PARABOLIC CHARACTER SHEAVES, II

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Dedicated to Boris Feigin on the occasion of his 50th birthday

ABSTRACT. The theory of character sheaves on a reductive group is extended to a class of varieties which includes the strata of the De Concini–Procesi completion of an adjoint group.


Key words and phrases. Reductive group, parabolic group, perverse sheaf, character sheaf.

INTRODUCTION

Let $G$ be a connected reductive algebraic group over an algebraically closed field $k$. Let $Z$ be the algebraic variety consisting of all triples $(P, P', U_P g U_P)$ where $P, P'$ run through some fixed conjugacy classes of parabolics in $G$ and $g$ is an element of $G$ that conjugates $P$ to a parabolic in a fixed “good” relative position $y$ with $P'$ (here $U_P, U_{P'}$ are the unipotent radicals of $P, P'$). The varieties $Z$ include more or less as a special case the boundary pieces of the De Concini–Procesi completion $\bar{G}$ of $G$ (assumed to be adjoint). They also include as a special case the varieties studied in the first part of this series $[L9]$ (where $y = 1$ that is, $g P g^{-1} = P'$). In this special case a theory of “character sheaves” on $Z$ was developed in $[L9]$. In the present paper we extend the theory of character sheaves to a general $Z$.

We now review the content of this paper in more detail. (The numbering of sections continues that of $[L9]$; we also follow the notation of $[L9]$.)

In Section 8 we introduce a partition of $Z$ similar to that in $[L9]$; as in $[L9]$, it is based on the combinatorics in Section 2. But whereas in $[L9]$ the combinatorics needed is covered by the results in $[B]$, for the present paper we actually need the slight generalization of $[B]$ given in Section 2. Now, it is not obvious that the partition of $Z$ defined in Section 8 reduces for $y = 1$ to that in Section 3; this needs an argument that is given in Section 9. In Section 10 we consider the example where $G$ is a general linear group. In Section 11 we define the “parabolic character sheaves” on $Z$. As in the case $y = 1$ (Section 4), we give two definitions for these; one uses the partition in Section 8 and allows us to enumerate the parabolic

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character sheaves; the other one imitates the definition of character sheaves in [L3].
(The two definitions are equivalent by 11.15 and 11.18.) The theory of character sheaves in Section 11 generalizes that in Section 4 (this is seen from the second definition). A consequence of the coincidence of the two definitions of parabolic character sheaves on \(Z\) is that a statement like 0.1(a) (concerning characteristic functions over a finite field) continues to hold in the generality of this paper. In Section 12 we define the notion of character sheaf on the completion \(\hat{G}\). We again have two definitions; one is based on the partition in Section 8, and the second one is reminiscent of the definition of character sheaves in [L3]. The character sheaves according to the first definition satisfy the analogue of 0.1(a) for \(\hat{G}\) over a finite field (this follows from the corresponding statement for the strata of \(\hat{G}\)). We expect that the two definitions coincide, but we cannot prove this. Our results also provide a finite partition of \(\hat{G}\) into \(G\)-stable pieces (a refinement of the usual partition of \(\hat{G}\) into \(G\times G\)-orbits) which allows us to give an explicit description of the set of \(G\)-conjugacy classes in \(\hat{G}\) (see 12.3(a)).

8. The Variety \(Z_{J,y,\delta}\) and its Partition

8.1. We preserve the setup of 3.1. (Thus, \(\hat{G}\) is a possibly disconnected reductive algebraic group over \(k\) with identity component \(G\) and \(G^1\) is a fixed connected component of \(\hat{G}\). Also \(W, I\) is the Weyl group of \(G\) and \(\delta: W \xrightarrow{\sim} W\).) Let \(P \in \mathcal{P}_J, Q \in \mathcal{P}_K, u = \text{pos}(P, Q)\). We have
\[
\dim((U_P \cap U_Q) \setminus U_P) = l(u) + \nu_J - \nu_{J \cap \text{Ad}(u)K}.
\]
(a) Here \(\nu_J\) is the number of reflections in \(W_J\).

