Abstract. We revisit the issue of counting all local fields of the restricted sine-Gordon model, in the case corresponding to a perturbation of minimal unitary conformal field theory. The problem amounts to the study of a quotient of certain space of polynomials which enter the integral representation for form factors. This space may be viewed as a $q$-analog of the space of conformal coinvariants associated with $U_q(\widehat{sl}_2)$ with $q = \sqrt{-1}$. We prove that its character is given by the restricted Kostka polynomial multiplied by a simple factor. As a result, we obtain a formula for the truncated character of the total space of local fields in terms of the Virasoro characters.


Key words and phrases. Form factor, restricted sine-Gordon model.

1. Introduction

Integrable perturbation of conformal field theory initiated in [26] has been a subject of intensive study over the last 15 years, and many rich structures have been revealed. From physics point of view, it is natural to expect that the space of local fields in a perturbed theory is ‘isomorphic’ to its conformal limit. Here, by ‘isomorphic’ we mean that their characters with respect to natural gradings coincide. The form factor bootstrap [19] offers an appropriate framework to examine the validity of this picture. Favorable results have been obtained for simple models where the $S$ matrix is a scalar [3], [10]. For models with internal degrees of freedom, the problem becomes far more complicated. Important progress in this direction has been made for the sine-Gordon (SG) model by Smirnov and Babelon–Bernard–Smirnov [20], [2]. Nevertheless, we think that the issue of determining the character of the

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1Actually, in massive theory, the character of the whole space does not literally make sense, and one needs to consider a certain truncation. See eq. (1.19) below and the discussion there.
space of local fields has not been settled in these works. Recently Nakayashiki [12] solved this problem (under certain assumptions) for the SU(2)-invariant Thirring model, which is a rational degeneration of the SG model. The aim of this paper is to perform a similar analysis for the SG model at a generic coupling, and the restricted sine-Gordon (RSG) model corresponding to the perturbation of minimal unitary series.

Let us describe the problem in more detail. Recall that in the bootstrap approach, a local operator $\mathcal{O}$ in the theory is specified by its form factor. A form factor is a tower $f = (f_n)_{n=0}^\infty$ of meromorphic functions $f_n = f_n(\beta_1, \ldots, \beta_n)$, satisfying certain axioms. In physical terms, these functions for real values of $\beta_i$’s are the matrix elements of $\mathcal{O}$ between the vacuum and the $n$-particle asymptotic states with rapidities $\beta_1, \ldots, \beta_n$. We will refer to $f_n$ as an $n$-particle form factor. General matrix elements between $m$- and $n$-particle states are obtained by analytic continuation from $f_{n+m}$ [19]. When the operator $\mathcal{O}$ has Lorentz spin $s$, the corresponding form factor has the homogeneity property

$$f_n(\beta_1 + \Lambda, \ldots, \beta_n + \Lambda) = e^{s\Lambda} f_n(\beta_1, \ldots, \beta_n) \quad \text{for any } \Lambda \in \mathbb{R} \text{ and } n \in \mathbb{Z}_{\geq 0}. \quad (1.1)$$

We say that $f$ has degree $s$ if (1.1) holds.

We consider the SG model with the coupling parameter $\xi > 1$, so that breathers do not appear in the space of physical states. The $n$-particle form factors $f_n$ then take values in $(\mathbb{C}^2)^{\otimes n}$. We regard $(\mathbb{C}^2)^{\otimes n}$ as a representation space of the quantum loop algebra $U_q(\mathfrak{sl}_2)$, wherein $e^{-\beta_i/\xi}, \ldots, e^{-\beta_n/\xi}$ play the role of spectral parameters. The parameter $q$ of the algebra and the parameter $\xi$ of the SG model are related by

$$q = e^{-\frac{2\pi}{\xi}}. \quad (1.2)$$

In this paper, we consider the SG model in a restricted sector: We impose the constraints that $f_n(\beta_1, \ldots, \beta_n) \in (\mathbb{C}^2)^{\otimes n}$ satisfies

$$h_1 f_n(\beta_1, \ldots, \beta_n) = m f_n(\beta_1, \ldots, \beta_n), \quad (1.3)$$

$$e_1 f_n(\beta_1, \ldots, \beta_n) = 0, \quad (1.4)$$

for some $m \in \mathbb{Z}_{\geq 0}$, where $e_i, f_i, t_i = q^{h_i}$ are the Chevalley generators of $U_q(\mathfrak{sl}_2)$. Note that we have chosen one of the two $U_q(\mathfrak{sl}_2)$ symmetries of the model in choosing the restricted sector. In Section 2, we formulate Smirnov’s axioms for form factors in the SG model along with these constraints. A large class of functions satisfying these axioms is afforded by the theory of hypergeometric integrals [24, 13, 23]. In counting form factors, we make the basic Ansatz that ‘all solutions’ to the form factor axioms is afforded by the theory of hypergeometric integrals.

Let $n, l$ be non-negative integers with $n \geq 2l$. Let $C_{n,l}$ denote the space of polynomials $P(X_1, \ldots, X_l; z_1, \ldots, z_n)$ which are skew-symmetric in $X_1, \ldots, X_l,$ symmetric in $z_1, \ldots, z_n$, and have degree less than $n$ in each variable $X_p$. For $P_i \in C_{n,l_i}$ ($i = 1, 2$) we will use the wedge product notation

$$(P_1 \wedge P_2)(X_1, \ldots, X_{l_1+l_2}) = \text{Skew}(P_1(X_1, \ldots, X_{l_1})P_2(X_{l_1+1}, \ldots, X_{l_1+l_2})), \quad (1.5)$$
where Skew stands for the skew symmetrization. The hypergeometric integral is a linear map which associates to each \( P \in \text{C}_{n,l} \) a meromorphic function \( \psi_P(\beta_1, \ldots, \beta_n) \) with values in \((\mathbb{C}^2)^\otimes n\). It has the form

\[
\psi_P(\beta_1, \ldots, \beta_n) = \int_{C} \prod_{p=1}^{l} d\alpha_p \prod_{p=1}^{l} \phi(\alpha_p; \beta_1, \ldots, \beta_n) \\
\times v(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_n)P(X_1, \ldots, X_l; z_1, \ldots, z_n),
\]

where \( \phi \) is a certain special function, \( v \) is a fixed vector-valued function, \( X_p = e^{-\alpha_p}, \quad z_j = e^{\beta_j}, \)

and the integral is over certain contour \( C \). We give the details in Section 2. We call an element \( P \) of \( \text{C}_{n,l} \) a deformed cycle, since in a certain limit the integral reduces to a hyperelliptic integral whose cycle of integration is determined by \( P \). The constraints (1.3), (1.4) are also satisfied by \( \psi_P \), where the \( \mathfrak{sl}_2 \)-weight \( m \) in (1.3) is related to \( n \) and \( l \) by

\[
m = n - 2l.
\]

To construct form factors, we choose \( P_{n,l} \in \text{C}_{n,l} \) for each \( n, l \) satisfying (1.6), and define \( f_n \) from \( \psi_{P_{n,l}} \) by multiplying a common scalar function. Up to a shift depending on \( n, l, \xi \), the degree \( s \) in (1.1) is equal to the degree of the polynomial \( P_{n,l} \) given by

\[
\deg X_p = -1, \quad \deg z_j = 1.
\]

Among the three main axioms for form factors, the first two give conditions on each \( f_n \) separately. The \( f_n \) constructed above automatically satisfies them for any choice of \( P_{n,l} \in \text{C}_{n,l} \). The third axiom relates the residue of \( f_n \) at \( \beta_n = \beta_{n-1} + \pi i \) to \( f_{n-2} \). In order to satisfy this condition, we must choose a tower of polynomials \( \{P_{n,l}\} \) properly. In this paper we do not address this question. Instead, we follow the approach of minimal form factors [20], [12].

We say a form factor \( f \) is \( N \)-minimal if

\[
f_n = 0 \quad \text{for all } n < N.
\]

The residue axiom then implies that

\[
\text{Res}_{\beta_{N} = \beta_{N-1} + \pi i} f_N = 0.
\]

We assume that, conversely, any function \( f_N \) satisfying (1.8) and the rest of the axioms can always be extended to a form factor \( f = (f_n)_{n=0}^{\infty} \) containing \( f_N \) as a member. Then the counting of local fields, or equivalently of form factors, is reduced to that of a single function \( f_N \). In the context of the SU(2)-invariant Thirring model, Nakayashiki [12] pointed out that

\[
\text{Res}_{\beta_{N} = \beta_{N-1} + \pi i} \psi_{P_{N,l}} = 0
\]

follows from the condition

\[
P_{N,l}|_{X_1^{-1} = z_1 = -z_2} = 0,
\]

and proposed the assumption that it is also necessary for (1.9). We make the same assumption in the SG model.
Let \( W_{N,l} \) be the subspace of \( C_{N,l} \) consisting of polynomials satisfying (1.10). Under the assumptions made above, the space of \( N \)-minimal form factors can be identified with the quotient space \( M_{N,l} = W_{N,l}/\text{Ker} \psi \cap W_{N,l} \). We note that the kernel of the hypergeometric map \( \psi : P \mapsto \psi_P \) is known explicitly [20], [23]. It is generated by two homogeneous cycles \( \Sigma_1(X) \) and \( \Sigma_2(X_1, X_2) \) in the sense of the wedge product. Both \( W_{N,l} \) and \( M_{N,l} \) are graded by the degree assignment (1.7). The counting of local fields in the SG model is reduced to the problem of determining the character of \( M_{N,l} \). Here and in what follows, by a character of a graded vector space \( V = \bigoplus_d V_d \) we mean the generating series 

\[
\ch q V = \sum_d q^d \dim V_d. \tag{1.11}
\]

The following results are proved in [12]:

\[
\ch q W_{N,l} = \frac{1}{(q)_N} \left( \begin{array}{c} N \\ l \end{array} \right), \tag{1.12}
\]

\[
\ch q M_{N,l} = \frac{1}{(q)_N} \left( \begin{array}{c} N \\ l \end{array} \right) - \frac{1}{(q)_{N-l}} \left( \begin{array}{c} N \\ l-1 \end{array} \right), \tag{1.13}
\]

where

\[
\left[ \begin{array}{c} N \\ l \end{array} \right] = \frac{(q)_N}{(q)_l(q)_{N-l}}, \quad (q)_l = \prod_{i=1}^l (1-q^i).
\]

In this paper we give an alternative proof of Nakayashiki’s results (1.13) by clarifying the algebraic structure of \( W_{N,l}, M_{N,l} \).

We mentioned the quantum algebra \( U_q(\tilde{\mathfrak{sl}}_2) \) where \( q = e^{-\pi i/\xi} \). This algebra describes the \((\mathbb{C}^2)^{\otimes n}\) structure of form factors. It controls the \((\mathbb{C}^2)^{\otimes n}\)-valued \( l \)-form (‘deformed cocycles’) in the hypergeometric integral. There appears another quantum algebra \( U_{\sqrt{-1}}(\tilde{\mathfrak{sl}}_2) \), which is the specialization of \( U_q(\tilde{\mathfrak{sl}}_2) \) at \( q = \sqrt{-1} \) in the sense of Lustzig. It acts on the space of deformed cycles [24], [23]. Set

\[
R_N = \mathbb{C}[z_1, \ldots, z_N]^{\otimes n},
\]

and denote by \( \hat{x}_n^\pm, a_n \) the Drinfeld generators of \( U_q(\tilde{\mathfrak{sl}}_2) \). Let \( \mathcal{F}_N \) be the \( R_N \)-subalgebra of \( U_{\sqrt{-1}}(\tilde{\mathfrak{sl}}_2) \otimes R_N \) generated by the currents which are obtained by the specialization at \( q = \sqrt{-1} \) from

\[
\chi(z) = \sum_{k=1}^\infty x_k^-(q^{-1}z)^k, \quad \chi(z)^{(2)} = \frac{\chi(z)^2}{q + q^{-1}}, \tag{1.14}
\]

along with \( x_0^- \) and \( (x_0^-)^{(2)} = (x_0^-)^2/(q + q^{-1}) \). At \( q = \sqrt{-1} \), the generators \( x_k^- \) anti-commute by the definition of \( U_{\sqrt{-1}}(\tilde{\mathfrak{sl}}_2) \).

Set

\[
W_N = \bigoplus_{l=0}^N W_{N,l}.
\]

This is an \( R_N \)-algebra by the product given by (1.5).

We will show that there exists an \( R_N \)-algebra homomorphism

\[
\rho_N : \mathcal{F}_N \to W_N, \tag{1.15}
\]
and determine the kernel \( J_N \) of the mapping \( \rho_N \) explicitly by applying a supersymmetric analog of the argument in [5]. As a byproduct, we obtain the first identity (1.12). The second identity (1.13) is obtained as follows. The generators \( \Sigma_1, \Sigma_2 \) of \( \text{Ker } \psi \) are given by \( \rho_N(x_0), \rho_N((x_0)^{(2)}) \) [2], [23]. To be precise, in [23] it is proved that \( \Sigma_1 \) and \( \Sigma_2 \) generate the kernel in the case of the SU(2)-invariant Thirring model. We assume that the same statement is valid in the sine-Gordon case.

By applying a super-symmetric version of the standard argument of filtration in the dual functional spaces [22] we obtain an estimate for \( \text{ch}_{q} M_{N,l} \) from above. By counting dimensions, which follows from Tarasov’s result [23], we obtain the estimate from below. These are the main results for the SG model.

When the parameter \( \xi \) is rational, a reduction takes place in the SG model. In this paper we consider the RSG model taking \( \xi \) to be an integer \( r \geq 3 \), which corresponds to the \( \phi_{1,3} \)-perturbation of minimal unitary series. We call a sequence \( J_n = (j_1, \ldots, j_n) \) of non-negative half integers \( r \)-restricted path if for all \( i \) we have \( j_{i+1} = j_i \pm 1 \) and \( 2j_i \leq r - 2 \). As explained in [16], asymptotic states of the RSG model are parametrized by \( r \)-restricted paths. In the SG model considered so far, \( n \)-particle form factors take values in the space \( \Omega_{n,l} \) of \((\mathbb{C}^2)^{\otimes n}\) consisting of highest weight vectors of \( U_q(\mathfrak{sl}_2) \) with weight \( m = n - 2l \). For convenience, we make a gauge transformation \( \tilde{f}_n = e^{(1/2r) \sum \beta_i} f_n \), so that the action of \( U_q(\mathfrak{sl}_2) \) on \( \tilde{f}_n \) becomes independent of \( \beta_1, \ldots, \beta_n \). The parameter \( q \) is now a root of unity \( \epsilon = e^{-\pi i \frac{1}{r}} \).

We have a decomposition

\[
(\mathbb{C}^2)^{\otimes n} = \mathcal{G}_n^{(r)} \oplus \mathcal{B}_n^{(r)}
\]

into ‘good’ subspace \( \mathcal{G}_n^{(r)} \) and ‘bad’ subspace \( \mathcal{B}_n^{(r)} \), the latter being a direct sum of modules with 0 quantum dimension [17]. Set

\[
\Omega_{n,l}^{(r)} = \Omega_{n,l}/\Omega_{n,l} \cap \mathcal{B}_n^{(r)},
\]

and denote by \( \mathcal{P}: \Omega_{n,l} \to \Omega_{n,l}^{(r)} \) the projection along (1.16). By definition, \( n \)-particle form factors of the RSG model are the projection \( \mathcal{P} \tilde{f}_n \) of the one \( f_n \) in the SG model.

Under similar assumptions as in the SG case, the space of the minimal form factors \( f_N \) is identified with the quotient space of \( M_{N,l} \) modulo those \( P_{N,l} \) which satisfy

\[
\mathcal{P} \tilde{f}_n P_{N,l} = 0.
\]

Set

\[
\mu = r - 1 - (N - 2l).
\]

Setting \( \Gamma_1 = \rho_N(x_1) \) and \( \Gamma_2 = \rho_N((x_1)^{(2)}) \), we show that (1.17) holds if \( P_{N,l} \) belongs to the subspace

\[
\Gamma_1 \wedge \left( \bigwedge^{\mu+1} \Gamma_2 \right) \wedge M_{N,l-2\nu-1} + \left( \bigwedge^{\nu+1} \Gamma_2 \right) \wedge M_{N,l-2\nu-2} \quad \text{if } \mu = 2\nu + 1,
\]
or

\[
\left( \bigwedge^{\nu} \Gamma_2 \right) \wedge M_{N,l-2\nu} \quad \text{if } \mu = 2\nu.
\]

We denote by \( M_{N,l}^{(r)} \) the quotient space of \( M_{N,l} \) by the above subspace. Again, we assume that the kernel of the mapping \( \tilde{P}_\psi \) is equal to this subspace.

From the definition, \( M_{N,l}^{(r)} \) is a \( q \)-analog of the space of conformal coinvariants in the level \( r - 2 \) SU(2) WZW conformal field theory, with the deformation parameter \( q \) being \( \sqrt{-1} \). We prove that

\[
\chi_q M_{N,l}^{(r)} = \frac{1}{(q)_N} K_{N-2l(1\nu)}^{(r-2)}(q),
\]

where \( K_{N-2l(1\nu)}^{(r-2)}(q) \) is the restricted Kostka polynomial corresponding to the tensor product \( (C^2)^{\otimes N} \). We note that the right hand side of (1.13) is equal to the Kostka polynomial \( K_{N-2l(1\nu)}^{(r-2)}(q) \) when \( r \to \infty \). Apart from the factor \( 1/(q)_N \), formula (1.18) is a direct analog of the corresponding result for the conformal case obtained in [6].

Our proof of the equality (1.18) consists of two parts; the estimate from above and the estimate from below. The argument for the former is a super-symmetric version of the conformal case [6]. To show the latter in the conformal case, we used in [6] the fusion rule proved by Tsuchiya–Ueno–Yamada [25]. In the present case, we employ results from Kashiwara’s theory of global basis for level 0-modules [9].

If we extend the minimal form factors to those obtained from the deformed cycles in the extended space \( C_{n,l}(z_1 \cdots z_n)^{-\frac{L}{2}} \) (\( L = 0, 1, 2, \ldots \)), we obtain an increasing sequence of the space of minimal form factors

\[
F_m^{(r)}[0] \subset F_m^{(r)}[1] \subset F_m^{(r)}[2] \subset \cdots.
\]

Under the assumptions made so far, the total space of form factors in the RSG model should be represented by the union of them. In the above, we have considered the case \( L = 0 \). From (1.18), it follows that

\[
\chi_q F_m^{(r)}[0] = q^{c/24} \chi_m^{(r,1\nu+1)}(q),
\]

where \( \chi_m^{(r,1\nu+1)}(q) \) denotes the irreducible character of the Virasoro minimal unitary series with central charge and highest weight

\[
c = 1 - \frac{6}{r(r + 1)}, \quad h_{b,a} = \frac{(r + 1)b - ra)^2 - 1}{4r(r + 1)}.
\]

For general \( L \), formula (1.20) generalizes to

\[
\chi_q F_m^{(r)}[L] = \sum_{a \equiv L-1 \mod 2} \chi_{m+1,a}^{(r,1\nu+1)}(q) \chi_{1,a}^{(r,1\nu+1)}(q^{-1}; L),
\]

where \( \chi_m^{(r,1\nu+1)}(q; L) \) is a polynomial first found in [1] as a finitization of \( \chi_m^{(r,1\nu+1)}(q) \). Formulas of this sort have been observed earlier in the RSG model corresponding to the \( (2, p) \) minimal series, where the \( S \) matrices are scalar [10]. The appearance
of $q$ and $q^{-1}$ is interpreted there as mixing of the two chiralities in massive field theory. Here we have confirmed the validity of this picture in a general setting.

Our counting is based on the assumption that any $N$-particle minimal form factor $f_N$ can be lifted to a tower $f = (f_n)_{n=0}^\infty$. We plan to address this problem in our next paper.

The plan of the paper is as follows. In Section 2, we review the integral formula for form factors of the SG model. After a brief review of the bootstrap approach, we give the hypergeometric pairing, describe null cycles and discuss the minimality condition. The materials in this section follow basically [13], [14] with minor modifications. In Section 3, we study the space $W_N$. We give the action of the algebra $F_N$ mentioned above, and determine the complete set of relations for the currents $X(z), X(z)^{(2)}$. The main results are stated in Theorems 3.1 and 3.2. In Section 4, we determine the structure of $M_{N,t}$. The formula 1.13 is proved in Theorem 4.1. The RSG model is discussed in Section 5. The formula (1.18) is proved in Theorem 5.4. Section 6 is devoted to the derivation of the character formula (1.21).

In the appendices we collect some facts concerning the quantum loop algebra and its representations. In Appendix A we give our convention concerning $U_q(\mathfrak{sl}_2)$. In Appendix B we discuss realization of the currents $X(z), X(z)^{(2)}$ using the creation part of the Jordan–Wigner fermions. Appendix C is an exposition of the action of $U_q(\mathfrak{sl}_2)$ on the trigonometric hypergeometric space of Tarasov–Varchenko [24]. Appendix D is concerned with representations of $U_\epsilon(\mathfrak{sl}_2)$ at $\epsilon^\ast = -1$.

2. Form Factors of the sine-Gordon Model

2.1. The bootstrap axiom. In this subsection, we review briefly the general setting for form factors of the SG model. Our aim is to present the main ‘axioms’ and motivate the subsequent discussions, thereby introducing our notation. For more details and the physical background, the reader is referred to [19].

Fix a real parameter $\xi > 1$ throughout. Consider the quantum loop algebra $U = U_q(\mathfrak{sl}_2)$ with $q = e^{-\pi i/\xi}$. We will use the convention concerning $U$ in Appendix A. Let

$$V = \mathbb{C}v_+ \oplus \mathbb{C}v_- \simeq \mathbb{C}^2.$$ 

For $\zeta \in \mathbb{C}^\times$, let $\pi_\zeta: U \rightarrow \text{End}(V)$ be the representation given by

$$e_0 \mapsto \zeta \sigma^-, \quad f_0 \mapsto \zeta^{-1} \sigma^+ , \quad e_1 \mapsto \zeta \sigma^+, \quad f_1 \mapsto \zeta^{-1} \sigma^- , \quad t_0 \mapsto q^{-\sigma_-^z}, \quad t_1 \mapsto q^{\sigma_-^z},$$

where $\sigma^\pm, \sigma^z$ are the Pauli matrices. We regard the tensor product

$$\pi_{\zeta_1} \otimes \cdots \otimes \pi_{\zeta_n}$$

as a $U$-module via the coproduct

$$\Delta'(e_i) = e_i \otimes t_i + 1 \otimes e_i, \quad \Delta'(f_i) = f_i \otimes 1 + t_i^{-1} \otimes f_i, \quad \Delta'(t_i) = t_i \otimes t_i. \quad (2.2)$$

This coproduct is opposite to the one given in (A.1). We will use the symmetric bilinear form $(\cdot, \cdot)$ such that the vectors $v_{\zeta_1} \otimes \cdots \otimes v_{\zeta_n}$ are orthonormal. We have

$$(xu, v) = (u, \alpha(x)v) \quad (\forall u \in \pi_{\zeta_{i-1}} \otimes \cdots \otimes \pi_{\zeta_n}, \quad \forall v \in \pi_{\zeta_i} \otimes \cdots \otimes \pi_{\zeta_n}),$$
where \( \alpha : U \to U \) denotes the anti-involution given by

\[
\alpha(e_i) = q t_i f_i, \quad \alpha(f_i) = q^{-1} e_i t_i^{-1}, \quad \alpha(t_i) = t_i.
\]

At a heuristic level, the space of physical states of the SG model is a ‘direct sum’ of \( \pi_{\zeta_1} \otimes \cdots \otimes \pi_{\zeta_n} \), where \( n \in \mathbb{Z}_{\geq 0}, \zeta_j = e^{\beta_j / \xi} \), and \( \beta_1, \ldots, \beta_n \) are real parameters called rapidities. We use the symbol \( | \beta_1, \ldots, \beta_n, \ldots, \rangle \) to represent the vector \( v_{\zeta_1} \otimes \cdots \otimes v_{\zeta_n} \) in the module \( \pi_{\zeta_1} \otimes \cdots \otimes \pi_{\zeta_n} \). There is an exchange relation which identifies the vectors when \( \beta_j \) and \( \beta_{j+1} \) are interchanged. This relation is given by the \( S \)-matrix.

The \( S \)-matrix of the SG model is

\[
S(\beta) = S_0(\beta) \hat{S}(\beta).
\]

Here \( \hat{S}(\beta) \) is the linear operator defined by

\[
\hat{S}(\beta)(v_{\epsilon_1} \otimes v_{\epsilon_2}) = \sum_{\epsilon_1', \epsilon_2'} S_{\epsilon_1, \epsilon_2; \epsilon_1', \epsilon_2'}(\beta)(v_{\epsilon_1'} \otimes v_{\epsilon_2'}),
\]

\[
\hat{S}_{\pm, \pm}^{\pm, \pm}(\beta) = 1, \quad \hat{S}_{\pm, \pm}^{\pm, \pm}(\beta) = \frac{\sh \frac{\beta}{\xi}}{\sh \frac{1}{\xi}(\beta - \pi i)}, \quad \hat{S}_{\pm, \pm}^{\pm, \pm}(\beta) = \frac{\sh \frac{-\beta}{\xi}}{\sh \frac{1}{\xi}(\beta - \pi i)},
\]

and \( S_0(\beta) \) is the normalization factor

\[
S_0(\beta) = \frac{S_2(-i\beta) S_2(\pi + i\beta)}{S_2(\pi - i\beta) S_2(i\beta)}, \tag{2.3}
\]

where \( S_2(\beta) = S_2(\beta|2\pi, \pi) \) is the double sine function. (See [8] for properties of the double sine function.) The exchange relation is given by

\[
| \ldots, \beta_j, \beta_{j+1}, \ldots \rangle_{\epsilon_1, \epsilon_{j+1}, \ldots} = \sum_{\epsilon_1', \epsilon_j', \ldots} | \ldots, \beta_j+1, \beta_j, \ldots \rangle_{\epsilon_1', \epsilon_j', \ldots} \times S_{\epsilon_1, \epsilon_j+1; \epsilon_1', \epsilon_j'}(\beta_{j+1} - \beta_j).
\]

The map

\[
PS(\beta_2 - \beta_1) : v_{\epsilon_1} \otimes v_{\epsilon_2} \mapsto \sum_{\epsilon_1', \epsilon_2'} v_{\epsilon_2'} \otimes v_{\epsilon_1'} \times S_{\epsilon_1, \epsilon_2; \epsilon_1', \epsilon_2'}(\beta_2 - \beta_1)
\]

is an intertwiner from \( \pi_{\zeta_1} \otimes \pi_{\zeta_1} \) to \( \pi_{\zeta_1} \otimes \pi_{\zeta_1} \), where \( P(v_1 \otimes v_2) = v_2 \otimes v_1 \). Hence the above relation is compatible with the action of \( U \).

