition 2.4.

2.3. Localization. Assume given an S -manifold X , a Harish-Chandra pair $\langle G, \mathfrak{h} \rangle$, and a $\langle G, \mathfrak{h} \rangle$ -torsor $\langle \mathcal{M}, \theta_{\mathcal{M}} \rangle$ over X/S . Let V be a finite-dimensional $\langle G, \mathfrak{h} \rangle$ -module. Then we have a map

$$f: \langle G, \mathfrak{h} \rangle \rightarrow \langle \mathrm{GL}(V), \mathfrak{gl}(V) \rangle$$

and the induced torsor $f_*\mathcal{M}$. If \mathcal{V} is the vector bundle on X associated to the G -module V , then $f_*\mathcal{M}$ coincides with the principal $\mathrm{GL}(V)$ -bundle of frames in \mathcal{V} . By construction it carries a flat connection. Thus \mathcal{V} also carries a canonical flat connection ∇ . Explicitly, let $\xi \in \Gamma(U, \mathcal{E}_{\mathcal{M}})$ be a local section of the Atiyah sheaf $\mathcal{E}_{\mathcal{M}}$, and let $a \in \Gamma(U, \mathcal{V})$ be a local section of the bundle \mathcal{V} . Then by construction both the Atiyah sheaf and the Lie algebra bundle $\mathfrak{h}_{\mathcal{M}}$ act on sections of the bundle \mathcal{V} , and the expression

$$\nabla_{\xi}(a) = \xi \cdot a - \theta_{\mathcal{M}}(\xi) \cdot a \in \Gamma(U, \mathcal{V}) \quad (2.2)$$

only depends on the image of ξ in the tangent sheaf \mathcal{T}_X . Thus it defines a connection on \mathcal{V} , which is exactly ∇ .

When the module V is only a projective limit of finite-dimensional vector spaces, the group $\mathrm{GL}(V)$ is not well-defined. However, we can still define a flat connection on the associated bundle \mathcal{V} by directly applying (2.2). Associated bundle in this case lies in the completed category of pro-coherent sheaves — just as the jet bundles considered in Section 1.1.

To sum up, given the torsor \mathcal{M} , for any module $\langle G, \mathfrak{h} \rangle$ -module V we obtain a flat bundle \mathcal{V} on the S -manifold X . In other words, the torsor \mathcal{M} defines a functor from the category of $\langle G, \mathfrak{h} \rangle$ -modules to the category of flat vector bundles on X/S . We will call this the *localization functor* associated to \mathcal{M} , and we will denote it by

$$\mathcal{V} = \mathrm{loc}(\mathcal{M}, V).$$

The functor of localization with respect to \mathcal{M} is obviously exact. In particular, it extends to derived categories and induces a canonical localization map

$$\mathrm{loc}(\mathcal{M}, -): H^\bullet(\langle G, \mathfrak{h} \rangle, V) \rightarrow H_{\mathrm{DR}}^\bullet(X, \mathcal{V}).$$

Moreover, localization is a tensor functor.

Remark 2.5. Most probably, the converse is also true: modulo the appropriate finiteness conditions, every tensor functor from the category of $\langle G, \mathfrak{h} \rangle$ -modules to the category of flat bundles on X comes from a $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} on X . Equivalent functors give isomorphic torsors. We do not develop this Tannakian-type formalism here to save space.

Localization can be also be described in a different language. Recall that the standard descent procedure induces an equivalence between the category of vector bundles on X equipped with a flat connection and the category of G -equivariant vector bundle on \mathcal{M} equipped with a flat connection which is compatible with the G -action. Compatibility here means that for every vector $\xi \in G$, the covariant derivative ∇_ξ with respect to the corresponding vector field on \mathcal{M} coincides with the action of ξ coming from the G -equivariant structure (it is well-known that this definition does not give the correct equivariant version of the derived category of flat vector bundles; however, if we stick to the abelian categories, the descent works just fine). Using this equivalence, one does the localization procedure in two steps. First, one considers the constant vector bundle $V \otimes \mathcal{O}_{\mathcal{M}}$ on \mathcal{M} with the trivial flat connection, and equips it with the product G -action. The connection and the G -action are not compatible. Then one corrects the connection on $V \otimes \mathcal{O}_{\mathcal{M}}$ by (2.2) — for this one needs the \mathfrak{h} -valued connection on \mathcal{M} and the \mathfrak{h} -action on V . After that, the localization $\mathrm{loc}(\mathcal{M}, V)$ is obtained by descent.

The descent procedure is of course quite general, it is by no means limited to vector bundles of type $V \otimes \mathcal{O}_{\mathcal{M}}$. We note that descent works especially well when the $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} is transitive in the sense of Definition 2.4. In this case, we have an \mathfrak{h} -action on \mathcal{M} which trivializes the tangent bundle $\mathcal{T}_{\mathcal{M}}$, $\mathcal{T}_{\mathcal{M}} \cong \mathfrak{h} \otimes \mathcal{O}_{\mathcal{M}}$ (the Lie algebra structure on $\mathfrak{h} \otimes \mathcal{O}_{\mathcal{M}}$ is not $\mathcal{O}_{\mathcal{M}}$ -linear, it is skew-linear with respect to the \mathfrak{h} -action on $\mathcal{O}_{\mathcal{M}}$). It follows immediately that giving a compatible flat connection ∇ on a G -equivariant vector bundle \mathcal{E} on \mathcal{M} is equivalent to extending the G -action on \mathcal{E} to a compatible \mathfrak{h} -action. Thus we have the following.

Lemma 2.6. *Let \mathcal{M} be a transitive $\langle G, \mathfrak{h} \rangle$ -torsor over an S -manifold X . Then the category of vector (pro)bundles on X equipped with a flat connection is equivalent to the category of $\langle G, \mathfrak{h} \rangle$ -equivariant vector (pro)bundles on \mathcal{M} . \square*

Here a $\langle G, \mathfrak{h} \rangle$ -equivariant vector bundle \mathcal{E} is a G -equivariant vector bundle equipped with an action of the Lie algebra \mathfrak{h} which is compatible with the \mathfrak{h} -action on \mathcal{M} and gives the differential of the G -action after restriction to $\mathfrak{g} \subset \mathfrak{h}$. For a constant $\langle G, \mathfrak{h} \rangle$ -module V , one simply takes the product $\langle G, \mathfrak{h} \rangle$ -action on $V \otimes \mathcal{O}_{\mathcal{M}}$; descent by Lemma 2.6 gives $\text{loc}(\mathcal{M}, V)$.

2.4. Harish-Chandra extensions. In the body of the paper, we will need to study the behavior of Harish-Chandra torsors under extensions. More precisely, we need what is usually referred to as the long exact sequence in the non-abelian cohomology. So as not to interrupt the exposition too much, we give all the statements here, and we postpone the proofs till Section 5.

Let $\langle G, \mathfrak{h} \rangle$ be a Harish-Chandra pair, and let V be a $\langle G, \mathfrak{h} \rangle$ -module. Consider V as an (additive) algebraic group. By an *extension*

$$1 \rightarrow V \xrightarrow{\rho} \langle G_1, \mathfrak{h}_1 \rangle \xrightarrow{\pi} \langle G, \mathfrak{h} \rangle \rightarrow 1 \quad (2.3)$$

of the pair $\langle G, \mathfrak{h} \rangle$ by the module V we will understand a Harish-Chandra pair $\langle G_1, \mathfrak{h}_1 \rangle$ equipped with a map $f: \langle G_1, \mathfrak{h}_1 \rangle \rightarrow \langle G, \mathfrak{h} \rangle$ such that $\text{Ker } f = \langle V, V \rangle \subset \langle G_1, \mathfrak{h}_1 \rangle$ is the tautological Harish-Chandra pair associated to V , and the adjoint action of $\langle G, \mathfrak{h} \rangle$ on V comes from the given module structure. In other words, we have an extension of groups compatible with the extension of the Lie algebras.

Given an S -manifold and a $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} over X/S , we denote by

$$H_{\mathcal{M}}^1(X, \langle G_1, \mathfrak{h}_1 \rangle)$$

the set of isomorphism classes of $\langle G_1, \mathfrak{h}_1 \rangle$ -torsors \mathcal{M}_1 on X/S equipped with an isomorphism $\pi_* \mathcal{M}_1 \cong \mathcal{M}$. We will call torsors of this type *liftings* of the torsor \mathcal{M} to the Harish-Chandra pair $\langle G_1, \mathfrak{h}_1 \rangle$ (if $\langle G_1, \mathfrak{h}_1 \rangle$ were to be a subobject $\langle G_1, \mathfrak{h}_1 \rangle \subset \langle G, \mathfrak{h} \rangle$, the common term would be “restriction”).

Let $\mathcal{V} = \text{loc}(\mathcal{M}, V)$ be the localization of the $\langle G, \mathfrak{h} \rangle$ -module V with respect to the torsor \mathcal{M} . The basic statement we need is the following one.

Proposition 2.7. *Let \mathcal{M} be a transitive $\langle G, \mathfrak{h} \rangle$ -torsor over an S -manifold X .*

- (i) *There exists a canonical cohomology class $c \in H^2(\langle G, \mathfrak{h} \rangle, V)$ with the following property: the set $H_{\mathcal{M}}^1(X, \langle G_1, \mathfrak{h}_1 \rangle)$ is non-empty if and only if the localization $\text{loc}(\mathcal{M}, c) \in H_{\text{DR}}^2(X/S, \mathcal{V})$ is trivial.*
- (ii) *If the class $\text{loc}(\mathcal{M}, c)$ is indeed trivial, then the set $H_{\mathcal{M}}^1(X, \langle G_1, \mathfrak{h}_1 \rangle)$ is naturally a torsor over the de Rham cohomology group $H_{\text{DR}}^1(X/S, \mathcal{V})$.*

We will also need a more involved statement, a certain compatibility result vaguely reminiscent of the octahedron axiom in homological algebra. Consider a Harish-Chandra pair $\langle G, \mathfrak{h} \rangle$, and let

$$0 \rightarrow U \xrightarrow{a} V \xrightarrow{b} W \rightarrow 0 \quad (2.4)$$

be a short exact sequence of $\langle G, \mathfrak{h} \rangle$ -modules. Assume given an extension

$$1 \rightarrow V \rightarrow \langle G_1, \mathfrak{h}_1 \rangle \xrightarrow{\pi} \langle G, \mathfrak{h} \rangle \rightarrow 1$$

of the Harish-Chandra pair $\langle G, \mathfrak{h} \rangle$ by the module V , and denote its cohomology class by $c \in H^2(\langle G, \mathfrak{h} \rangle, V)$. Let $\langle G_0, \mathfrak{h}_0 \rangle = \langle G_1, \mathfrak{h}_1 \rangle / U$ be the associated extension of $\langle G, \mathfrak{h} \rangle$ by W . By definition, $\langle G_1, \mathfrak{h}_1 \rangle$ is an extension of $\langle G_0, \mathfrak{h}_0 \rangle$ by the module U . Denote its cohomology class by $c_0 \in H^2(\langle G_0, \mathfrak{h}_0 \rangle, U)$.

Assume given a $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} over X/S , and let \mathcal{U} , \mathcal{V} and \mathcal{W} be the localizations of the $\langle G, \mathfrak{h} \rangle$ -modules U , V and W . We have a long exact sequence of de Rham cohomology groups

$$H_{\text{DR}}^2(X/S, \mathcal{U}) \xrightarrow{a} H_{\text{DR}}^2(X/S, \mathcal{V}) \xrightarrow{b} H_{\text{DR}}^2(X/S, \mathcal{W}) \rightarrow \quad (2.5)$$

Assume that we are in the following situation: the localization

$$\text{loc}(\mathcal{M}, c) \in H_{\text{DR}}^2(X/S, \mathcal{V})$$

is *not* trivial, but its restriction

$$b(\text{loc}(\mathcal{M}, c)) \in H_{\text{DR}}^2(X/S, \mathcal{W})$$

is trivial. Then the $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} does not lift to a $\langle G_1, \mathfrak{h}_1 \rangle$ -torsor over X/S , but it does lift to a $\langle G_0, \mathfrak{h}_0 \rangle$ -torsor. For every such lifting $\mathcal{M}_1 \in H_{\mathcal{M}}^1(X/S, \langle G_0, \mathfrak{h}_0 \rangle)$, we obtain a lifting of the obstruction cohomology class $\text{loc}(\mathcal{M}, c) \in H_{\text{DR}}^2(X/S, \mathcal{V})$ to a cohomology class in $H_{\text{DR}}^2(X/S, \mathcal{U})$, namely, the class $\text{loc}(\mathcal{M}_1, c_0) \in H_{\text{DR}}^2(X/S, \mathcal{U})$. By Proposition 2.7, we know that the set $H_{\mathcal{M}}^1(X/S, \langle G_0, \mathfrak{h}_0 \rangle)$ is a torsor over the group $H_{\text{DR}}^1(X/S, \mathcal{W})$. On the other hand, by the exact sequence (2.5) the group $H_{\text{DR}}^1(X/S, \mathcal{W})$ acts on the set $H_{\text{DR}}^2(X/S, \mathcal{U})$.