8.2. Let \(P, P'\) be two parabolics of \(G\). The following hold.
(a) \(PP', P'P\) are in good position and \(\text{pos}(P, P') = \text{pos}(PP', PP)\);
(b) if \(B \in B, B \subset PP'\) then for some \(B' \in B, B' \subset P'\) we have \(\text{pos}(B, B') = \text{pos}(P, P')\).

To prove (b) we may replace \(P, P'\) by \(PP', PP\). It suffices to prove: if \(P, P'\) are in good position and \(B \subset P\), then for some \(B' \subset P'\) we have \(\text{pos}(B, B') = \text{pos}(P, P')\).

8.3. If \(P, Q\) are parabolics in good position, we have a bijection
\[
\{\text{parabolics contained in } P\} \xrightarrow{\sim} \{\text{parabolics contained in } Q\}
\]
given by \(P' \leftrightarrow Q', Q' = QP', P' = PQ'\). (Then \(P', Q'\) are in good position and \(\text{pos}(P', Q') = \text{pos}(P, Q)\).)

Lemma 8.4. Let \(P, Q, R\) be parabolic subgroups with a common Levi \(L\). Then
\[
\text{pos}(P, Q) \text{pos}(Q, R) = \text{pos}(P, R).
\]

Let \(\beta\) be a Borel of \(L\). Then \(B = U_P\beta, B' = U_Q\beta, B'' = U_R\beta\) are Borels of \(P, Q, R\) respectively and we have
\[
\text{pos}(P, Q) = \text{pos}(B, B'), \text{pos}(Q, R) = \text{pos}(B', B''), \text{pos}(P, R) = \text{pos}(B, B'').
\]
It suffices to show that $\text{pos}(B, B') \cap \text{pos}(B', B'') = \text{pos}(B, B'')$. This holds since $B, B', B''$ contain $\beta$ hence have a common maximal torus.

**Lemma 8.5.** Let $P, Q, R$ be parabolics with a common Levi $L$; let $P', Q', R'$ be parabolics with a common Levi $L'$. Assume that $\text{pos}(P, Q) = \text{pos}(P', Q')$, $\text{pos}(Q, R) = \text{pos}(Q', R')$. Assume also that $P, P'$ have the same type; $Q, Q'$ have the same type; $R, R'$ have the same type. Then there exists $x \in G$ that conjugates $P, Q, R$ to $P', Q', R'$.

Clearly, we can assume that $P = P', Q = Q', L = L'$. Then we use the following fact: if $Q, R, R'$ are parabolics with a common Levi $L$ such that $\text{pos}(Q, R) = \text{pos}(Q', R')$ then $R = R'$. (This can be reduced to the case where $Q, R, R'$ are Borels, which is clear.)

**Lemma 8.6.** Let $Q, Q'$ be parabolics with a common Levi. Then $Q \cap U_Q' = U_Q' \cap U_{Q'}$.

It suffices to show that $\text{Lie} Q \cap \text{Lie} U_{Q'} = \text{Lie} U_Q \cap \text{Lie} U_{Q'}$. Let $L$ be a common Levi of $Q, Q'$. Consider the weight decomposition $\text{Lie} G = \bigoplus \alpha \text{Lie} G_{\alpha}$ with respect to the connected centre of $L$. Then $\text{Lie} G_0 = \text{Lie} L$ and $\text{Lie} U_Q, \text{Lie} U_{Q'}$ are direct sums of various $\text{Lie} G_{\alpha}$ with $\alpha \neq 0$. Let $x \in \text{Lie} Q \cap \text{Lie} U_{Q'}$. We have $x = x_0 + x'$ with $x_0 \in \text{Lie} L$ and $x' \in \text{Lie} U_Q$ is in $\bigoplus_{\alpha \neq 0} \text{Lie} G_{\alpha}$. Since $x \in \text{Lie} U_{Q'}$, we have $x \in \bigoplus_{\alpha \neq 0} \text{Lie} G_{\alpha}$. Hence $x_0 = 0$ and $x \in \text{Lie} U_Q$. Thus, $\text{Lie} Q \cap \text{Lie} U_{Q'} \subset \text{Lie} U_Q \cap \text{Lie} U_{Q'}$. The reverse inclusion is obvious.