We denote the vacuum vector by \( |\text{vac}\rangle \). This is a generator of the space of 0 particle state, i.e., \( n = 0 \) in (2.1). We denote its dual vector by \( \langle \text{vac}| \). A local operator \( \mathcal{O} \) is an operator acting on the space of physical states. It is uniquely specified by the matrix element [19]

\[
f_n(\beta_1, \ldots, \beta_n)_{\epsilon_1, \ldots, \epsilon_n} = \langle \text{vac}| \mathcal{O}| \beta_1, \ldots, \beta_n, \ldots \rangle_{\epsilon_1, \ldots, \epsilon_n}.
\]

They are encapsulated into a dual vector

\[
f_n(\beta_1, \ldots, \beta_n) = \sum_{\epsilon_1, \ldots, \epsilon_n} f_n(\beta_1, \ldots, \beta_n)_{\epsilon_1, \ldots, \epsilon_n} v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_n},
\]
on which $U$ acts by $\pi_{\gamma^{-1}} \otimes \cdots \otimes \pi_{\epsilon^{-1}}$. We call the tower of vector-valued functions $f = (f_n(\beta_1, \ldots, \beta_n))_{n \geq 0}$ the form factor of the local operator $\mathcal{O}$.

In this paper we consider only form factors satisfying the conditions

$$h_1 f_n(\beta_1, \ldots, \beta_n) = m f_n(\beta_1, \ldots, \beta_n),$$

$$e_1 f_n(\beta_1, \ldots, \beta_n) = 0$$

for some $m \in \mathbb{Z}_{\geq 0}$. In other words, the vector $f_n(\beta_1, \ldots, \beta_n)$ is the highest weight vector of the $m+1$ dimensional sub-representation of $V^{\otimes n}$ as $U_q(sl_2)$-module, where $U_q(sl_2)$ signifies the subalgebra of $U$ generated by $e_1, f_1, t_1$. As it is explained in [16], this amounts to considering a certain subsector of local operators. This class is further subdivided into two subsectors labeled by the index $\varepsilon \in \{0, 1\}$, which enters the axioms (A2), (A3) below.

Physical considerations for local operators lead to several conditions on form factors. They are required to have the following analyticity and asymptotic properties.

(a) $f_n(\beta_1, \ldots, \beta_n)$ extend to meromorphic functions in $\beta_1, \ldots, \beta_n$ on the whole complex plane,

(b) When $\beta_j \in \mathbb{R} (j \neq n)$, they are holomorphic in the domain $0 < \text{Im} \beta_n < 2\pi$ except for possible simple poles at $\beta_n = \beta_j + \pi i$,

(c) When $\beta_j \to \pm \infty$ we have $|f_n(\beta_1, \ldots, \beta_n)| = O(e^{K|\beta_j|})$ for some $K > 0$.

In addition, the following main ‘axioms’ are imposed [19]:

(A1) $f_n(\ldots, \beta_{j+1}, \beta_j, \ldots) = P_{\beta_{j+1}} S_{\beta_j+1}(\beta_j - \beta_{j+1})f_n(\ldots, \beta_j, \beta_{j+1}, \ldots)$,

(A2) $f_n(\beta_1, \ldots, \beta_{n-1}, \beta_n + 2\pi i) = e^{(2i+\varepsilon)\pi i} e^{\frac{m\pi i}{\beta_n}} \sigma^1 P_{\beta_{n-1}} \cdots P_2 f_n(\beta_n, \beta_1, \ldots, \beta_{n-1})$,

(A3) $\text{res}_{\beta_n = \beta_{n-1} + \pi} f_n(\beta_1, \ldots, \beta_n) = (I + e^{(2i+\varepsilon)\pi i} s_{\beta_{n-1}} - \beta_{n-2}) \cdots S_{\beta_{n-1}}(\beta_{n-1} - \beta_1) e^{\frac{m\pi i}{\beta_n}} \sigma^1 \times f_{n-2}(\beta_1, \ldots, \beta_{n-2}) \otimes (v_+ \otimes v_- - v_- \otimes v+)$. 

In the above, the subscripts of the operators refer to the components in the tensor product $V^{\otimes n}$ on which the operators act.

A large family of functions with these properties can be constructed in terms of the hypergeometric integrals [19], [24] (see (2.38) below). We expect that ‘all’ form factors can be obtained in this way. Motivated by these considerations, henceforth we restrict ourselves to form factors obtained by hypergeometric integrals.

In the next section, we define the hypergeometric integral following [19], [24], and using it we construct $N$-minimal form factors of the $sl_2$ weight $N - 2l$. We fix $N$ and $l$ until we start the discussion on Virasoro characters in Section 6 (except in the proof of Proposition 3.15).

The hypergeometric integral consists of three ingredients; the phase function $\phi(\alpha; \beta_1, \ldots, \beta_N)$, a deformed cocycle $w(a_1, \ldots, a_l; \beta_1, \ldots, \beta_N)$ and a deformed cycle $P(X_1, \ldots, X_l; z_1, \ldots, z_N)$. It gives a pairing between deformed cocycles and deformed cycles. Here and in what follows, we set

$$X_p = e^{-\omega_p}, \quad z_j = e^{\beta_j}, \quad a_p = e^{2\pi i}, \quad b_j = e^{2\pi i}, \quad \omega = e^{2\pi i}. \quad (2.4)$$
The variables $X_p$ and $z_j$ are $2\pi i$ periodic with respect to $\alpha_p$ and $\beta_j$, respectively, while the variables $a_p$ and $b_j$ are $\xi \pi i$ periodic.

2.2. Phase function. We use the function

$$\varphi(\alpha) = \frac{1}{S_2(\frac{\pi}{2} - i\alpha)S_2(\frac{\pi}{2} + i\alpha)}.$$  

Define the phase function

$$\phi(\alpha; \beta_1, \ldots, \beta_N) = \prod_{j=1}^{N} (\frac{e^{\xi^{-1}(\alpha - \beta_j)} \varphi(\alpha - \beta_j + \frac{3\pi i}{2})}{\frac{e^{\xi^{-1}(\alpha - \beta_j)} \varphi(\alpha - \beta_j + \frac{3\pi i}{2})}}).$$  

(2.5)

We have

$$\frac{\phi(\alpha; \beta_1, \ldots, \beta_N + 2\pi i)}{\phi(\alpha; \beta_1, \ldots, \beta_N)} = \frac{\omega a - b_N}{a - b_N},$$  

(2.6)

$$\frac{\phi(\alpha - 2\pi i; \beta_1, \ldots, \beta_N)}{\phi(\alpha; \beta_1, \ldots, \beta_N)} = \prod_{j=1}^{N} \frac{\omega a - b_j}{a - b_j},$$  

(2.7)

$$\frac{\phi(\alpha - \xi \pi i; \beta_1, \ldots, \beta_N)}{\phi(\alpha; \beta_1, \ldots, \beta_N)} = \prod_{j=1}^{N} \frac{e^{-(\alpha - \beta_j) + 1}}{e^{-(\alpha - \beta_j) + \xi \pi i - 1}},$$  

(2.8)

where $a = e^{2\alpha/\xi}$. We have the estimates,

$$|\phi(\alpha)| = \begin{cases} O(e^{-\frac{N\alpha}{\xi}}) & \text{when } \alpha \to \infty; \\ O(e^{N\alpha}) & \text{when } \alpha \to -\infty. \end{cases}$$  

(2.9)

For each integer $l$ such that $0 \leq 2l \leq N$, we define the space of deformed cocycles and that of deformed cycles.

2.3. Deformed cocycles. A deformed cocycle $w$ is a function of the variables $a_1, \ldots, a_l$ and $\beta_1, \ldots, \beta_N$ such that

$$w(a_1, \ldots, a_l; \beta_1, \ldots, \beta_N) = \frac{Q(a_1, \ldots, a_l; \beta_1, \ldots, \beta_N)}{\prod_{p=1}^{l} \prod_{j=1}^{N} (a_p - b_j)},$$  

(2.10)

where $Q$ is a polynomial in $a_1, \ldots, a_l$ satisfying the conditions,

$$Q \text{ is skew-symmetric in } a_1, \ldots, a_l,$$  

(2.11)

$$\deg_{a_p} Q \leq N + l - 1,$$  

(2.12)

$$Q|_{a_p=0} = 0 \text{ for any } p,$$  

(2.13)

$$Q|_{a_p=\omega a_p, b_j=0} = 0 \text{ for any } p, p', j.$$  

(2.14)

We define special cocycles $w_M$ indexed by a subset $M \subset \{1, \ldots, N\}$ such that $M = \{m_1 < \cdots < m_l\}$, and we mainly use them:

$$w_M = \text{Skew}_{a_1, \ldots, a_l} g_M.$$  

(2.15)
Here the function \( g_M \) is defined by

\[
g_M(a_1, \ldots, a_l; \beta_1, \ldots, \beta_N) = e^{N-2l} \sum_{j=1}^{N} \frac{1}{j} \prod_{p=1}^{l} \beta_{m_p} \\
\times \prod_{p=1}^{l} a_p \prod_{j=m_p}^{l} (\omega a_p - b_j) \prod_{p>q} (a_p - a_q) \\
\times \prod_{j=1}^{N} (a_p - b_j) \prod_{p<p'} (\omega a_p - a_{p'}) ,
\]  

(2.16)

and Skew_{a_1, \ldots, a_l} is the skew-symmetrization with respect to \( a_1, \ldots, a_l \):

\[
\text{Skew}_{a_1, \ldots, a_l} g_M = \sum_{\sigma \in S_l} (\text{sgn} \, \sigma) \, g_M(a_{\sigma(1)}, \ldots, a_{\sigma(l)}),
\]

where \( S_l \) stands for the symmetric group on \( l \) letters.

In the construction of the form factors, we use the special vector \( v_{N,l} \) in \((V \otimes N)_l = \{ v \in V \otimes N : h_1 v = (N-2l) v \} \) given by

\[
v_{N,l} = \sum_{#M=l} q^{\nu(M)} w_M v_M,
\]

(2.17)

where \( N, l \) are fixed in the right hand side, and

\[
\nu(M) = \sum_{p=1}^{l} (m_p - 1),
\]

(2.18)

\[
v_M = v_{\varepsilon_1} \otimes \cdots \otimes v_{\varepsilon_N},
\]

(2.19)

where

\[
M = \{ j : 1 \leq j \leq N, \, \varepsilon_j = - \}. \tag{2.20}
\]

When we specialize \( \beta_1, \ldots, \beta_N \) to generic values, the vector space of the deformed cocycles is \( \binom{N}{l} \)-dimensional, and it is spanned by \( w_M \). This is known in [24].

2.4. Deformed cycles. A deformed cycle \( P \) is a polynomial of \( X_1, \ldots, X_l \) and \( z_1, \ldots, z_N \) satisfying the conditions

\[
P \text{ is skew-symmetric in } X_1, \ldots, X_l,
\]

(2.21)

\[
P \text{ is symmetric in } z_1, \ldots, z_N,
\]

(2.22)

\[
\text{deg}_{X_p} P \leq N - 1.
\]

(2.23)

Denote by \( C_{N,l} \) the space of deformed cycles with fixed \( N, l \).

2.5. Hypergeometric pairing. The hypergeometric pairing of a deformed cocycle \( w \) and a deformed cycle \( P \) is given by

\[
I(w, P) = \int_{C} \prod_{p=1}^{l} d\alpha_p \prod_{p=1}^{l} \phi(\alpha_p; \beta_1, \ldots, \beta_N) \\
\times w(a_1, \ldots, a_l; \beta_1, \ldots, \beta_N) P(X_1, \ldots, X_l; z_1, \ldots, z_N),
\]

(2.24)

where the integration contour \( C \) goes along real axis except that the simple poles of the integrand at

\[
\alpha_p = \beta_j - 2\pi i \mathbb{Z}_{\geq 0} - \xi \pi i \mathbb{Z}_{\geq 0}
\]

(2.25)
are located below $C$, and those at
\begin{align}
\alpha_p &= \beta_j - \pi i + 2\pi i Z_{\geq 0} + \xi \pi i Z_{\geq 0} \\
&= \beta_j - \pi i + 2\pi i Z_{\geq 0} + \xi \pi i Z_{\geq 0} + 2 \pi i Z_{\geq 0} + \xi \pi i Z_{\geq 0} \tag{2.26}
\end{align}
above $C$. These are the only poles of the integrand.

Recall that we have the restriction $0 \leq 2l \leq N$ from the axiom (A5). The convergence of the integral follows from the following estimates when $\alpha_p \to \pm \infty$.

\[ |w| = \begin{cases} O(e^{2(l-1)\alpha_p}) & \text{when } \alpha_p \to \infty; \\ O(e^{2\alpha_p}) & \text{when } \alpha_p \to -\infty, \end{cases} \tag{2.27} \]

\[ |P| = \begin{cases} O(1) & \text{when } \alpha_p \to \infty; \\ O(e^{-(N-1)\alpha_p}) & \text{when } \alpha_p \to -\infty. \end{cases} \tag{2.28} \]

**Remark 2.1.** The convergence of the integral $I(w, P)$ is valid even if we weaken the condition (2.23) to
\[ \deg_{X_p} P \leq N. \tag{2.29} \]
This is because of the condition (2.13) for $w$. If we drop this condition for $w$, the convergence is still valid under the assumption (2.23) for $P$.

**Theorem 2.2.** For each deformed cycle $P$, the hypergeometric integral
\[ \psi_P(\beta_1, \ldots, \beta_N) = I(v_{N,l}, P) \tag{2.30} \]
satisfies the following:
\begin{align}
h_1 \psi_P(\beta_1, \ldots, \beta_N) &= (N - 2l) \psi_P(\beta_1, \ldots, \beta_N), \\
\psi_P(\beta_1, \ldots, \beta_{j+1}, \beta_j, \ldots) &= P_{j,j+1} \hat{S}_{j,j+1}(\beta_j - \beta_{j+1}) \psi_P(\ldots, \beta_j, \beta_{j+1}, \ldots), \\
\psi_P(\beta_1, \ldots, \beta_{N-1}, \beta_N + 2\pi i) &= e^{\frac{(\alpha_N - 2\pi i)\sigma}{\alpha_i}} P_{N,N-1} \cdots P_{2,1} \psi_P(\beta_N, \beta_1, \ldots, \beta_{N-1}). \tag{2.32} \end{align}

The proof is similar to that of Theorem 6.3 in [13]. We omit the proof here.

When we specialize $\beta_1, \ldots, \beta_N$, the space of the deformed cycles is nothing but the space of skew-symmetric polynomials in $X_1, \ldots, X_l$ satisfying the degree restriction (2.23). This is an $\binom{N}{l}$-dimensional vector space. If the values of $\beta_1, \ldots, \beta_N$ are generic, the space spanned by $w_{M}$ is also $\binom{N}{l}$-dimensional. The integral $I(w, P)$ defines a pairing between these two spaces. This pairing is degenerate. There exist cocycles $w_0$ such that $I(w_0, P) = 0$ for all $P$, and vice versa. We call them null (co)cycles.

We define the twisted difference operator $\nabla_{\alpha,c}$ acting on a function $f(\alpha)$ by
\[ \nabla_{\alpha,c}(f) = f(\alpha) - f(\alpha + c) \times \frac{\phi(\alpha + c)}{\phi(\alpha)}. \tag{2.34} \]

Similarly, for two variables $\alpha_1, \alpha_2$, we define
\[ \nabla_{\alpha_1,\alpha_2,c}(f) = f(\alpha_1, \alpha_2) - f(\alpha_1 + c, \alpha_2 + c) \times \frac{\phi(\alpha_1 + c)\phi(\alpha_2 + c)}{\phi(\alpha_1)\phi(\alpha_2)}. \tag{2.35} \]
If the integral \( \int_C \frac{d\alpha \phi f}{\nabla_{\alpha,c}(f)} \) is convergent and the integrand has no pole between the contours \( C \) and \( C + c \), then
\[
\int_C \frac{d\alpha \phi f}{\nabla_{\alpha,c}(f)} = \left( \int_C - \int_{C+c} \right) \frac{d\alpha \phi f}{\nabla_{\alpha,c}(f)} = 0.
\]

We use \( c = -2\pi i \) for the construction of null cocycles, and \( c = -\xi \pi i \) for null cycles. The ratio \( \phi(\alpha + c)/\phi(\alpha) \) for these \( c \) is given by (2.7) and (2.8). We call this type of argument for the vanishing of the integral ‘the twisted difference method’.

2.6. Null cocycles. We can extend the definition of the pairing \( I(w, P) \) to a wider class of functions \( w \) by dropping the skew-symmetry of \( Q \) and the conditions (2.13) and (2.14). In this case, we must restrict \( P \) by (2.23) for convergence. Since \( P \) is skew-symmetric, the integral \( I(w, P) \) is multiplied by \( l! \) if we replace \( w \) by its skew-symmetrization with respect to \( a_1, \ldots, a_l \).

Lemma 2.3. Let \( w \) be of the form (2.10) where \( Q \) is a polynomial in \( a_1, \ldots, a_l \) satisfying (2.12). We assume that \( Q \) has zero at \( a_1 = b_j \) for all \( 1 \leq j \leq N \). We have
\[
\nabla_{\alpha_1,-2\pi i}(w) = w(\alpha_1) - w(\alpha_i + 2\pi i) \times \prod_{j=1}^{N} \omega a_1 - b_j
\]
and
\[
I(\nabla_{\alpha_1,-2\pi i}(w), P) = 0
\]
for any deformed cycle \( P \).

Proof. We apply the twisted difference method. The integrand of \( I(w, P) \) has no pole in \( \alpha_1 \) between the contours \( C \) and \( C - 2\pi i \). The deformed cycle \( P \) is \( 2\pi i \) periodic, and the phase function \( \phi \) satisfies (2.7). The assertion follows from these properties.

Lemma 2.4. Fix \( N \) and \( l \), and let
\[
\tilde{g}_M = (\omega - 1)e^{-\frac{N-1}{l+1} \sum_{j=1}^{N} \beta_j - \frac{1}{l} \sum_{p=1}^{l} \beta_{mp}} \tilde{g}_M
\]
where \( \tilde{g}_M \) is given by (2.16). For each \( k \notin M \), we similarly define \( \tilde{g}_{M \cup \{k\}} \) with \( l \) replaced by \( l + 1 \). We define \( \varepsilon_j \) by (2.20) for \( M \), and \( \nu(M \cup \{k\}) \) by (2.18) for \( M \cup \{k\} \). Set
\[
w(\alpha, a_1, \ldots, a_l) = - \prod_{p=1}^{l} (\omega a - a_p) \times \tilde{g}_M(\alpha_1, \ldots, a_l; \beta_1, \ldots, \beta_N).
\]

Then, we have
\[
\nabla_{\alpha_1,-2\pi i}(w) \equiv q^{-\nu(M)-1} \sum_{k \notin M} q^{\sum_{j \leq k} \varepsilon_j + \nu(M \cup \{k\})} \tilde{g}_{M \cup \{k\}},
\]
where the equivalence relation \( A \equiv B \) means \( \text{Skew}_{a_1, \ldots, a_l}(A - B) = 0 \).

Proof. This is a straightforward generalization of (3.5) in Lemma 3.5 of [13].
By these lemmas we conclude that the deformed cocycles
\[ \sum_{k \in \mathbb{M}} q^{\sum_{j \leq k} \epsilon_j + \nu(M(k))} e^{-\frac{i}{2} \beta_k w_{M(k)}} \]  
(2.36)
with \( \#(M) = l - 1 \) are null cocycles.

Consider an action of \( U_q(sl_2) \) with the canonical generators \( e_1, f_1, t_1 \) and \( q = e^{-\frac{2\pi}{i}} \), on \( V^{\otimes N} \) through the opposite coproduct \( \Delta' \) given by (2.2). The function \( \psi_P \) takes values in the representation space \( V^{\otimes N} \) of \( \pi_{\zeta_1} \otimes \cdots \otimes \pi_{\zeta_N} \) with \( \zeta_j = e^{3j/\xi} \).

**Corollary 2.5.** The vector \( \psi_P = I(v_{N,l}, P) \) is a highest weight vector for any deformed cycle \( P \):

\[ e_1 \psi_P(\beta_1, \ldots, \beta_N) = 0. \]  
(2.37)

Now we introduce the function \( \zeta(\beta) \) defined by
\[ \zeta(\beta) = \frac{S_3(-i\beta + 2\pi)S_3(i\beta)}{S_3(-i\beta + 3\pi)S_3(i\beta + \pi)} \]
\[ S_3(\beta) = S_3(\beta|2\pi, 2\pi, \pi). \]
Here \( S_3(\beta) \) is the triple sine function (see [8] for the definition and properties).

For a deformed cycle \( P \in C_{N,l} \) we set
\[ f_P(\beta_1, \ldots, \beta_N) = e^{(\frac{3}{2} - \frac{2}{\xi}) \sum_{j=1}^N \beta_j} \prod_{1 \leq j < p \leq N} \zeta(\beta_j - \beta_p) \cdot \psi_P(\beta_1, \ldots, \beta_N). \]  
(2.38)

Here we write down the formula for the function \( \psi_P \):
\[ \psi_P(\beta_1, \ldots, \beta_N) = \sum_{\#M = l} v_M \int_{C_l} \prod_{p=1}^l d\alpha_p \prod_{p=1}^l \phi(\alpha_p; \beta_1, \ldots, \beta_N) \]
\[ \times (\text{Skew}_{\alpha_1, \ldots, \alpha_l} g'_M(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_N)) P(X_1, \ldots, X_l; z_1, \ldots, z_N). \]

The phase function \( \phi(\alpha; \beta_1, \ldots, \beta_N) \) is defined by (2.1). See (2.24) below for the definition of the contour \( C \). We have set \( g'_M = q^{\nu(M)} g_M \) where \( g_M \) given by (2.16).

More explicitly,
\[ g'_M(\alpha_1, \ldots, \alpha_l) = e_{l \xi}^{\sum_{j=1}^N \beta_j + \frac{i}{2} \sum_{p=1}^l \alpha_p} \]
\[ \times \prod_{p=1}^l \left( \frac{1}{\text{sh} \frac{1}{2} (\alpha_p - \beta_{m_p})} \prod_{j < m_p} \frac{\text{sh} \frac{1}{2}(\alpha_p - \beta_j + \pi i)}{\text{sh} \frac{1}{2}(\alpha_p - \beta_j)} \right) \prod_{1 \leq p < p' \leq l} \text{sh} \frac{1}{\xi} (\alpha_p - \alpha_{p'} + \pi i), \]
where \( e_{l \xi} := 2^{l(\xi-1)/2} \) is a constant.

From Theorem 2.2 and Corollary 2.5 we find

**Proposition 2.6.** For any deformed cycle \( P \in C_{N,l} \) the function \( f_P \) satisfies the axioms (A0), (A1) and (A2) with \( m = N - 2l \).

**2.7. Minimality condition.** Let us consider the \( N \)-minimality condition:
\[ \text{res}_{\beta_N = \beta_{N-1} + \pi} f_P(\beta_1, \ldots, \beta_N) = 0. \]
A deformed cycle \( P \in C_{N,l} \) is called minimal if
\[ P|_{z_1 = -z_2 = X_{l-1}} = 0. \]  
(2.39)
We denote by \( W_{N,l} \) the space of the minimal deformed cycles with fixed \( N, l \).
**Theorem 2.7.** Let $\xi > 1$ be a generic value. For each minimal deformed cycle $P$, the hypergeometric integral $\psi_P = I(v_{N,l}, P)$ satisfies

$$\text{res}_{\beta_N = \beta_{N-1} + \pi i} \psi_P(\beta_1, \ldots, \beta_N) = 0.$$ 

Hence the function $f_P$ associated with a minimal deformed cycle $P \in W_{N,l}$ gives an $N$-particle minimal form factor of the SG model.

**Proof.** It is enough to show the cancellation of the residues at $\beta_N = \beta_{N-1} + \pi i$. The proof is given by repeating the argument in [12]. □

### 2.8. Null cycles

Our aim in this section is to construct minimal deformed cycle $P$ such that

$$I(v_{N,l}, P) = 0.$$ 

If $P_1 \in C_{N,l}$ is a null cycle, then for any $P_2 \in C_{N,l}$, the deformed cycle $P_1 \land P_2 \in C_{N,l}$ is a null cycle. Following [23], we find the following minimal null cycles.

Set

$$\Theta(X) = \prod_{j=1}^N (1 - z_j X), \quad (2.40)$$

$$\Theta(X_1, X_2) = \Theta(X_1)\Theta(X_2) - \Theta(-X_1)\Theta(-X_2). \quad (2.41)$$

and

$$\Sigma_1(X) = \Theta(-X) - (-1)^N \Theta(X), \quad (2.42)$$

$$\Sigma_2(X_1, X_2) = \frac{X_1 - X_2}{X_1 + X_2} \Theta(X_1, X_2) + (-1)^N \Theta(X_1, -X_2). \quad (2.43)$$

Here we use $X = e^{-\alpha}$, $z_j = e^{\beta_j}$, etc. The polynomial $\Theta(X_1, X_2)$ is divisible by $X_1 + X_2$. The degree $N$ terms cancel in both $\Sigma_1$ and $\Sigma_2$. Note also that both $\Sigma_1$ and $\Sigma_2$ are minimal cycles.

**Proposition 2.8 [2, 23].** The deformed cycles $\Sigma_1$ and $\Sigma_2$ are null cycles. In fact, $I(w, \Sigma_1) = 0$ for all $w(a)$ which is given by (2.10) ($l = 1$) with $Q(a)$ satisfying (2.12) and (2.13). We have also $I(w, \Sigma_2) = 0$ if $w = w(a_1, a_2)$ is given by (2.10) ($l = 2$) with $Q(a_1, a_2)$ satisfying (2.12), (2.13) and (2.14).

**Proof.** Set $P_1(X) = \Theta(X)$. Note that $\Sigma_1 = \nabla_{\alpha, -\xi \pi i}(P_1)$. By Remark 2.1, the integral $I(w, P_1)$ is convergent for any $w$ as given in the statement of the lemma. The choice of $P_1$ is such that the poles of the integrand is canceled between the contours $C$ and $C - \xi \pi i$. Using (2.8), we obtain $I(w, \nabla_{\alpha, -\xi \pi i}(P_1)) = 0$.