Lemma 2.8. *The map $H_{\mathcal{M}}^1(X/S, \langle G_0, \mathfrak{h}_0 \rangle) \rightarrow H_{\text{DR}}^2(X/S, \mathcal{U})$ given by*

$$\mathcal{M}_1 \mapsto \text{loc}(\mathcal{M}_1, c_0)$$

is compatible with the $H_{\text{DR}}^1(X/S, \mathcal{W})$ -action on both sides.

The reader will find the proofs of Proposition 2.7 and Lemma 2.8 in Section 5. We note that the condition of transitivity is needed only in the proof of the “if” part of Proposition 2.7 (i). The Proposition is also true for general $\langle G, \mathfrak{h} \rangle$ -torsors, but the proof is slightly longer; since in our applications all torsors will be transitive, we impose this assumption right from the start to save space.

3. QUANTIZATION VIA FORMAL GEOMETRY

3.1. The bundle of coordinate systems. Formal geometry is a technique of dealing with various questions in differential geometry by solving them first in the universal context, — that is, over a formal polydisc, — and equivariantly with respect to the Lie algebra of vector fields on the polydisc. It dates back at least to the papers [GK] by I. Gelfand and D. Kazhdan and/or [Bt] by R. Bott. However, there are no convenient general references. We have learned what we know of this technique at B. Feigin’s Moscow seminar. Since we do need to use it, — and in the relative setting, to make things worse, — we give here a self-contained outline of the basic setup.

Fix a dimension n , at this point not necessarily even. Consider the formal power series algebra $\mathcal{A} = k[[x_1, \dots, x_n]]$. Denote by \mathcal{W} the Lie algebra of derivations

of the algebra \mathcal{A} —in other words, the Lie algebra of vector fields on the formal polydisc $\mathrm{Spf} \mathcal{A}$. Consider the subalgebra $W_0 \subset W$ of vector fields vanishing at the closed point (equivalently, derivations preserving the maximal ideal in \mathcal{A}). Then the Lie algebra W_0 is naturally the Lie algebra of a proalgebraic group $\mathrm{Aut} \mathcal{A}$ of automorphisms of the local k -algebra \mathcal{A} . In the language of Section 2, we have a Harish-Chandra pair $\langle \mathrm{Aut} \mathcal{A}, W \rangle$.

Let X be an S -manifold of dimension n with projection map $\pi: X \rightarrow S$. For any scheme T , giving a map $\eta: T \rightarrow X$ is equivalent to giving a map $p(\eta) = \pi \circ \eta: T \rightarrow S$ and a section $\sigma(\eta): T \rightarrow T \times_S X$ of the canonical projection $T \times_S X \rightarrow T$. Formal germs of functions on $T \times_S X$ near the closed subscheme $\sigma(\eta)(T) \subset T \times_S X$ form a sheaf of (topological) \mathcal{O}_T -algebras on T which we denote by $\widehat{\mathcal{O}}_{X,\eta}$. If the scheme T is affine, the sheaf $\widehat{\mathcal{O}}_{X,\eta}$ is non-canonically isomorphic to the completed tensor product $\mathcal{O}_T \widehat{\otimes} \mathcal{A}$. Let $\mathcal{M}(T)$ be the set of all pairs

$$\langle \eta: T \rightarrow X, \varphi: \widehat{\mathcal{O}}_{X,\eta} \cong \mathcal{O}_T \widehat{\otimes} \mathcal{A} \rangle,$$

where φ is an isomorphism of sheaves of topological \mathcal{O}_T -algebras. Geometrically, such a pair corresponds to a commutative diagram

$$\begin{array}{ccc} \mathrm{Spf} \mathcal{A} \times T & \xrightarrow{\varphi} & X \\ \downarrow & & \downarrow \pi \\ T & \xrightarrow{p(\eta)} & S \end{array} \quad (3.1)$$

which induces an identification between the formal neighborhood of $\sigma(\eta)(T)$ in $T \times_S X$ and the product $\mathrm{Spf} \mathcal{A} \times T$ —loosely speaking, a family of formal coordinate systems on X/S parametrized by T . Setting $T \mapsto \mathcal{M}(T)$ defines a functor from the category of affine schemes to the category of sets. We leave it to the reader to check that this functor is represented by a (non-Noetherian) scheme $\mathcal{M}_{\mathrm{coord}}$, smooth and affine over X . In fact, $\mathcal{M}_{\mathrm{coord}}$ is the projective limit of a family of S -manifolds, and it is a torsor over the group $\mathrm{Aut} \mathcal{A}$ with respect to the natural action. Moreover, the torsor $\mathcal{M}_{\mathrm{coord}}$ carries a structure of a transitive Harish-Chandra torsor over $\langle \mathrm{Aut} \mathcal{A}, W \rangle$. Indeed, the Lie algebra W also acts on \mathcal{A} , hence on $\mathcal{M}_{\mathrm{coord}}$, and the action map descends to a map

$$a: \mathcal{W}_{\mathcal{M}} \rightarrow \mathcal{E}_{\mathcal{M}},$$

where $\mathcal{W}_{\mathcal{M}}$ is the vector bundle on X associated to W , and $\mathcal{E}_{\mathcal{M}}$ is the Atiyah bundle of the torsor $\mathcal{M}_{\mathrm{coord}}$. It is elementary to check that the map a is in fact an isomorphism. To define a W -valued flat connection $\theta_{\mathcal{M}}: \mathcal{E}_{\mathcal{M}} \rightarrow \mathcal{W}_{\mathcal{M}}$ on $\mathcal{M}_{\mathrm{coord}}$, it suffices to take the inverse isomorphism $\theta_{\mathcal{M}} = a^{-1}$. Since a is obtained from a Lie algebra map $W \rightarrow \mathcal{T}_{\mathcal{M}}$, the connection $\theta_{\mathcal{M}}$ is flat.

Definition 3.1. The $\langle \mathrm{Aut} \mathcal{A}, W \rangle$ -torsor $\langle \mathcal{M}_{\mathrm{coord}}, \theta_{\mathcal{M}} = a^{-1} \rangle$ over X is called the *bundle of formal coordinate systems* on the S -manifold X .

The bundle of formal coordinate systems is the main object of formal geometry. It is completely canonical, and it allows one to do the following two things:

- (i) Obtain various canonical sheaves on X , such as sheaves of sections of different symmetric and tensor powers of the tangent bundle $\mathcal{T}(X)$, as

sheaves of flat sections of localizations of appropriate representations of the Harish-Chandra pair $\langle \text{Aut } \mathcal{A}, W \rangle$.

- (ii) Describe various differential-geometric structures on X as reductions of the torsor $\mathcal{M}_{\text{coord}}$ to different subgroups in $\langle \text{Aut } \mathcal{A}, W \rangle$.

Usual applications revolve around (i). More precisely, the construction one uses is the following one. The simplest module over the Harish-Chandra pair $\langle \text{Aut } \mathcal{A}, W \rangle$ is the algebra \mathcal{A} itself. It is easy to check that its localization with respect to the $\langle \text{Aut } \mathcal{A}, W \rangle$ -torsor $\mathcal{M}_{\text{coord}}$ coincides with the jet bundle $J^\infty \mathcal{O}_X$:

$$\text{loc}(\mathcal{M}_{\text{coord}}, \mathcal{A}) \cong J^\infty \mathcal{O}_X.$$

The sheaf of its flat sections is the structure sheaf \mathcal{O}_X of the variety X . Analogously, one can take the $\langle \text{Aut } \mathcal{A}, W \rangle$ -module W of vector fields on \mathcal{A} , or the module $\Omega^p \mathcal{A}$ of p -forms on \mathcal{A} for some $p \leq n$, or, more generally, the $\langle \text{Aut } \mathcal{A}, W \rangle$ -module Ξ of tensors of some type on \mathcal{A} . Then its localization is $J^\infty \mathcal{T}$, the jet bundle of the tangent sheaf \mathcal{T}_X , resp. $J^\infty \Omega_X^p$, resp. the jet bundle of the sheaf of tensors on X of the same type as Ξ . As usual, one recovers the sheaf from its jet bundle by taking flat sections.

One can use this construction, for instance, to obtain characteristic classes of the variety X starting from cohomology classes of the $\langle \text{Aut } \mathcal{A}, W \rangle$ -module Ξ . In the present paper, we leave this subject completely alone. Our applications of formal geometry are related to (ii).

The following is the motivating example. Assume that the dimension $n = 2d$ is even, and equip the formal polydisc $\mathcal{A} = k[[x_1, \dots, x_d, y_1, \dots, y_d]]$ with the symplectic form $\omega = \sum dx_i \wedge dy_i$. Denote by $H \subset W$ the Lie subalgebra of Hamiltonian vector fields — in other words, the vector fields that preserve the symplectic form. As before, the subalgebra $W_0 \cap H \subset H$ is naturally integrated to a pro-algebraic group $\text{Symp } \mathcal{A}$, and we have a Harish-Chandra pair $\langle \text{Symp } \mathcal{A}, H \rangle$.

Lemma 3.2. *Let X be an S -manifold of dimension $n = 2d$. There is a one-to-one correspondence between symplectic structures on X/S and reductions of the $\langle \text{Aut } \mathcal{A}, W \rangle$ -torsor $\mathcal{M}_{\text{coord}}$ to $\langle \text{Symp } \mathcal{A}, H \rangle \subset \langle \text{Aut } \mathcal{A}, W \rangle$.*

Proof. Given a symplectic form, one takes the subvariety $\mathcal{M}_s \subset \mathcal{M}_{\text{coord}}$ of formal coordinate systems $\varphi: \text{Spf } \mathcal{A} \rightarrow X$ compatible with symplectic forms on both sides, and notices that by the formal Darboux Theorem, \mathcal{M}_s is a torsor over $\text{Symp } \mathcal{A} \subset \text{Aut } \mathcal{A}$.

Conversely, given such a reduction $\mathcal{M}_s \subset \mathcal{M}_{\text{coord}}$, one recalls that by definition, the $\langle \text{Aut } \mathcal{A}, W \rangle$ -module $\Omega^2 \mathcal{A}$ contains an H -invariant vector ω . By localization, ω gives a flat section of the jet bundle $J^\infty \Omega_X^2$, thus a symplectic form on X/S . \square

Given a symplectic S -manifold, we will use the term *bundle of symplectic formal coordinate systems* for the associated $\langle \text{Symp } \mathcal{A}, H \rangle$ -torsor \mathcal{M}_s on X .

The cohomology class $[\Omega] \in H_{\text{DR}}^2(X)$ of the symplectic form Ω also has a natural interpretation in terms of the torsor \mathcal{M}_s . Namely, the de Rham complex of the polydisc gives a resolution of the trivial $\langle \text{Symp } \mathcal{A}, H \rangle$ -module k , and the standard symplectic form on the polydisc defines a cohomology class $[\omega] \in H^2(\langle \text{Symp } \mathcal{A}, H \rangle, k)$. The class $[\Omega] \in H_{\text{DR}}^2(X)$ is the localization of this class $[\omega]$ with respect to the

torsor \mathcal{M}_s . The class $[\omega]$ corresponds to the central extension

$$0 \rightarrow k \rightarrow \mathcal{A} \rightarrow H \rightarrow 0$$

of the Lie algebra H of Hamiltonian vector fields on the polydisc (or rather, to the corresponding extension of Harish-Chandra pairs).