**Lemma 8.7.** Let $P, P', Q$ be parabolics such that $\text{pos}(P', P) = y$, that $P', Q$ are in good position and that $Q$ contains a Levi of $P \cap P'$. Let $Q' = Q^{(P', y)}$. Then

(a) $\text{pos}(P P', P' P) = z^{-1}$;

(b) $\text{pos}(P P', Q') = y$;

(c) $\text{pos}(P P', Q') = z^{-1} y$ and $P P', Q'$ are in good position.

Statement (a) follows from 8.2; (b) follows from 8.3. We prove (c). Let $L_0$ be a common Levi of $P P', P' P$ that is contained in $Q$. Then $P P', P' P, Q'$ have a common Levi $L_0$. By Lemma 8.4 we have $\text{pos}(P P', Q') = \text{pos}(P P', P' P) \text{pos}(P P', Q') = z^{-1} y$ as required.

**8.8.** Let $J, J' \subset I$ and $y \in W$ be such that $\text{Ad}(y) \delta(J) = J'$, $y \in J' W^{\delta(J)}$. For $P, P'$ in $P_{J, J'}$ let

$A_y(P, P') = \{g \in G^1; \text{pos}(P, g P) = y\}$,

$A'_y(P, P') = \{g \in A_y(P, P'); g \text{ contains some Levi of } P \cap P'\}$.

**Lemma 8.9.** (a) $A_y(P, P')$ is a single $P'$, $P$ double coset and also a single $P$, $U_P$ double coset.

(b) $A_y(P, P') = U_P A'_y(P, P')$.

(c) $A'_y(P, P')$ is a single $(P \cap P')$, $P$ double coset and also a single $U_{P \cap P'}$, $P$ double coset.

We prove (a). We can find $Q \in P_{I, J'}$ such that $\text{pos}(P', Q) = y$; moreover, we can find $g \in G^1$ such that $g P = Q$. Thus $A_y(P, P') \neq \emptyset$. Let $g, g' \in A_y(P, P')$. We prove (a). We can find $Q \in P_{I, J'}$ such that $\text{pos}(P', Q) = y$; moreover, we can find $g \in G^1$ such that $g P = Q$. Thus $A_y(P, P') \neq \emptyset$. Let $g, g' \in A_y(P, P')$. We prove (a). We can find $Q \in P_{I, J'}$ such that $\text{pos}(P', Q) = y$; moreover, we can find $g \in G^1$ such that $g P = Q$. Thus $A_y(P, P') \neq \emptyset$. Let $g, g' \in A_y(P, P')$.
Clearly, $g' = xgp$ with $x \in P'$, $p \in P$. Now $gP$, $P'$ are in good position; let $L$ be a common Levi of them. Since $g^{-1}L$ is a Levi of $P$, we have $p \in g^{-1}LU_{P'}$. Thus $p = g^{-1}lu$ with $l \in L$, $u \in U_{P'}$ and $g' = xgp = xlgu \in P'gU_{P'}$.

We prove (b). Let $g \in A'_g(P, P')$. Let $L_0$ be a Levi of $P \cap P'$. Then $L_0$ is contained in a Levi $L_1$ of $P'$. Let $L_2$ be a common Levi of $P', gP$. Then $L_1 = L_2$ for some $u \in U_{P'}$. We have $ugP = u(gP) \supset uL_2 = L_1 \supset L_0$ hence $ug \in A'_g(P, P')$.

We prove (c). Let $g, g' \in A'_g(P, P')$. Then

- $P'P', P''P', (gP)(P'')$ have a common Levi;
- $P'P', P''P', (gP')(P'')$ have a common Levi;
- $\text{pos}(P', (gP)(P'')) = \text{pos}(P'P', (gP')(P'')) = y$.

By Lemma 8.5, there exists $x \in G$ which conjugates $P'P', P''P', (gP)(P'')$ to $P''P', P'P', (gP)(P')$. Then $x \in P'P' \cap P''P' = P \cap P'$ and $x$ conjugates $gP$ to $gP'$, since $gP$ and $g'P$ are parabolics of type $g(J)$ containing $(gP)(P')$, $(gP')(P'')$. Hence $xg \in gP'$ if $g' \in gP$. Let $M$ be a Levi of $P \cap P'$ with $M \subset gP$. We can write $x = vmp$ with $v \in U_{P'P'}$, $m \in M$. Then $g^{-1}mg \in P$, $xgP = vmgP = vgg^{-1}mgP = vgp$. The lemma is proved.