The proof for $\Sigma_2$ is slightly more involved. Since

$$\Theta(X_1, -X_2) = \Sigma_1(X_1) \prod_{j=1}^N (z_j X_2 + 1) - \prod_{j=1}^N (z_j X_1 + 1) \cdot \Sigma_1(X_2),$$

we have $I(w, \Theta(X_1, -X_2)) = 0$. It remains to show that $I(w, \frac{X_1 - X_2}{X_1 + X_2} \Theta(X_1, X_2)) = 0$. Set

$$P_2(X_1, X_2) = \frac{X_1 - X_2}{X_1 + X_2} \Theta(X_1)\Theta(X_2),$$

(2.44)
denote the integrand of $I(w, P_2)$ by $F(\alpha_1, \alpha_2) = \phi(\alpha_1)\phi(\alpha_2)w(a_1, a_2)P_2(X_1, X_2)$. It has poles at $\alpha_p = \beta_j - 2\pi i \mathbb{Z}_{\geq 0} - \xi \pi i \mathbb{Z}_{\geq 0}$, $\alpha_p = \beta_j - \pi i + 2\pi i \mathbb{Z}_{\geq 0} + \xi \pi i \mathbb{Z}_{\geq 0}$, and $\alpha_1 = \alpha_2 + \pi i + 2\pi i \mathbb{Z}$. Let $C_1$ be the union of small circles going clockwise around $\beta_j - \xi \pi i$ ($1 \leq j \leq N$), and let $C_2$ be the contour going from $-\infty$ to $\infty$, such that $\beta_j - \xi \pi i$ is above and $\beta_j - \pi i - \xi \pi i$ is below $C_2$. Since the poles at $\alpha_p = \beta_j - 2\pi i \mathbb{Z}$ is cancelled by $P_2$, we can deform the contour $C \times C$ to $(C_1 + C_2) \times (C_1 + C_2)$. If we move $C_2$ further below $\beta_j - \pi i - \xi \pi i$, the contour becomes $(C - \xi \pi i) \times (C - \xi \pi i)$, and we obtain

$$
\int_{C_1+C_2} \int_{C_1+C_2} F(\alpha_1, \alpha_2)d\alpha_1 d\alpha_2 = \int_{C - \xi \pi i} \int_{C - \xi \pi i} F(\alpha_1, \alpha_2)d\alpha_1 d\alpha_2 - 2\pi i(I_{12} + I_{21}),
$$

where $I_{jk} = \int_C d\alpha_j \text{res}_{\alpha_k = \alpha_j - \pi i} F(\alpha_1, \alpha_2)$. Because of the condition (2.14), we have $I_{jk} = 0$. Therefore,

$$
\int_C \int F(\alpha_1, \alpha_2) - F(\alpha_1 - \pi i, \alpha_2 - \xi \pi i) d\alpha_1 d\alpha_2 = 0.
$$

Noting that

$$
\nabla_{\alpha_1, \alpha_2, -\xi \pi i}(P_2) = \frac{X_1 - X_2}{X_1 + X_2} \Theta(X_1, X_2),
$$

we have $I(w, \frac{X_1 - X_2}{X_1 + X_2} \Theta(X_1, X_2)) = 0$. This completes the proof of $I(w, \Sigma_2) = 0$. □

2.9. The degree of minimal deformed cycles. Define the degree on $W_{N,I}$ by

$$
\text{deg } X_a = -1, \quad \text{deg } Z_j = 1.
$$

(2.45)

Set

$$
M_{N,I} = W_{N,I}/(\Sigma_1 \wedge W_{N,I-1} + \Sigma_2 \wedge W_{N,I-2}).
$$

(2.46)

We identify $M_{N,I}$ with the space of $N$-particle minimal form factors by the map $P \mapsto f_P$ (2.38). Note that $\text{deg } \Sigma_1 = 0$ and $\text{deg } \Sigma_2 = 0$, and hence $M_{N,I}$ is also graded by the degree.

In the following sections we calculate the characters

$$
\text{ch}_q W_{N,I} = \sum_d q^d \dim \mathbb{C}(W_{N,I})_d, \quad \text{ch}_q M_{N,I} = \sum_d q^d \dim \mathbb{C}(M_{N,I})_d,
$$

where $(W_{N,I})_d$ and $(M_{N,I})_d$ are the homogeneous components with the degree $d$ of $W_{N,I}$ and $M_{N,I}$, respectively.

3. Algebraic Structure of the Space of Deformed Cycles

In this section we determine the algebraic structure of the space of minimal deformed cycles

$$
W_N = \bigoplus_{I=0}^N W_{N,I}.
$$

This space is naturally embedded in the associative algebra $A_N \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$, where $A_N$ is the exterior algebra generated by the space $\bigoplus_{j=0}^{N-1} \mathbb{C}X^j$ of polynomials in $X$ of degree less than $N$. We denote the ring of symmetric polynomials in $N$ variables by $R_N$:

$$
R_N = \mathbb{C}[z_1, \ldots, z_N]^{\otimes N}.
$$

(3.1)
Note that if $P_1, P_2$ are minimal deformed cycles, then $P_1 \wedge P_2$ is also a minimal deformed cycle. Hence $W_N$ is an $R_N$-algebra.

First we outline the content of this section. We consider the quantum algebra $U_{\sqrt{1}}$, and its action $\varpi_N$ on $V \otimes N \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ given by the coproduct $\Delta$ (A.1)

$$\varpi_N : U_{\sqrt{1}} \rightarrow \text{End}(V \otimes N \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]).$$

Following [23], we define an embedding (see (3.12))

$$\mathcal{E}_N : V \otimes N \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \rightarrow A_N \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}].$$

In particular, we have

$$\mathcal{E}_N V \otimes N = 1.$$

Let $F'(+) \equiv \mathbb{C}F \otimes \mathbb{C}[t, t^{-1}]$. The action of an element $x \in F'(+) \equiv \mathbb{C}[x, y]$ is given by left multiplication by $\mathcal{E}_N(\varpi_N(x)(v_1^N \otimes 1))$ in $A_N \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ and the space $W_N$ is invariant by this action.

Define the $R_N$-algebra $\mathcal{F}_N = F'(+) \otimes R_N$. There is an $R_N$-algebra homomorphism

$$\rho_N : \mathcal{F}_N \rightarrow W_N,$$

through which $\mathcal{F}_N$ acts on $W_N$ by left multiplication. We will define a two-sided ideal $\mathcal{J}_N$ of $\mathcal{F}_N$ which belongs to the kernel of $\rho_N$ (see the end of Section 3.2). The main result in this section is

**Theorem 3.1.** The map (3.4) is surjective, and we have the isomorphism

$$\mathcal{F}_N/\mathcal{J}_N \cong W_N.$$

This is a super-symmetric analog of the result by Feigin–Feigin [5]. In the process of its proof, we rederive Nakayashiki’s result on the character of $W_N$.

**Theorem 3.2** [12]. The space of the minimal deformed cycles $W_{N, l}$ is a free $R_N$-module. Its character is given by

$$\text{ch}_{q} W_{N, l} = \frac{1}{(q_N)^l} \left[ {N \atop l} \right].$$

### 3.1. Action of the algebra $F^{(+)}$ on the space of the deformed cycles.

We have the action of $U_{\sqrt{1}} \otimes R_N$ on $(V^{\text{aff}})^{\otimes N} \simeq V \otimes N \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$. We abuse the notation to denote it by the same letter

$$\varpi_N : U_{\sqrt{1}} \otimes R_N \rightarrow \text{End}((V^{\text{aff}})^{\otimes N}).$$

In fact, the image $\varpi_N(U_{\sqrt{1}} \otimes R_N)$ is contained in $\text{End}(V \otimes N) \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ where $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ acts as multiplication. We rewrite the action $\varpi_N$ of the subalgebra $\mathcal{F}_N$ by using the Jordan–Wigner transformation. The details are given in Appendix B. Here we summarize the results which we use in the further discussion.
We have the Grassmann variables $\psi_1, \ldots, \psi_N$. We denote by $\Lambda_N$ the exterior algebra generated by them over $\mathbb{C}$. It acts on $V^\otimes N$. There is an inclusion $\Lambda_N \subset \text{End}(V^\otimes N)$ of the algebra induced from this action, and an isomorphism $\Lambda_N \simeq V^\otimes N$ of vector spaces given by $\psi \mapsto \psi v^\otimes N$. We make these identifications throughout the paper. For the algebra $F_N$ we have

$$\varpi_N(F_N) \subset \Lambda_N \otimes \mathbb{C}[z_1, \ldots, z_N].$$

Explicitly, the actions of the generators are given by

**Proposition 3.3.** We have

$$\varpi_N(x_0^{-}) = \sum_{a=1}^{N} (-1)^{N-a} \psi_a, \quad (3.8)$$

$$\varpi_N((x_0^{-})^{(2)}) = -i \sum_{1 \leq a < b \leq N} (-1)^{a+b} \psi_a \psi_b, \quad (3.9)$$

$$\varpi_N(\chi(z)) = \sum_{a=1}^{N} c_a(z) \psi_a, \quad (3.10)$$

$$\varpi_N(\chi^{(2)}(z)) = i \sum_{1 \leq a < b \leq N} c_{a,b}(z) \psi_a \psi_b, \quad (3.11)$$

where

$$c_a(z) = \frac{z_a z}{1 - z_a^2} \prod_{j=a+1}^{N} \frac{1 + z_j z}{1 - z_j^2}, \quad c_{a,b}(z) = \frac{z_a z}{1 - z_a^2} \prod_{j=a+1}^{b-1} \frac{1 + z_j z}{1 - z_j^2} \frac{1 - z_b z}{1 - z_b^2}.$$

Let $A_{N,l}$ be the space of skew-symmetric polynomials in the variables $X_1, \ldots, X_l$ of degree less than $N$ in each variable $X_p$. We identify

$$A_N \simeq \bigoplus_{l=0}^{N} A_{N,l}.$$

Note that the space of the deformed cycles $W_{N,l}$ is a subspace of $A_{N,l} \otimes \mathbb{C}[z_1, \ldots, z_N]$.

The map (3.2) is $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$-linear and is given by

$$e_N(\psi_{m_1} \cdots \psi_{m_l} v^\otimes N) = \text{Skew}(G_{m_1}(X_1) \cdots G_{m_l}(X_l)), \quad (3.12)$$

where $G_m(X)$ denotes the polynomial

$$G_m(X) = \prod_{j=1}^{m-1} (1 + z_j X) \prod_{j=m+1}^{N} (1 - z_j X).$$

If we write $G_m(X) = \sum_{j=0}^{N-1} G_{mj} X^j$, then it is easy to see that

$$\det(G_{mj})_{1 \leq m \leq N, 0 \leq j \leq N-1} = \prod_{1 \leq i < j \leq N} (z_i + z_j).$$

From this it follows that (3.12) is an embedding.

By the mapping $e_N$, the action of $\psi \in \Lambda_N$ on $V^\otimes N \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ is intertwined with the wedge product $e_N(\psi v^\otimes N) \wedge$ on $\Lambda_N \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$. 
Let \( U_{\Lambda(N)}^+ \) denote the subalgebra of \( U_{\Lambda(N)} \) generated by \( t_i^{\pm 1} \) and \( e_i^{(s)}, f_i^{(s)} \) \((s \geq 0)\). The algebra \( \mathcal{F}_N \) is a subalgebra of \( U_{\Lambda(N)}^+ \otimes \mathbb{R}_N \). In Appendix C we give a proof of the following proposition along with its background following \([24]\).

**Proposition 3.4.** We have

\[
\mathcal{E}_N(\mathbb{R}_N(U_{\Lambda(N)}^+(v_0^N) \otimes 1)) \subset \mathcal{W}_N.
\]

We set

\[
\rho_N: \mathcal{F}_N \to \mathcal{W}_N, \quad x \mapsto \mathcal{E}_N(\mathbb{R}_N(x)(v_0^N) \otimes 1).
\]  (3.13)

Combining Proposition 3.3 and the formula (3.12), we obtain

**Proposition 3.5.** The image of the generators of \( F^{(+)} \simeq F^{(+)} \otimes 1 \subset \mathcal{F}_N \) by (3.13) is given as follows.

\[
x_0^- \mapsto \frac{1}{2} \Sigma_1(X),
\]  (3.14)

\[
(x_n^+)^{(2)} \mapsto \frac{i}{4} \Sigma_2(X_1, X_2),
\]  (3.15)

\[
\chi(z) \mapsto \frac{1}{\Theta(z)} \frac{z}{2(X - z)} \Theta(z, -X),
\]  (3.16)

\[
(\chi(z))^{(2)} \mapsto \frac{i}{4} \left\{ \frac{X_1 - X_2}{X_1 + X_2} \frac{z}{X_1 + z X_2 + z} \Theta(X_1, X_2)
\right.
\]

\[
\left. + \text{Skew}_{X_1, X_2} \frac{\Theta(-X_2)}{\Theta(z)} \frac{z}{X_2 + z X_1 - z} \Theta(z, -X_1) \right\}
\]  (3.17)

where \( \Sigma_1(X), \Sigma_2(X_1, X_2), \Theta(X_1, X_2) \) are given by (2.42), (2.43), (2.41), respectively.

**3.2. Relations of the super-symmetric currents.** Our goal is to determine the relations satisfied by \( \mathcal{F}_N \) when it acts on \( V^\otimes N \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \) through \( \mathbb{R}_N \). For this purpose we introduce formal symbols which represent the generators of \( F^{(+)} \).

Consider the Grassmann variables \( \xi_n \ (n \geq 0) \) and denote by \( \Lambda(\xi_0, \xi_1, \xi_2, \ldots) \) the exterior algebra generated by them over \( \mathbb{C} \). We also consider the commuting variables \( \eta_n \ (n \geq 0) \) and set

\[
Z = \Lambda(\xi_0, \xi_1, \xi_2, \ldots) \otimes \mathbb{C}[\eta_0, \eta_1, \eta_2, \ldots].
\]  (3.18)

We define the degree and weight of the generators by

\[
\deg \xi_0 = n, \quad \deg \eta_n = n, \quad \text{wt} \xi_n = 1, \quad \text{wt} \eta_n = 2.
\]  (3.19)

We introduce the currents in \( Z \):

\[
\xi(z) = \sum_{n=1}^{\infty} \xi_n z^n, \quad \eta(z) = \sum_{n=1}^{\infty} \eta_n z^n.
\]  (3.20)

Recall Proposition 3.3 which describes the action of \( \mathcal{F}_N \) on \( \Lambda_N \otimes \mathbb{C}[z_1, \ldots, z_N] \).

Let us consider the algebra homomorphism over \( R_N \)

\[
\rho'_N: Z \otimes R_N \to \Lambda_N \otimes \mathbb{C}[z_1, \ldots, z_N]
\]  (3.21)
given by

\[ \rho'_N(\xi) = \sum_{a=1}^{N} (-1)^{a-1} \psi_a, \quad (3.22) \]
\[ \rho'_N(\eta) = \sum_{1 \leq a < b \leq N} (-1)^{a+b} \psi_a \psi_b, \quad (3.23) \]
\[ \rho'_N(\xi(z)) = \sum_{a=1}^{N} c_a(z) \psi_a, \quad (3.24) \]
\[ \rho'_N(\eta(z)) = \sum_{1 \leq a < b \leq N} c_{a,b}(z) \psi_a \psi_b. \quad (3.25) \]

If we count the degree and weight in \( \Lambda_N \) and \( \mathbb{C}[z_1, \ldots, z_N] \) by

\[ \deg \psi_j = 0, \quad wt \psi_j = 1, \quad \deg z_j = 1, \quad wt z_j = 0, \quad (3.26) \]

then the mapping \( \rho'_N \) respects them.

Let us consider the kernel of \( \rho'_N \).

**Proposition 3.6.** We have the relations

\[ \xi(z) \xi(-z) + \eta(z) - \eta(-z) \in \text{Ker} \rho'_N. \quad (3.27) \]

The proof is straightforward. These relations stem from an identity between the currents \( X(z)^{(s)} \) proved in [4], Proposition 4.1.

Note, in particular, that

\[ \eta_1 \in \text{Ker} \rho'_N. \]

For \( A = \{a_1, \ldots, a_\mu\} \) \( (a_1 < \cdots < a_\mu) \), we set

\[ \psi_A = \psi_{a_1} \cdots \psi_{a_\mu}, \]

and

\[ c_A(z) = \begin{cases} c_{a_1, a_2}(z) \cdots c_{a_{2\nu-3}, a_{2\nu-2}}(z) c_{a_{2\nu-1}}(z), & \text{if } \mu = 2\nu - 1; \\ c_{a_1, a_2}(z) \cdots c_{a_{2\nu-3}, a_{2\nu-2}}(z) c_{a_{2\nu-1}, a_{2\nu}}(z), & \text{if } \mu = 2\nu. \end{cases} \]

The following is also straightforward.

**Lemma 3.7.** We have

\[ \frac{1}{\nu!} \rho'_N(\xi(z) \eta(z)^\nu) = \sum_{#A=2\nu+1} \psi_A c_A(z), \quad \frac{1}{\nu!} \rho'_N(\eta(z)^\nu) = \sum_{#A=2\nu} \psi_A c_A(z). \]

In particular, we see that

\[ \prod_{j=1}^{N} (1 - z_j) \cdot \rho'_N(\xi(z) \eta(z)^\nu) \quad \text{and} \quad \prod_{j=1}^{N} (1 - z_j) \cdot \rho'_N(\eta(z)^\nu) \]

are polynomials in \( z \).
Consider the case

Proposition 3.8. The currents \( I_{\mu}^{(N)}(z) \) \((\mu = 1, 2, \ldots)\) given by

\[
I_{2\nu+1}^{(N)}(z) = \prod_{j=1}^{N} (1 - z_j z) \cdot (\xi_0 + \xi(\eta_0 + \eta(z))^\nu,
\]

\[
I_{2\nu}^{(N)}(z) = \prod_{j=1}^{N} (1 - z_j z) \cdot \{(\eta_0 + \eta(z))^\nu + \nu \xi_0 \xi(\eta_0 + \eta(z))^{\nu-1} \}\.
\]

From Lemma 3.7, these are mapped by \( \rho_1' \) to polynomials in \( z \). In fact, we have the following.

**Proposition 3.8.** The currents \( I_{\mu}^{(N)}(z) \) \((\mu = 1, 2, \ldots)\) satisfy that

\[
[I_{\mu}^{(N)}(z)]_{\geq N-\mu+1} \in \text{Ker} \rho_N' \otimes \mathbb{C}[z].
\]

**Proof.** We prove the assertion by induction on \( N \). Set

\[
a_0^{(N)}(z) = \prod_{j=1}^{N} (1 - z_j z),
\]

\[
a_{2\nu+1}^{(N)}(z) = \prod_{j=1}^{N} (1 - z_j z) \cdot \frac{1}{\mu!} \rho_N'(\xi(\eta(z))^\nu),
\]

\[
a_{2\nu}^{(N)}(z) = \prod_{j=1}^{N} (1 - z_j z) \cdot \frac{1}{\mu!} \rho_N'(\eta(z)^\nu).
\]

Consider the case \( N = 1 \). We have

\[
a_0^{(1)}(z) = 1 - z_1 z, \quad a_1^{(1)}(z) = z_1 z \psi_1, \quad a_{\mu}^{(1)}(z) = 0 \quad (\mu > 1),
\]

\[
\rho_1'(\xi_0) = \psi_1, \quad \rho_1'(\eta_0) = 0.
\]

From these formulas it is easy to check that \( \rho_1'(I_1^{(1)}(z)) = \psi_1 \) and \( \rho_1'(I_{\mu}^{(1)}(z)) = 0 \) \((\mu \geq 2)\).

Next consider the case of \( N > 1 \). We make a natural identification of \( \Lambda_{N-1} \otimes \mathbb{C}[z_1, \ldots, z_{N-1}] \) with the subspace of \( \Lambda_N \otimes \mathbb{C}[z_1, \ldots, z_N] \). From Lemma 3.7 we see that

\[
a_{2\nu+1}^{(N)}(z) = (1 + z_N z) a_{2\nu+1}^{(N-1)}(z) + a_{2\nu}^{(N-1)}(z) z_N z \psi_N,
\]

\[
a_{2\nu}^{(N)}(z) = (1 - z_N z) a_{2\nu}^{(N-1)}(z) + a_{2\nu-1}^{(N-1)}(z) z_N z \psi_N,
\]

\[
\rho_N'(\xi_0) = -\rho_{N-1}'(\xi_0) + \psi_N, \quad \rho_N'(\eta_0) = \rho_{N-1}'(\eta_0) - \rho_{N-1}'(\xi_0) \psi_N.
\]

From these recursions we find

\[
\rho_N'(I_{2\nu+1}^{(N)}(z)) = (1 + z_N z) \rho_{N-1}'(I_{2\nu+1}^{(N-1)}(z)) + (\psi_N - 2\pi_{N-1}(\xi_0)) \rho_{N-1}'(I_{2\nu-1}^{(N-1)}(z)),
\]

\[
\rho_N'(I_{2\nu}^{(N)}(z)) = (1 - z_N z) \rho_{N-1}'(I_{2\nu}^{(N-1)}(z)) + \nu(\psi_N - 2\pi_{N-1}(\xi_0)) \rho_{N-1}'(I_{2\nu-1}^{(N-1)}(z)).
\]
From these relations and the induction assumption $\rho'_{N-1}(I_{\mu}^{(N-1)}(z))_{\geq N-\mu} = 0$, we get $\rho'_{N}(I_{\mu}^{(N)}(z))_{\geq N-\mu+1} = 0$. \qed

**Definition 3.9.** Let $\mathcal{I}_N \subset Z \otimes R_N$ be the two-sided ideal generated by the coefficients of

$$\xi(z)\xi(-z) + \eta(z) - \eta(-z)$$

and

$$[I_{\mu}^{(N)}(z)]_{\geq N-\mu+1} \quad (\mu = 1, 2, \ldots).$$

We define the $R_N$-algebra $Z_N$ by

$$Z_N = (Z \otimes R_N)/\mathcal{I}_N'.$$

Let $\mathcal{J}_N$ be the two-sided ideal of $\mathcal{I}_N$ generated by the coefficients of (3.29) where $\xi_0, \xi(z), \eta_0, \eta(z)$ are replaced by $x_0^-, -ix(z), (x_0^-)^{(2)}, -ix(z)^{(2)}$, respectively. By the definition and the remark after Proposition 3.6, we have surjections

$$Z_N \twoheadrightarrow \mathcal{I}_N/\mathcal{J}_N \twoheadrightarrow \mathcal{I}_N(\mathcal{F}_N).$$

In Section 3.4, we will prove that these are isomorphisms (Corollary 3.21 and (3.54)).

**3.3. Isomorphism of $R_N$ algebras.** The algebra $Z \otimes R_N$ is bi-graded by degree and weight. The ideal $\mathcal{I}_N$ is also bi-graded, and, therefore, the algebra $Z_N$ is bi-graded. Notice that the variables $z_j$ do not enter the definition of the algebra $Z$.

We denote the degree $s$ component of $Z$ by $Z_s$, and set $Z_{\leq s} = \bigoplus_{t=0}^s Z_t$. Then we define the $R_N$-submodule $F_s(Z_N)$ of $Z_N$ by

$$F_s(Z_N) = (Z_{\leq s} \otimes R_N)/((Z_{\leq s} \otimes R_N) \cap \mathcal{I}_N').$$

These submodules satisfy

$$Z_N = \bigcup_{s=0}^\infty F_s(Z_N), \quad 0 = F_{-1}(Z_N) \subset F_0(Z_N) \subset F_1(Z_N) \subset F_2(Z_N) \subset \cdots.
$$

Hence $Z_N$ is a filtered $R_N$-module. We consider the associated graded module

$$\text{gr } Z_N = \bigoplus_{s=0}^\infty F_s(Z_N)/F_{s-1}(Z_N).$$

**Definition 3.10.** We denote by $J_N$ the ideal of $Z$ generated by the coefficients of

$$\xi(z)\xi(-z) + \eta(z) - \eta(-z) \quad \text{and} \quad [I_{\mu}^{(N)}(z)]_{\geq N-\mu+1} \quad (\mu = 1, 2, \ldots),$$

where

$$J_{2\mu-1}(z) = (\xi_0 + \xi(z))(\eta_0 + \eta(z))^{\mu-1},$$

$$J_{2\mu}(z) = (\eta_0 + \eta(z))^{\mu} + \nu \xi_0 \xi(z)(\eta_0 + \eta(z))^{\mu-1}.$$  

For $s \geq N - \mu + 1$ the generator $[I_{\mu}^{(N)}(z)]_s$ of the ideal $\mathcal{I}_N'$ belongs to $F_s(Z_N)$ and satisfies $[I_{\mu}^{(N)}(z)]_s = [I_{\mu}(z)]_s$ in $F_s(Z_N)/F_{s-1}(Z_N)$.

We set $Z_\nu = Z/J_\nu$. This is a bi-graded $\mathbb{C}$-algebra. We denote by $(Z_\nu)_{s,l}$ the component of degree $s$ and weight $l$. 
**Proposition 3.11.** There exists a surjective $R_N$-module homomorphism
\[ \tilde{Z}_N \otimes R_N \to \text{gr} \tilde{Z}_N \to 0. \]  
(3.32)

**Proof.** We have the exact sequence of $R_N$-modules:
\[ 0 \to J_N \otimes R_N \to Z \otimes R_N \to (Z/J_N) \otimes R_N \to 0. \]
Note that the associated graded module $\text{gr} \tilde{Z}_N$ is canonically isomorphic to the quotient
\[ \text{gr} \tilde{Z}_N \cong (Z \otimes R_N)/J_N^\text{top}, \]
where
\[ J_N^\text{top} = \text{span}_{R_N} \{ b_s |^2 b = b_0 + b_1 + \cdots + b_s \in J_N \text{ where } b_i \in Z_i \otimes R_N \}. \]
We have $J_N \otimes R_N \subset J_N^\text{top}$. Hence we get
\[ (Z/J_N) \otimes R_N \cong (Z \otimes R_N)/(J_N \otimes R_N) \to (Z \otimes R_N)/J_N^\text{top} \cong \text{gr} \tilde{Z}_N. \]
(3.36)

We will prove that the above mapping is an isomorphism (see Proposition 3.18). In other words, we have $J_N \otimes R_N = J_N^\text{top}$.

Let us calculate the character of $\tilde{Z}_N$,
\[ \chi_N(q, z) = \sum_{s,l \geq 0} q^s z^l \dim_{\mathbb{C}}(\tilde{Z}_N)_{s,l}. \]  
(3.33)

Our strategy is to follow the idea in [5]. First we find the upper bound of the character. To this end we prove the following two lemmas.

**Lemma 3.12.** Let $\iota: Z \to Z$ be the $\mathbb{C}$-algebra homomorphism defined by
\[ \iota(\xi_0 + \xi(z)) = \frac{\xi(z)}{z}, \]  
(3.34)
\[ \iota(\eta_0 + \eta(z)) = \frac{1}{z^2}(\eta(z) + \eta_1 z + \xi(z)\xi_1 z). \]  
(3.35)

This map induces the $\mathbb{C}$-algebra homomorphism $\iota_N: \tilde{Z}_N-1 \to \tilde{Z}_N$.