Remark 3.3. V. Drinfeld has explained to us that the torsor $\mathcal{M}_{\text{coord}}$ of formal coordinate systems on a manifold X over an algebraically closed field k can be in fact characterized by a universal property (see [BD], especially Remark 2.6.4 and Example 2.6.5). Firstly, for any Harish-Chandra pair $\langle G, \mathfrak{h} \rangle$, there exists a canonical formal manifold, carrying a transitive $\langle G, \mathfrak{h} \rangle$ -torsor (in the language of [BD], our transitive $\langle G, \mathfrak{h} \rangle$ -torsors correspond to (\mathfrak{h}, G) -structures). This manifold is $X = \text{Spf } k[[\mathfrak{h}/\mathfrak{g}^*]]$, the formal completion at 0 of the quotient $\mathfrak{h}/\mathfrak{g}$. Secondly, for any Harish-Chandra pair $\langle G, \mathfrak{h} \rangle$ and a transitive $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} on a manifold X of dimension n , there exists a unique map $\langle G, \mathfrak{h} \rangle \rightarrow \langle \text{Aut } \mathcal{A}, W \rangle$ of Harish-Chandra pairs and a unique compatible map $\tau: \mathcal{M} \rightarrow \mathcal{M}_{\text{coord}}$ from \mathcal{M} into the torsor $\mathcal{M}_{\text{coord}}$. To construct the map τ , one fixes an isomorphism between $\text{Spf } k[[\mathfrak{h}/\mathfrak{g}^*]]$ and the standard formal polydisc $\text{Spf } \mathcal{A}$ and notices that a point $m \in \mathcal{M}$ lying over a point $x \in X$ induces an isomorphism between $\text{Spf } k[[\mathfrak{h}/\mathfrak{g}^*]]$ and the formal neighborhood of $x \in X$. In particular, the torsor $\mathcal{M}_{\text{coord}}$ is the unique, up to a unique isomorphism, transitive $\langle \text{Aut } \mathcal{A}, W \rangle$ -torsor over X .

3.2. Automorphisms of the formal Weyl algebra. We can now reformulate the quantization problem in the language of formal geometry; additionally, we will recall some standard facts on automorphisms of the formal Weyl algebra D .

The basic statement is very straightforward; essentially, it is a quantized version of Lemma 3.2. Consider the Lie algebra $\text{Der } D$ of $k[[h]]$ -linear derivations of the $k[[h]]$ -algebra D . Since the derivations in $\text{Der } D$ are $k[[h]]$ -linear, the algebra $\text{Der } D$ it preserves the ideal $hD \subset D$ and acts on the quotient $\mathcal{A} = D/hD$. The action map $a: (\text{Der } D) \rightarrow W$ factors through the quotient $(\text{Der } D)_0$ which is isomorphic to the Lie algebra $H \subset W$ of Hamiltonian vector fields on the polydisc. The subalgebra $(\text{Der } D)^0 = a^{-1}(W^0)$ is naturally integrated to the pro-algebraic group $\text{Aut } D$, namely, the group of $k[[h]]$ -linear automorphisms of the Weyl algebra D preserving the two-sided ideal $\mathfrak{m}_{\mathcal{A}} + hD \subset D$. Thus we have a natural Harish-Chandra pair $\langle \text{Aut } D, \text{Der } D \rangle$.

Lemma 3.4. *Let X be an S -manifold of dimension $n = 2d$ equipped with a symplectic form Ω . Denote by \mathcal{M}_s the bundle of symplectic formal coordinate systems on $\langle X, \Omega \rangle$. Then there exists a natural bijection between the set $Q(X, \Omega)$ of isomorphism classes of quantizations of the symplectic S -manifold X , and the set $H^1_{\mathcal{M}_s}(X, \langle \text{Aut } D, \text{Der } D \rangle)$ of the isomorphism classes of liftings of the symplectic coordinate system bundle \mathcal{M}_s to a $\langle \text{Aut } D, \text{Der } D \rangle$ -torsor with respect to the canonical map of Harish-Chandra pairs $\langle \text{Aut } D, \text{Der } D \rangle \rightarrow \langle \text{Symp } \mathcal{A}, H \rangle$.*

Proof. To pass from a lifting \mathcal{M}_q to a quantization, one takes the localization $\text{loc}(\mathcal{M}_q, D)$ of the $\langle \text{Aut } D, \text{Der } D \rangle$ -module D , and considers the sheaf \mathcal{D} of its flat sections. Since D is a $k[[h]]$ -algebra, $\mathcal{A} = D/hD$, and both these facts are G -equivariant, the sheaf \mathcal{D} is a quantization of the symplectic manifold $\langle X/S, \Omega \rangle$ in the sense of Definition 1.3.

Conversely, given a quantization \mathcal{D} and an affine scheme T , one follows Definition 3.1 and defines $\mathcal{M}_q(T)$ to be the set of all pairs of a map $\eta: T \rightarrow X$ and an isomorphism

$$\Phi: \mathcal{O}_T \hat{\otimes} D \cong \hat{\mathcal{D}}_{X,\eta},$$

where $\hat{\mathcal{D}}_{X,\eta}$ is the completion of the quantization $p(\eta)^*\mathcal{D}$ of the T -manifold $T \times_S X$ obtained by pullback with respect to the composition $p(\eta): T \rightarrow S$ of the map $\eta: T \rightarrow X$ and the projection $X \rightarrow S$; the completion is taken with respect to the ideal spanned by $h(p(\eta)^*\mathcal{D})$ and the ideal $\mathcal{J}_\eta \subset \mathcal{O}_{T \times_S X}$ of the closed subscheme $\sigma\eta(T) \subset T \times_S X$. We claim that the functor $T \mapsto \mathcal{M}_q(T)$ is represented by a (non-Noetherian) scheme \mathcal{M}_q .

Indeed, consider the jet bundle $J^\infty\mathcal{D}$ of the quantization \mathcal{D} . By definition, the completion $\hat{\mathcal{D}}_{X,\eta}$ coincides with the pullback $\eta^*J^\infty\mathcal{D}$. Then every isomorphism $\Phi: \mathcal{O}_T \otimes D \rightarrow \hat{\mathcal{D}}_{X,\eta} \cong \eta^*J^\infty\mathcal{D}$ is defined by the $2n$ elements $\Phi(x_1), \dots, \Phi(x_n), \Phi(y_1), \dots, \Phi(y_n) \in H^0(T, \eta^*J^\infty\mathcal{D})$ which come from the generators $x_1, \dots, x_n, y_1, \dots, y_n \in D$. Conversely, such a set of $2n$ elements give a map if and only if they satisfy the defining relations for D . Moreover, by the very nature of these defining relations, every such map Φ induces a symplectic map $\bar{\Phi}: \hat{\mathcal{O}}_{X,\eta} \rightarrow \mathcal{O}_T \hat{\otimes} \mathcal{A}$, and since $\bar{\Phi}$ is symplectic, its codifferential must be surjective. By Nakayama Lemma, this means that both $\bar{\Phi}$ and Φ are automatically isomorphisms. Therefore the correspondence $\Phi \mapsto \langle \Phi(x_1), \dots, \Phi(x_n), \Phi(y_1), \dots, \Phi(y_n) \rangle$ identifies the set $\mathcal{M}_q(T)$ with the functor represented by a closed subscheme in the total space of the $2n$ -fold sum $(J^\infty\mathcal{D})^{\oplus 2n}$ of the bundle $J^\infty\mathcal{D}$. This is our representing scheme \mathcal{M}_q .

It remains to note that \mathcal{M}_q is naturally a $\langle \text{Aut } D, \text{Der } D \rangle$ -torsor (to check that \mathcal{M}_q is not only flat over X but faithfully flat, one uses Lemma 1.5). Moreover, setting $\Phi \mapsto \bar{\Phi}$ gives a natural map $\mathcal{M}_q \rightarrow \mathcal{M}_s$ compatible with the map $\langle \text{Aut } D, \text{Der } D \rangle \rightarrow \langle \text{Symp } \mathcal{A}, H \rangle$. \square

Remark 3.5. We note that the equivalence between torsors and quantizations given in Lemma 3.4 in fact goes through objects of a third type: quantum-type deformations of the jet bundle $J^\infty\mathcal{O}_X$ in the tensor category of pro-vector bundles on X equipped with a flat connection. This might be useful, for instance, in comparing our approach with that of A. Yekutieli—any isomorphism between two jet-bundle deformations by definition induces a gauge equivalence in the sense of [Y], so that the jet bundle deformations by definition satisfy the local differential triviality condition of [Y].

To make use of Lemma 3.4, we need some information of the structure of the Lie algebra $\text{Der } D$. Recall that every derivation $d \in \text{Der } D$ is almost inner—namely, it can be obtained as the commutator with an element

$$\tilde{d} \in h^{-1}D \subset D \otimes_{k[[h]]} k((h)).$$

The vector space $h^{-1}D$ is closed under the commutator bracket and forms a Lie algebra. Denote this Lie algebra by G . Its center coincides with the scalars $h^{-1}k[[h]] \cdot 1 \subset h^{-1}D$, and we have a central extension of Lie algebras

$$0 \rightarrow k[[h]] \xrightarrow{h^{-1}} G \rightarrow \text{Der } D \rightarrow 0. \quad (3.2)$$

For every $p \geq 0$, let $(\operatorname{Der} D)_{>p} = h^{p+1} \operatorname{Der} D \subset \operatorname{Der} D$ be the subspace of $d \in \operatorname{Der} D$ such that for all $a \in D$, $d(a) = 0 \pmod{h^{p+1}}$. Then $(\operatorname{Der} D)_{>p} \subset \operatorname{Der} D$ is a Lie algebra ideal. Denote the quotient $\operatorname{Der} D / (\operatorname{Der} D)_{>p}$ by $(\operatorname{Der} D)_p$. For every p , denote

$$G_p = G/h^p D = G/h^{p+1} G.$$

The extension (3.2) is compatible with these quotients — for every $p \geq 0$, the Lie algebra G_p is a central extension of the Lie algebra $(\operatorname{Der} D)_p$ by the vector space $k[h]/h^{p+1}$. The kernel of the surjective map $G_{p+1} \twoheadrightarrow G_p$ is the space $\mathcal{A} = D/hD$ of functions on the standard symplectic polydisc. The kernel of the map $(\operatorname{Der} D)_{p+1} \twoheadrightarrow (\operatorname{Der} D)_p$ is the vector space $H = \mathcal{A}/k \cdot 1$ of Hamiltonian vector fields on the polydisc. All in all, for every $p \geq 0$ we have a commutative diagram of the following type:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & h^p \cdot k & \longrightarrow & h^p \cdot \mathcal{A} & \longrightarrow & h^p \cdot H \longrightarrow 0 \\
 & & \downarrow h & & \downarrow & & \downarrow \\
 0 & \longrightarrow & k[h]/h^{p+2} & \longrightarrow & G_{p+1} & \longrightarrow & (\operatorname{Der} D)_{p+1} \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & k[h]/h^{p+1} & \longrightarrow & G_p & \longrightarrow & (\operatorname{Der} D)_p \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array} \tag{3.3}$$

The second and the third row, as well as the second and the third column are extensions of Lie algebras. The first row and the first column are extensions of the Lie algebra modules. Moreover, the modules and the extension in the first column are trivial (and the Lie algebra extensions in rows two and three are central).

Denote by p the composition map $p: \operatorname{Aut} D \rightarrow \operatorname{Symp} \mathcal{A} \rightarrow \operatorname{Sp}(2d)$; this is a projection from $\operatorname{Aut} D$ onto the symplectic group. Its kernel $\operatorname{Ker} p \subset \operatorname{Aut} D$ is unipotent, and the projection itself admits a splitting — we have a semi-direct product decomposition

$$\operatorname{Aut} D \cong \operatorname{Ker} p \rtimes \operatorname{Sp}(2d).$$

This is in fact just the Levi splitting; to construct it explicitly, one interprets the Weyl algebra D as the completed universal enveloping algebra of the Heisenberg Lie algebra. The group $\operatorname{Sp}(2d)$ obviously acts on the Heisenberg algebra, hence on D . Moreover, since $\operatorname{Sp}(2d)$ is semisimple, the central extension (3.2) splits when restricted to $\operatorname{Sp}(2d) \subset \operatorname{Aut} D$, so that the semi-direct product decomposition lifts to the central extension. This immediately allows us to integrate this central extension to an extension of Harish-Chandra pairs. Indeed, it suffices to integrate the extension of the kernel $\operatorname{Ker} p \subset \operatorname{Aut} D$ and take the semi-direct product with $\operatorname{Sp}(2d)$; since $\operatorname{Ker} p$ is unipotent, constructing the extension of the group $\operatorname{Ker} p$ presents no problems (the center of this extension is the vector space $k[[h]]$ considered as an additive group). By abuse of notation, denote the whole extended Harish-Chandra pair by G (earlier this was used to denote its Lie algebra part). We will also need

to consider the quotient G/k , where $k \subset k[[h]]$ is the 1-dimensional subspace of scalars; this quotient will be denoted by $\overline{G} = G/k$.