8.10. Let $P \in \mathcal{P}_J$, $P' \in \mathcal{P}_J$ be such that $\text{pos}(P', P) = z$. Let

$$J_1 = J \cap \delta^{-1}\text{Ad}(g^{-1}z)J, \quad J'_1 = J \cap \text{Ad}(z^{-1}y)\delta(J).$$

Then $\text{Ad}(z^{-1}y)\delta(J_1) = J'_1$. Let $g \in A'_g(P, P')$. We set

$$P_1 = g^{-1}(gP)(P''), \quad P'_1 = P''P', \quad (P_1, P'_1) = \alpha(P', P, g).$$

We have $P_1 \subset P$, $P'_1 \subset P$ and, by Lemma 8.7, $P_1, gP_1$ are in good position,

$$P_1 \in \mathcal{P}_J, \quad P'_1 \in \mathcal{P}_J', \quad \text{pos}(P'_1, gP_1) = z^{-1}y.$$

(We have also $\text{pos}(P'_1, P''P') = z^{-1}$, $\text{pos}(P''P', gP_1) = y$.) Thus, $z^{-1}y \in J'_1W_\delta^{J_1}$ and $g \in A_{z^{-1}y}(P_1, P'_1)$.

Lemma 8.11. Let $g, g' \in A'_g(P, P')$, $u' \in U_{P'}$, $u \in U_{P}$ with $g' = u'gu$. Then

(a) $g' = u'_1gu_1$ with $u'_1 \in U_{P'_1}$, $u_1 \in U_{P_1}$;

(b) we have $\alpha(P, P', g') = (P_1, P'_1)$.

We prove (a). Since $P_1 \subset P$ we have $U_{P_1} \subset U_{P}$. Hence we may assume that $u = 1$. By Lemma 8.9(c) we have $g' = vgp$ with $v \in U_{P''P'}$, $p \in P$. Thus $g' = vgp = u'g$. Hence $v^{-1}u' = vgp^{-1}$. Now $v^{-1}u' \in U_{P''P'}U_{P'} \subset U_{P''P'}$. Thus $v^{-1}u' \in U_{P''P'} \cap gP$. We have

$$U_{P''P'} \cap gP = U_{P''P'} \cap gP_1 = U_{P''P'} \cap U_{P'_1}. \quad \text{(In general, if } R, S \text{ are parabolics in good position and } R' \subset R, S' = S'^R, \text{ then } U_R \cap S = U_{R'} \cap S', \text{ thereby } U_{R'} \cap S \subset S', \text{ since } R', S' \text{ are in good position, we have } U_{R'} \cap S' = U_{R'} \cap U_S, \text{ by Lemma 8.6.) Thus, } ggp^{-1} = v^{-1}u' \in U_{P'_1} \text{ so that } p \in U_{P'_1}. \text{ Now } v \in U_{P''P'} \subset U_{P''P'} = U_{P'_1}. \text{ We see that } g' = vgp \text{ with } v \in U_{P'_1}, p \in U_{P'_1}. \text{ This proves (a).}

We prove (b). We must show that

$$u^{-1}g^{-1}u'^{-1}(u'guP)(P'')u'^{-1}gu = g^{-1}(gP)(P'')g.$$
or that $u^{-1}g^{-1}(qP)(P')gu = g^{-1}(qP)(P')g$. This holds since $g^{-1}(qP)(P')g$ is a parabolic subgroup of $P$ hence it contains $u$.

**Lemma 8.12.** Let $g, g' \in A'_y(P, P')$. Assume that $\alpha(P, P', g) = \alpha(P, P', g') = (P_1, P'_1)$ and that $g' \in U_{P_1}gU_{P_1}$. Then there exist $x \in U_{P'} \cap P'$, $w' \in U_{P'}$, $w \in U_P$ such that $g' = w'xgw$.