**Proof.** It is enough to prove that
\[ \iota(\xi(z)\xi(-z) + \eta(z) - \eta(-z)) = 0 \quad \text{and} \quad \iota([J_\mu(z)]_{\geq N-\mu}) = 0 \quad \text{in } \tilde{Z}_N. \]  
(3.36)
(3.37)

We can check (3.36) easily. Let us prove (3.37). The image of the current $J_\mu(z)$ is given as follows:
\[ \iota(J_{2\nu-1}(z)) = z^{-2(\nu-1)}(\xi(z)(\eta_1 z + \eta_1 z)^{\nu-1}), \]
\[ \iota(J_{2\nu}(z)) = z^{-2\nu}(\eta(z) + \eta_1 z)^{\nu}. \]

From the relation $-\eta(z) + \eta(-z) = \xi(z)\xi(-z)$ we have
\[ \eta_1 = 0 \quad \text{and} \quad \xi(z)\eta(z) = \xi(z)\eta(z) \quad \text{in } \tilde{Z}_N. \]

Hence (3.37) is equivalent to
\[ [\xi(z)\eta(z)^{\nu-1}]_{\geq N} = 0 \quad \text{and} \quad [\eta(z)^{\nu}]_{\geq N} = 0. \]
We have $[J_\mu(z)]_{2N-\mu+1} = 0$ in $\tilde{Z}_N$. Note that

$$J_{2\nu}(z) - \nu \xi_0 J_{2\nu-1}(z) = (\eta_0 + \eta(z))' \nu.$$  

Hence we find

$$[(\eta_0 + \eta(z))']_{2N-2\nu+2} = 0 \text{ in } \tilde{Z}_N. \quad (3.38)$$

From this we can prove that $[\eta(z)]_{2N} = 0$ by induction on $\nu$. This result and $[J_{2\nu-1}(z)]_{2N-2\nu+2} = 0$ in $\tilde{Z}_N$ imply that $[\xi(z)\eta(z)^{2\nu-1}]_{2N} = 0$.

We denote by $\mathcal{Z}'$ the subalgebra of $\mathcal{Z}$ generated by $\xi, \eta, \nu (\nu \geq 1)$.

**Lemma 3.13.** The image of the subalgebra $\mathcal{Z}'$ in $\tilde{Z}_N$ belongs to $\iota_N(\tilde{Z}_N)$. Proof. Since $\iota_N$ is an algebra homomorphism, it is enough to prove that $\eta(z)$ and $\xi(z)$ belong to $\iota_N(\tilde{Z}_N)$. The latter is obvious from the definition (3.34). Therefore, $\xi(z)\xi_1$ belongs to $\iota_N(\tilde{Z}_N)$. Note that $\eta_1 = 0$ in $\tilde{Z}_N$. Therefore, from (3.35) we have $\eta(z) \in \iota_N(\tilde{Z}_N)$. □

**Lemma 3.14.** Let $\varphi: \mathcal{Z} \to \mathcal{Z}'$ be the $\mathcal{Z}'$-linear map defined by

$$\varphi(\eta_0^k) = \xi_0 \eta_0^k, \quad \varphi(\xi_0 \eta_0^k) = \frac{k+1}{k+1}.$$  

This map induces a surjection $\varphi_N: \tilde{Z}_N \to \tilde{Z}_N/\iota_N(\tilde{Z}_N)$. Proof. It is enough to prove that

$$\varphi(\xi_0^{k+1} \eta_0^k (\xi(z) \xi(-z) + \eta(z) - \eta(-z))) = 0 \quad \text{and} \quad (3.39)$$

$$\varphi(\xi_0^{k+1} \eta_0^k (J_\mu(z))]_{2N-\mu} = 0 \text{ in } \tilde{Z}_N/\iota_N(\tilde{Z}_N) \quad (3.40)$$

for $\delta = 0, 1$ and $k \geq 0$.

Since $\xi(z)\xi(-z) + \eta(z) - \eta(-z) \in \mathcal{Z}'$, it is clear that (3.39) holds. Here we prove (3.40) for odd $\mu$. The proof for even $\mu$ is similar.

First consider the case $\delta = 0$. Expanding the factor $(\eta_0 + \eta(z))^{2\nu-1}$ in $J_{2\nu-1}(z)$ we have

$$\eta_0^k J_{2\nu-1}(z) = \sum_{s=0}^{\nu-1} \left( \begin{array}{c} \nu-1 \\ s \end{array} \right) \left( \xi_0 \eta_0^{k+s} \eta(z)^{2\nu-s-1} + \eta_0^{k+s} \xi(z) \eta(z)^{2\nu-s-1} \right).$$

Therefore, we have

$$\varphi(\eta_0^k J_{2\nu-1}(z)) = \sum_{s=0}^{\nu-1} \left( \begin{array}{c} \nu-1 \\ s \end{array} \right) \left( \frac{\eta_0^{k+s+1}}{\xi_0 \eta_0^{k+s} + \eta(z)^{2\nu-s-1}} \right)$$

$$= \sum_{s=0}^{\nu-1} \left( \begin{array}{c} \nu-1 \\ s \end{array} \right) \frac{\eta_0^{k+s+1}}{\eta_0^{k+s} \xi(z) + \eta(z)^{2\nu-s-1}}.$$ 

Now we apply the following identity to the first term.

$$\sum_{a=0}^{n} \binom{n}{a} \frac{-a}{a+b} y^{n-a} = \sum_{j=0}^{b-1} (-1)^j \frac{(b-1) \cdots (b-j)}{(n+1) \cdots (n+j+1)} x^{b-j-1}(x+y)^{n+j+1}$$

$$+ (-1)^b \frac{(b-1)!}{(n+1) \cdots (n+b)} y^{n+b}.$$
where \( x \) and \( y \) are commuting variables and \( b = 1, 2, \ldots \). Then we get

\[
\varphi(\gamma_0^k J_{2
u-1}(z)) = \frac{1}{k!} \gamma_0^k J_{2\nu}(z) + \sum_{j=1}^{k} (-1)^j \frac{k(k-1) \cdots (k-j+1)}{\nu^j} \gamma_0^{k-j} ((\eta_0 + \eta(z))^{\nu+j})
\]

\[
+ (-1)^{k+1} \frac{k!}{\nu^j} \eta(z)^{\nu+k}.
\]

We have \((3.38)\) and also \([J_{2
u}(z)]_{\geq N-2\nu+1} = 0\) in \( \bar{Z}_N \). Moreover the coefficients of \( \eta(z)^{\nu+k} \) are in \( \iota(\bar{Z}_{N-1}) \) by Lemma 3.13. Therefore we find \((3.40)\) with \( \delta = 0 \) and odd \( \mu \).

The proof for \( \delta = 1 \) is similar. Using \((\xi_0 + \xi(z))J_{2\nu-1}(z) = 0\), we have

\[
-\varphi(\xi_0^k J_{2\nu-1}(z)) = \sum_{j=0}^{k} (-1)^j \frac{k(k-1) \cdots (k-j+1)}{\nu^j} \gamma_0^{k-j} \xi(z)(\eta_0 + \eta(z))^{\nu+j}
\]

\[
+ (-1)^{k+1} \frac{k!}{\nu^j} \eta(z)^{\nu+k}.
\]

Note that

\[
\xi(z)(\eta_0 + \eta(z))^{\nu+j} = J_{2(\nu+j)+1}(z) - \xi_0 J_{2(\nu+j)}(z).
\]

From this equality and \([J_\mu(z)]_{\geq N-\mu+1} = 0\) in \( \bar{Z}_N \), we see that

\[
[\xi(z)(\eta_0 + \eta(z))^{\nu+j}]_{\geq N-2\nu+1} = 0.
\]

The coefficients of \( \xi(z)(\eta(z)^{\nu+k} \) are in \( \iota(\bar{Z}_{N-1}) \) by Lemma 3.13. Thus we get \((3.40)\) with \( \delta = 1 \) and odd \( \mu \).

From Lemma 3.12 and Lemma 3.14 we get the following diagram:

\[
\begin{array}{ccc}
\bar{Z}_{N-1} & \xrightarrow{\iota_N} & \bar{Z}_N \\
\varphi_N \downarrow & & \varphi_N \\
0 & \xrightarrow{\iota_N(\bar{Z}_{N-1})} & \bar{Z}_N/\iota_N(\bar{Z}_{N-1}) & \xrightarrow{0}.
\end{array}
\]

Here and after, for formal series

\[
f = \sum_{\alpha_1, \ldots, \alpha_\nu} f_{\alpha_1, \ldots, \alpha_\nu} z^{\alpha_1} \cdots z^{\alpha_\nu}, \quad g = \sum_{\alpha_1, \ldots, \alpha_\nu} g_{\alpha_1, \ldots, \alpha_\nu} z^{\alpha_1} \cdots z^{\alpha_\nu}
\]

with integer coefficients, we write \( f \leq g \) if \( f_{\alpha_1, \ldots, \alpha_\nu} \leq g_{\alpha_1, \ldots, \alpha_\nu} \) holds for all \( \alpha_1, \ldots, \alpha_\nu \). For a homogeneous element \( b \in Z \) we have

\[
\deg \iota(b) = \deg b + wt b, \quad wt \iota(b) = wt b,
\]

\[
\deg \varphi(b) = \deg b, \quad wt \varphi(b) = wt b + 1.
\]

Hence we find

\[
\chi_N(q, z) \leq \chi_{N-1}(q, qz) + z\chi_{N-1}(q, z).
\]

It is easy to see that \( \chi_1(q, z) = 1 + z \). Starting from this and using \((3.42)\) repeatedly, we get the following upper bound for the character \( \chi_N \).
Proposition 3.15. We have
\[ \chi_N(q, z) \leq \sum_{l=0}^{N} \binom{N}{l} z^l. \]  
(3.43)

Let us prove the equality in (3.43). It is enough to prove that
\[ \chi_N(1, 1) = \dim_{\mathbb{C}} \tilde{Z}_N \geq \sum_{l=0}^{N} \binom{N}{l} = 2^N. \]  
(3.44)

For \( c = (c_1, \ldots, c_N) \in \mathbb{C}^N \), introduce the evaluation map
\[ e_c : \mathbb{C}[z_1, \ldots, z_N] \to \mathbb{C}, \quad P(z_1, \ldots, z_N) \mapsto P(c_1, \ldots, c_N). \]  
(3.45)

This map induces
\[ \tilde{Z}_N \otimes R_N \to \hat{Z}_N, \]
\[ \Lambda_N \otimes \mathbb{C}[z_1, \ldots, z_N] \to \Lambda_N, \]
\[ A_N \otimes R_N \to A_N. \]

We denote these induced maps by the same letter \( e_c \).

The space \( \varpi_N(F_N) \) contains the coefficients of
\[ \prod_{j=1}^{N} (1 - z_j z) \cdot \varpi_N(\mathcal{X}(z)) = \sum_{a=1}^{N} \psi_a \prod_{j=1}^{a-1} (1 - z_j z) \cdot z_a z \cdot \prod_{j=a+1}^{N} (1 + z_j z). \]

Hence if \( c = (c_1, \ldots, c_N) \) satisfies
\[ \prod_{j=1}^{N} c_j \prod_{1 \leq j < j' \leq N} (c_j + c_{j'}) \neq 0, \]  
(3.46)
we have
\[ e_c(\varpi_N(F_N)) = \Lambda_N. \]  
(3.47)

We have the surjections (3.32) and
\[ Z_N \to \varpi_N(F_N) \subset \Lambda_N \otimes \mathbb{C}[z_1, \ldots, z_N]. \]
Evaluating these maps at \( c \) satisfying (3.46), we get the exact sequences
\[
\begin{array}{ccc}
\hat{Z}_N & \to & \text{gr } Z_N \\
\downarrow e_c & & \downarrow e_c \\
\tilde{Z}_N & \to & e_c(\text{gr } Z_N) \\
& & 0
\end{array}
\]
and
\[
\begin{array}{ccc}
Z_N & \to & \varpi_N(F_N) \\
\downarrow e_c & & \downarrow e_c \\
e_c(Z_N) & \to & e_c(\varpi_N(F_N)) \\
& & 0
\end{array}
\]
where the vertical arrows are surjective.

The space \( e_c(Z_N) \) is given by \( e_c(Z_N) = Z/e_c(\mathcal{Y}_N') \). Introduce the filtration \( \{ F_s(Z/e_c(\mathcal{Y}_N')) \} \) on \( Z/e_c(\mathcal{Y}_N') \) in the same way as (3.31), that is \( F_s(Z/e_c(\mathcal{Y}_N')) = \)
$Z_{\leq s}/(Z_{\leq s} \cap e_c(I_N'))$. Then the associated graded space $\text{gr}(Z/e_c(I_N')) = e_c(Z_N)$ is isomorphic to $e_c(\text{gr}Z_N)$. Hence we get

$$\dim \mathbb{C} Z_N \geq \dim \mathbb{C} e_c(Z_N) = \dim \mathbb{C} e_c(Z_N) \geq \dim \mathbb{C} \Lambda_N = 2^N.$$ 

We have proved (3.44). Thus we obtain

**Theorem 3.16.** The character $\chi_N(q, z)$ of the bi-graded algebra $\bar{Z}_N$ is given by

$$\chi_N(q, z) = \sum_{l=0}^{N} \left[ \begin{array}{l} N \\ l \end{array} \right] z^l.$$ 

**Corollary 3.17.** We have the following isomorphism as $\mathbb{C}$-algebras:

$$\bar{Z}_N \simeq e_c(\text{gr}Z_N)$$

at any point $c$ satisfying (3.46).

**Proposition 3.18.** We have the isomorphism of $R_N$-modules:

$$\bar{Z}_N \otimes \mathbb{C} R_N \simeq \text{gr}Z_N.$$ (3.48)

**Proof.** We have the following exact sequences.

$$\bar{Z}_N \otimes \mathbb{C} R_N \xrightarrow{\theta} \text{gr}Z_N \xrightarrow{e_c} 0$$

Take a $\mathbb{C}$-basis $\{Q_i\}_{i=1,...,2^N}$ of $\bar{Z}_N$. Then the set $\{\theta(Q_i \otimes 1)\}$ generates $\text{gr}Z_N$ over $R_N$. Let us prove that it is linearly independent over $R_N$.

Suppose that $\sum_i r_i \theta(Q_i \otimes 1) = 0$ for some $r_i \in R_N$. Evaluating this equality at $c$ we get

$$e_c \left( \sum_i r_i \theta(Q_i \otimes 1) \right) = \sum_i e_c(r_i)(e_c \circ \theta)(Q_i \otimes 1)$$

$$= \sum_i e_c(r_i)(\theta' \circ e_c)(Q_i \otimes 1) = \sum_i e_c(r_i) = 0.$$ 

Since $\theta'$ is an isomorphism and $\{Q_i\}$ is a basis, we have $e_c(r_i) = 0$ for all $i$. Thus $r_i$ satisfies $e_c(r_i) = 0$ for any generic point $c$, and this implies $r_i = 0$. \qed

### 3.4. Proof of Theorems 3.1, 3.2.

We conclude this section by proving Theorems 3.1 and 3.2. Recall that we fix $N$, $l$ satisfying $0 \leq l \leq N$. We set

$$\Delta_+ = \prod_{1 \leq i < j \leq N} (z_i + z_j).$$

The following two Lemmas are proved in [12].

**Lemma 3.19.** Let $P_a$ ($1 \leq a \leq \binom{N}{l}$) be arbitrary elements of $W_{N,l}$, and set

$$P_a = \sum_j P_{a,j} X_1^{j_1} \wedge \cdots \wedge X_l^{j_l},$$ (3.50)
where \( J = (j_1, \ldots, j_l) \) runs over the set of indices satisfying \( 0 \leq j_1 < \cdots < j_l \leq N - 1 \). Then \( \det(P_{aJ}) \) is divisible by \( \Delta_{+}^{(N-1)+(N-2)} \).

**Proof.** If \( N = 1 \), there is nothing to prove. Suppose \( N \geq 2 \), and consider the matrix

\[
M = (P_{aJ}).
\]

It is sufficient to show that \( \det M \) is divisible by \( (z_1 + z_2)^m \) with \( m = (N-1) + (N-2) \). Let \( L \) be the \( n \)-dimensional space of column vectors over the field \( \mathbb{C}(z_1, \ldots, z_N) \). Set \( a = (1, z_1^{-1}, \ldots, z_2^{-N+1}), b = (1, (-z_2)^{-1}, \ldots, (-z_2)^{-N+1}) \). By the minimality condition (2.39), the subspace \( L' = a \wedge (N_l^{-1} L) + b \wedge (N_l^{-1} L) \) of \( \Lambda^j L \) belongs to the kernel of \( M_{1}^{j_1=j_2} \). Since \( a \wedge b \neq 0 \), we have \( \dim L' = m \). The assertion follows from the fact that, if \( X(z) \) is a square matrix whose entries are polynomials in \( z \) and if \( \text{corank} X(0) = k \), then \( \det X(z) \) is divisible by \( z^k \). \( \square \)

**Lemma 3.20.** Let \( \{Q_{a}\}_{1 \leq a \leq N} \) be a set of homogeneous elements of \( W_{N,l} \) with the following properties:

\[
\sum_{a=1}^{N} q_{a} = \begin{bmatrix} N \\ l \end{bmatrix} \quad (d_a = \deg Q_a), \tag{3.51}
\]

\( \{e_{c}(Q_{a})\} \) is linearly independent for some \( c \in \mathbb{C}^N \) satisfying (3.46). \( \tag{3.52} \)

In the second line, \( e_c \) denotes the evaluation map (3.45). Then \( \{Q_{a}\}_{1 \leq a \leq N} \) is a free \( R_N \)-basis of \( W_{N,l} \).

**Proof.** As in (3.50), denote by \( Q_{aJ} \) the transition coefficients between the \( \{Q_a\} \) and the monomials \( X^J = X_1^{j_1} \wedge \cdots \wedge X_l^{j_l}, J = (j_1, \ldots, j_l) \). We have \( \deg Q_{aJ} = d_a + |J| \) where \( |J| = j_1 + \cdots + j_l \). Using (3.51) one computes easily

\[
\sum_{a=1}^{N} d_a = \binom{N}{l} \frac{l(N-l)}{2},
\]

\[
\sum_{J} |J| = \binom{N}{l} \left( \binom{l}{2} + \frac{l(N-l)}{2} \right),
\]

and hence

\[
\deg \det(Q_{aJ}) = \sum_{a=1}^{N} d_a + \sum_{J} |J| = \binom{N}{2} \left( \binom{N-1}{l-1} + \binom{N-2}{l-1} \right) \cdot
\]

By Lemma 3.19, it follows that \( \det(Q_{aJ}) = C \cdot \Delta_{+}^{(N-1)+(N-2)} \) for some \( C \in \mathbb{C} \). Evaluating both sides at \( z = c \) and using (3.52) we find that \( C \neq 0 \). In particular, \( \{Q_a\} \) is a linearly independent set over the field \( \mathbb{C}(z_1, \ldots, z_N)^{\otimes N} \). Let us show that it is a basis of \( W_{N,l} \) over \( R_N \). Take \( P = \sum_{J} P_{J} X^J \in W_{N,l} \). By the linear independence of \( \{Q_a\} \), there exist unique elements \( p_a \in \mathbb{C}(z_1, \ldots, z_N)^{\otimes N} \) such that \( P = \sum_{a=1}^{N} p_a Q_{a} \) holds. Use Cramer’s rule to solve the equation \( P_{J} = \sum_{a=1}^{N} p_a Q_{a,J} \) for \( p_a \). By Lemma 3.19, we see that \( p_a \) are polynomials. This proves the lemma. \( \square \)
Proof of Theorems 3.1 and 3.2. According to Theorem 3.16, there exist homogeneous elements $\zeta_a$ ($1 \leq a \leq \binom{N}{l}$) of $\mathbb{Z}$, satisfying $\text{wt} \zeta_a = l$, $\sum_a q^{d\zeta_a} \zeta_a = [\binom{N}{l}]$, and whose equivalence classes give a basis of $\bar{\mathbb{Z}}_N$. By Corollary 3.17, we may assume that $\{e_c(\zeta_a)\}$ is a linearly independent set for some $c \in \mathbb{C}^N$. Proposition 3.18 shows that $\{\zeta_a \otimes 1\}$ span $\mathbb{Z}_N$ over $\mathbb{R}$. Let $\bar{\rho}_N$ denote the composition map

$$\mathbb{Z}_N \rightarrow \mathcal{F}_N/\mathcal{I}_N \rightarrow \varpi_N(\mathcal{F}_N) \rightarrow W_N.$$  

(3.53)

Let $Q_a = \pi(\zeta_a \otimes 1)$. Then (3.51) is satisfied. From the remark below (3.12), the map

$$\psi_{m_1} \cdots \psi_{m_l} \mapsto e_c(\text{Skew}(G_{m_1}(X_1) \cdots G_{m_l}(X_l)))$$

is injective, so that $\{e_c(Q_a)\}$ is linearly independent over $\mathbb{C}$. Therefore Lemma 3.20 applies, and $W_{N,l}$ is a free $R_N$-module with free basis $\{Q_a\}$. This completes the proof of Theorems 3.1 and 3.2. □

From the proof we obtain

Corollary 3.21. The natural map

$$\bar{\rho}_N : \mathbb{Z}_N \twoheadrightarrow W_N$$  

(3.54)

is an isomorphism.

4. The Character of $M_{N,l}$

Recall that the space $M_{N,l}$ is defined as the quotient of $W_{N,l}$ by the subspace generated by $\Sigma_1(X)$ and $\Sigma_2(X_1, X_2)$. The goal of this section is the following result.

Theorem 4.1. The space $M_{N,l}$ is a free $R_N$-module with the character

$$\text{ch}_q M_{N,l} = \frac{1}{(q)_N} \left( \binom{N}{l} - \binom{N}{l-1} \right).$$  

(4.1)

Formula (4.1) was obtained by Nakayashiki [12]. We will give here an alternative proof based on the results of the previous section.

4.1. Estimate from both ends. In the previous section, we proved the identity in Theorem 3.16 by showing two inequalities: estimate from above (3.43) and estimate from below (3.44). We prove Theorem 4.1 in the same way.

In order to obtain an estimate from above, we make use of the isomorphism (3.54),

$$\bar{\rho}_N : \mathbb{Z}_N = (Z \otimes R_N)/\mathcal{I}_N' \twoheadrightarrow W_N,$$

where $Z$ is defined in (3.18), and $\mathcal{I}_N'$ is given in Definition 3.9. Let $\pi : W_{N,l} \rightarrow M_{N,l}$ be the canonical projection. The filtration (3.31) on $\mathbb{Z}_N$ induces a filtration $\{\pi(\bar{\rho}_N(F_s))\}$ on $M_{N,l}$.

Definition 4.2. Let $\mathcal{I}_N$ be the ideal of $Z$ generated by $J_N$ and $\xi_0, \eta_0$, where $J_N$ is the ideal given in Definition 3.10. We define

$$\mathbb{Z}_{N,l} = \mathbb{Z}_l/(Z_l \cap \mathcal{I}_N),$$

where $Z_l$ denotes the subspace of weight $l$. 

From the correspondence in Proposition 3.5, we have a surjection
\[ \mathcal{Z}_{N,l} \otimes R_N \rightarrow \text{gr} \ M_{N,l} \rightarrow 0. \] (4.2)
In the following subsections we will prove that

**Proposition 4.3.**
\[ \text{ch}_q \mathcal{Z}_{N,l} \leq \left[ \frac{N}{l} \right] - \left[ \frac{N}{l-1} \right]. \] (4.3)

For the estimate from below, we use the following result of Tarasov ([23], in the proof of Theorem 4.4). It can also be derived using the knowledge of crystal basis (see Section 5.6).

**Proposition 4.4** [23]. Let \( e_c : W_{N,l} \rightarrow A_{N,l} \) be the evaluation map, and denote the map induced on the quotient space \( M_{N,l} \) by the same letter \( e_c \). Then for a generic \( c \) we have
\[ \dim_{\mathbb{C}} e_c(M_{N,l}) = \left( \frac{N}{l} \right) - \left( \frac{N}{l-1} \right). \]

Theorem 4.1 follows from Proposition 4.3 and Proposition 4.4, by reasoning in a similar manner as in Subsection 3.3.

### 4.2. Kostka polynomial for type \( A_1 \)

Before proceeding to the proof of Proposition 4.3, it is useful to rephrase (4.3) in terms of Kostka polynomials. The Kostka polynomial for type \( A_1 \) is a polynomial \( K_{m,\nu}(q) \) in \( q \), depending on an integer \( m \) and an array \( \nu = (\nu_1, \ldots, \nu_N) \) of non-negative integers. The following fermionic formula is available.
\[ K_{m,\nu}(q) = \sum_n q^{c(n)} \prod_j \left[ P_j + n_j \right]. \] (4.4)

Here we set \( m_a = \# \{ j : \nu_j = a \} \) and the sum ranges over all the sequences \( n = (n_j)_{j=1,2,\ldots} \) of non-negative integers \( n_j \in \mathbb{Z}_{\geq 0} \) such that \( 2 \sum_{j\geq 1} j n_j = \sum_{j\geq 1} j m_j - m \). The quantities \( c(n) \) and \( P_j \) are defined by
\[ c(n) = \sum_{j,j' \geq 1} A_{jj'} n_j n_j', \] (4.5)
\[ A_{jj'} = \min(j, j'), \quad P_j = \sum_{j' \geq 1} A_{jj'} (m_j' - 2n_j'). \] (4.6)

In the simplest case where \( \nu = (1^N) = (1, \ldots, 1) \), the Kostka polynomial \( K_{m,(1^N)}(q) \) is given by a simple formula:
\[ K_{m,(1^N)}(q) = \left\{ \begin{array}{ll} \left[ \frac{N-m}{2} \right] - \left[ \frac{N-m-2}{2} \right], & \text{if } m \equiv N \mod 2, \\
0, & \text{otherwise}. \end{array} \right. \] (4.7)

Hence (4.3) is equivalently stated as
\[ \text{ch}_q \mathcal{Z}_{N,l} \leq K_{N-2l,(1^N)}(q). \] (4.8)
4.3. Dual space. To show (4.8), it is convenient to pass to the dual space. Let $\mathbb{C}[x_1, \ldots, x_s]^{\text{Symm}}$ be the space of symmetric polynomials in $x_1, \ldots, x_s$, and $\mathbb{C}[y_1, \ldots, y_t]^{\text{Symm}}$ the space of symmetric polynomials in $y_1, \ldots, y_t$. Denote by $\Lambda(\xi_0, \xi_1, \xi_2, \ldots)$, or $\mathbb{C}[\eta_0, \eta_1, \eta_2, \ldots]$, the homogeneous component of weight $t$ with respect to the weight defined by (3.19). Set

$$\hat{\xi}(z) = \xi_0 + \xi(z) = \sum_{n=0}^{\infty} \xi_n z^n, \quad \hat{\eta}(z) = \eta_0 + \eta(z) = \sum_{n=0}^{\infty} \eta_n z^n.$$ 

There are non-degenerate bilinear pairings

$$\Lambda(\xi_0, \xi_1, \xi_2, \ldots)_s \times \mathbb{C}[x_1, \ldots, x_s]^{\text{Skew}} \to \mathbb{C}, \quad (4.9)$$

$$\mathbb{C}[\eta_0, \eta_1, \eta_2, \ldots]_t \times \mathbb{C}[y_1, \ldots, y_t]^{\text{Symm}} \to \mathbb{C}, \quad (4.10)$$

classified by

$$\langle \hat{\xi}(x_1) \cdots \hat{\xi}(x_s), f \rangle = f(x_1, \ldots, x_s), \quad (4.11)$$

$$\langle \hat{\eta}(y_1) \cdots \hat{\eta}(y_t), g \rangle = g(y_1, \ldots, y_t), \quad (4.12)$$

for $f \in \mathbb{C}[x_1, \ldots, x_s]^{\text{Skew}}$, $g \in \mathbb{C}[y_1, \ldots, y_t]^{\text{Symm}}$. The pairings (4.9), (4.10) respect degrees with the assignment $\deg x_a = 1$, $\deg y_b = 1$.