By construction we have a surjective map $\overline{G} \rightarrow \langle \text{Symp } \mathcal{A}, H \rangle$; denote its kernel by D^\dagger . The Lie algebra of the group D^\dagger is D itself with the commutator bracket. The group D^\dagger is unipotent; it is the product of the additive group k and the group D^\times of invertible elements in D of the form $1+f$, where f lies in the ideal $1 + \mathfrak{m}_{\mathcal{A}} + hD \subset D$.

This integration procedure is compatible with the filtration by degrees of h , so that we obtain the quotient Harish-Chandra pairs $\langle (\text{Aut } D)_p, (\text{Der } D)_p \rangle$, G_p , and \overline{G}_p , $p \geq 0$.

The extensions in the right-hand side column of (3.3) are non-trivial, both on the level of groups and on the level of Lie algebras, with the following important exception.

Lemma 3.6. *The Harish-Chandra extension*

$$0 \rightarrow H \rightarrow (\text{Der } D)_1 \rightarrow (\text{Der } D)_0 \rightarrow 0$$

splits into a semidirect product, $(\text{Der } D)_1 \cong H \rtimes (\text{Der } D)_0$.

Proof. Notice that the Weyl algebra D has a canonical antiinvolution ι defined by $\iota(x_i) = x_i$, $\iota(y_i) = y_i$, $\iota(h) = -h$. Consider the algebra $D_1 = D/h^2D$. This is an associative algebra. Decomposing it with respect to the eigenvalues on ι , we obtain a canonical ι -equivariant vector space identification $D_1 \cong \mathcal{A} \oplus h\mathcal{A}$, $\iota(h) = -h$ and $\iota = \text{id}$ on \mathcal{A} . Since ι is an antiinvolution, under this identification the product in D_1 is equal to

$$a * b = ab + h\{a, b\}, \quad (3.4)$$

where $\{-, -\}$ is the Poisson bracket in \mathcal{A} . Moreover, the Poisson bracket extends to a Lie bracket in D_1 defined by

$$\{a, b\} = \frac{1}{h}(\tilde{a}\tilde{b} - \tilde{b}\tilde{a}) \mod h^2$$

for any $a, b \in D_1$ lifted to $\tilde{a}, \tilde{b} \in D$. Again, since ι is an antiinvolution, the bracket in D_1 coincides with the Poisson bracket on \mathcal{A} extended to $\mathcal{A} \oplus h\mathcal{A} \cong \mathcal{A}[h]/h^2$ in the h -linear way.

Denote by $(\text{Der } D)_1'$ the Lie algebra of derivations of the algebra D_1 which are also derivations with respect to the Lie bracket. We have a natural projection $(\text{Der } D)_1' \rightarrow (\text{Der } D)_0$, and its kernel is isomorphic to the vector space H . By definition, all derivation of the algebra D_1 which come from derivations of the algebra D do preserve the Lie bracket, so that we have a natural injective map $(\text{Der } D)_1 \rightarrow (\text{Der } D)_1'$.

Since both $(\text{Der } D)_1$ and $(\text{Der } D)_1'$ project to $(\text{Der } D)_0$ with the same kernel, an injective map $(\text{Der } D)_1 \rightarrow (\text{Der } D)_1'$ must be an isomorphism. Thus to prove the Lemma, it suffices to construct a splitting $(\text{Der } D)_0 \rightarrow (\text{Der } D)_1'$. This is immediate: the algebra $(\text{Der } D)_0 \cong H$ acts naturally on $D_1 \cong \mathcal{A} \oplus h\mathcal{A}$, and this action preserves both the bracket and the product (3.4). \square

The same argument also provides a splitting on the level of groups and on the level of Harish-Chandra pairs. It is also easy to see that this splitting extends to a splitting of the projection $\overline{G}_1 \rightarrow \overline{G}_0$.

3.3. Categorical quantization. At a suggestion of V. Drinfeld, we conclude this section with some remarks on categorical aspects of quantization (this will not be used in the rest of the paper; similar results were obtained earlier in the analytic setting by P. Polesello and P. Schapira [PS]). Note that since the ideal $\mathfrak{m}_A + \hbar D \subset D$ is invariant with respect to the group part $\mathrm{Aut} D$ of the Harish-Chandra pair $\langle \mathrm{Aut} D, \mathrm{Der} D \rangle$, the projection $\mathcal{D}^\dagger \rightarrow k = D^\dagger/D^u$ is also preserved by $\mathrm{Aut} D$. Therefore the group part $\overline{G}_{\mathrm{gr}}$ of the Harish-Chandra pair \overline{G} admits a canonical product decomposition

$$\overline{G}_{\mathrm{gr}} \cong k \times (\overline{G}_{\mathrm{gr}}/k).$$

Then starting from \overline{G} , we can define a different Harish-Chandra pair \overline{G}' : it has the same Lie algebra part, but the group part is replaced with the product $\overline{G}'_{\mathrm{gr}} = k^* \times (\overline{G}_{\mathrm{gr}}/k)$. In other words, $\overline{G}'_{\mathrm{gr}}$ is the extension of the group $\mathrm{Aut} D$ by the group D^* of invertible elements in the algebra D .

Definition 3.7. A quantization \mathcal{D} of an S -manifold X is called *integral* if the corresponding $\langle \mathrm{Aut} D, \mathrm{Der} D \rangle$ -torsor \mathcal{M}_q lifts to a torsor \mathcal{M}'_q over the Harish-Chandra pair \overline{G}' .

Remark 3.8. We will see (in the end of Section 4) that a canonical quantization in the sense of Definition 1.9 is always integral; so, integral quantizations do exist, at least for manifolds which admit a canonical quantization (such as, for instance, manifolds with $H^i(X, \mathcal{O}_X) = 0$ for $i = 1, 2$).

Assume given a symplectic S -manifold X and an integral quantization \mathcal{D} of X , and denote by \mathcal{M}'_q the lifting of the corresponding torsor \mathcal{M}_q to a \overline{G}' -torsor. By construction, the sheaf \mathcal{D} is the sheaf of flat sections of the algebra bundle $J^\infty \mathcal{D}$, so that the category of sheaves of left \mathcal{D} -modules is equivalent to the category of pro-coherent sheaves of \mathcal{O}_X -modules equipped with a structure of a left module over $J^\infty \mathcal{D}$ and a compatible flat connection. But it is easy to see that the torsor \mathcal{M}'_q is transitive in the sense of Definition 2.4. Therefore by Lemma 2.6 the latter category is in turn equivalent to the category of \overline{G}' -equivariant pro-coherent sheaves of $\mathcal{O}_{\mathcal{M}'_q}$ -modules on \mathcal{M}'_q equipped with a \overline{G}' -equivariant structure of a left module over the Weyl algebra D .

However, it turns out that there exists a different description of this category which does not use the torsor \mathcal{M}'_q , nor the torsor \mathcal{M}_q (nor indeed the quantization \mathcal{D}). Namely, we have the following.

Proposition 3.9. *Let \mathcal{D} be a canonical quantization of a symplectic S -manifold X . Let \mathcal{M}_s be the torsor of formal symplectic coordinate systems of X , and let the Harish-Chandra pair \overline{G}' act on \mathcal{M}_s through the quotient map $\overline{G}' \rightarrow \langle \mathrm{Symp} \mathcal{A}, \mathrm{H} \rangle$. Then the category of sheaves of left \mathcal{D} -modules on X is equivalent to the category of \overline{G}' -equivariant pro-coherent sheaves \mathcal{E} of $\mathcal{O}_{\mathcal{M}_s}$ -modules on \mathcal{M}_s such that the action of the kernel $D^* \subset \overline{G}'$ of the projection $\overline{G}' \rightarrow \langle \mathrm{Symp} \mathcal{A}, \mathrm{H} \rangle$ on \mathcal{E} extends to a structure of a left module over D .*

To establish this equivalence, consider the natural action of the Harish-Chandra pair \overline{G}' on D^* , and let

$$\tilde{G} = D^* \rtimes \overline{G}'$$

be the semi-direct product. We have a natural projection $\tilde{G} \rightarrow \langle \text{Symp } \mathcal{A}, H \rangle$, and its kernel is the semidirect product of the group D^* with itself, with action by conjugation; this kernel is of course canonically isomorphic to the product $D^* \times D^*$, so that \tilde{G} is an extension of $\langle \text{Symp } \mathcal{A}, H \rangle$ by $D^* \times D^*$. We denote the left and right copies of $D^* \subset D^* \times D^* \subset \tilde{G}$ by D_l^* and D_r^* .

We have a natural embedding $\overline{G}' \rightarrow \tilde{G}$ (it restricts to the diagonal embedding $D^* \rightarrow D^* \times D^*$ on $D^* \subset \overline{G}'$); we also have two projections $\tau_L, \tau_R: \tilde{G} \rightarrow \overline{G}'$, whose kernels are $D_r^*, D_l^* \subset D^* \times D^* \subset \tilde{G}$. The natural \overline{G}' -action on the formal Weyl algebra D naturally extends to a \tilde{G} -module structure such that $D^* \times D^* \subset \tilde{G}$ acts by multiplication on the left and on the right. We will denote D with this \tilde{G} -module structure by D_o . We also need to consider two other \tilde{G} -module structures on D : D_L will be D on which \tilde{G} acts through the projection $\tau_L: \tilde{G} \rightarrow \overline{G}'$, and D_R will be D on which \tilde{G} acts through $\tau_R: \tilde{G} \rightarrow \overline{G}'$. The \overline{G}' -action on D is always the standard one. Moreover, the action of \tilde{G} on D_L, D_R and D_o always gives the standard action when restricted to $\overline{G}' \subset \tilde{G}$.

Finally, we note that the \tilde{G} -module structure on D_R and D_L is compatible with the algebra structure in D . The \tilde{G} -module D_o is a $D_L - D_R$ bimodule, D_L acts on the left, D_R acts on the right, both actions compatible with \tilde{G} .

Proof of Proposition 3.9. As noted above, the category of sheaves of left \mathcal{D} -modules on X is equivalent to the category of \overline{G}' -equivariant pro-coherent sheaves of $\mathcal{O}_{\mathcal{M}'_q}$ -modules on the torsor \mathcal{M}'_q equipped with a \overline{G}' -equivariant structure of a left module over the Weyl algebra D . Let the Harish-Chandra pair \tilde{G} act on \mathcal{M}'_q through the projection $\tau_R: \tilde{G} \rightarrow \overline{G}'$, so that the right-hand copy $D_r^* \subset D^* \times D^* \subset \tilde{G}$ acts trivially. Then this category is tautologically the same as the category \mathcal{C}_0 of \tilde{G} -equivariant pro-coherent sheaves of $\mathcal{O}_{\mathcal{M}'_q}$ -modules on \mathcal{M}'_q on which D_r^* acts trivially, and equipped with a \tilde{G} -equivariant structure of left D_R -module.

Taking tensor product over D_R with the bimodule D_o sends left D_R -modules into left D_L -modules, and it is immediate to check that left D_R -modules on which D_l^* acts trivially are sent into left D_L -modules on which D_l^* acts by left multiplication. This tensor product functor is obviously an equivalence. We conclude that the category \mathcal{C}_0 is naturally equivalent to the category \mathcal{C}_1 of \tilde{G} -equivariant pro-coherent sheaves of $\mathcal{O}_{\mathcal{M}'_q}$ -modules on \mathcal{M}'_q equipped with a \tilde{G} -equivariant structure of a left D_L -module, such that $D_l^* \subset \tilde{G}$ acts by left multiplication.

Consider now the right-hand copy $D_r^* \subset D^* \times D^* \subset \tilde{G}$. This subgroup acts freely on \mathcal{M}'_q , and we have $\mathcal{M}'_q/D_r^* \cong \mathcal{M}_s$, so that \mathcal{M}'_q is a principal D_r^* -bundle over \mathcal{M}_s . On the other hand, D_r^* acts trivially on D_L . Applying descent with respect to D_r^* , we identify \mathcal{C}_1 with the category of $\overline{G}' = \tilde{G}/D_r^*$ -equivariant pro-coherent sheaves of $\mathcal{O}_{\mathcal{M}_s}$ -modules on \mathcal{M}_s equipped with a structure of a left D -module such that $D^* \subset \overline{G}'$ acts by left multiplication. Since the D -module structure is uniquely defined by the D^* -action, this proves the proposition.