By Lemma 8.9(c) we have $g' = vgp$ with $v \in U_{P' \cap P'}$, $p \in P$. By assumption, $p^{-1}g^{-1}v^{-1}(vgp)(P')vgp = g^{-1}(qP)(P')g$ that is $\gamma^{-1}P_1 = P_1$, or $p \in P_1$. Also $g' = u'gu$ with $u' \in U_{P'_1}$, $u \in U_{P_1}$. Thus, $g' = vgp = u'gu$. Setting $\pi = p^{-1}u' \in P_1$ we have $vg\pi = u'g'$ and $v^{-1}u' \in U_{P' \cap P'}$. Thus $v^{-1}u' \in U_{P' \cap P}$. Since $P'P$ and $P$ are in good position, we have $v^{-1}u' \in U_{P' \cap P} \cap U_{P_1}$. Then $\gamma u' \in U_{P_1}$ and $\pi \in U_{P_1}$. Since $u \in U_{P_1}$, we have $u \in P_1$. Thus $g' \in U_{P_1}U_{P_1}U_{P_1}$. Now $\gamma P_1 = \gamma (qP \cap U_{P' \cap P})U_{P_1}$ so that

$$g' \in U_{P_1}U_{P_1}U_{P_1} = U_{P_1}U_{P_1}U_{P_1}gU_{P_1} \subset U_{P_1}gU_P = U_{P_1}U_{P_1}U_{P_1}gU_P.$$  

Thus, $g' = u'xgw$ with $w' \in U_{P'}$, $x \in P' \cap U_{P'}$, $w \in U_P$, as desired.

**8.13.** We fix $z \in J'W_{J'}$. Let $J_1 = J \cap \delta^{-1}\Ad(y^{-1}z)J$, $J'_1 = J \cap \Ad(z^{-1})\delta(J)$, so that $\Ad(z^{-1})\delta(J_1) = J'_1$. Let $Q, Q'$ in $\mathcal{P}_{J_1}, \mathcal{P}_{J'_1}$ be such that $\pos(Q', Q) \in W_{J_1}$. Let $\gamma_1$ be a $U_{Q'}, U_Q$ double coset in $A_{z^{-1}}(Q, Q')$. Let $F'$ be the set of all $(P, P', g)$ with $P \in \mathcal{P}_J$, $P' \in \mathcal{P}_{J'}$, $\pos(P', P) = z$, $g \in A'_y(P, P')$ such that $\alpha(P, P', g) = (Q, Q')$ and $g \in \gamma_1$. (Since $Q' \subset P$, $P$ is uniquely determined.) Let $F$ be the set of all $(P, P', \gamma)$ with $P \in \mathcal{P}_J$, $P' \in \mathcal{P}_{J'}$, $\pos(P', P) = z$, $\gamma \in U_{P'} \backslash A_0(P, P') \cap U_P$ such that for some $g \in \gamma \cap A'_y(P, P')$ we have $\alpha(P, P', g) = (Q, Q')$ and $g \in \gamma_1$. (The equivalence of “some/any” follows from Lemma 8.11.) Again, $P$ is uniquely determined. The map $F' \to F$, $(P, P', g) \mapsto (P, P', U_{P'}gU_P)$ is surjective by Lemma 8.9(b).