We have thus a non-degenerate pairing

$$(\Lambda[\xi] \otimes \mathbb{C}[\eta])_s \times \bigoplus_{s+2t=I} (\mathbb{C}[x_1, \ldots, x_s]^{\text{Skew}} \otimes \mathbb{C}[y_1, \ldots, y_t]^{\text{Symm}}) \to \mathbb{C}. \quad (4.13)$$

The dual space of $\mathbb{Z}_{N,I}$ is identified with (4.13) with the space $\mathbb{D}_{N,I}$, which consists of sets of polynomials $f = (f_{s,t})_{s+2t=I}$ orthogonal to the ideal $\mathcal{J}_{N,I}$. Explicitly the generators of $\mathcal{J}_{N,I}$ are

$$\xi_0, \quad \eta_0, \quad (4.14)$$

$$\xi(z)\xi(-z) + \eta(z) - \eta(-z), \quad (4.15)$$

$$[\eta(z)^\nu]_{\geq N-2\nu+1} \quad (\nu \geq 1), \quad (4.16)$$

$$[\xi(z)\eta(z)^{\nu-1}]_{\geq N-2\nu+2} \quad (\nu \geq 1).$$

From the rules (4.11), (4.12) of the pairing, we find the following conditions defining $\mathbb{D}_{N,I}$.

**Lemma 4.5.** A set of polynomials $f = (f_{s,t})_{s+2t=I}$ belongs to $\mathbb{D}_{N,I}$ if and only if the following conditions are satisfied.

(i) $f_{s,t}(x_1, \ldots, x_s; y_1, \ldots, y_t)$ is skew-symmetric with respect to $x_1, \ldots, x_s$ and symmetric with respect to $y_1, \ldots, y_t$.

(ii) There exists a polynomial $g_{s,t}$ such that

$$f_{s,t} = x_1 \cdots x_s (y_1 \cdots y_t)^2 g_{s,t}.$$ 

(iii) The polynomials $f_{s,t}$ satisfy the relations

$$f_{s+2,t}(z, -z; x_1, \ldots, x_s; y_1, \ldots, y_t) + f_{s,t+1}(x_1, \ldots, x_s; z, y_1, \ldots, y_t)$$

$$- f_{s,t+1}(x_1, \ldots, x_s; -z, y_1, \ldots, y_t) = 0,$$
(iv) For all \( \nu \geq 1 \) we have

\[
\deg_z f_{s,t}(x_1, \ldots, x_s; z, \ldots, z, y_{\nu+1}, \ldots, y_t) \leq N - 2\nu,
\]

\[
\deg_z f_{s,t}(z, x_2, \ldots, x_s; z, \ldots, z, y_{\nu+1}, \ldots, y_t) \leq N - 2\nu + 1.
\]

The inequality (4.8) is equivalent to

**Proposition 4.6.** We have

\[
\text{ch}_q D_{N,l} \leq K_{N-2l, (1^n)}(q).
\]

In the next subsection we will prove Proposition 4.6, using the fermionic formula of \( K_{N-2l, (1^n)}(q) \).

### 4.4. Proof of Proposition 4.6

Let \( P_l \) be the set of partitions of \( l \). For \( \lambda = (\lambda_1, \ldots, \lambda_n) = (1^{n_1}, 2^{n_2}, \ldots, k^{n_k}) \in P_l \), we have \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0 \), \( n_\alpha \geq 0 \) and \( \sum_{i=1}^n \lambda_i = \sum_{\alpha=1}^k a_n \alpha = l \). We define a total order of \( P_l \): \( \lambda > \lambda' \) if and only if there is some \( j \) such that \( \lambda_i = \lambda'_i \) if \( 1 \leq i < j \) and \( \lambda_j > \lambda'_j \).

We define a polynomial \( \varphi_\lambda(f) \) of the variables \( v_1, \ldots, v_n \). We set

\[
\lambda_i = \begin{cases} 
2\nu_i + 1 & \text{if } i = i_1, \ldots, i_s; \\
2\nu_i & \text{otherwise}. 
\end{cases}
\]

(4.18)

We define a filtration \( \Gamma_\lambda \) of the vector space \( D_{N,l} \):

\[
\Gamma_\lambda = \bigcap_{\lambda' > \lambda} \ker \varphi_{\lambda'}.
\]

(4.19)

The associated graded space is defined as \( \Gamma_\lambda/\Gamma'_\lambda \) where \( \Gamma'_\lambda = \Gamma_\lambda \cap \ker \varphi_\lambda \). Since \( \varphi_\lambda(f) = 0 \) if \( f \in \ker \varphi_\lambda \), the specialization \( f \mapsto \varphi_\lambda(f) \) is an injective mapping defined on the graded component \( \Gamma_\lambda/\Gamma'_\lambda \). Our aim is to determine the image of this mapping.

**Proposition 4.7.** Let \( f \in \Gamma_\lambda/\Gamma'_\lambda \) and \( F = \varphi_\lambda(f) \). The polynomial \( F(v_1, \ldots, v_n) \) is divisible by

\[
\prod_a v_a^{\lambda_a} \prod_{a > b} (v_a^2 - v_b^2)^{\lambda_a}.
\]

(4.20)

Before the proof, let us see that Proposition 4.6 follows from this proposition.

Set

\[
\varphi_\lambda(f) = \left( \prod_a v_a^{\lambda_a} \prod_{a > b} (v_a^2 - v_b^2)^{\lambda_a} \right) f_\lambda.
\]
Then \( f_\lambda \) is symmetric with respect to \( v_a \) and \( v_b \) such that \( \lambda_a = \lambda_b \). The degree of \( f_\phi \) with respect to \( v_b \) is at most \( P_{\lambda_b} \), where \( P_{\lambda} \) is given by \((4.6)\) with \( m = N - 2l \) and \( \nu = (1^N) \). Hence we have

\[
\text{ch}_q \varphi_\lambda(\Gamma_{\lambda}/\Gamma'_{\lambda}) \leq q^{c(n)} \prod_j \left[ \frac{P_j + n_j}{n_j} \right],
\]

where \( c(n) \) is given by \((4.5)\), and \( n = (n_j)_{j=1,2,\ldots} \) is determined from \( \lambda = (1^{n_1}, 2^{n_2}, \ldots) \). Thus we get the upper estimate in Proposition \(4.6\).

The proof of Proposition \(4.7\) is divided into several lemmas.

**Lemma 4.8.** Let \( f_{s,t} \) be a member of \( f \in \mathbb{D}_{N,l} \). We have the equality

\[
f_{s,t}(v, \ldots; v, \ldots) = f_{s,t}(v, \ldots; -v, \ldots).
\]

*Proof.* This is a consequence of the skew-symmetry of \( f_{s,t} \) in the first set of variables and the relation \((iii)\) in Lemma \(4.5\). \( \square \)

**Lemma 4.9.** Let \( \lambda = (\lambda_1, \ldots, \lambda_n) \) be a partition such that for some \( a > b \) we have \( \lambda_a = 2m + 1, \lambda_b = 2\nu \). We denote by \( \kappa \) the partition obtained from \( \lambda \) by splitting \( \lambda_a \) to \((1, 2^m) \). We use the variables \( v_1, \ldots, v_n \) and \( w_1, \ldots, w_m \) to represent \( \varphi_{\kappa}(f) \); in particular, the variables \( v_a \) and \( v_b \) correspond to the parts \( 1 \) and \( 2\nu \). The polynomial \( \varphi_{\kappa}(f) \) where \( f \in \Gamma_{\lambda} \) is divisible by \( v_a^2 - v_b^2 \).

*Proof.* Let \( \kappa' \) be the partition obtained from \( \kappa \) by splitting the part \( 2\nu \) to \( 2\nu \). We introduce the variables \( y_1, \ldots, y_\nu \) in place of \( v_b \) to represent \( \varphi_{\kappa'}(f) \). Let us denote this polynomial by \( f(v_a; y_1, \ldots, y_\nu) \) forgetting the dependence on the other variables. The assertion follows if we show two equalities \( f(v; v, \ldots, v) = 0 \) and \( f(v; -v, \ldots, -v) = 0 \). Let \( \kappa'' \) be a partition obtained from \( \kappa \) by merging the parts \( 1, 2\nu \) in a new part \( 2\nu + 1 \). Then, we have \( \kappa'' > \lambda \), and therefore

\[
f(v; v, \ldots, v) = \varphi_{\kappa''}(f) = 0.
\]

The second equality follows from this and Lemma \(4.8\). \( \square \)

**Lemma 4.10.** Let \( f^{(0)}(x_1, x_2, x_3; y_1, \ldots, y_{\nu-2}), f^{(1)}(x_1, x_2; y_1, \ldots, y_{\nu-2}) \) and \( f^{(2)}(y_1, \ldots, y_\nu) \) be polynomials, which are skew-symmetric in \( \{x_a\} \), symmetric in \( \{y_b\} \), and satisfy the relations

\[
\begin{align*}
f^{(0)}(x_1, x_2, v, -v; y_1, \ldots, y_{\nu-2}) &= -f^{(1)}(x_1, x_2; v, y_1, \ldots, y_{\nu-2}) + f^{(1)}(x_1, x_2; -v, y_1, \ldots, y_{\nu-2}), \quad (4.21) \\
f^{(1)}(v, -v; y_1, \ldots, y_{\nu-1}) &= -f^{(2)}(v, y_1, \ldots, y_{\nu-1}) + f^{(2)}(-v, y_1, \ldots, y_{\nu-1}). \quad (4.22)
\end{align*}
\]

If the polynomial \( f^{(2)}(v, \ldots, v) \) of \( v \) is identically 0, then for all \( 0 \leq \nu' \leq \nu \) we have \( f_{\nu'}^{(\nu)}(v, \ldots, v, -v, \ldots, -v) = 0 \).
Proof. Lemma 4.8 implies that $f^{(1)}(v, -v; v, \ldots, v, -v, \ldots, -v)$ are independent of $0 \leq \nu' \leq \nu - 1$. Using (4.22), we write these $f^{(1)}$ in terms of $f^{(2)}$. Let

$$A_{a,b}^{(\nu)} = \begin{cases} 2 & \text{if } a = b; \\ -1 & \text{if } a = b \pm 1; \\ 0 & \text{otherwise}. \end{cases}$$

Under the condition $f^{(2)}(v, \ldots, v) = f^{(2)}(-v, \ldots, -v) = 0$, we have

$$\sum_{\nu' = 1}^{\nu - 1} A_{a,\nu'}^{(\nu)} f^{(\nu)} = 0$$

for all $1 \leq a \leq \nu - 1$. Since $A^{(\nu)}$ is non-degenerate, the assertion follows.

Lemma 4.11. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition such that for some $a > b$ we have $\lambda_a = 2\nu' + 1$, $\lambda_b = 2\nu - 1$. We denote by $\kappa$ a partition obtained from $\lambda$ by splitting $\lambda_b$ to $(1, 2\nu')$. We use the variables $v_1, \ldots, v_n$ and $w_1, \ldots, w_{\nu'}$ to represent $\varphi_\kappa(f)$; in particular, the variables $v_a$ and $v_b$ correspond to the parts $1$ and $2\nu - 1$. The polynomial $\varphi_\kappa(f)$ where $f \in \Gamma_\lambda$ is divisible by $v_a^2 - v_b^2$.

Proof. Let $\lambda^{(0)}, \lambda^{(1)}, \lambda^{(2)}$ be partitions obtained from $\kappa$ by replacing the parts $1, 2\nu - 1$ with new parts $(1^2, 2^{\nu - 2}), (1^2, 2^{\nu - 1}), (2^\nu)$, respectively. We write

\[
\varphi_{\lambda^{(0)}}(f) = f^{(0)}(x_1, x_2, x_3, x_4; y_1, \ldots, y_{\nu - 2}), \\
\varphi_{\lambda^{(1)}}(f) = f^{(1)}(x_1, x_2; y_1, \ldots, y_{\nu - 1}), \\
\varphi_{\lambda^{(2)}}(f) = f^{(2)}(y_1, \ldots, y_\nu),
\]

forgetting the variables other than those which correspond to the new parts. It suffices to prove that $f^{(1)}(v, -v; v, \ldots, v) = 0$ and $f^{(1)}(v, -v; -v, \ldots, -v) = 0$. The former follows from the skew-symmetry of $f_{x,t}$ in the first set of variables. Let $\kappa'$ be a partition obtained from $\kappa$ by merging $1, 2\nu - 1$ in $2\nu$. Then, we have $\kappa' > \lambda$, and therefore, $f^{(2)}(v, v, \ldots, v) = \varphi_{\kappa'}(f) = 0$. By Lemma 4.10, it then follows that $f^{(2)}(v, -v, \ldots, -v) = f^{(2)}(-v, -v, \ldots, -v) = 0$. Because of (4.22), these equalities imply $f^{(1)}(v, -v; v, -v, \ldots, -v) = 0$.

Lemma 4.12. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition. Take a pair of indices $a > b$ such that $\lambda_a \geq 2$. We denote by $\kappa$ a partition obtained from $\lambda$ by splitting $\lambda_a$ to $(1, 2\nu')$ if $\lambda_a = 2\nu' + 1$, or to $2\nu'$ if $\lambda_a = 2\nu'$. If $\lambda_a = 2\nu' + 1$, we use the variables $v_1, \ldots, v_n$ and $w_1, \ldots, w_{\nu'}$, to represent $\varphi_\kappa(f)$; or if $\lambda_a = 2\nu'$, we use $v_1, \ldots, v_n$ except for $v_a$ and $w_1, \ldots, w_{\nu'}$. In particular, the variables $w_1, \ldots, w_{\nu'}$ and $v_b$ correspond to the parts $2\nu'$ and $\lambda_b$. The polynomial $\varphi_\kappa(f)$ where $f \in \Gamma_\lambda$ is divisible by $(w_a^2 - v_b^2)^2$.

Case $\lambda_b = 2\nu - 2$. We follow the same steps as in the proof of Lemma 4.11 with the parts $1, 2\nu - 1$ replaced with $2, 2\nu - 2$, and define $f^{(0)}, f^{(1)}, f^{(2)}$, and $\kappa'$. Note that $\kappa' > \lambda$. We now want to prove that both $f^{(2)}(w, w, \ldots, w)$ and $f^{(2)}(-v, w, \ldots, w)$
are divisible by $(v-w)^2$. It is sufficient to prove that $f^{(2)}(v, v, \ldots, v) = 0$, $\frac{\partial f^{(2)}}{\partial y_1}(v, v, \ldots, v) = 0$, $f^{(2)}(-v, v, \ldots, v) = 0$ and $\frac{\partial f^{(2)}}{\partial y_1}(-v, v, \ldots, v) = 0$. The first equality follows from the condition $f \in \Gamma_\lambda \subset \text{Ker} \varphi_{\kappa'}$, the second, then follows from the symmetry of $f^{(2)}$ in the variables $y_1, \ldots, y_\nu$. The third follows from Lemma 4.10.

Let $\kappa''$ be a partition obtained from $\kappa$ by replacing two parts $2$ and $2\nu - 2$ by new parts $1$ and $2\nu - 1$. We have $\kappa'' > \lambda$. Because of (4.22) and the skew-symmetry of $f^{(1)}$ in $x_1, x_2$, the last equality follows from $\frac{\partial f^{(1)}}{\partial x_2}(v, -v; v, \ldots, v) = 0$, which is a consequence of $f \in \Gamma_\lambda \subset \text{Ker} \varphi_{\kappa''}$.

Case $\lambda_0 = 2\nu - 1$. Let $\kappa'$ be a partition obtained from $\kappa$ by splitting $2\nu - 1$ to $(1, 2^{\nu-1})$. Along with a part $2$ coming from the splitting of $\lambda_0 \geq 2$, it contains the parts $(1, 2^{\nu})$. We introduce the variables $x$ and $y_1, \ldots, y_\nu$, corresponding to these parts, and write $\varphi_{\kappa'}(f) = f(x_1; y_1, \ldots, y_\nu)$ forgetting the other variables. Set $h(v, w) = f(v; w, v, \ldots, v)$. We want to show that $h(v, w)$ is divisible by $(v-w)^2$ and $(v+w)^2$, or equivalently, $f(v; v, v, \ldots, v) = 0$, $\frac{\partial f}{\partial y_1}(v; v, v, \ldots, v) = 0$, $f(v; -v, v, \ldots, v) = 0$ and $\frac{\partial f}{\partial y_1}(v; -v, v, \ldots, v) = 0$. Let $\kappa''$ be a partition obtained from $\kappa$ by replacing two parts $2$ and $2\nu - 1$ by new parts $1$ and $2\nu$. We have $\kappa'' > \lambda$. The first equality follows from $f(v; w, \ldots, w) = 0$, which is a consequence of $f \in \Gamma_\lambda \subset \text{Ker} \varphi_{\kappa''}$. The second then follows from the symmetry of $f(x_1; y_1, \ldots, y_\nu)$ in the variables $y_1, \ldots, y_\nu$. By Lemma 4.8 we have the third, $f(v; -v, v, \ldots, v) = 0$, and also $f(v; w, v, \ldots, v) = f(v; w, -v, \ldots, -v)$. In particular, we have $\frac{\partial f}{\partial y_1}(v; -v, v, \ldots, v) = \frac{\partial f}{\partial y_1}(v; -v, \ldots, -v)$. This is in fact zero because $f(v; w, \ldots, w) = 0$.

Proof of Proposition 4.7. We use Lemmas 4.9, 4.11 and 4.12 to show that $F$ has the factor $(\nu_0^2 - \nu_\nu^2)^{\lambda_0}$. Let $\kappa$ be a partition defined in Lemma 4.12. Since the number of odd parts, i.e., $s$, in $\lambda$ and in $\kappa$ are the same, the same function $f_{s, t}$ is used both in $\varphi_{\kappa'}(f)$ and $\varphi_{\lambda}(f)$. Therefore, the latter is obtained by specialization of the former. The assertion follows from this observation.

5. Form Factors of the Restricted Sine-Gordon Model

5.1. Formulation. When the coupling constant $\xi$ becomes rational, a ‘reduction’ of the SG theory takes place [16]. In this subsection, we formulate form factors of the restricted sine-Gordon (RSG) model in the case where $\xi$ is an integer $r \geq 3$. Set $\epsilon = e^{-\pi i/r}$, and denote by $U_{\epsilon}(\mathfrak{sl}_2)$ the subalgebra of $U_{\epsilon}$ generated by $E = e_1$, $F = f_1$, $T^{\pm 1} = t^{\pm 1}$. We use the opposite coproduct

$$\Delta'(E) = E \otimes T + 1 \otimes E, \quad \Delta'(F) = F \otimes 1 + T^{-1} \otimes F, \quad \Delta'(T) = T \otimes T. \quad (5.1)$$

We introduce the gauge transformation

$$\tilde{f}_n(\beta_1, \ldots, \beta_n) = G(\beta_1, \ldots, \beta_n) f_n(\beta_1, \ldots, \beta_n), \quad (5.2)$$

where

$$G(\beta_1, \ldots, \beta_n) := \epsilon \sum_i \beta_i \sigma_i^{-1}.$$
which makes the action of $U_q(sl_2)$ on $V^\otimes N$ to be the standard one, i.e., without spectral parameters. We will consider the decomposition of the space of highest weight vectors in this action. This procedure leads us to the $r$-restricted paths.

First let us summarize some facts about representations of $U_q(sl_2)$. The details are given in appendix D. Besides $V = \mathbb{C}v_+ \oplus \mathbb{C}v_-$, we use three types of $U_q(sl_2)$-modules [17],

$$V^s, \quad X^s(\alpha) \quad (0 \leq s \leq r-2, \alpha = \pm 1),$$

$$W^s(\alpha) \quad (0 \leq s \leq r-1, \alpha = \pm 1).$$

The modules $V^s (0 \leq s \leq r-2)$ are specializations of irreducible modules at $q = \epsilon$. They are irreducible and $\dim V^s = s + 1$. In particular, $V^1 = V$. The modules $X^s(\alpha)$ and $W^s(\alpha)$ are indecomposable, and have dimensions $2r$ and $r$, respectively.

The tensor product $V^\otimes n$ is a direct sum of ‘good’ and ‘bad’ subspaces (see Definition D.1)

$$V^\otimes n = G_n(r) \oplus B_n(r).$$

The ‘good’ subspace $G_n(r)$ is a direct sum of $V^s (0 \leq s \leq r-2)$, while the ‘bad’ subspace $B_n(r)$ is a direct sum of $X^s(\alpha) (0 \leq s \leq r-2)$ and $W^{r-1}(\alpha)$. The decomposition (5.4) is orthogonal relative to the standard symmetric bilinear form $(\ , \ )$ on $V^\otimes n$.

In the case of generic $\xi$, we considered form factors $f = (f_n)_n^{\infty}$ taking values in $\Omega_{n,l} = \text{Ker} e_1 \cap (V^\otimes n)$, in the action $\pi_{\xi}^{-1} \otimes \cdots \otimes \pi_{\xi_n}^{-1}$. This space is invariant under the action of the operators which enter the axioms for form factors of the SG model:

$$P_{j,j+1}S_{j,j+1}(\beta_j - \beta_{j+1})$$

$$= G(\ldots, \beta_{j+1}, \beta_j, \ldots)^{-1}S_{j,j+1}(\beta_j - \beta_{j+1})G(\ldots, \beta_j, \beta_{j+1}, \ldots)$$

and

$$e^{\frac{(n-2i)(n+1)}{4}}P_{n,n-1} \cdots P_{2,1} = G(\beta_1, \ldots, \beta_{n-1}, \beta_n + 2\pi i)^{-1}\Pi_{n,l} G(\beta_n, \beta_1, \ldots, \beta_{n-1}).$$

Here the operator $\tilde{S}(\beta)$ is given by

$$\tilde{S}(\beta) = S_0(\beta) \frac{R^\pm \zeta - (R^\pm)^{-1} \zeta^{-1}}{q\zeta - q^{-1}\zeta^{-1}}, \quad \zeta = e^\frac{\beta}{\epsilon}.$$

See (D.5) and (D.6) for the definition of $R^\pm$ and $\Pi_{n,l}$.

In the restricted case, we consider the gauge transformed action. Though the subspaces $G_n(r)$, $B_n(r)$ are not invariant under the actions of $\tilde{S}_{j,j+1}(\beta)$ and $\Pi_{n,l}$, it can be shown that the space $\Omega_{n,l} \cap B_n(r)$ is invariant (see Lemma D.3). This observation makes the following definition of the form factors of the RSG model well defined.

We define

$$\Omega_{n,l}^{(r)} = \Omega_{n,l}/\Omega_{n,l} \cap B_n(r).$$

The operator $\tilde{S}_{j,j+1}(\beta)$ and $\Pi_{n,l}$ are well defined on $\Omega_{n,l}^{(r)}$. Let

$$\mathcal{P}: \Omega_{n,l} \to \Omega_{n,l}^{(r)}$$

be the projection.
Definition 5.1. For a form factor \( f = (f_n)_{n=0}^\infty \) of the SG model satisfying the axioms (A0)–(A3), we call the tower of the \( \Omega^{(k)}_{n,l} \)-valued functions \( (\Omega f_n)_{n=0}^\infty \) a form factor of the RSG model.

A form factor \( (\Omega f_n)_{n=0}^\infty \) is called \( N \)-minimal if
\[
(\Omega f_n)_{n=0}^\infty \quad \text{for } n < N.
\]
From the axiom (A3) the \( N \)-particle minimal form factor satisfies
\[
\text{res}_{\beta N=\beta N-1+\pi i} (\Omega f_n)_{n=0}^\infty = 0.
\]

5.2. The qKZ equation of face type. In this section we introduce the basis of the tensor product \( V^\otimes N \) parametrized by certain combinatorial objects called paths. For the time being, let the parameter \( \xi \in \mathbb{C} \) be generic and set
\[
[n] = (q^n - q^{-n})/(q - q^{-1}),
\]
where \( q = e^{-\pi i/\xi} \). We consider the action of \( U_q(\mathfrak{sl}_2) \) as given in Section 2.6. For a non-negative half integer \( j \in \frac{1}{2}\mathbb{Z} \), we take a basis \( \psi^j_m \) \((-j \leq m \leq j, j - m \in \mathbb{Z})\) of the irreducible \((2j + 1)\)-dimensional representation such that
\[
E \psi^j_m = q^{m+\frac{1}{2}} \sqrt{j + m} \psi^j_{m+1}.
\]

This basis \( \{ \psi^j_m \} \) is related to \( \{ e^{2\pi j} \} \) used in Appendix D by a scalar multiple.)