We note that one can also consider the bundles $\mathcal{D}_R, \mathcal{D}_L$ and \mathcal{D}_o on \mathcal{M}_s obtained by descent from the \tilde{G} -modules D_R, D_L and D_o . They are naturally equivariant

with respect to $\overline{G}' = \widetilde{G}/D_r^*$; \mathcal{D}_R and \mathcal{D}_L are bundles of algebras, while \mathcal{D}_o is a $\mathcal{D}_L - \mathcal{D}_R$ -bimodule. The bundle \mathcal{D}_L is constant, and the group $D^* \subset \overline{G}'$ acts on it by conjugation. The bundle \mathcal{D}_R is not constant, but the group $D^* \subset \overline{G}'$ acts on it trivially. The categories of \overline{G}' -equivariant left \mathcal{D}_R and \mathcal{D}_L -modules are equivalent, and the equivalence is given by tensoring with \mathcal{D}_o . \square

The reason Proposition 3.9 is interesting is that the right-hand side of the established equivalence is defined *a priori*, without any reference either to \mathcal{D} or to \mathcal{M}_q . Thus it gives a perfectly well-defined abelian category $Quan(X)$ for *any* symplectic S -manifold X , not only for an admissible one. Loosely speaking, in terms of this category, the problem of finding an integral quantization of a symplectic S -manifold X becomes the problem of finding an object of rank 1 in $Quan(X)$ (the object \mathcal{D}_o described above). Given such an object, the $\langle \text{Symp } \mathcal{A}, \mathbb{H} \rangle$ -equivariant algebra \mathcal{D}_R on \mathcal{M}_s is recovered as the endomorphism algebra of \mathcal{D}_o , and the quantization \mathcal{D} is obtained by localizing \mathcal{D}_R with respect to $\langle \text{Symp } \mathcal{A}, \mathbb{H} \rangle$. This gives an alternative approach to the quantization problem (which is however pretty close in essence, if not in form, to our Lemma 3.4).

Remark 3.10. As pretty much everything in this paper, the argument in this subsection is not terribly original. In this particular case, we learned the idea from V. Drinfeld; its origins are attributed to J. Bernstein, P. Deligne and M. Kontsevich. The influence of Kontsevich of course looms large over the whole subject, although in this text we have deliberately used old-fashioned arguments independent from the Formality Theorem. Notice also that [PS] is based on earlier work on contact manifolds done by the Japanese school, see e.g. [Kash, Section 8.2] or [KS, Chapter 7].

Remark 3.11. If one uses the existence of canonical quantization for affine manifolds (which follows from Theorem 1.8, since for an affine X we have $H^i(X, \mathcal{O}_X) = 0$), then one can recast the construction of the category $Quan(X)$ in a Čech-cocycle style. Namely, the quantizations exist locally; being canonical, they are isomorphic on intersections. The compatibility isomorphisms do not agree on triple intersections, but the difference between them is an *inner* automorphism $\gamma_{i,j,k}$ of the algebra \mathcal{D} . Moreover, if instead of just quantizations one considers pairs of a quantization \mathcal{D} and a lifting of the corresponding torsor \mathcal{M}_q to a \overline{G} -torsor, then the automorphisms $\gamma_{i,j,k}$ come equipped with a lifting to an element of \mathcal{D}^* . One then uses these elements to glue together the local categories by a standard construction. The argument in this subsection is essentially the same, but a Čech covering is replaced with the torsor \mathcal{M}_s ; this allows to avoid using Theorem 1.8.

4. THE NON-COMMUTATIVE PERIOD MAP

We can now define the non-commutative period map and prove Theorem 1.8. Fix an S -manifold X of dimension $n = 2d$ equipped with a symplectic form Ω . Let \mathcal{M}_s be the bundle of symplectic formal coordinate systems on X . By Lemma 3.4, the set $Q(X, \Omega)$ of quantizations of $\langle X, \Omega \rangle$ is in one-to-one correspondence with the set $H^1_{\mathcal{M}_s}(X, \langle \text{Aut } D, \text{Der } D \rangle)$ of the liftings of \mathcal{M}_s to a $\langle \text{Aut } D, \text{Der } D \rangle$ -torsor \mathcal{M}_q .

Recall that the central extension (3.2) is integrated to a central extension

$$1 \rightarrow k[[h]] \rightarrow G \rightarrow \langle \text{Aut } D, \text{Der } D \rangle \rightarrow 1$$

of Harish-Chandra pairs. For any $\langle \text{Aut } D, \text{Der } D \rangle$ -torsor \mathcal{M}_q , the localization of the trivial $\langle \text{Aut } D, \text{Der } D \rangle$ -module $k[[h]]$ is the constant local system $\mathcal{O}_X[[h]]$ on X . By Proposition 2.7, an obstruction map from $Q(X, \Omega) \cong H_{\mathcal{M}_s}^1(X, \langle \text{Aut } D, \text{Der } D \rangle)$ to the second cohomology group $H_{\text{DR}}^2(X)[[h]]$ sends a torsor \mathcal{M} to the class which obstructs the lifting of \mathcal{M} to a torsor over G .

Definition 4.1. The obstruction map

$$\text{Per}: Q(X, \Omega) \rightarrow H_{\text{DR}}^2(X, A[[h]]) \cong H_{\text{DR}}^2(X)[[h]]$$

associated to the extension (3.2) is called the *non-commutative period map*.

Proof of Theorem 1.8. For any $p \geq 0$, integrate the central extension (3.2) to a Harish-Chandra extension of the quotient Harish-Chandra pair $\langle (\text{Aut } D)_p, (\text{Der } D)_p \rangle$ by the trivial module $h^{-1}k[h]/h^p k[h]$. By abuse of notation, denote the whole extended Harish-Chandra pair by G_p . The commutative diagram (3.3) can be considered as a diagram of Harish-Chandra pairs and their extensions.

We have to prove that the period map $\text{Per}: Q(X, \Omega) \rightarrow H_{\text{DR}}^2(X)[[h]]$ is injective, maps any quantization to a power series with constant term $[\Omega]$, and that any splitting $F: H_{\text{DR}}^2(X) \rightarrow H_F^2(X)$ induces an isomorphism

$$Q(X, \Omega) \cong F([\Omega]) + hH_F^2(X)[[h]].$$

Fix such a splitting $F: H_{\text{DR}}^2(X) \rightarrow H_F^2(X)$. To simplify notation, denote the set $H_{\mathcal{M}_s}^1(X, \langle (\text{Aut } D)_p, (\text{Der } D)_p \rangle)$ by Q_p , and denote by Per_p the obstruction map $Q_p \rightarrow H_{\text{DR}}^2(X, A[h]/h^p)$. Since $(\text{Der } D)_0$ is simply the algebra H of Hamiltonian vector fields, the set Q_0 consists of one point, namely, the $\langle \text{Symp } \mathcal{A}, H \rangle$ -torsor \mathcal{M}_s . By the remarks after Lemma 3.2, we have

$$\text{Per}_0(Q_0) = \text{loc}(\mathcal{M}_s, [\omega]) = [\Omega] \in H_{\text{DR}}^2(X).$$

By induction, it suffices to prove that for every $l > 0$, the map Per_l is injective, and the projection F identifies its image with $H_F^2(X) \otimes_k k[h]/h^{l+1}$. We may assume the claim proved for all $l \leq p$ and consider the case $l = p+1$. Moreover, we may fix a torsor $\mathcal{M} \in Q_p$. Once we do it, it suffices to prove that the period map Per_{p+1} is injective on the set

$$H_{\mathcal{M}}^1(X, \langle (\text{Aut } D)_{p+1}, (\text{Der } D)_{p+1} \rangle),$$

and that it sends this set to a torsor over $H_F^2(X) \cdot h^{p+1} \subset H_{\text{DR}}^2(X)[h]/h^{p+2}$.

By (3.3), the Harish-Chandra pair G_{p+1} is an extension of the Harish-Chandra pair $(\text{Der } D)_p$ by the module

$$V = (k[h]/h^{p+2} \oplus \mathcal{A} \cdot h^p)/k \cdot h^p.$$

Consider the submodule $U = k[h]/h^{p+2} \subset V$, and denote the quotient module by $W = V/U = \mathcal{A}/k = H$. Thus we have a short exact sequence (2.4) and a Harish-Chandra extension of the type considered in Lemma 2.8, with $\langle G, \mathfrak{h} \rangle = \langle (\text{Aut } D)_p, (\text{Der } D)_p \rangle$. The intermediate Harish-Chandra extension $\langle G_0, \mathfrak{h}_0 \rangle$ is given by $\langle (\text{Aut } D)_{p+1}, (\text{Der } D)_{p+1} \rangle$. As in Lemma 2.8, denote the localizations of the

$\langle (\mathrm{Aut} D)_p, (\mathrm{Der} D)_p \rangle$ -modules U , V and W with respect to the torsor \mathcal{M} by \mathcal{U} , \mathcal{V} and \mathcal{W} .

Lemma 4.2. *Assume that the manifold X is admissible.*

- (i) *The canonical map $H_{\mathrm{DR}}^2(X, \mathcal{V}) \rightarrow H_{\mathrm{DR}}^2(X, \mathcal{W})$ is trivial.*
- (ii) *The canonical map $H_{\mathrm{DR}}^1(X, \mathcal{W}) \rightarrow H_{\mathrm{DR}}^2(X, \mathcal{U})$ is injective.*

Proof. By the long exact sequence associated to (2.4), (ii) is equivalent to saying that the canonical map $H_{\mathrm{DR}}^1(X, \mathcal{V}) \rightarrow H_{\mathrm{DR}}^1(X, \mathcal{W})$ is trivial. In other words, we have to prove that the map $H_{\mathrm{DR}}^l(X, \mathcal{V}) \rightarrow H_{\mathrm{DR}}^l(X, \mathcal{W})$ is trivial for $l = 1, 2$.

Note that the $\langle (\mathrm{Aut} D)_p, (\mathrm{Der} D)_p \rangle$ -module structure on U , V and W is obtained by restriction from a $\langle \mathrm{Symp} \mathcal{A}, \mathrm{H} \rangle$ -module structure by the canonical map $\langle (\mathrm{Aut} D)_p, (\mathrm{Der} D)_p \rangle \rightarrow \langle \mathrm{Symp} \mathcal{A}, \mathrm{H} \rangle$. Therefore the localizations \mathcal{U} , \mathcal{V} and \mathcal{W} do not depend on the torsor \mathcal{M} . Moreover, since $k[h]/h^{p+2} \cong k \cdot h^{p+1} \oplus k[h]/h^{p+1}$ as vector spaces, we have $V \cong \mathcal{U} \oplus k[h]/h^{p+1}$, and the map $V \rightarrow W$ is trivial on the second summand. Therefore it suffices to prove that the surjection $\mathcal{A} \rightarrow \mathrm{H} \cong W$ induces a trivial map

$$H_{\mathrm{DR}}^l(X, \mathrm{loc}(\mathcal{M}_s, \mathcal{A})) \rightarrow H_{\mathrm{DR}}^l(X, \mathcal{W}) \cong H_{\mathrm{DR}}^l(X, \mathrm{loc}(\mathcal{M}_s, \mathrm{H}))$$

for $l = 1, 2$. Since $\mathrm{H} = \mathcal{A}/k$, this is in turn equivalent to saying that the map

$$H_{\mathrm{DR}}^l(X) \cong H_{\mathrm{DR}}^l(X, \mathrm{loc}(\mathcal{M}_s, k)) \rightarrow H_{\mathrm{DR}}^l(X, \mathrm{loc}(\mathcal{M}_s, \mathcal{A}))$$

is surjective for $l = 1, 2$. But we know that $\mathrm{loc}(\mathcal{M}_s, \mathcal{A}) \cong J^\infty \mathcal{O}_X$. Therefore $H_{\mathrm{DR}}^l(X, \mathrm{loc}(\mathcal{M}_s, \mathcal{A})) \cong H^l(X, \mathcal{O}_X)$, and the claim becomes the definition of admissibility. \square

Let $c \in H_{\mathrm{DR}}^2(X, \mathcal{V})$ be the obstruction class associated to the torsor \mathcal{M} and the extension G_{p+1} . By Lemma 4.2, the class c restricts to zero in $H_{\mathrm{DR}}^2(X, \mathcal{W})$. Thus the assumptions of Lemma 2.8 are satisfied. We conclude that the period map

$$\mathrm{Per}_{p+1}: H_{\mathcal{M}}^1(X, \langle (\mathrm{Aut} D)_{p+1}, (\mathrm{Der} D)_{p+1} \rangle) \rightarrow H_{\mathrm{DR}}^2(X, \mathcal{U}) = H_{\mathrm{DR}}^2(X, \mathcal{A}[h]/h^{p+2})$$

is compatible with the $H_{\mathrm{DR}}^1(X, \mathcal{W})$ -action on both sides. By Proposition 2.7, the left-hand side is a $H_{\mathrm{DR}}^1(X, \mathcal{W})$ -torsor. Moreover, we have $H_{\mathrm{DR}}^1(X, \mathcal{W}) \cong H_F^2(X)$. Thus to prove the inductive step and the Theorem, it suffices to prove that the $H_{\mathrm{DR}}^1(X, \mathcal{W})$ -action on the right-hand side is free. This is exactly Lemma 4.2 (ii). \square

We finish this section with some observations on the canonical and integral quantizations. By our definition of the non-abelian period map, a symplectic S -manifold X is canonical in the sense of Definition 1.9 if and only if the corresponding $\langle \mathrm{Aut} D, \mathrm{Der} D \rangle$ -torsor \mathcal{M}_q over X lifts to a $\bar{\mathrm{G}}$ -torsor. One can try to construct such a lifting by going step-by-step through the quotient groups $\bar{\mathrm{G}}_p$. Due to Lemma 3.6, there are no obstructions to this at the first step — the symplectic coordinate torsor \mathcal{M}_s always lifts to a $\bar{\mathrm{G}}_1$ -torsor, and a preferred lifting is given by the torsor \mathcal{M}'_1 induced by means of the splitting map $\langle \mathrm{Symp} \mathcal{A}, \mathrm{H} \rangle \cong \bar{\mathrm{G}}_0 \rightarrow \bar{\mathrm{G}}_1$. There may be other liftings, but by Theorem 1.8 they all become isomorphic after taking quotient by the center $h \cdot k \subset \bar{\mathrm{G}}_1$.

At every further step, there is an obstruction class lying in the group $H^2(X, \mathcal{O}_X)$. However, we note that these obstructions do not depend on the entire \overline{G}_1 -torsor \mathcal{M}'_1 , but only on its reduction $\mathcal{M}_1 = \mathcal{M}'_1/k$ to \overline{G}_1/k . Indeed, we have

$$\overline{G} \cong \overline{G}_1 \times_{\overline{G}_1/k} \overline{G}/(h \cdot k),$$

where $h \cdot k \subset hk[[h]]$ lies in the center $hk[[h]]$ of the Harish-Cahndra pair \overline{G} . Then every \overline{G} -torsor \mathcal{M}'_q is of the form

$$\mathcal{M}'_q \cong \mathcal{M}'_1 \times_{\mathcal{M}_1} \overline{\mathcal{M}}_q,$$

where $\mathcal{M}'_1 = \mathcal{M}_q/\overline{G}_{>1}$ is the reduction of \mathcal{M}'_q to \overline{G}_1 , $\overline{\mathcal{M}}_q = \mathcal{M}'_q/(h \cdot k)$ is its reduction to $\overline{G}/(h \cdot k)$, and $\mathcal{M}_1 = \mathcal{M}'_1/k = \overline{\mathcal{M}}_q/\overline{G}_{>1}$ is their common reduction to $\overline{G}/((h \cdot k) \times \overline{G}_{>1})$. This means that in order to lift a $\langle \text{Aut } D, \text{Der } D \rangle$ -torsor \mathcal{M}_q to \overline{G} , it is sufficient, and necessary, to lift it to $\overline{G}/(h \cdot k)$, and to lift its reduction $\mathcal{M}_1 = \mathcal{M}_q/\overline{G}_{>1}$ to \overline{G}_1 .

Analogously, one can try to construct step-by-step an integral quantization in the sense of Definition 3.7. Again, there is no obstruction at the first step, and there is a preferred lifting given by induction with respect to the splitting map. We leave it to the reader to check that all other liftings are parametrized by line bundles L on X , and the leading two terms of the period map for an integral quantization is of the form $[\Omega] + hc_1(L)$, where $c_1(L)$ is the first Chern class. To construct an integral quantization, it suffices to have a $\langle \text{Aut } D, \text{Der } D \rangle$ -tors \mathcal{M}_q which lifts to \overline{G}'/k^* , and whose reduction $\mathcal{M}_1/\overline{G}'_{>1}$ to \overline{G}'_1/k^* lifts to \overline{G}'_1 .

In particular, if we are given a canonical quantization, then the corresponding $\langle \text{Aut } D, \text{Der } D \rangle$ -torsor \mathcal{M}_q comes equipped with a lifting to \overline{G}'/k^* — indeed, by definition we have $\overline{G}'/k^* \cong \overline{G}/(h \cdot k)$ and $\overline{G}'_1/k^* \cong \overline{G}_1/k$. Moreover, the reduction $\mathcal{M}_1 = \mathcal{M}_q/\overline{G}_{>1}$ is the preferred lifting of the symplectic coordinate torsor \mathcal{M}_s , so that it lifts to \overline{G}'_1 . We conclude that a canonical quantization is integral in the sense of Definition 3.7.

Remark 4.3. One can actually compute the first of the obstruction classes, namely, the obstruction to lifting a \overline{G}_1 -torsor to \overline{G}_2 . This amounts to computing the cocycle that represents the extension $\overline{G}_2 \rightarrow \overline{G}_1$; we will not do it here to save space, and only state the answer — up to a non-zero constant, the obstruction is given by the so-called Rozansky–Witten class corresponding to the trivalent graph with two vertices connected by three edges (for Rozansky–Witten classes, see e.g. [Kap2]; for the proof in the holomorphic setting, see in [NT]).

5. NON-ABELIAN COHOMOLOGY

We now return to the basics and prove the results announced in Section 2.4. In the case of ordinary torsors, even in a very general topos, everything is completely standard ([Gi], or, for example, a much shorter and nicer exposition in [Ga]). Unfortunately, we need to work with flat connections. One can probably obtain all the results for free by passing to the crystalline topos, but this raises the amount of high science used to a completely disproportionate degree. For the convenience of the reader, and for our own peace of mind, we will give a proof of all the facts we

need in down-to-earth terms. To save space, the more standard parts of the proofs are left to the reader.

5.1. Linear algebra. Recall that for any two objects A, B in a fixed abelian category, we can form the *extension groupoid* $\mathcal{E}xt^1(B, A)$ whose objects are short exact sequences

$$0 \rightarrow A \rightarrow \bullet \rightarrow B \rightarrow 0,$$

and whose morphisms are isomorphisms of the exact sequences identical on A and on B . The set of isomorphism classes of objects in the groupoid $\mathcal{E}xt^1(B, A)$ is the first Ext-group $\text{Ext}^1(B, A)$. The groupoid $\mathcal{E}xt^1(B, A)$ has an additional structure of a symmetric monoidal category: the sum is given by the Baer sum of extensions.

Fix objects A, B , and let $c \in \text{Ext}^2(B, A)$ be an element in the second Ext-group. Represent c by a four-term exact sequence, Yoneda-style

$$0 \rightarrow A \rightarrow E_1 \rightarrow E_2 \rightarrow B \rightarrow 0. \quad (5.1)$$

In other words, we have a two-term complex $E_1 \rightarrow E_2$ whose cohomology objects are A and B . Recall that $c = 0$ if and only if there exist a complex

$$A \xrightarrow{a} E \xrightarrow{b} B \quad (5.2)$$

such that $\text{Ker } b \cong E_1$, $\text{Coker } a \cong E_2$, and the natural sequence

$$0 \rightarrow A \rightarrow \text{Ker } b \rightarrow \text{Coker } a \rightarrow B \rightarrow 0$$

is exact and isomorphic to (5.1). Diagrams of the form (5.2) form a groupoid, which we will denote by $\mathcal{S}pl(c)$ (we require maps between diagrams to be identical on $\text{Ker } b \cong E_1$ and $\text{Coker } a \cong E_2$). The groupoid $\mathcal{S}pl(c)$ is naturally a gerb over the symmetric monoidal groupoid $\mathcal{E}xt^1(B, A)$ — this means that we have a sum functor $\mathcal{E}xt^1(B, A) \times \mathcal{S}pl(c) \rightarrow \mathcal{S}pl(c)$, a difference functor $\mathcal{S}pl(c) \times \mathcal{S}pl(c) \rightarrow \mathcal{E}xt^1(B, A)$, and natural compatibility morphisms between these functors which turn $\mathcal{S}pl(c)$ into a “torsor” over $\mathcal{E}xt^1(B, A)$ in the obvious sense. Both the sum and the difference functor are again given by the Baer sum construction.

This construction is functorial in the following way: every exact functor F between abelian categories induces a functor $F: \mathcal{S}pl(c) \rightarrow \mathcal{S}pl(F(c))$ between groupoids $\mathcal{S}pl(c)$ and $\mathcal{S}pl(F(c))$.

Taking a different Yoneda representation (5.1) for the same $c \in \text{Ext}^2(B, A)$ gives an equivalent groupoid $\mathcal{S}pl(c)$. To make this quite canonical, one has to consider all possible representations and treat objects of $\mathcal{S}pl(c)$ as certain diagrams of sheaves on the category of these representations. This is very beautiful but too technical to describe here, see [Gr]. For our purposes, it suffices to carry a fixed Yoneda representation in all the constructions.

Of course, if the class c is not trivial, the groupoid $\mathcal{S}pl(c)$ is empty. But it is important to define it anyway.

Finally, in proving Lemma 2.8 we will need to consider the following situation. Assume that the object A in (5.1) is the middle term of a short exact sequence

$$0 \rightarrow A_0 \xrightarrow{a} A \xrightarrow{b} A_1 \rightarrow 0. \quad (5.3)$$

We have the canonical Yoneda representation $E_1/A_0 \rightarrow E_2$ of the class $b(c) \in \text{Ext}^2(B, A_1)$. Assume in addition that we have $b(c) = 0$. Then we claim that every object $s \in \mathcal{S}pl(b(c))$ canonically defines a class $c_0 \in \text{Ext}^2(B, A_0)$ such that $c = a(c_0) \in \text{Ext}^2(B, A)$. Indeed, we can take the class represented by the exact sequence

$$0 \rightarrow A_0 \rightarrow E_1 \rightarrow E \rightarrow B \rightarrow 0,$$

where $A_1 \rightarrow E \rightarrow B$ represents the object $s \in \mathcal{S}pl(b(c))$. If we twist the object s by an extension $e \in \text{Ext}^1(B, A_1)$, then the corresponding class c_0 is replaced by $c_0 + d(e)$, where $d: \text{Ext}^1(B, A_1) \rightarrow \text{Ext}^2(B, A_0)$ is the differential in the long exact sequence associated to (5.3). To prove it, it suffices to notice that the twisting is by definition done via Baer sum with the sequence

$$0 \rightarrow A_0 \rightarrow A \rightarrow F \rightarrow B \rightarrow 0,$$

the Yoneda product of the sequence (5.3) and the sequence $0 \rightarrow A_1 \rightarrow F \rightarrow B \rightarrow 0$ which represents $e \in \text{Ext}^1(B, A_1)$.

5.2. Extensions. Assume given a Harish-Chandra pair $\langle G, \mathfrak{h} \rangle$ and a $\langle G, \mathfrak{h} \rangle$ -module V . Consider V as an (additive) algebraic group. Let $\langle V, V \rangle$ be the tautological Harish-Chandra pair $\langle V, V \rangle$. The Harish-Chandra cohomology groups $H^\bullet(\langle V, V \rangle, V)$ coincide with the cohomology $H^\bullet(V, V)$ of the group (or Lie algebra) V . In particular, classes in the group $H^1(\langle V, V \rangle, V)$ correspond to Lie algebra derivations $d: V \rightarrow V$. Denote by $\tau_V \in H^1(\langle V, V \rangle, V)$ the tautological class — namely, the one corresponding to the identity map $\text{id}: V \rightarrow V$.

Fix a Harish-Chandra extension of the type (2.3), and consider the Hochschild–Serre spectral sequence

$$H^\bullet(\langle G, \mathfrak{h} \rangle, H^\bullet(V, V)) \Rightarrow H^\bullet(\langle G_1, \mathfrak{h}_1 \rangle, V)$$

which computes the cohomology groups $H^\bullet(\langle G_1, \mathfrak{h}_1 \rangle, V)$. The E^2 -term of this sequence contains in particular the group $H^1(\langle V, V \rangle, V)$, and the differential gives a map

$$d: H^1(\langle V, V \rangle, V) \rightarrow H^2(\langle G, \mathfrak{h} \rangle, V).$$

Applying d to the tautological class τ_V gives an element

$$c = d\tau_V \in H^2(\langle G, \mathfrak{h} \rangle, V)$$

canonically associated to the extension.