We set
\[
\begin{bmatrix}
 j & 1/2 \\
 m & \pm 1/2
\end{bmatrix} = q^{\pm j m} \sqrt{j + m + 1}.
\]

These are called the \( q \)-deformed 3j symbols. If \( j, m \) do not satisfy the conditions stated above, they are defined to be zero. The decomposition of the tensor product \( \mathbb{C}^{2j+1} \otimes \mathbb{C}^2 = \mathbb{C}^{2j+2} \oplus \mathbb{C}^{2j} \) is given by
\[
\psi^j \otimes \frac{1}{2} = \begin{bmatrix}
 j & 1/2 \\
 m & \pm 1/2
\end{bmatrix} \psi^j_m \otimes \frac{1}{2} + \begin{bmatrix}
 j + 1/2 \\
 m + 1/2
\end{bmatrix} \psi^j_{m+1} \otimes \frac{1}{2}.
\]

A path of length \( N \) is a sequence \( J = (j_1, \ldots, j_N) \) satisfying \( j_n \in (1/2)\mathbb{Z} \), \( j_1 = 1/2 \) and \( j_{n+1} = j_n \pm 1/2 \). The integer \( 2j_N \) is called the weight of the path \( J \). A path \( J \) is called classically restricted if \( j_n \geq 0 \) for all \( 0 \leq n \leq N \). For a classically restricted path \( J \) and \( m_N \in \mathbb{Z} + j_N \) with \( -j_N \leq m_N \leq j_N \), we define a vector \( u_{J,m_N} \in \mathbb{C} \otimes \cdots \otimes \mathbb{C} \) by
\[
u_M = \sum_{\varepsilon_1, \ldots, \varepsilon_N = \pm 1 \atop \varepsilon_1 + \cdots + \varepsilon_N = 2m_n} \begin{bmatrix}
 j_1 & 1/2 \\
 m_1 & \varepsilon_1/2
\end{bmatrix} \cdots \begin{bmatrix}
 j_{N-1} & 1/2 \\
 m_{N-1} & \varepsilon_{N-1}/2
\end{bmatrix} \begin{bmatrix}
 j_N & 1/2 \\
 m_N & \varepsilon_N/2
\end{bmatrix} v_M.
\]

Here the vector \( v_M \) is specified by the subset \( M = \{ n : \varepsilon_n = -1 \} \), and the sum is restricted by the condition \( \varepsilon_1 + \cdots + \varepsilon_N = 2m_N \). For \( n < N \) the equation \( \varepsilon_1 + \cdots + \varepsilon_N = 2m_n \) defines \( m_n \) in the summation.
The vectors \( u_{J,n} \) constitute an orthonormal basis of the tensor product \( V^{\otimes N} \). In particular, the vectors \( u_J = u_{J,J_N} \) constitute a basis of \( \ker E \). Since the integral

\[
I(G(v_{N,1}), P) = \sum_{J,J_N=(N-2l)/2} \tilde{\psi}_{P,J} u_J,
\]

(5.13)

\[
\tilde{\psi}_{P,J} = \sum_{M, \#(M)=l} q^{\rho(M)} e^{\frac{1}{\epsilon} \left( \sum_{k \in M} \beta_k - \sum_{m \in M} \beta_m \right)} C_{J,M} I(w_M, P),
\]

(5.14)

\[
C_{J,M} = \left[ \begin{array}{ccc} j_1 m_1 & 1/2 & j_2 m_2 \\ \varepsilon_2/2 & m_{N-1} & \varepsilon N/2 \\ j N & \varepsilon N/2 & j N \end{array} \right],
\]

(5.15)

where \( \varepsilon_1 + \cdots + \varepsilon_n = 2m_n \) for \( 1 \leq n \leq N-1 \), and \( M = \{ u \mid \varepsilon_n = -1 \} \).

By a standard argument [15], it follows from Theorem 2.2 and Corollary 2.5 that \( \tilde{\psi}_{P,J}(\beta_1, \ldots, \beta_N) \) satisfies the qKZ equation of face type:

\[
\tilde{\psi}_{P(\ldots,j_{i-1},j_i,j_{i+1},\ldots)}(\ldots, \beta_{i+1}, \ldots) = \sum_{j^i} \tilde{\psi}_{P(\ldots,j_{i-1},j_i',j_{i+1},\ldots)}(\ldots, \beta_i, \beta_{i+1}, \ldots) W \begin{bmatrix} j_{i-1} & j_i \\ j_i' & j_{i+1} \end{bmatrix} (\beta_i - \beta_{i+1}),
\]

(5.16)

where the coefficients are the Boltzmann weights of the RSOS model [1]

\[
W \begin{bmatrix} j + 1/2 & j \\ j & j + 1/2 \end{bmatrix} (\beta) = 1,
\]

\[
W \begin{bmatrix} j & j + 1/2 \\ j + 1/2 & j \end{bmatrix} (\beta) = \frac{[2j+1][1+u]}{[2j+1][1+u]},
\]

\[
W \begin{bmatrix} j & j + 1/2 \\ j + 1/2 & j \end{bmatrix} (\beta) = \frac{\sqrt{[2j][2j+2][u][u]}}{[2j+1][1+u]},
\]

and \( u = -\frac{\beta}{1-\beta} \).

Now let us consider the case where \( \xi = r \geq 3 \) is an integer. Accordingly \( q \) is specialized to the root of unity \( \epsilon = e^{-\pi i/r} \). A classically restricted path \( J \) is called \( r \)-restricted if \( 2j_n \leq r-2 \) for all \( 1 \leq n \leq N \). It is known that the set of the vectors \( \{ \nu J \} \) associated with \( r \)-restricted paths give a basis of \( \Omega_N^r \).

The q integer \([j] = \frac{r-j}{r-1} \) vanishes if \( j \) is a multiple of \( r \). However, the Boltzmann weights appearing in the qKZ equation of face type (5.16) are non-zero and finite if the paths in the equations are \( r \)-restricted. Consider the special \( r \)-restricted path

\[
J_{N,l} = \left( \frac{1}{2}, 0, \ldots, \frac{1}{2}, 0, \frac{3}{2}, \ldots, \frac{N-2l}{2} \right).
\]

(5.17)

Solving the qKZ equation of face type, one can obtain \( \tilde{\psi}_{P,J} \) for an arbitrary \( r \)-restricted path \( J \) in terms of the one for \( J_{N,l} \). Therefore, if \( \tilde{\psi}_{P,J_{N,l}} = 0 \) then \( \tilde{\psi}_{P,J} = 0 \) for all paths \( J \) of length \( N \) and weight \( N - 2l \). In the next section, we construct minimal deformed cycles with this property.
5.3. Restricted cocycles and restricted null cycles. We fix \( N, l \). Let \( J \) be an \( r \)-restricted path of length \( N \) and weight \( N - 2l \). We call a linear combination of the deformed cocycles \( \tilde{w}_J = \text{Skew}_{a_1, \ldots, a_l} \tilde{g}_J \), where

\[
\tilde{g}_J = \sum_{\#M = l} e^{\pi i \left( \sum_{k \in M} \beta_k - \sum_{m \in M} \beta_m \right) e^\nu(M)} C_{J,M} g_M,
\]  

(5.18)
an \( r \)-restricted cocycle. Here \( C_{J,M} \) are defined in (5.15). A deformed cycle \( P \) is called an \( r \)-restricted null cycle if

\[
I(\tilde{w}, P) = 0
\]

for all \( r \)-restricted cocycles. As discussed at the end of the previous section, this is equivalent to \( I(\tilde{w}_{J_{N,l}}, P) = 0 \).

Lemma 5.2. Let \( Q \) be the polynomial corresponding to \( \tilde{w}_{J_{N,l}} \) in (2.10). Then, we have

\[
\deg_Q Q \leq N + l - 2.
\]  

(5.19)

Proof. For the path \( J_{N,l} \), the coefficient \( C_{J_{N,l},M} \) is non-zero if and only if

\[
M = (\epsilon_1, -\epsilon_1, \ldots, \epsilon_l, -\epsilon_l, \ldots, +)
\]

\( N - 2l \).

From (5.10) and (5.11), we have

\[
C_{J_{N,l},M} = (-1)^{\frac{l}{2}} \sum_{p=1}^{l} (\epsilon_p + 1) \frac{\epsilon^{\frac{1}{2}} \sum_{p=1}^{l} \epsilon_p}{[2]^\frac{l}{2}}.
\]

Taking the summation over each \( \epsilon_p \) in the right hand side of (5.18), we obtain (5.19).

\[\square\]

Set

\[
\mu = r - 1 - (N - 2l).
\]  

(5.20)

We assume that \( \mu \geq 1 \). This is equivalent to

\[
N - 2l \leq r - 2.
\]

Namely, unless \( \mu \geq 1 \), there exists no \( r \)-restricted path of length \( N \) and weight \( N - 2l \). We also assume that \( \mu \leq l \). This is equivalent to

\[
N - l \geq r - 1.
\]

Namely, unless \( \mu \leq l \), all the classically restricted paths of length \( N \) and weight \( N - 2l \) are \( r \)-restricted. The aim of this section is to find as many minimal \( r \)-restricted null cycles as there are classically restricted paths that are not \( r \)-restricted.

We define minimal cycles \( \Gamma_1 \) (when \( r \equiv N \mod 2 \)) and \( \Gamma_2 \) by

\[
\Gamma_1(X) = X^{-1} \left( \Theta(X) - (-1)^{N+r} \Theta(-X) \right),
\]  

(5.21)

\[
\Gamma_2(X_1, X_2) = X_1^{-1} X_2^{-1} \left( \frac{X_1 - X_2}{X_1 + X_2} \Theta(X_1, X_2) - \Theta(X_1, -X_2) \right),
\]  

(5.22)

where \( \Theta(X) \) and \( \Theta(X_1, X_2) \) are given by (2.40) and (2.41), respectively. Observe that \( \Gamma_1 \) is a polynomial of \( X \) of degree less than \( N \) if \( r \equiv N \mod 2 \), and \( \Gamma_2 \) is a skew-symmetric polynomial of \( X_1, X_2 \) of degree less than \( N \) in each variable.

Moreover, they satisfy the condition of minimality (2.39).
Proposition 5.3. (i) Suppose that $\mu = 2\nu + 1$. For $\bar{P} \in W_{N,l-\mu}$, we set
$$P = \Gamma_1 \wedge (\Lambda^{\nu+1} \Gamma_2) \wedge \bar{P}.$$ This is an $r$-restricted null cycle.

(ii) Suppose in addition that $\mu + 1 \leq l$. For $\bar{P} \in W_{N,l-\mu-1}$, we set
$$P = (\Lambda^{\nu+1} \Gamma_2) \wedge \bar{P}.$$ This is an $r$-restricted null cycle.

(iii) Suppose that $\mu = 2\nu$. For $\bar{P} \in W_{N,l-\mu}$, we set
$$P = (\Lambda^{\nu} \Gamma_2) \wedge \bar{P}.$$ This is an $r$-restricted null cycle.

Proof. We prove (i). Note that $r \equiv N \mod 2$. We have the following equalities.

$$\nabla_{\alpha, -r \pi i} \left( X^{-1} \Theta(X) \right) = \Gamma_1(X) \text{ if } r \equiv N \mod 2,$$

$$\nabla_{\alpha_1, \alpha_2, -r \pi i} \left( \frac{X^2-1}{X_1 + X_2} \Theta(X_1) \Theta(X_2) \right) = \frac{X^2-1}{X_1 + X_2} \Theta(X_1, X_2),$$

$$X_1^{-1} X_2^{-1} \Theta(X_1, -X_2) = \Gamma_1(X_1) X_2^{-1} \prod_{j=1}^{N} (z_j X_2 + 1) - X_2^{-1} \prod_{j=1}^{N} (z_j X_1 + 1) \Gamma_1(X_2).$$

If the convergence of the integrals is assured, we can show that $I(w, P) = 0$ for each of (i), (ii), (iii) by repeating a similar argument as in the proof of Proposition 2.8. The present case is different from that by the factor $X^{-1}$ (or $X_1^{-1} X_2^{-1}$). Therefore, the convergence of the integral when $\alpha \to \infty$ ($\alpha_1, \alpha_2 \to \infty$) only matters.

Consider the integral
$$I(w, P) = \int_C d\alpha \phi(\alpha) w(a) P(X).$$

Here $P = X^{-1} \Theta(X)$ and $w$ is of the form (2.10) with $l = 1$, where $Q$ is a polynomial of $a$ satisfying
$$\deg_a Q \leq \kappa.$$ The integral is convergent if
$$r - 3N + 2\kappa < 0. \quad (5.23)$$

The same estimate holds for the double integral $I(w, P)$ with
$$P = \frac{X_2^{-1} - X_1^{-1}}{X_1 + X_2} \Theta(X_1) \Theta(X_2).$$

We show that $I(w, P) = 0$ for the case (i). Note that (5.23) is equivalent to
$$\kappa < N + l - \nu - 1. \quad (5.24)$$

We take
$$Q(a_1, \ldots, a_l) = a_1^\lambda_1 \cdots a_l^\lambda_l.$$ Because of the skew-symmetry of $P$ one can assume that $\lambda_1 > \cdots > \lambda_l$. Because of Lemma 5.2, we can also assume that $\lambda_1 \leq N + l - 2$. It is enough to show that
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for any permutation of the variables $a_1, \ldots, a_l$ by $\sigma \in S_l$ we have $I(\sigma(Q), P_0) = 0$ where

$$P_0(X_1, \ldots, X_l) = \Gamma_1(X_1)\Gamma_2(X_2, X_3) \cdots \Gamma_2(X_{\mu-1}, X_{\mu})\tilde{P}(X_{\mu+1}, \ldots, X_l).$$

The integral is zero if we can apply the twisted difference method to one of the factors $\Gamma_1(X_1), \Gamma_2(X_2, X_3), \ldots, \Gamma_2(X_{\mu-1}, X_{\mu}).$ It is the case if (5.24) is valid for

$$\kappa = \min(\lambda_\sigma(1), \max(\lambda_\sigma(2), \lambda_\sigma(3)), \ldots, \max(\lambda_\sigma(\rho-1), \lambda_\sigma(\rho))).$$

If this is not the case, we have $\lambda_{\nu+1} \geq N + l - \nu - 1.$ This implies

$$\lambda_1 \geq \lambda_{\nu+1} + \nu \geq N + l - 1.$$ This is a contradiction.

The proofs for the cases (ii) and (iii) are similar. □

Now we define the space of $r$-restricted null cycles $D^{(r)}_{N,l}$ as follows. If $\mu = 2\nu$ we set

$$D^{(r)}_{N,l} = \Sigma_1 \wedge W_{N,l-1} + \Sigma_2 \wedge W_{N,l-2} + \left(\bigwedge^{\nu} \Gamma_2\right) \wedge W_{N,l-2\nu},$$

and if $\mu = 2\nu + 1$ then

$$D^{(r)}_{N,l} = \Sigma_1 \wedge W_{N,l-1} + \Sigma_2 \wedge W_{N,l-2} + \Gamma_1 \wedge \left(\bigwedge^{\nu} \Gamma_2\right) \wedge W_{N,l-2\nu-1} + \left(\bigwedge^{\nu+1} \Gamma_2\right) \wedge W_{N,l-2\nu-2}.$$

Set

$$M^{(r)}_{N,l} = W_{N,l}/D^{(r)}_{N,l}.$$ We identify $M^{(r)}_{N,l}$ with the space of $N$-particle minimal form factors of the RSG model at $\xi = r$ by the map $P \mapsto \tilde{P}P.$

Recall the definition (2.45) of the degree on $W_{N,l}.$ The generators $\Sigma_1, \Sigma_2, \Gamma_1$ and $\Gamma_2$ of $D^{(r)}_{N,l}$ are homogeneous elements. Hence $M^{(r)}_{N,l}$ is graded. Consider the character

$$\text{ch}_q M^{(r)}_{N,l} = \sum_d q^d \dim \langle M^{(r)}_{N,l} \rangle_d,$$

where $(M^{(r)}_{N,l})_d$ is the homogeneous component of degree $d.$ We will see in the next subsection (Theorem 5.4) that it is represented in terms of the restricted Kostka polynomial.

5.4. Restricted Kostka polynomial. The level-restricted Kostka polynomial $K^{(k)}_{m,\nu}(q)$ is given by the same type of fermionic formula as in (4.4):

$$K^{(k)}_{m,\nu}(q) = \sum_{n_1, \ldots, n_k} q^{(n_1, \ldots, n_k)} \prod_{j=1}^k \left[ \frac{P_j + n_j}{n_j} \right].$$ (5.25)
Here we set \( m_a = \{ j : \nu_j = a \} \),

\[ c(n_1, \ldots, n_k) = \sum_{j,j'}^k A_{jj'} n_j n_{j'} + \sum_{j=1}^k v_j n_j, \]

\[ A_{jj'} = \min(j, j'), \quad v_j = \max(0, j - k + m), \]

\[ P_j = \sum_{j'=1}^k A_{jj'} (m_{j'} - 2n_{j'}) - v_j, \]

and the sum ranges over \( n_j \in \mathbb{Z}_{\geq 0} \) such that \( 2 \sum_{j=1}^k j n_j = \sum_{j=1}^k j m_j - m \). We will need the simplest case where \( \nu = (1^N) = (1, \ldots, 1) \). Note that \( K_{m,(1^N)}(q) = 0 \) if \( m \not\equiv N \mod 2 \).

We are now in a position to state the main result of this paper.

**Theorem 5.4.** The space \( M^{(r)}_{N, l} \) is a free \( R_N \)-module with the character

\[ \text{ch}_q M^{(r)}_{N, l} = \frac{1}{(q)_N} K^{(r-2)}_{N-2l,(1^N)}(q). \]

Theorem 5.4 is reduced to the following two statements.

**Proposition 5.5.** There exist homogeneous elements \( Q_j \in M^{(r)}_{N, l} \) \((1 \leq j \leq d)\) such that

\[ M^{(r)}_{N, l} = \sum_{j=1}^d R_N Q_j \]

and

\[ \sum_{j=1}^d q^\deg Q_j \leq K^{(r-2)}_{N-2l,(1^N)}(1). \]

**Proposition 5.6.** Let \( e_c : W_{N, l} \to A_{N, l} \) \((c \in \mathbb{C}^N)\) be the evaluation map, and denote the map induced on the quotient space \( M^{(r)}_{N, l} \) by the same letter \( e_c \). Then for a generic \( c \) we have

\[ \dim_{\mathbb{C}} e_c(M^{(r)}_{N, l}) \geq K^{(r-2)}_{N-2l,(1^N)}(1). \]

Theorem 5.4 follows from these propositions by the same reasoning as given in Section 3.3. Below we prove Proposition 5.5 in Section 5.5 and Proposition 5.6 in Section 5.6.

### 5.5. Estimate from above

The proof of Proposition 5.5 is quite similar to that of Proposition 4.3. We make use of the isomorphism \( \tilde{\rho}_N : Z_N \to W_N \). Let \( \pi^{(r)} : W_{N, l} \to M^{(r)}_{N, l} \) be the canonical projection. The filtration (3.31) on \( Z_N \) induces a filtration \( \{ \pi^{(r)}(\tilde{\rho}_N(F_i)) \} \) on \( M^{(r)}_{N, l} \).

We denote by \( \mathcal{J}^{(r)}_{N, l} \) the ideal of \( Z = \Lambda[\xi] \otimes \mathbb{C}[\eta] \) generated by the elements (4.14)–(4.16) and

\[
\begin{cases}
\eta_2^\nu, & \text{if } \mu = 2\nu \text{ is even}, \\
\xi_1 \eta_2^\nu, \quad \eta_2^{\nu+1}, & \text{if } \mu = 2\nu + 1 \text{ is odd}.
\end{cases}
\]

\[ \mathcal{J}^{(r)}_{N, l} = Z_l / (Z_l \cap \mathcal{J}^{(r)}_{N, l}). \]
Recall the correspondence in Proposition 3.5. In particular, we have
\[ x_1^+ \mapsto -\frac{1}{2} \Gamma_1(X), \quad -\frac{1}{2} i (x_1^+)^{(2)} \mapsto \Gamma_2(X_1, X_2), \]  
(5.30)
where \( \Gamma_1(X) \) and \( \Gamma_2(X_1, X_2) \) are given by (5.21) and (5.22), respectively. From
this fact and the definition of \( \mathcal{Z}^{(r)}_{N,l} \), we have a surjection
\[ \mathcal{Z}^{(r)}_{N,l} \otimes R_N \rightarrow \text{gr} M^{(r)}_{N,l} \rightarrow 0. \]  
(5.31)
Therefore, Proposition 5.5 will follow if we show the estimate
\[ \text{ch}_q \mathcal{Z}^{(r)}_{N,l} \leq K^{(r-2)}_{N-2l(1N)}(q). \]  
(5.32)
To show (5.32) we consider the dual space of \( \mathcal{Z}^{(r)}_{N,l} \) determined by the pairing
\( (4.13) \). Then the dual space \( \mathcal{D}^{(r)}_{N,l} \) of \( \mathcal{Z}^{(r)}_{N,l} \) is described as follows:

**Lemma 5.7.** A set of polynomials \( f = (f_{st})_{s+t=1} \) belongs to \( \mathcal{D}^{(r)}_{N,l} \) if and only if
the conditions (i)–(iv) in Lemma 4.5 and the following condition are satisfied:

\( \text{(v)} \) If \( \mu = 2\nu \) is even, then
\[ g_{s,t}(x_1, \ldots, x_s; 0, \ldots, 0, y_{\nu+1}, \ldots, y_t) = 0. \]

If \( \mu = 2\nu + 1 \) is odd, then
\[ g_{s,t}(0, x_2, \ldots, x_s; 0, \ldots, 0, y_{\nu+1}, \ldots, y_t) = 0, \]
\[ g_{s,t}(x_1, \ldots, x_s; 0, \ldots, 0, y_{\nu+2}, \ldots, y_t) = 0. \]

The statement (5.32) is equivalent to

**Proposition 5.8.** We have
\[ \text{ch}_q \mathcal{D}^{(r)}_{N,l} \leq K^{(r-2)}_{N-2l(1N)}(q). \]

This result is a super-symmetric version of a result in [6].

We prove Proposition 5.8 in the same way as in the proof of Proposition 4.6. For
a partition \( \lambda \) of \( l \), we define the map \( \varphi_\lambda \) by (4.17). Introduce the filtration \( \Gamma_\lambda \) of
the vector space \( \mathcal{D}^{(r)}_{N,l} \) by (4.19) and consider the associated graded space \( \Gamma_\lambda / \Gamma_\lambda' \),
where \( \Gamma_\lambda' = \Gamma_\lambda \cap \text{Ker} \varphi_\lambda \). Then the map \( f \mapsto \varphi_\lambda(f) \) is an injective mapping defined
on the graded component \( \Gamma_\lambda / \Gamma_\lambda' \).

We use \( (x)_+ = \max(x, 0) \). Proposition 5.8 follows from the following proposition
and the fermionic formula (5.25) for the restricted Kostka polynomial.

**Proposition 5.9.** Let \( f \in \Gamma_\lambda / \Gamma_\lambda' \) and \( F = \varphi_\lambda(f) \). The polynomial \( F(v_1, \ldots, v_n) \)
is divisible by
\[ \prod_a \lambda_a^{x_a + (\lambda_a - \mu_x + 1)} \prod_a \prod_b (v_a^2 - v_b^2)^{\lambda_a}. \]  
(5.33)
Proof. From Proposition 4.7 it is enough to prove that $F$ has the factor $v^{\alpha+\lambda}$ where $\alpha = \lambda_\alpha$. Because of (ii) in Lemma 4.5, it is enough to show that the function

$$h(v) = \begin{cases} g_{s,t}(v, \ldots, v, \ldots) & \text{if } \alpha = 2\nu - 1; \\ g_{s,t}(\ldots, v, \ldots, v, \ldots) & \text{if } \alpha = 2\nu, \end{cases}$$

satisfies

$$\partial^j h(0) = 0 \quad \text{for } 0 \leq j \leq \alpha - \mu. \quad (5.34)$$

If $\alpha < \mu$, there is nothing to show, and if $\alpha = \mu$, the assertion follows from (v) in Lemma 5.7. We consider the case $\alpha > \mu$ in the following.

The following lemmas are straightforward.

Lemma 5.10. Let $g^{(0)}(x_1, x_2, x_3, x_4)$, $g^{(1)}(x_1, x_2; y_1)$ and $g^{(2)}(y_1, y_2)$ be polynomials which are skew-symmetric in the variables $x_i$ and satisfy the relations

$$g^{(0)}(v_1, -v_1, v_2, -v_2) = g^{(1)}(v_2, -v_2; v_1) - g^{(1)}(v_2, -v_2; -v_1),$$

$$g^{(1)}(v_2, -v_2; v_1) = g^{(2)}(v_1, v_2) - g^{(2)}(v_1, -v_2).$$

We have

$$\frac{\partial^2 g^{(2)}}{\partial v_1 \partial v_2}(0, 0) = 0.$$

Lemma 5.11. Let $g^{(0)}(x_1, x_2, x_3)$ and $g^{(1)}(x_1; y_1)$ be polynomials which are skew-symmetric in the variables $x_i$ and satisfy the relation

$$g^{(0)}(v_1, -v_1, v_2) = g^{(1)}(v_2; v_1) - g^{(1)}(v_2; -v_1).$$

We have

$$\frac{\partial g^{(1)}}{\partial v_1}(0; 0) = 0.$$

Lemma 5.12. Let $g^{(0)}(x_1, x_2)$ and $g^{(1)}(y_1)$ be polynomials satisfying the relations

$$g^{(0)}(v, -v) = g^{(1)}(v) - g^{(1)}(-v),$$

$$g^{(0)}(0, v) = g^{(0)}(v, 0) = 0.$$

We have

$$\frac{\partial g^{(1)}}{\partial v}(0) = 0.$$

Now let us prove (5.34). Recall that $\alpha > \mu$. We divide the proof in four cases.

Case $\alpha = 2\nu, \mu = 2\nu - 2\kappa$. We have

$$h(v) = g_{s,t}(\ldots, v, \ldots). \quad (5.35)$$

We use

$$g_{s,t}(\ldots, y_1, \ldots, y_{\kappa}, 0, \ldots, 0) = 0.$$
The derivative $\partial^j_i h(0)$ is a linear combination of

$$\partial^j g_{s,t}/\partial y^{\delta_1}_1 \cdots \partial y^{\delta_\nu}_\nu |_{y_1=\cdots=y_\nu=0},$$

(5.36)

where $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_\nu \geq 0$. By using $j \leq 2\kappa$, one can check the following.

(i) Unless there are two or more 1 in $\{\delta_1, \ldots, \delta_\nu\}$, we have $\delta_i = 0$ for $i > \kappa$, and therefore, (5.36) is zero;

(ii) If there are two or more 1 in $\{\delta_1, \ldots, \delta_\nu\}$, then by Lemma 5.10 we can deduce that (5.36) is zero.

Case $\alpha = 2\nu + 1, \mu = 2\nu + 1 - 2\kappa$. We have

$$h(v) = g_{s,t}(v, \ldots; v, \ldots, v).$$

(5.37)

We use

$$g_{s,t}(0, \ldots; y_1, \ldots, y_\kappa, 0, \ldots, 0) = 0,$$

(5.38)

$$g_{s,t}(x_1, \ldots; y_1, \ldots, y_{\kappa-1}, 0, \ldots, 0) = 0.$$  

(5.39)

The derivative $\partial^j_i h(0)$ is a linear combination of

$$\partial^j g_{s,t}/\partial x^{\gamma_1}_1 \partial y^{\delta_1}_1 \cdots \partial y^{\delta_\nu}_\nu |_{x_1=y_1=\cdots=y_\nu=0},$$

(5.40)

where $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_\nu \geq 0$. By using $j \leq 2\kappa$, one can check the following.

(i) If $\gamma_1 = 0$, unless there are two or more 1 in $\{\delta_1, \ldots, \delta_\nu\}$, we have $\delta_i = 0$ for $i > \kappa$, and, therefore, because of (5.38), the derivative (5.40) is zero;

(ii) If $\gamma_1 = 1$, unless there are one or more 1 in $\{\delta_1, \ldots, \delta_\nu\}$, we have $\delta_i = 0$ for $i > \kappa - 1$, and, therefore, because of (5.39), the derivative (5.40) is zero;

(iii) If $\gamma_1 \geq 2$, unless there are two or more 1 in $\{\delta_1, \ldots, \delta_\nu\}$, we have $\delta_i = 0$ for $i > \kappa - 1$, and, therefore, because of (5.39), the derivative (5.40) is zero;

(iv) If there are two or more 1 in $\{\delta_1, \ldots, \delta_\nu\}$, or if $\gamma_1 = 1$ and there are one or more 1 in $\{\delta_1, \ldots, \delta_\nu\}$, then by Lemmas 5.10 and 5.11 we can deduce that the derivative (5.40) is zero.

Case $\alpha = 2\nu, \mu = 2\nu - (2\kappa - 1)$. We have (5.35). We use (5.38) and (5.39). The derivative $\partial^j_i h(0)$ is a linear combination of (5.36). By using $j \leq 2\kappa - 1$, one can check the following.

(i) Unless there are one or more 1 in $\{\delta_1, \ldots, \delta_\nu\}$, we have $\delta_i = 0$ for $i > \kappa - 1$, and, therefore, because of (5.39), the derivative (5.36) is zero;

(ii) If there are one or more 1 in $\{\delta_1, \ldots, \delta_\nu\}$, then because of (5.38) we can apply Lemma 5.12 and deduce that the derivative (5.36) is zero.

Case $\alpha = 2\nu + 1, \mu = 2(\nu - \kappa)$. We have (5.37). We use

$$g_{s,t}(x_1, \ldots; y_1, \ldots, y_{\kappa}, 0, \ldots, 0) = 0.$$  

(5.41)

The derivative $\partial^j_i h(0)$ is a linear combination of (5.40). By using $j \leq 2\kappa + 1$, one can check the following.
(i) If \( \gamma_1 = 0 \), unless there are one or more 1 in \( \{ \delta_1, \ldots, \delta_\nu \} \), we have \( \delta_i = 0 \) for \( i > \kappa \), and, therefore, because of (5.41), the derivative (5.40) is zero;
(ii) If \( \gamma_1 \geq 1 \), unless there are two or more 1 in \( \{ \delta_1, \ldots, \delta_\nu \} \), we have \( \delta_i = 0 \) for \( i > \kappa \), and, therefore, because of (5.41), the derivative (5.40) is zero;
(iii) If there are two or more 1 in \( \{ \delta_1, \ldots, \delta_\nu \} \), or if \( \gamma_1 = 0 \) and there are one or more 1 in \( \{ \delta_1, \ldots, \delta_\nu \} \), then by Lemmas 5.10 and 5.11, we can deduce that the derivative (5.40) is zero. \( \square \)

5.6. Estimate from below. For the proof of Proposition 5.6, we use the results of Kashiwara [9] on the existence of global basis for level zero representations. In this subsection we fix non-negative integers \( m, k \) with \( 0 \leq m \leq k \).

First we note a simple fact. Take \( a = (a_1, \ldots, a_N) \in (C^\times)^N \). Set \( A = C[q, q^{-1}], \varepsilon = \sqrt{-1} \), and

\[
\begin{align*}
(V_K)_a &= (V_K)_{a_1} \otimes \cdots \otimes (V_K)_{a_N}, \\
(V_A)_a &= (V_A)_{a_1} \otimes \cdots \otimes (V_A)_{a_N}, \\
(V)_{a} &= (V)_{a_1} \otimes \cdots \otimes (V)_{a_N}.
\end{align*}
\]

Lemma 5.13. Let \( Y_{k,m} \) denote the right ideal of \( U_A \) generated by \( f_1 \), \( (e_0^{(2)})^\nu \) if \( k - m + 1 = 2\nu \), \( f_1, e_0 (e_0^{(2)})^\nu, (e_0^{(2)})^{\nu+1} \) if \( k - m + 1 = 2\nu + 1 \). Then we have

\[
\dim_C((V)_{a_1}/(Y_{k,m}(V))_{a_1})_m \geq \dim_K((V_K)_a/(f_1(V_K)_a + e_0^{k-m+1}(V_K)_a)_m.
\]

Proof. This follows from the specialization argument. In fact,

\[
\dim_C((Y_{k,m}(V))_{a_1})_m \leq \text{rank}_A(Y_{k,m}(V)_a)_m = \dim_K(Y_{k,m}(V_K)_a)_m.
\]

Since \( Y_{k,m}(V_K)_a = f_1(V_K)_a + e_0^{k-m+1}(V_K)_a \), the assertion follows. \( \square \)

By Lemma 5.13, the proof of Proposition 5.6 is reduced to the following statement.

Proposition 5.14.

\[
\dim_K((V_K)_a/(f_1(V_K)_a + e_0^{k-m+1}(V_K)_a))_m = K_{m,1}^{(k)}(1).
\]  

Proof. Let \( A \) be the subring of \( K = C(q) \) consisting of rational functions which are regular at \( q = 0 \). The two-dimensional module \( V_K \) is a level 0 fundamental representation of \( U \) in the sense of [9]. It has the crystal base \( (L, B) \) with \( L = Av_+ \oplus Av_- = B = \{ v_+, v_- \} \subset L/qL \). Let \( (L^{\text{aff}}, B^{\text{aff}}) \) be the affinization, and set

\[
L = L^{\otimes N}, \quad B = B^{\otimes N}, \quad L^{\text{aff}} = (L^{\text{aff}})^{\otimes N}, \quad B^{\text{aff}} = (B^{\text{aff}})^{\otimes N}.
\]

Let further \( e^{\text{norm}} \) be the involution introduced in [9, (8.9) and a few lines above]. In the below, we use the standard notation in crystal theory such as \( \bar{e}_i, \bar{f}_i, \bar{z}_i(b) = \max\{ n : \bar{e}_i^n(b) \neq 0 \} \) and \( \varphi_i(b) = \max\{ n : \bar{f}_i^n(b) \neq 0 \} \), \( i = 0, 1 \). The following is a special case of the general results proved in [9, Theorem 8.5 and Theorem 6.2].

(i) For any \( b \in L^{\text{aff}}/qL^{\text{aff}} \), there is a unique element \( G(b) \in L^{\text{aff}} \cap (V_A)^{\text{aff}} \) such that \( e^{\text{norm}} G(b) = G(b) \) and \( G(b) \equiv b \mod qL^{\text{aff}} \). The set \( \{ G(b) \}_{b \in B^{\text{aff}}} \) is a \( K \)-basis of \( V^{\text{aff}} \).
(ii) For any \( i = 0, 1 \) and \( n \geq 0 \), we have

\[
\epsilon^\alpha_i^{\text{aff}} = \bigoplus_{b \in B^\text{aff}} K \psi(b), \quad \epsilon_i^{\alpha^{\text{aff}}} = \bigoplus_{b \in B^\text{aff}, \varphi_0(b) \geq n} KG(b).
\]

From the uniqueness of \( G \), we have

\[
G(z_\nu b) = z_\nu G(b) \quad (b \in B^\text{aff}).
\]

Let \( a_1, \ldots, a_N \in K^\times \) be such that \( a_\nu/a_{\nu+1} \in A \) for \( 1 \leq \nu \leq N - 1 \). Let \( \psi : V^\text{aff} \to V_a \) be the canonical surjection. By the argument \([9]\) (in the proof of Theorem 9.1), \( \{\psi(G(b))\}_{b \in B} \) is a \( K \)-basis of \( V_a \). Since \( \psi \) is \( U \)-linear and \( \psi(G(z_\nu b)) = a_\nu \psi(G(b)) \), we have for any \( i = 0, 1 \) and \( n \geq 0 \)

\[
f^\alpha_i V_a = \bigoplus_{b \in B, \varphi_1(b) \geq n} K \psi(G(b)), \quad \epsilon_i^{\alpha} V_a = \bigoplus_{b \in B, \varphi_0(b) \geq n} K \psi(G(b)).
\]

Consider the subspace

\[
f^\alpha_i V_a + \epsilon_i^{-m+1} V_a.
\]

From the reasoning above, it has the set \( \{\psi(G(b)) : b \in B, \varphi_1(b) \geq 1 \text{ or } \varphi_0(b) \geq k - m + 1\} \) as a basis. Therefore the left hand side of \((5.42)\) is equal to

\[
\#\{b \in B : \varphi_1 b = 0, \varphi_0 b = 0, \text{wt } b = m\}.
\]

It is well known (see e.g. \([7]\)) that the last line is given by the restricted Kostka number \( K_{m, (1^N)}^{(k)}(1) \). This proves Proposition 5.14. \( \square \)

6. Virasoro Characters

So far we have dealt with the space of minimal deformed cycles for each fixed \( N \), assuming that they are polynomials in \( z_1, \ldots, z_N \). In this section we consider the total space

\[
M^{(r)}_m = \bigoplus_{N \geq 0, \, 2l = m} M^{(r)}_{N, l} \otimes C[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]^{\Theta_N}.
\]

This space is \( \mathbb{Z} \)-graded with the assignment \( \deg X_i = -1, \deg z_j = 1 \). However its character does not make sense because each graded component is infinite dimensional. Following \([10]\), we consider instead a family of subspaces for which the character is well-defined. Recall that the axioms (A2), (A3) contain the parameter \( \varepsilon = 0, 1 \). For each \( L = 0, 1, 2, \ldots \) with \( L \equiv \varepsilon \mod 2 \), define

\[
M_m^{(r)}[L] = \left\{ P \in M_m^{(r)} : \left( \prod_{j=1}^N z_j \right)^{\frac{k-1}{r}} P \text{ is a polynomial} \right\}, \quad (6.1)
\]

so that \( M_m^{(r)} = \bigcup_{L \equiv \varepsilon \mod 2} M_m^{(r)}[L] \). We use the grading defined for \( P \in M_m^{(r)} \otimes C[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]^{\Theta_N} \) as

\[
\deg' P = \deg P + \frac{N^2}{4} - \frac{\varepsilon N}{2} + \frac{m(m+2)}{4r}, \quad (m = N - 2l).
\]
This is nothing but the degree of the $N$-particle minimal form factor $\mathcal{F}_P$ of the RSG model, that is
$$\mathcal{F}_P(\beta_1 + \Lambda, \ldots, \beta_N + \Lambda) = e^{\Lambda \text{deg} P} \mathcal{F}_P(\beta_1, \ldots, \beta_N).$$

The corresponding character is
$$\text{ch}_q M^{(r)}_m[L] = \sum_{N \geq 0} \frac{q^{n^2 - \frac{(m+2)(m+1)}{2} - \frac{L}{2}}} {(q)_N^N} K^{(r-2)}_{m, (1^N)}(q).$$

We show below (Theorem 6.1) that it is expressible in terms of characters of the Virasoro algebra.

Consider the irreducible characters of the Virasoro minimal unitary series
$$\chi^{(r+1)}_{b,a}(q) = q^{-\frac{h_{b,a}}{2}} \chi^{(r+1)}_{b,a}(q).$$

Here $1 \leq b \leq r - 1$, $1 \leq a \leq r$,
$$c = 1 - \frac{6}{r(r+1)}, \quad h_{b,a} = \frac{((r+1)b - ra)^2 - 1}{4r(r+1)},$$

and $\chi^{(r+1)}_{b,a}(q) = 1 + O(q)$. Explicitly we have
$$\chi^{(r+1)}_{b,a}(q) = \frac{1}{(q)_\infty} \left( \sum_{n \in \mathbb{Z}} q^{r(r+1)n^2 + ((r+1)b-ra)n} - \sum_{n \in \mathbb{Z}} q^{r(r+1)n^2 + ((r+1)b+ra)n+6b} \right),$$

where $(q)_\infty = \prod_{j=1}^\infty (1 - q^j)$. We shall use also their finitization due to [1]. For $L \in \mathbb{Z}_{\geq 0}$ such that $L \equiv b - a \mod 2$, define a polynomial
$$\chi^{(r+1)}_{b,a}(q; L) = \sum_{n \in \mathbb{Z}} q^{r(r+1)n^2 + ((r+1)b-ra)n} \left[ \frac{L-b+a}{2} - (r+1)n \right]$$
$$- \sum_{n \in \mathbb{Z}} q^{r(r+1)n^2 + ((r+1)b+ra)n+6b} \left[ \frac{L-b+a}{2} - (r+1)n \right].$$

We set $\chi^{(r+1)}_{b,a}(q; L) = 0$ if $L \not\equiv b - a \mod 2$. In the notation of [1] we have
$$\chi^{(r+1)}_{b,a}(q; L) = q^{-(a-b)(a-b-1)/4} X_L(a, b, b+1), r = r_{ABF} - 1,$$

where $r_{ABF}$ signifies the parameter $r$ used in [1]. With the definition
$$\chi^{(r+1)}_{b,a}(q; L) = q^{-\frac{h_{b,a}}{2}} \chi^{(r+1)}_{b,a}(q; L),$$

it is obvious that
$$\lim_{L \to a+b \mod 2} \chi^{(r+1)}_{b,a}(q; L) = \chi^{(r+1)}_{b,a}(q; L).$$

**Theorem 6.1.** Let $0 \leq m \leq r - 2$, $L \in \mathbb{Z}_{\geq 0}$. Then the following identity holds:
$$\text{ch}_q M^{(r)}_m[L] = \sum_{a \equiv L - 1 \mod 2} \chi^{(r+1)}_{m+1,a}(q) \chi^{(r+1)}_{1,a}^{-1}(q^{-1}; L).$$
Note that
\[ \chi_{1,a}^{(r,r+1)}(q^{-1}; L) = q^{\frac{N}{2} - b_{1,a} - \frac{(r)}{2} + \frac{(L-r)}{2}} \times K_{a-1,(1^L)}^{(r-1)}(q). \]

For the proof we need

**Lemma 6.2.** For any \( N \in \mathbb{Z}_{\geq 0} \) we have
\[ \sum_{p \geq 0} \frac{(zq^{p+1}; q)^\infty}{(q)_p} z^p q^{p(p-N)} = \sum_{s \geq 0} \left[ \frac{N}{s} \right] q^{-s} \quad (6.5) \]
where \((z; q)_n = \prod_{j=1}^n (1 - q^{-j+1}z)\) and \( \left[ \frac{N}{s} \right] \) signifies \( \frac{N}{s} \) with \( q \) replaced by \( q^{-1} \).

**Proof.** Let \( L_N(z) \) (resp. \( R_N(z) \)) stand for the left (resp. right) hand side. It is straightforward to check that \( L_N(z) = zL_{N-1}(z) + L_{N-1}(q^{-1}z) \) and the same relation for \( R_N(z) \). Clearly \( R_0(z) = 1 \). That \( L_0(z) = 1 \) follows from the identity \( \sum_{n \geq 0} q^{n(n-1)} z^n / ((q)_n(z)_n) = 1 / (z)_\infty \).

**Proof of Theorem 6.1.** We prove the assertion in the equivalent form
\[ \sum_{N \geq 0} \frac{q^{N^2-m^2}}{q^N} L_N K_{m,(1^L)}^{(r-2)}(q) \]
\[ = \sum_{1 \leq a \leq r \atop a \equiv L-1 \mod 2} \chi_{m+1,a}^{(r,r+1)}(q) q^{(a-1)m} \chi_{1,a}^{(r+1)}(q^{-1}; L). \quad (6.6) \]
Let us start from the right hand side. Substituting (6.3) and using the relations
\[ \chi_{b,-a}^{(r,r+1)}(q) = -q^{ba} \chi_{b,a}^{(r,r+1)}(q), \]
\[ \chi_{b,a-2(r+1)n}^{(r,r+1)}(q) = q^{-r(r+1)n^2 - ((r+1)b-ra)n} \chi_{b,a}^{(r,r+1)}(q), \]
we obtain
\[ \sum_{a \in \mathbb{Z} \atop a \equiv L-1 \mod 2} \chi_{m+1,a}^{(r,r+1)}(q) q^{(a-1)m} \left[ \frac{L}{L+q-1} \right]_{q^{-1}}. \]

Applying (6.5) we transform this expression into
\[ \sum_{n \in \mathbb{Z}} \left( q^{rn^2 + (m+1)n + (p-rn - \frac{m}{2})^2 - \frac{m^2}{4}} - \frac{1}{(q)_p - 2rn - m} \right) \]
\[ - q^{rn^2 + (m+1)n + m + 1 + (p + rn + \frac{m^2}{4})^2 - \frac{(m+2)^2}{4}} \frac{1}{(q)_p + 2rn + m + 2} \left( q^{p+2rn+m+2}(q)_p \right). \]
Here we set \( 1/(q)_a = 0 \) for a negative integer \( a \). Changing the variable \( p \) to \( N \) where \( p = \frac{N+m}{2} + rn \) for the first term and \( p = \frac{N-m-2}{2} - rn \) for the second, we obtain
\[ \sum_{N \geq 0} \frac{q^{N^2-m^2}}{q^N} K_{m+2rn,(1^N)}(q), \quad (6.7) \]
where \( K_{m+2r, (1^\infty)}(q) \) is the (non-restricted) Kostka polynomial given by (4.7). We now make use of the following alternating sum formula for the restricted Kostka polynomial \([18, (6.8)]\)

\[
K_{m,\nu}^{(r-2)}(q) = \sum_{i \geq 0} q^{i^2 + (m+1)i} K_{2ri+m,\nu}(q) = \sum_{i > 0} q^{-i^2 -(m+1)i} K_{2r_1-m-2,\nu}(q) \tag{6.8}
\]

and the property

\[
K_{m,(1^\infty)}(q) = -K_{-m-2,(1^\infty)}(q).
\]

Then (6.7) becomes the left hand side of (6.6) and the proof is over. \(\square\)

For small \(L\), the right hand side of (6.4) is given by

\[
L = 0: \quad q^\hat{a} \chi_{m+1,1}^{(r,r+1)}(q),
L = 1: \quad q^\hat{a} - h_{12} \chi_{m+1,2}^{(r,r+1)}(q),
L = 2: \quad q^\hat{a} \chi_{m+1,1}^{(r,r+1)}(q) + q^\hat{a} - h_{13} \chi_{m+1,3}^{(r,r+1)}(q),
L = 3: \quad q^\hat{a} - h_{12} (1 + q^{-2}) \chi_{m+1,2}^{(r,r+1)}(q) + q^\hat{a} - h_{14} \chi_{m+1,4}^{(r,r+1)}(q),
\]

and so forth. As long as \(0 \leq L \leq r - 1\), each time \(L\) is increased, a new term \(\chi_{m+1,L+1}^{(r,r+1)}(q)\) appears. This agrees with the formula for the two-particle form factors of the exponential operator [11].

If the argument \(q^{-1}\) in the second factor in (6.4) were an independent variable \(\tilde{q}\), then in the limit \(L \to \infty\) we would obtain

\[
\sum_{\substack{1 \leq a \leq n \leq N \\ a \neq \varepsilon \mod 2}} \chi_{m+1,a}^{(r,r+1)}(q) \chi_{1,a}^{(r,r+1)}(1/q) \quad (\varepsilon = 0, 1),
\]

a structure reminiscent of a modular invariant partition function of CFT. In the massive theory the two chiralities are not separated. This point was first observed in [10] in a few simple cases including the non-unitary models \((p, p') = (2, 5), (3, 5)\).

**Appendix A. Quantum Affine Algebra**

We summarize here our convention concerning the quantum loop algebra \(U_q(\hat{\mathfrak{sl}}_2)\). Let \(K = C(q)\) be the field of rational functions in indeterminate \(q\). The quantum loop algebra \(U = U_q(\hat{\mathfrak{sl}}_2)\) is a Hopf algebra over \(K\) generated by \(e_i, f_i, t_i \ (i = 0, 1)\) under the following defining relations.

\[
t_0 t_1 = t_1 t_0 = 1,
\]

\[
t_i e_j t_i^{-1} = q^{a_{i j}} e_j, \quad t_i f_j t_i^{-1} = q^{-a_{i j}} f_j,
\]

\[
[e_i, f_j] = \delta_{ij} t_i - t_i^{-1},
\]

\[
e_i^{(3)} e_j - e_i^{(2)} e_j e_i + e_i e_j e_i^{(2)} - e_j e_i^{(3)} = 0 \quad (i \neq j),
\]

\[
f_i^{(3)} f_j - f_i^{(2)} f_j f_i + f_i f_j f_i^{(2)} - f_j f_i^{(3)} = 0 \quad (i \neq j).
\]
Here $a_{ii} = 2$, $a_{ij} = -2$ ($i \neq j$), and we have set

$$x^{(n)} = \frac{x^n}{[n]!}, \quad [n]! = \prod_{j=1}^{n} [j], \quad [j] = \frac{q^j - q^{-j}}{q - q^{-1}}.$$ 

We choose the coproduct

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i, \quad \Delta(t_i) = t_i \otimes t_i.$$  \hspace{1cm} (A.1)

The algebra $U$ has an alternative presentation in terms of the Drinfeld generators $x^\pm_k$ $(k \in \mathbb{Z})$, $a_n$ $(n \in \mathbb{Z} \setminus \{0\})$ and $t^{\pm 1}_1$. They satisfy the relations

$$t_1 x^\pm_k t_1^{-1} = q^{\pm 2} x^\pm_k,$$  

$$[a_n, x^\pm_k] = \pm \frac{2n}{n} x^\pm_{k+n},$$  

$$x^\pm_{k+1} x^\pm_1 - q^{\pm 2} x^\pm_k x^\pm_{k+1} = q^{\pm 2} x^\pm_k x^\pm_{k+1} - x^\pm_1 x^\pm_k,$$  

$$[x^+_k, x^-_l] = \frac{\varphi^+_k - \varphi^+_{k+l}}{q - q^{-1}}.$$  

where

$$\sum_{k \geq 0} \varphi^\pm_k z^k = t^\pm_1 \exp \left( \pm (q - q^{-1}) \sum_{n=1}^{\infty} a_{\pm n} z^n \right)$$

and $\varphi^\pm_k = 0$ for $k < 0$. The two sets of generators are related by

$$x^+_0 = e_1, \quad x^-_1 = e_0 t_1, \quad x^-_0 = f_1, \quad x^+_1 = t_1^{-1} f_0.$$

We shall deal also with the algebra with $q$ specialized to a complex number. Let $U_A$ be the subalgebra of $U$ generated by the elements $e^{(s)}_i, f^{(s)}_i$ $(i = 0, 1, s \in \mathbb{Z}_{\geq 0})$ and $t^{\pm 1}_1$ over $A = \mathbb{C}[q, q^{-1}]$. $U_A$ is also generated by the elements $(x^\pm_k)^{(s)}$ $(k \in \mathbb{Z}, s \in \mathbb{Z}_{\geq 0})$ and $t^{\pm 1}_1$. For a non-zero complex number $\epsilon \in \mathbb{C}$, consider the ring homomorphism $A \to \mathbb{C}$ which sends $q$ to $\epsilon$. We define $U_{\epsilon} = U_A \otimes A \mathbb{C}$. Specialization of modules is defined in a similar manner. Let $W$ be a $U$-module equipped with a free $A$-submodule $W_A$ satisfying $U_A W_A \subset W_A$ and $W = W_A \otimes_A K$. We say that $W$ is defined over $A$, and that the $U_{\epsilon}$-module $W_A \otimes A \mathbb{C}$ is the specialization of $W$ to $\epsilon$.

In this paper we shall mainly consider the modules

$$V_K = K v_+ \oplus K v_-, \quad V^\text{aff}_K = V_K \otimes K[z^{\pm 1}],$$

$$V_A = Av_+ \oplus Av_-, \quad V^\text{aff}_A = V_A \otimes A[z^{\pm 1}],$$

$$V = Cv_+ \oplus Cv_-, \quad V^\text{aff} = V \otimes \mathbb{C}[z^{\pm 1}].$$

The space $V^\text{aff}_K$ is a module over $U \otimes K[z^{\pm 1}]$ defined by the assignment

$$e_0 \mapsto z \sigma^-, \quad e_1 \mapsto \sigma^+, \quad f_0 \mapsto z^{-1} \sigma^+, \quad f_1 \mapsto \sigma^-, \quad t_1 \mapsto \tau,$$

where $\sigma^a$ $(a = \pm, z), \tau$ denote linear operators given in the above basis by

$$\sigma^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \tau = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}.$$
We identify the $N$-fold tensor product $(V^\text{aff}_K)^\otimes N$ with $V^\otimes N_K \otimes K[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$. Similarly $(V^\text{aff}_K)^\otimes N$ (resp. $(V^\text{aff}_K)^\otimes N$) is a module over $U_A \otimes A[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$ (resp. $U_A \otimes \mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$). For a $U$-module $W$, we write $W[m] = \{v \in W \mid tv = q^m v\}$ and call it the subspace of weight $m$. This gives rise to a grading on $(V^\text{aff}_K)^\otimes N$ and $(V^\text{aff}_K)^\otimes N$ for any $\epsilon \in \mathbb{C}^\times$. Fixing $N$, we often write $(V^\text{aff}_K)^\otimes N[N - 2l]$ as $(\langle V^\text{aff}_K \rangle)^{N}_{N - 2l}$.

Appendix B. Fermionic Realization

In this appendix we give the details about Proposition 3.3.

Introduce the Jordan–Wigner fermions

$$\psi_a^+ = \sigma_a^{+1}(-i\sigma_{a+1}^z) \cdots (-i\sigma_N^z),$$

$$\psi_a^\dagger = \sigma_a^{1+}(-i\sigma_{a+1}^z) \cdots (i\sigma_N^z).$$

Here the subscript $a = 1, \ldots, N$ of $\sigma_a^\dagger$ indicates that it acts on the $a$-th tensor component of $V^\otimes N$. We have $[\psi_a^+, \psi_b^\dagger] = [\psi_a^+, \psi_b^\dagger] = 0$, $[\psi_a^+, \psi_b^\dagger] = \delta_{ab} (1 \leq a, b \leq N)$ and $\sigma_a^{1+} = -2\psi_a^+ \psi_a^\dagger + 1$. We make an identification $\Lambda_N \cong V^\otimes N$.