This class c is of course just the usual 2-cocycle known both in the theory of algebraic groups and in the theory of Lie algebras. Out of the myriad equivalent ways to construct it, this particular one has the advantage of only using the Hochschild–Serre spectral sequence. Therefore it generalizes to Harish-Chandra pairs without any additional work. We record explicitly one degenerate case: when the group G is trivial, the cocycle

$$c \in H^2(\langle G, \mathfrak{h} \rangle, V) = H^2(\mathfrak{h}, V) = \text{Hom}(\Lambda^2 \mathfrak{h}, V)$$

is just the commutator map in the Lie algebra \mathfrak{h}_1 .

For every map f between Harish-Chandra pairs, denote by f^* the restriction functor on the categories of modules. Fix a particular Yoneda representation of the class $c \in H^2(\langle G, \mathfrak{h} \rangle, V) = \text{Ext}^2(k, V)$, and consider the groupoid $\mathcal{S}pl(c)$. Fix

a splitting of the exact sequence (5.1) considered as a sequence of vector spaces. Since the composition $\pi \circ \rho$ factors through the map $\eta: H \rightarrow 1$, the fixed vector-space splitting defines a canonical object in the groupoid $\mathcal{S}pl(\rho^*\pi^*(c))$. This gives a trivialization of the corresponding gerbe, that is, an equivalence

$$\mathcal{S}pl(\rho^*\pi^*(c)) \cong \mathcal{E}xt^1(k, \rho^*\pi^*(V)).$$

Note now that by construction, the class π^*c is trivial. Therefore the groupoid $\mathcal{S}pl(\pi^*(c))$ is non-empty. Recall that it is also a gerbe over the extension groupoid $\mathcal{E}xt^1(k, \pi^*(V))$. Say that an object $s \in \mathcal{S}pl(\pi^*(c))$ is a *good splitting* if ρ^*s is the tautological extension represented by the cohomology class τ_V . Analyzing the Hochschild-Serre spectral sequence, we see that, since $c = d(\tau_V)$, good objects exist, and that pairs $\langle s, f: \rho^* \cong \tau_V \rangle$ form a gerbe over $\mathcal{E}xt^1(k, V)$. Since our goal is not to construct a general theory of group extensions but rather, to have a skeleton theory sufficient for applications to torsors, we will simply ignore this ambiguity and fix a good splitting $s \in \mathcal{S}pl(\pi^*(c))$ for every Harish-Chandra extension (2.3).

5.3. Torsors. Fix an S -manifold X . Assume given a Harish-Chandra extension (2.3) and a $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} over X . Denote by $\mathcal{V} = \text{loc}(\mathcal{M}, V)$ the localization of the $\langle G, \mathfrak{h} \rangle$ -module V with respect to \mathcal{M} . Consider the groupoid $\mathcal{H}_{\mathcal{M}}^1(X, \langle G_1, \mathfrak{h}_1 \rangle)$ of $\langle G_1, \mathfrak{h}_1 \rangle$ -torsors \mathcal{M}_0 over X equipped with an isomorphism $\pi_*\mathcal{M}_0 \cong \mathcal{M}$.

Let $c(\mathcal{M}) \in H_{\text{DR}}^2(X, \mathcal{V})$ be the localization of the cohomology class c with respect to the torsor \mathcal{M} . This class comes equipped with a Yoneda representation (obtained by the localization of the fixed Yoneda representation of the class c). Moreover, for any torsor $\mathcal{M}_0 \in \mathcal{H}_{\mathcal{M}}^1(X, \langle G_1, \mathfrak{h}_1 \rangle)$, the localization $\text{loc}(\mathcal{M}_0, \pi^*c)$ canonically coincides with $c(\mathcal{M})$. We can set

$$\mathcal{M}_0 \mapsto \text{loc}(\mathcal{M}_0, s)$$

and obtain a functor $\text{Lin}: \mathcal{H}_{\mathcal{M}}^1(X, \langle G_1, \mathfrak{h}_1 \rangle) \rightarrow \mathcal{S}pl(c(\mathcal{M}))$. The crucial part of both Proposition 2.7 and Lemma 2.8 is the following fact.

Lemma 5.1. *Assume that the $\langle G, \mathfrak{h} \rangle$ -torsor \mathcal{M} is transitive in the sense of Definition 2.4. Then the functor*

$$\text{Lin}: \mathcal{H}_{\mathcal{M}}^1(X, \langle G_1, \mathfrak{h}_1 \rangle) \rightarrow \mathcal{S}pl(c(\mathcal{M})) \quad (5.4)$$

is an equivalence of categories.

Proof. We will define an inverse equivalence. Assume given a splitting $s_X \in \mathcal{S}pl(c(\mathcal{M}))$. Consider the fixed Yoneda representation

$$0 \rightarrow \mathcal{V} \rightarrow \mathcal{E}_1 \rightarrow \mathcal{E}_2 \rightarrow \mathcal{O}_X \rightarrow 0$$

of the class $c(\mathcal{M})$ split by s_X . Denote by $\sigma: \mathcal{M} \rightarrow X$ the projection. By definition of the localization functor, the diagram

$$0 \rightarrow \sigma^*\mathcal{V} \rightarrow \sigma^*\mathcal{E}_1 \rightarrow \sigma^*\mathcal{E}_2 \rightarrow \mathcal{O}_{\mathcal{M}} \rightarrow 0$$

is a diagram of *constant* vector bundles on \mathcal{M} : it is isomorphic to

$$0 \rightarrow \rho^*V \rightarrow \rho^*E_1 \rightarrow \rho^*E_2 \rightarrow \rho^*k \rightarrow 0,$$

where $\rho: \mathcal{M} \rightarrow \text{Spec } k$ is the projection to the point, and

$$0 \rightarrow V \rightarrow E_1 \rightarrow E_2 \rightarrow k \rightarrow 0, \quad (5.5)$$

is the diagram representing the class c of the Harish-Chandra extension $\langle G_1, \mathfrak{h}_1 \rangle$. We have two splittings of this diagram: one is given by the fixed $\langle G_1, \mathfrak{h}_1 \rangle$ -equivariant splitting s of the diagram (5.5), the other is given by $\sigma^*(s_X)$.

For every point $m \in \mathcal{M}$, denote by A_m the set of all isomorphisms

$$\varphi: s_m \cong \sigma^*(s_X)_m \quad (5.6)$$

between the fibers of these splittings at the point m (which we consider as splittings in the category of vector spaces). Denote by \mathcal{M}_s the set of pairs

$$\langle m, \varphi \in A_m \rangle$$

of a point $m \in \mathcal{M}$ and an isomorphism $\varphi \in A_m$. Each of the sets A_m is naturally a torsor over the vector space $\text{Hom}(k, V) = V$. Therefore $\mathcal{M}_s/\mathcal{M}$ is a torsor over the constant bundle $\sigma^*\mathcal{V}$ on \mathcal{M} . This makes it into a scheme over \mathcal{M} , in fact into a (pro) S -manifold.

Since the splitting s is in fact a splitting in the category of $\langle G_1, \mathfrak{h}_1 \rangle$ -modules, the group G_1 acts naturally on the S -manifold \mathcal{M}_s : it acts on m through the quotient $G = G_1/V$, and it acts on φ by acting on the left-hand side of (5.6). Since the stabilizer of a point $m \in \mathcal{M}$ is the subgroup $V \subset G_1$, and the set A_m is a V -torsor, the whole \mathcal{M}_s is a G_1 -torsor. Moreover, the natural G_1 -action naturally extends to a transitive \mathfrak{h}_1 action, and turns \mathcal{M}_s into a well-defined transitive $\langle G_1, \mathfrak{h}_1 \rangle$ -torsor $\mathcal{M}_s \in \mathcal{H}_{\mathcal{M}}^1(X, \langle G_1, \mathfrak{h}_1 \rangle)$.

The correspondence $s_X \mapsto \mathcal{M}_s$ is obviously functorial in s_X and gives a functor $\text{Spl}(c(\mathcal{M})) \rightarrow \mathcal{H}_{\mathcal{M}}^1(X, \langle G_1, \mathfrak{h}_1 \rangle)$. This is the desired inverse equivalence, more or less by definition; the proof is easy, and we leave it to the reader. \square

This lemma immediately yields Proposition 2.7 and Lemma 2.8. Indeed, it allows to rewrite both statements as claims about the groupoid $\text{Spl}(c(\mathcal{M}))$, and these claims immediately follow from the homological considerations of Section 5.1.

6. GENERALIZATIONS

Theorem 1.8 admits two immediate generalizations—indeed, both could have been incorporated directly into its statement, and we did not do so only out of desire to keep the statement down to a reasonable size. We record both here, together with a result on comparison with symplectic deformation theory of [KV].

6.1. Equivariant situation. Let X be an S -manifold equipped with an action of a reductive group G . Note that G acts naturally on the de Rham cohomology $H_{\text{DR}}^\bullet(X)$ and on the coherent cohomology $H^\bullet(X, \mathcal{O}_X)$.

Definition 6.1. The S -manifold X equipped with the G -action is called *admissible in the G -equivariant sense* if the canonical map

$$(H_{\text{DR}}^i(X))^G \rightarrow (H^i(X, \mathcal{O}_X))^G$$

between the G -invariant parts of the respective cohomology groups is surjective for $i = 1, 2$.

Proposition 6.2. *Let X be a symplectic S -manifold equipped with a G -action which preserves the symplectic form, and assume that X is admissible in the G -equivariant sense. Then X has a G -equivariant quantization.*

Proof. The proof of Theorem 1.8 works without any changes, save for adding “ G -equivariant” in appropriate places. Note that G -equivariant local systems should be understood in a “stupid” way — as G -equivariant vector bundles equipped with a G -invariant flat connection. In particular, genuine G -equivariant cohomology groups $H_G^\bullet(X)$ do not enter into the picture. \square

The canonical quantization, being canonical, is equivariant with respect to any possible group action. This allows to define and construct quantizations of admissible global quotients by a finite group — indeed, for a finite group G , a quotient $X = Y/G$ is admissible if and only if Y is admissible in the G -equivariant sense. Quantization of arbitrary admissible Deligne–Mumford stacks is more delicate, and we prefer to postpone this investigation to a future paper.

Another situation when Proposition 6.2 might be useful is when we want to quantize a symplectic manifold which is not admissible in the sense of Definition 1.1. For example, given an S -manifold X with $H^1(X, \mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ and a line bundle L on X , one can consider the total space Z of the associated \mathbb{G}_m -torsor over X . Typically tensor powers L^k , $k \in \mathbb{Z}$ of the bundle L will have non-trivial cohomology groups, so that $H^i(Z, \mathcal{O}_Z)$ would be large and Z would not have a chance of being admissible. However, since

$$(H^i(Z, \mathcal{O}_Z))^{\mathbb{G}_m} = H^i(X, \mathcal{O}_X) = 0, \quad i = 1, 2,$$

the manifold Z is always admissible in the \mathbb{G}_m -equivariant sense.

6.2. Comparison with symplectic deformations. Theorem 1.8 holds literally, with the same proof, when either S , or X , or both are allowed to be formal schemes — indeed, all we ever used of a scheme was a formal neighborhood of its closed point. This allows for comparison with [KV]. The main result of [KV] is the following.

Theorem 6.3 [KV, Theorem 1.1]. *Let X be an admissible manifold over the field k . Assume that X is equipped with a nondegenerate symplectic form Ω_0 . Then the pair $\langle X, \Omega \rangle$ admits a universal formal deformation \mathfrak{X}/S . Moreover, the cohomology class*

$$[\Omega] \in H_{\text{DR}}^2(\mathfrak{X}/S) \cong H_{\text{DR}}^2(X) \otimes \mathcal{O}_S$$

of the relative symplectic form $\Omega \in \Omega^2(\mathfrak{X}/S)$ defines an embedding $S \rightarrow H_{\text{DR}}^2(X)$, and every splitting $H_{\text{DR}}^2(X) \rightarrow H_F^2(X)$ of the natural embedding $H_F^2(X) \rightarrow H_{\text{DR}}^2(X)$ identifies S with the formal completion of the affine space $H_F^2(X)$ at the point $[\Omega_0] \in H_F^2(X)$. \square

In general, the universal deformation \mathfrak{X}/S exists only as a formal scheme. The precise meaning of universality will not be important for us, see [KV]. What is important is that \mathfrak{X} is smooth and symplectic over S , so that we can apply Theorem 1.8 and construct a quantization of \mathfrak{X}/S . Having done this, we obtain

a non-commutative multiparameter deformation \mathcal{D}_S of the structure sheaf of the symplectic manifold $X/\mathrm{Spec} k$. The base of this deformation is

$$\bar{S} = \Delta \times S \subset \Delta \times H_{\mathrm{DR}}^2(X),$$

where $\Delta = \mathrm{Spf} k[[h]]$ is the formal disc, and $H_{\mathrm{DR}}^2(X)$ is considered as an affine space. For any section $s: \Delta \rightarrow \bar{S}$ of the natural projection $\bar{S} \rightarrow \Delta$, the pullback $s^*\mathcal{D}_S$ is a quantization of the manifold X . Algebraically, every such section s is given by a formal power series $P_s \in H_{\mathrm{DR}}^2(X)[[h]]$.

Lemma 6.4. *Assume that the chosen quantization \mathcal{D}_S of the S -manifold \mathfrak{X} is canonical. Then the non-commutative period map sends the quantization $s^*\mathcal{D}_S$ to the formal power series P_s .*

Proof. This is immediate from the definitions. Indeed, since \mathcal{D}_S is the canonical quantization of \mathfrak{X}/S , its non-commutative period is simply the class $[\Omega]$ of the symplectic form $\Omega \in \Omega^2(X/S)$, and it is easy to check that the non-commutative period map is compatible with the base change. \square

Comparing Theorem 1.8 and Theorem 6.3, we see that if the quantization \mathcal{D}_S of the universal deformation \mathfrak{X}/S is canonical, then *all* quantizations of the symplectic manifold $X/\mathrm{Spec} k$ can be obtained in a unique way by pullback from \mathcal{D}_S . Thus $Q(X, \Omega_0)$ is identified with the set of sections

$$s = \mathrm{id} \times s': \Delta \rightarrow \bar{S} = \Delta \times S$$

of the canonical projection $\bar{S} \rightarrow \Delta$. The canonical quantization of X corresponds to the constant section $s = \mathrm{id} \times [\Omega_0]$. Analogously, by Theorem 6.3 every symplectic deformation X'/Δ can be obtained by a pullback with respect to a map $s: \Delta \rightarrow \bar{S}$ of the type

$$s = \{0\} \times s': \Delta \rightarrow \bar{S} = \Delta \times S.$$

In the case when the quantization \mathcal{D}_S is not canonical, Lemma 6.4 no longer holds. However, it is easy to check that the basic picture is still the same: any quantization of X can be obtained in a unique way by pullback from \mathcal{D}_S . Thus even in the case when \mathfrak{X}/S does not admit a canonical quantization, it is still possible to fit all the quantizations into a single multi-parameter deformation \mathcal{D}_S .

7. DISCUSSION

7.1. The main difference between our approach and that of Fedosov is that Fedosov works in C^∞ setting, where all principal bundles with respect to nilpotent groups are trivial. Therefore the group part of our Harish-Chandra torsor \mathcal{M}_q reduces to the bundle of symplectic frames in $\mathcal{T}(X)$. Fedosov does this reduction implicitly, by choosing a symplectic connection on X . After that, the only non-trivial part of the quantization procedure is the construction of the flat \mathfrak{h} -valued connection. This can be done directly in the jet bundle $J^\infty(D)$ associated to D . The principal bundle \mathcal{M}_q itself does not enter into the picture at any point.

7.2. Nest and Tsygan [NT] generalize the Fedosov construction to the holomorphic setting. In this case the principal bundle can be quite non-trivial. However, it still reduces to the symplectic frame bundle in the C^∞ category. Nest and Tsygan analyze the holomorphic non-triviality by considering the Dolbeault complexes, and encoding everything into the $(0, 1)$ -part of the Fedosov connection. Again, everything is done in the jet bundle, and the principal bundle does not appear explicitly at any point.

7.3. De Wilde and Lecomte use a different approach — they choose an open cover of the manifold X and glue together local quantizations by explicit group-valued Čech cocycles. Moreover, Deligne re-tells their construction in the language of nonabelian cohomology and gerbes. This is further from our approach in that it does not use the jet bundles, but it is closer in that it does mention groups and torsors more explicitly. The main difference is in how to set up the induction process — in other words, how to filter the deformation problem by the powers of the Planck constant \hbar . Since De Wilde and Lecomte do not use jet bundles, they cannot work directly with groups — for all the groups and Lie algebras that appear, they have to find an explicit description as automorphisms and derivations of this or that algebraic object. Unfortunately, it seems that it is not obvious how to interpret our groups $(\text{Der } D)_p$ in this way. In particular, setting $D_p = D/\hbar^{p+1}$ gives an embedding

$$(\text{Der } D)_p \rightarrow \text{Der } D_p,$$

but this embedding is *not* an isomorphism (the difference appears in the \hbar^p part — the left-hand side contains only the Hamiltonian vector fields there, while the right-hand side contain all vector fields). This can be corrected at the first step by imposing an additional bracket operation on D_1 , the way we do in Lemma 3.6. However, listing axioms for this bracket in higher orders and constructing the deformation theory for such objects seems to be quite cumbersome. De Wilde–Lecomte and Deligne also use some additional algebraic data, and this is partially successful, at least in the C^∞ setting where they work. But it does introduce some complications into the proofs, and there are some extra parasitic obstructions which one has to kill by hand.

7.4. The central role played by the central extension (3.2) is fully realized both by Fedosov and by De Wilde–Lecomte. In De Wilde–Lecomte (retold by Deligne), it is used to add necessary rigidity to D_p . In Fedosov, and even more so in Nest–Tsygan, it appears in the connections themselves — in our notation, they are not $(\text{Der } D)$ -valued but G -valued. To compensate for this, the connections are allowed to have non-trivial curvature with values in the center of the Lie algebra G . It is this curvature that parametrizes the quantizations. In our approach, this appears coupled with the possible group-theoretic obstructions in the guise of our non-commutative period map.

7.5. In general, the quantizations we construct are purely formal. However, among manifolds admissible in our sense, one finds compact smooth projective manifolds over \mathbb{C} . In this situation, it would be very interesting to try to use compactness and obtain some sort of quantization which is analytic in \hbar in some appropriate sense. We would like to note, though, that brute force does not work:

it is not possible to obtain a deformation of the sheaf \mathcal{O}_X of holomorphic functions which is defined over an actual small disc with coordinate h . Indeed, for every small open disc $U \subset X$, the power series in h which define the quantized product of holomorphic functions on U do converge. However, by looking at the Weyl algebra it is elementary to check that the radius of convergence roughly coincides with the size of U . Therefore it goes to 0 when U is shrunk to a point.

7.6. One final word concerns a more explicit description of the set $Q(X, \Omega)$ of isomorphism classes of quantizations. We have embedded it canonically into $H_{\text{DR}}^2(X)[[h]]$, and we have proved that $Q(X, \Omega)$ is non-canonically isomorphic to $H_F^2(X)$. This is really weak — essentially, we just say that $Q(X, \Omega) \subset H_{\text{DR}}^2(X)[[h]]$ is a smooth algebraic subvariety with correct transversality properties with respect to $H_F^2(X)[[h]] \subset H_{\text{DR}}^2(X)[[h]]$, and use the implicit function theorem to get an identification $Q(X, \Omega) \cong H_F^2(X)[[h]]$. In the commutative symplectic case considered in [KV], the final answer is analogous (if the universal symplectic deformation \mathfrak{X}/S admits a canonical quantization, the answer is in fact literally the same, see Section 6). However, at least for projective X the full answer is also known, due to the pioneering work of F. Bogomolov [Bg]. The period domain for commutative deformations of irreducible holomorphic symplectic manifolds is a globally, not infinitesimally defined quadric in $H_{\text{DR}}^2(X)$. This is a deep result; in particular, we get a non-trivial and completely canonical quadratic form on $H_{\text{DR}}^2(X)$, known as the Bogomolov–Beauville form. What happens for non-commutative deformations? Nest and Tsygan asked the same question, in their language. Moreover, they were able to compute the “first-order” part of $Q(X, \Omega) \subset H_{\text{DR}}^2(X)[[h]]$. The answer is expressed in terms of the so-called *Rozansky–Witten* characteristic classes of the symplectic manifold X ; the reader can find it (without proof) in Remark 4.3. It would be very interesting to obtain a full answer. This would probably involve some non-linear combinations of Rozansky–Witten classes — hopefully no more than quadratic, but possibly not.

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REFERENCES

- [BB] A. Beilinson and J. Bernstein, *A proof of Jantzen conjectures*, I. M. Gelfand Seminar, Adv. Soviet Math., vol. 16, Amer. Math. Soc., Providence, RI, 1993, pp. 1–50. MR [95a:22022](#)
- [BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchin’s integrable system and Hecke eigensheaves*. Preprint version available at <http://www.math.uchicago.edu/~arinkin/langlands/>

- [BFM] A. Beilinson, B. Feigin, and B. Mazur, *Conformal field theory on curves*, preprint, 1992.
- [Bg] F. A. Bogomolov, *Hamiltonian Kählerian manifolds*, Dokl. Akad. Nauk SSSR **243** (1978), no. 5, 1101–1104. MR [80c:32024](#)
- [Bt] R. Bott, *Lectures on characteristic classes and foliations*, Lectures on algebraic and differential topology (Second Latin American School in Math., Mexico City, 1971), Springer, Berlin, 1972, pp. 1–94. Lecture Notes in Math., Vol. 279. MR [50 #14777](#)
- [D1] P. Deligne, appendix to R. Hartshorne, *Residues and duality*, Lecture notes of a seminar on the work of A. Grothendieck, given at Harvard 1963/64. Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR [36 #5145](#)
- [D2] P. Deligne, *Déformations de l'algèbre des fonctions d'une variété symplectique: comparaison entre Fedosov et De Wilde*, Lecomte, Selecta Math. (N.S.) **1** (1995), no. 4, 667–697. MR [97a:58068](#)
- [DWL] M. De Wilde and P. B. A. Lecomte, *Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys. **7** (1983), no. 6, 487–496. MR [85j:17021](#)
- [F] B. V. Fedosov, *A simple geometrical construction of deformation quantization*, J. Differential Geom. **40** (1994), no. 2, 213–238. MR [95h:58062](#)
- [Ga] D. Gaitsgory, *Grothendieck topologies and deformation theory. II*, Compositio Math. **106** (1997), no. 3, 321–348. MR [2000e:14021](#)
- [GK] I. M. Gelfand and D. A. Kazhdan, *Certain questions of differential geometry and the computation of the cohomologies of the Lie algebras of vector fields*, Dokl. Akad. Nauk SSSR **200** (1971), 269–272 (Russian). MR [44 #4770](#)
- [Gi] J. Giraud, *Cohomologie non abélienne*, Springer-Verlag, Berlin, 1971. MR [49 #8992](#)
- [Gr] A. Grothendieck, *Catégories cofibrées additives et complexe cotangent relatif*, Lecture Notes in Mathematics, No. 79, Springer-Verlag, Berlin, 1968. MR [39 #2835](#)
- [KV] D. Kaledin and M. Verbitsky, *Period map for non-compact holomorphically symplectic manifolds*, Geom. Funct. Anal. **12** (2002), no. 6, 1265–1295. MR [2004e:32011](#)
- [Kap1] M. Kapranov, *Noncommutative geometry based on commutator expansions*, J. Reine Angew. Math. **505** (1998), 73–118. MR [2000b:14003](#)
- [Kap2] M. Kapranov, *Rozansky–Witten invariants via Atiyah classes*, Compositio Math. **115** (1999), no. 1, 71–113. MR [2000h:57056](#)
- [Kash] M. Kashiwara, *D-modules and microlocal calculus*, Translations of Mathematical Monographs, vol. 217, American Mathematical Society, Providence, RI, 2003. MR [2003i:32018](#)
- [KS] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 292, Springer-Verlag, Berlin, 1994. MR [95g:58222](#)
- [Kon] M. Kontsevich, *Deformation quantization of algebraic varieties*, Lett. Math. Phys. **56** (2001), no. 3, 271–294. MR [2002j:53117](#). Preprint version available at <http://arxiv.org/math.AG/0106006>.
- [NT] R. Nest and B. Tsygan, *Deformations of symplectic Lie algebroids, deformations of holomorphic symplectic structures, and index theorems*, Asian J. Math. **5** (2001), no. 4, 599–635. MR [2003k:53126](#)
- [PS] P. Polesello and P. Schapira, *Stacks of quantization-deformation modules on complex symplectic manifolds*, Int. Math. Res. Not. (2004), no. 49, 2637–2664. MR [2 077 680](#). Preprint version available at <http://arxiv.org/math.AG/0305171>.
- [Y] A. Yekutieli, *Deformation quantization in algebraic geometry*, preprint, available at <http://arxiv.org/math.AG/0310399>.

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