Proof of Proposition 3.3. Set $\epsilon = -i(q - i) + (q - i)^2/2$, so that $q \equiv 1 + \epsilon + \epsilon^2/2) \text{ mod } (q - i)^2$. Using the same letter $\varpi_N$ to denote the representation $U_A \rightarrow \text{End}_A((V^\text{aff}_K)^\otimes N)$, we set

$$\varpi_N(iq^{-1}a_1) \equiv \alpha_0 + \epsilon \alpha_1 + \epsilon^2 \alpha_2 \text{ mod } \epsilon^3,$$

$$\varpi_N(x^a_0) \equiv \beta_0 + \epsilon \beta_1 \text{ mod } \epsilon^2,$$

$$\varpi_N(\chi(z)) \equiv \gamma_0(z) + \epsilon \gamma_1(z) \text{ mod } \epsilon^2.$$

Explicitly we have

$$\alpha_0 = \sum_{a=1}^{N} z_a, \quad \alpha_1 = -\sum_{a=1}^{N} z_a \sigma_a^z + 4 \sum_{a<b} z_a \psi_a^\dagger \psi_b^\dagger,$$

$$\alpha_2 = \frac{1}{2} \alpha_0 = 4 \sum_{a<b} z_a \psi_a^{\dagger} \sigma_b^z \psi_b^\dagger,$$

$$\beta_0 = \sum_{a=1}^{N} (-1)^{N-a} \psi_a, \quad \beta_1 = \sum_{a<b} (-1)^{N-a-1} \psi_a \sigma_b^z.$$

Substituting these into the relation

$$\chi^{\dagger}_{k+1} = \frac{1}{[2]^2} [a_1, \chi^{\dagger}_k],$$

we obtain

$$2\gamma_0(z) = z[a_1, \beta_0 + \gamma_0(z)], \quad (B.1)$$

$$2\gamma_1(z) = z([a_1, \beta_1 + \gamma_1(z)] + [a_2, \beta_0 + \gamma_0(z))]. \quad (B.2)$$
The formal series $\gamma_0(z), \gamma_1(z)$ are uniquely determined by these relations. We seek them in the form

$$
\gamma_0(z) = \sum_{a=1}^{N} A_a(z) \psi_a,
$$

$$
\gamma_1(z) = \sum_{a<b} B_{ab}(z) \psi_a \sigma_b^z + \sum_{a<b<c} C_{abc}(z) \psi_a \psi_b \psi_c^z.
$$

Then (B.1), (B.2) are rewritten as the relations for the coefficients:

$$
(1 - z_a z) A_a(z) = z_a \left( 1 + 2 \sum_{b=a+1}^{N} A_b(z) \right),
$$

$$
(1 - z_a z) B_{ab}(z) = 2z_a z \sum_{a<c<b} B_{abc}(z) + \frac{z_a (1 - z_b z)}{z_b} A_b(z) \quad (a < b),
$$

$$
(1 - z_a z - z_b z + z_c z) C_{abc}(z) = 4z_a z (A_b(z) + B_{bc}(z)) + 4z_b z (B_{ab}(z) - B_{ac}(z))
$$

$$
+ 2z_a z \sum_{a<c<b} C_{abc}(z) - 2z_a z \sum_{b<p<c} C_{apc}(z) - 2z_b z C_{abc}(z) \quad (a < b < c).
$$

A direct computation shows that the following is the solution.

$$
A_a(z) = \frac{z_a z}{1 - z_a z} \prod_{j=a+1}^{N} \frac{1 + z_j z}{1 - z_j z},
$$

$$
B_{ab}(z) = \frac{z_a z}{1 - z_a z} \prod_{a<j<b \neq a}^{N} \frac{1 + z_j z}{1 - z_j z} \quad (a < b),
$$

$$
C_{abc}(z) = 8 \frac{z_a z}{1 - z_a z} \left( \prod_{a<j<b} \frac{1 + z_j z}{1 - z_j z} \right) \frac{z_b z}{1 - z_b z} \frac{1}{1 - z_c z} \prod_{c<j\leq N} \frac{1 + z_j z}{1 - z_j z} \quad (a < b < c).
$$

Calculating

$$\varpi_N((x_0^-)^{(2)}) \equiv -\frac{i}{2} [\beta_0, \beta_1]_+ \bmod \varepsilon,$$

$$\varpi_N((X^-)^{(2)}) \equiv -\frac{i}{2} [\gamma_0(z), \gamma_1(z)]_+ \bmod \varepsilon,$$

we arrive at the desired formulas. □

**Remark.** In a similar manner, one can show that the elements $a_n$ act as multiplication by a scalar,

$$
a_n = \begin{cases} 
\frac{j^{n-1}}{n} \sum_{j=1}^{N} z_j^n, & \text{if } n \text{ is odd} \\
0, & \text{if } n \text{ is even}
\end{cases}
$$

The action of $a_n$ with odd $n$ coincide with that of integrals of motion of the SG model.
Appendix C. Trigonometric Hypergeometric Space

In this appendix we give an account of the connection between the space of polynomials \( W_N = \bigoplus_{l=0}^N W_{N,l} \) and the tensor product \( V_K^{\otimes N} \otimes K[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \) specialized to \( q = \sqrt{-1} \). Our exposition is based on [24] with a slight modification.

Let us recall a well-known construction in the algebraic Bethe Ansatz. Consider the \( R \) matrix

\[
R(z) = \begin{pmatrix}
1 & (1-z)q & 1-q^2 \\
1-q^2z & 1 & 1-q^2z \\
(1-q^2z) & (1-z)q & 1-q^2z \\
1-q^2z & 1 & 1-q^2z \\
1 & 
\end{pmatrix}.
\]

Let \( R_{i,j}(z) \) stand for the matrix \( R(z) \) acting on the \((i, j)\)-th tensor component of \( V_K^{\otimes (N+1)} \) \((0 \leq i, j \leq N)\). Define operators \( A(z), B(z), C(z), D(z) \) by

\[
\begin{pmatrix}
A(z) \\
B(z) \\
C(z) \\
D(z)
\end{pmatrix} = \frac{\Theta(q^{-2}z^{-1})}{1-q^{-2}} R_0 N(z/zN) \cdots R_0 1(z/z1),
\]

where \( \Theta(X) = \prod_{j=1}^N (1-z_j X) \) is given by (2.40). Then \( C(z) \) is a polynomial in \( z^{-1} \) of degree \( N-1 \). It is also a polynomial in \( z_1, \ldots, z_N \) and a Laurent polynomial in \( q \). We take the basis \( \{ v^*_+, v^- \} \subset V_K^* \) dual to \( \{ v_+, v_- \} \), and set for \( v \in (V_K^\otimes N)_1 \)

\[
\mathcal{C}_N(v) = \langle v^*_+ \otimes \cdots \otimes v^*_+ , C(X_1^{-1}) \cdots C(X_l^{-1}) v \rangle. \tag{C.1}
\]

Since \( C(z)C(w) = C(w)C(z) \), the right hand side is symmetric in \( X_1, \ldots, X_l \). For the vectors (2.19) we have

\[
\mathcal{C}'_N(v_M) = \text{Sym} \left( G'_{m_1}(X_1) \cdots G'_{m_l}(X_l) \prod_{j < j'} \frac{q^{-1}X_j - qX_{j'}}{X_j - X_{j'}} \right), \tag{C.2}
\]

where

\[
G'_m(X) = q^{-m-N} \prod_{j=1}^{m-1} (1-q^{-2}z_j X) \prod_{j=m+1}^N (1-z_j X).
\]

Let \( F_{N,l} \) denote the space of symmetric polynomials \( P \) in \( X_1, \ldots, X_l \) with coefficients in \( K[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \), which have degree at most \( N-1 \) in each \( X_i \) and satisfies the condition

\[
P(z_k^{-1}, q^2z_k^{-1}, X_3, \ldots, X_l) = 0 \quad \text{for} \quad k = 1, \ldots, N.
\]

From (C.2), it is easy to see that \( \mathcal{C}'_N(v_M) \in F_{N,l} \). Set \( F_N = \bigoplus_{l=0}^N F_{N,l} \). The space \( F_N \) is a version of the trigonometric hypergeometric space introduced in [24].

Extending the definition (C.1) by linearity, we obtain a map

\[
\mathcal{C}_N' : V_K^{\otimes N} \otimes K[z_1^{\pm 1}, \ldots, z_N^{\pm 1}] \rightarrow F_N. \tag{C.3}
\]
Proposition C.1 (cf. [24], [23]). The space $F_N$ is endowed with the structure of a $U$-module such that (C.3) is an intertwiner. The action of $U$ on $P \in F_{N,l}$ is given as follows:

$$(e_1 P)(X_1, \ldots, X_{l-1}) = q^{N-l} P(X_1, \ldots, X_{l-1}, 0),$$

$$(f_0 P)(X_1, \ldots, X_{l-1}) = (-1)^{N-1} q^{2N-1} \prod_{j=1}^{N} z_j^{-1}(z^{N-1} P(X_1, \ldots, X_{l-1}, z^{-1}))|_{z=0},$$

$$(f_1 P)(X_1, \ldots, X_{l+1}) = \frac{q^{-N+l}}{1-q^2} \sum_{\nu}^l P(X_1, \ldots, X_{l+1})$$

$$\times \left( q^N \Theta(q^{-2} X_{\nu}) \prod_{1 \leq k \leq l+1, k \neq \nu}^l \frac{q^2 X_k - X_{\nu}}{X_k - X_{\nu}} - q^{-N} \Theta(X_{\nu}) \prod_{1 \leq k \leq l+1, k \neq \nu}^l \frac{q^2 X_k - X_{\nu}}{X_k - X_{\nu}} \right),$$

$$(e_0 P)(X_1, \ldots, X_{l+1}) = \frac{q^{-N+l}}{1-q^2} \sum_{\nu}^l P(X_1, \ldots, X_{l+1})$$

$$\times X_{\nu}^{-1} \left( \Theta(q^{-2} X_{\nu}) \prod_{1 \leq k \leq l+1, k \neq \nu}^l \frac{X_k - q^2 X_{\nu}}{X_k - X_{\nu}} - \Theta(X_{\nu}) \prod_{1 \leq k \leq l+1, k \neq \nu}^l \frac{X_k - q^2 X_{\nu}}{X_k - X_{\nu}} \right),$$

$$(t_1 P)(X_1, \ldots, X_l) = q^{N-2l} P(X_1, \ldots, X_l),$$

and $e_1 P = f_0 P = 0$ for $l = 0$.

Proposition C.2. Set

$$F_{A,N,l} := F_{N,l} \cap A[z_1^{\pm 1}, \ldots, z_N^{\pm 1}][X_1, \ldots, X_l],$$

and $F_{A,N} := \bigoplus_{l=0}^{N} F_{A,N,l}$. Then for $i = 0, 1$ and $s \geq 0$ we have

$$e_i^{(s)} F_{A,N} \subset F_{A,N}, \quad f_i^{(s)} F_{A,N} \subset F_{A,N}.$$  

Proof. Take $P \in F_{A,N,l}$. We are to show that, for any $s \geq 0$ and $x = e_i, f_i, x^s P$ is divisible by $\prod_{j=1}^{l} (q^j - q^{-j})/(q^j - q^{-1})$. For $x = f_1$ or $e_0$, this follows from the formula for the action given above and the identity

$$\text{Sym} \prod_{1 \leq j < j' \leq s} q^{-1} X_j - q X_{j'} = \prod_{j=1}^{s} q^j - q^{-j}.$$ 

The assertion for $x = e_1$ or $f_0$ can be shown from the Lemma below by choosing $t = q^2$ and specializing $X = 0$ or $X = \infty$. 

Lemma C.3. Let $Q(X_1, \ldots, X_l)$ be a symmetric polynomial in $X_1, \ldots, X_l$ with coefficients in $\mathbb{C}[z_1^{\pm 1}, \ldots, z_N^{\pm 1}]$, which has degree at most $N-1$ in each $X_i$. Assume further that $Q(z_k^{-1}, t z_k^{-1}, X_3, \ldots, X_l) = 0$ for $k = 1, \ldots, N$. Then, for any $2 \leq m \leq l$, $Q(X, t X, \ldots, t^{m-1} X, X_{m+1}, \ldots, X_l)$ is divisible by $\prod_{j=1}^{m} (t^j - 1)/(t - 1)$.

Proof. Let $\epsilon$ be a primitive $d$-th root of unity with $2 \leq d \leq m$, and consider

$$Q(X, t X, \ldots, t^{d-1} X, X_{d+1}, \ldots, X_l)|_{t=\epsilon}.$$  

(C.4)
We obtain the map \((C.4)\) has degree at most \(d(N-1)\). By the assumption it has \(dN\) distinct zeroes \(e^jz_k^{-1}\) \((0 \leq j \leq d - 1, 1 \leq k \leq N)\). Hence \((C.4)\) vanishes identically.

More generally,

\[
Q(X_1, tx_1, \ldots, t^{d-1}x_1, \ldots, X_r, tX_r, \ldots, t^{d-1}X_r, X_{rd+1}, \ldots, X_l)
\]

has \(r\)-fold zeroes at \(t = \epsilon\), as shown by differentiation with respect to \(t\). Therefore, if \(rd \leq m\), then

\[
Q(X, tx, \ldots, t^{m-1}X, X_{m+1}, \ldots, X_l)
\]

is divisible by \(\varphi_d(t)^r\), where \(\varphi_d(t)\) stands for the \(d\)-th cyclotomic polynomial. Noting that

\[
\prod_{j=1}^{m} \frac{t_j - 1}{t - 1} = \prod_{2 \leq j \leq m} \prod_{2 \leq d \leq m} \varphi_d(t) = \prod_{2 \leq d \leq m} \varphi_d(t)^{m/d},
\]

where \([x]\) is the greatest integer not exceeding \(x\), we obtain the assertion. \(\square\)

Let \(U_A^\pm\) be the subalgebra of \(U_A\) generated by \(e_0^{(s)}, e_1^{(s)}, f_i^{(s)}\) \((s \geq 0)\) and \(t_1^\pm\). Let further \(W_{A,N}'\) denote the subspace of \(F_{A,N}\) consisting of elements \(P\) which are symmetric polynomials in \(z_1, \ldots, z_N\) and satisfy the condition

\[
P = 0 \quad \text{if} \quad X_i^{-1} = q^{-2}z_{N-1} = z_N.
\]

**Lemma C.4.**

\[
C_N'(U_A^+ v_0^{\otimes N}) \subset W_{A,N}'.
\]

**Proof.** By Proposition C.2, we have \(U_A^+ F_{A,N} \subset F_{A,N}\). Since \(C_N'(v_0^{\otimes N}) = 1 \in F_{A,N}\), 

\[
x \cdot v_0^{\otimes N} \quad (x \in U_A^+) \text{ is a polynomial in } z_1, \ldots, z_N \text{ with coefficients in } A \text{. If we write}
\]

\[
C(z) = C(z|z_1, \ldots, z_N) \quad \text{and} \quad Y_i = P_{i+1}R_i(z_i/z_{i+1}),
\]

then we have

\[
Y_iC(z|\ldots, z_i, z_{i+1}, \ldots)Y_i^{-1} = C(z|\ldots, z_{i+1}, z_i, \ldots).
\]

The operator \(Y_i\) \(V_K[z_i, z_i^{-1}] \otimes V_K[z_{i+1}, z_{i+1}^{-1}] \rightarrow V_K[z_{i+1}, z_{i+1}^{-1}] \otimes V_K[z_i, z_i^{-1}]\) commutes with the action of \(U\), and leaves \(v_0^\otimes N\) and \(v_0^{\otimes N}\) invariant. The symmetry in \(z_1, \ldots, z_N\) follows from these properties. The condition \((C.5)\) is a consequence of the property \((v_0^\otimes \cdots \otimes v_0^\otimes, C(z|\ldots, q^2z, z)v) = 0\) for any \(v\), which can be verified easily from the definition. \(\square\)

When \(q\) is specialized to \(\sqrt{-1}\), the image of \(C_N'\) becomes divisible by \(\prod_{j \neq p} (X_j + X_p')\), as is seen from \((C.2)\). Redefining

\[
C_N(v) = \prod_{j \neq p} \frac{X_j - X_p'}{i(X_j + X_p')} : C_N'(v) \quad (v \in (V_K^\otimes N)_{l})
\]

we obtain the map \((3.12)\). Proposition 3.4 follows from Lemma C.4.
Appendix D. Representations of $\mathcal{U}_r(\mathfrak{sl}_2)$ at Roots of 1

We collect here some facts about representations of $\mathcal{U}_r(\mathfrak{sl}_2)$ used in the text. Recall that $\epsilon = e^{-\pi i/r}$ with $r \geq 3$.

The $\mathcal{U}_r(\mathfrak{sl}_2)$-modules $V^s(\alpha)$, $X^s(\alpha)$ ($0 \leq s \leq r - 2$, $\alpha = \pm 1$) and $W^s(\alpha)$ ($0 \leq s \leq r - 1$, $\alpha = \pm 1$) are defined as follows. The module $V^s(\alpha)$ has basis $\{v^s_k(\alpha)\}_{0 \leq k \leq s}$ with the action of $\mathcal{U}_r(\mathfrak{sl}_2)$ given by

$$Ev^s_k(\alpha) = \alpha[k][s + 1 - k]v^s_{k-1}(\alpha),$$
$$Fv^s_k(\alpha) = v^s_{k+1}(\alpha),$$
$$Tv^s_k(\alpha) = \alpha \epsilon^{s-2k}v^s_k(\alpha),$$

where $[n] = \frac{\epsilon^n - \epsilon^{-n}}{\epsilon - \epsilon^{-1}}$ and $v^s_{-1}(\alpha) = v^s_1(\alpha) = 0$. We abbreviate $V^s(1)$ to $V^s$ and $V^1$ to $V$.

The module $W^s(\alpha)$ has basis $\{w^s_k(\alpha)\}_{0 \leq k \leq r-1}$ with the action of $\mathcal{U}_r(\mathfrak{sl}_2)$ given by

$$Ew^s_k(\alpha) = \alpha[k][s + 1 - k]w^s_{k-1}(\alpha),$$
$$Fw^s_k(\alpha) = w^s_{k+1}(\alpha),$$
$$Tw^s_k(\alpha) = \alpha \epsilon^{s-2k}w^s_k(\alpha),$$

where $w^s_{-1}(\alpha) = w^s_0(\alpha) = 0$. The module $X^s(\alpha)$ has basis

$$\{x^s_k(\alpha), y^s_k(\alpha)\}_{0 \leq k \leq s} \cup \{a^s_k(\alpha), b^s_k(\alpha)\}_{0 \leq k \leq r-2-s},$$

and the action of $\mathcal{U}_r(\mathfrak{sl}_2)$ is given as follows.

$$Ex^s_k(\alpha) = \alpha[k][s + 1 - k]x^s_{k-1}(\alpha) \ (0 \leq k \leq s),$$
$$Ey^s_k(\alpha) = \begin{cases} \alpha[k][s + 1 - k]y^s_{k-1}(\alpha) \ (1 \leq k \leq s), \\ a^s_{r-2-s}(\alpha) \ (k = 0), \end{cases}$$
$$Ea^s_k(\alpha) = -\alpha[k][r - 1 - s - k]a^s_{k-1}(\alpha) \ (0 \leq k \leq r - 2 - s),$$
$$Eb^s_k(\alpha) = \begin{cases} -\alpha[k][r - 1 - s - k]b^s_{k-1}(\alpha) + a^s_{k-1}(\alpha) \ (1 \leq k \leq r - 2 - s), \\ x^s_k(\alpha) \ (k = 0), \end{cases}$$
$$Fx^s_k(\alpha) = x^s_{k+1}(\alpha),$$
$$Fy^s_k(\alpha) = y^s_{k+1}(\alpha) \ (0 \leq k \leq s),$$
$$Fa^s_k(\alpha) = a^s_{k+1}(\alpha),$$
$$Fb^s_k(\alpha) = b^s_{k+1}(\alpha) \ (0 \leq k \leq r - 2 - s),$$
$$Tx^s_k(\alpha) = \alpha \epsilon^{s-2k}x^s_k(\alpha),$$
$$Ty^s_k(\alpha) = \alpha \epsilon^{s-2k}y^s_k(\alpha) \ (0 \leq k \leq s),$$
$$Ta^s_k(\alpha) = -\alpha \epsilon^{r-2-s-2k}a^s_k(\alpha),$$
$$Tb^s_k(\alpha) = -\alpha \epsilon^{r-2-s-2k}b^s_k(\alpha) \ (0 \leq k \leq r - 2 - s).$$

Here we have set $x^s_{-1}(\alpha) = a^s_{-1}(\alpha) = 0$, $x^s_{s+1}(\alpha) = a^s_0(\alpha)$, $y^s_{s+1}(\alpha) = 0$, $a^{r-1-s}_0(\alpha) = 0$, $b^{r-1-s}_0(\alpha) = 0$.

The modules $V^s(\alpha)$ ($0 \leq s \leq r - 2$) and $W^{r-1}(\alpha)$ are irreducible. The others are indecomposable and we have

$$0 \to V^{r-2-s}(-\alpha) \to W^s(\alpha) \to V^s(\alpha) \to 0 \ (0 \leq s \leq r - 2),$$
$$0 \to W^s(\alpha) \to X^s(\alpha) \to W^{r-2-s}(-\alpha) \to 0 \ (0 \leq s \leq r - 2).$$
Upon tensoring with $V$, these modules decompose as \cite{17}

\begin{align}
V^s(\alpha) \otimes V &= V^{s+1}(\alpha) \oplus V^{s-1}(\alpha), \quad (0 \leq s \leq r - 2), \\
W^s(\alpha) \otimes V &= W^{s+1}(\alpha) \oplus W^{s-1}(\alpha), \quad (0 \leq s \leq r - 2), \\
W^{r-1}(\alpha) \otimes V &= X^0(\alpha), \\
X^s(\alpha) \otimes V &= X^{s+1}(\alpha) \oplus X^{s-1}(\alpha), \quad (0 \leq s \leq r - 2).
\end{align}

In the above, we set

\begin{align*}
V^{-1}(\alpha) &= 0, \quad V^{r-1}(\alpha) = W^{r-1}(\alpha), \quad W^{-1}(\alpha) = W^{r-1}(-\alpha), \\
X^{-1}(\alpha) &= W^{r-1}(-\alpha) \oplus W^{r-1}(\alpha), \quad X^{r-1}(\alpha) = W^{r-1}(\alpha) \oplus W^{r-1}(\alpha).
\end{align*}

Applying the above rule repeatedly, we see that $V^{\otimes n}$ decomposes as a direct sum of $V^s$, $X^s(\alpha)$ ($0 \leq s \leq r - 2$) and $W^{r-1}(\alpha)$, $\alpha = \pm 1$.

Define subspaces $G^{(r)}_n$, $B^{(r)}_n$ of $V^{\otimes n}$ inductively as follows.

**Definition D.1.** We set $G^{(r)}_1 = V = V^1$, $B^{(r)}_1 = 0$. For $n \geq 2$, $G^{(r)}_n$ is the direct sum of the $V^s$ ($0 \leq s \leq r - 2$) appearing in the decomposition of $G^{(r)}_{n-1} \otimes V$. $B^{(r)}_n$ is the sum of $B^{(r)}_{n-1} \otimes V$ and the direct sum of $W^{r-1}(1)$'s appearing in $G^{(r)}_{n-1} \otimes V$.

We have

\[ V^{\otimes n} = G^{(r)}_n \oplus B^{(r)}_n. \]

The decomposition (D.1) is orthogonal with respect to the standard symmetric bilinear form $\langle \ , \ \rangle$ on $V^{\otimes n}$. Hence $G^{(r)}_n$ and $B^{(r)}_n$ are orthogonal. Note that $F^{r-1}G^{(r)}_n = 0$.

Let

\[ \Omega_{n,t} = \text{Ker} E \cap (V^{\otimes n}). \]

**Lemma D.2.** If $u \in \Omega_{n,t} \cap B^{(r)}_n$ and $F^{r-1}v = 0$, then $(u, v) = 0$.

**Proof.** We may assume that $v$ belongs to one of the subspaces isomorphic to (5.3). From the structure of these modules, we see that the condition $F^{r-1}v = 0$ implies $v \in \text{Im} F$ or $v \in V^s$. If $v = f v'$ for some $v'$, then we have $(u, v) = (u, F v') = (E T^{-1}u, v') = 0$. Otherwise $v \in G^{(r)}_n$, and the assertion follows from the orthogonality of $G^{(r)}_n$ and $B^{(r)}_n$. \qed

We denote the basis of $V = V^1$ by $v_+ = v^1_0(1)$ and $v_- = v_1(1)$. Let $R^+$ be the linear operator on $V \otimes V$ given by

\begin{align}
R^+ v_+ \otimes v_+ &= \epsilon v_+ \otimes v_-, \\
R^+ v_+ \otimes v_- &= v_- \otimes v_+, \\
R^+ v_- \otimes v_+ &= (\epsilon - \epsilon^{-1}) v_- \otimes v_+ + v_+ \otimes v_-.
\end{align}

Denote by $R^+_{i+1} \in \text{End}(V^{\otimes n})$ ($1 \leq i \leq n - 1$) the operator acting as $R^+$ on the $(i, i+1)$ tensor factor and as identity elsewhere. They commute with the action of $U_q(\mathfrak{sl}_2)$ on $V^{\otimes n}$ defined by the opposite coproduct $\Delta'$ (5.1). Define further

\[ \Pi_{n,t} = P_{n-1} P_{n-2} \cdots P_{n} \cdot D^{1-n/2-1}_1, \]
where $D \in \text{End}(V)$, $D^{1/2}v_{\pm} = e^{\mp \pi i/2}v_{\pm}$. It is easy to check that

$$R_{i+1}^+\Omega_{n,l} \subset \Omega_{n,l}, \quad \Pi_{n,l}\Omega_{n,l} \subset \Omega_{n,l}.$$  

The subspaces $\mathcal{G}^{(r)}_n$, $\mathcal{B}_n^{(r)}$ are not invariant under the actions of $R_{i+1}^+$ and $\Pi_{n,l}$. Nevertheless we have

**Lemma D.3.** The space $\Omega_{n,l} \cap \mathcal{B}_n^{(r)}$ is invariant by the operators $R_{i+1}^+$ ($1 \leq i \leq n-1$), $\Pi_{n,l}$.

**Proof.** Let $u \in \Omega_{n,l} \cap \mathcal{B}_n^{(r)}$. Since $\mathcal{B}_n^{(r)}$ is the orthogonal complement of $\mathcal{G}^{(r)}_n$, the assertion will follow if we show that

$$\langle R_{i+1}^+u, \mathcal{G}^{(r)}_n \rangle = 0, \quad (\Pi_{n,l}u, \mathcal{G}^{(r)}_n) = 0.$$  

The equation (D.7) is a consequence of the relation $(R_{i+1}^+u, v) = (u, R_{i+1}^+v)$, $F^{r-1}R_{i+1}^+\mathcal{G}^{(r)}_n = R_{i+1}^+F^{r-1}\mathcal{G}^{(r)}_n = 0$ and Lemma D.2.

Let us verify (D.8). Take $v \in \mathcal{G}^{(r)}_n$ and set $\tilde{v} = D^{1-n/2-1}P_{12}P_{23}\cdots P_{n-1,n}v$, so that $(\Pi_{n,l}u, v) = (u, \tilde{v})$. We have $v \in V^s \otimes V$ for some $0 \leq s \leq r - 2$, and $\tilde{v} \in V \otimes V^s$. Therefore we have either $F^{r-1}\tilde{v} = 0$, or else $s = r - 2$ and $\tilde{v} \in v_+ \otimes v'$. $Ev' = 0$. The latter does not take place. Indeed, since $v_+$ is an eigenvector of $D$, it would mean that $v = P_{n-1,n}\cdots P_{12}D_1^{1+n/2+2}v$ is proportional to $v' \otimes v_+$, which belong to the irreducible component $W^{r-1}(1)$ of $V^{r-2} \otimes V$. This is a contradiction. Hence Lemma D.2 implies (D.8). \qed

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