**Lemma 8.14.** $F'$ is non-empty.

Let $g \in \gamma_1$. Then $\pos(Q', Q) = z^{-1}y$. Define $P \in \mathcal{P}_J$ by $Q \subset P$. Since $\pos(Q', Q) \in W_{J_1}$, we have also $Q' \subset P$. Let $\tilde{P}, P'$ in $\mathcal{P}_J, \mathcal{P}_{J'}$ with $\pos(P', P) = z$. By Lemma 8.9(c), we can find $\tilde{g} \in A'_y(\tilde{P}, P')$. Now $\alpha(\tilde{P}, P', \tilde{g}) = (Q, Q')$ with $\tilde{Q} \in \mathcal{P}_{J_1}, \tilde{Q}' \in \mathcal{P}_{J'_1}$ contained in $\tilde{P}$ and $\pos(\tilde{Q}', \tilde{Q}) = z^{-1}y$. We have $Q' = h\tilde{Q}'$ for some $h \in G$. Replacing $\tilde{P}, P', \tilde{g}$ by $h\tilde{P}, hP', h\tilde{g}h^{-1}$, we may assume that $\tilde{Q}' = Q'$. Then $\tilde{P} \in \mathcal{P}_{J_1}$ contains $Q'$ hence $\tilde{P} = P$. Now $\tilde{Q}, \tilde{Q}'$ are contained in $P$ and have the same type hence $\tilde{Q} = \tilde{Q}'$ for some $p \in P$. We have $\tilde{g}p\tilde{Q} = (\tilde{g}P)(P')$ and $\tilde{g}p\tilde{Q} = (Q', Q)$ hence $\alpha(P, P', \tilde{g}p\tilde{Q}) = (Q, Q')$. Since $\pos(Q', Q) = \pos(\tilde{Q}', \tilde{Q})$ and $Q', \tilde{Q}$ are in good position, we have $\tilde{g}p = u'gq$ with $q \in Q, u' \in U_{Q'}$. Thus $u'gq \in A'_y(P, P')$ and $\alpha(P, P', u'gq) = (Q, Q')$. We have $u'gq \in A'_y(P, P')$ and $\alpha(P, P', u'gq) = (Q, Q')$ (we use that $\gamma \in \gamma_1$). Since $u'g \in \gamma_1$, we see that $F' \neq \emptyset$.

**Lemma 8.15.** Let $(P, P', g) \in F'$. Then $(u, v) \mapsto (P, uP', U_{uP'}uvgU_P)$ is a well defined, surjective map $\kappa: U_P \times (U_P \cap P') \to F$. 
Let $u \in U_P$, $v \in U_P \cap P'$. Clearly, $(P, \tilde{P}', \gamma) \in F'$. Hence $\kappa$ is well defined.

We show that $\kappa$ is surjective. Let $(\tilde{P}, \tilde{P}', \gamma) \in F$. We can find $\tilde{g} \in \gamma$ such that $(\tilde{P}, \tilde{P}', \tilde{g}) \in \tilde{F}'$. We have $\tilde{g} \in \gamma_1$, $\tilde{g} \in \gamma_1$ hence $\tilde{g} \in U_{Q'} U_{Q'}$. Since pos$(\tilde{P}', \tilde{P}) = \text{pos}(\tilde{P}', \tilde{P})$, we have $\tilde{P}' = \tilde{P}'$ for some $p \in P$. Since $\tilde{P}' = \tilde{P}'$, $\tilde{P}' = \tilde{P}'(\tilde{P}'') = \tilde{P}'(\tilde{P}'')$, we have $p \in P''$. Thus, $p = u \pi$ with $\pi \in P \cap P'$, $u \in U_P$. Since $\tilde{P}' = \tilde{P}'$, we may assume that $p = u \in U_P$ so that $\tilde{P}' = \tilde{P}'$. Applying Lemma 8.12 to $(P, \tilde{P}', \tilde{g}), (P, \tilde{P}', \tilde{g}) \in \tilde{F}'$ (instead of $(P, \tilde{P}', \tilde{g}), (P, \tilde{P}', \tilde{g})$) we see that $\tilde{g} = \pi' x u g w$ for some $x \in U_P \cap U_P$, $w' \in U_{P'}$, $w \in U_P$. Let $v = u^{-1} x u \in U_P \cap P'$. Then $\tilde{g} = \pi' x u g w$. Thus, $(P, \tilde{P}', \gamma) = \kappa(u, v)$. The lemma is proved.

**Lemma 8.16.** In the setup of Lemma 8.15, the following two conditions for $(u, v)$, $(u, v')$ in $U_P \times (U_P \cap P')$ are equivalent:

(i) $\kappa(u, v) = \kappa(u', v')$;

(ii) $u' = u f, v' = f^{-1} d v$ for some $f \in U_P \cap P'$, $d \in U_P \cap U_P$.

Assume that (i) holds. We have $U_P = U_P'$, $u v g u v' \in U_{P'}$. Thus $u' = u f$ with $f \in P'$ (hence $f \in U_P \cap P'$) and $u v g u v' \in U_P f v' g u v' \in U_P$, that is, $v \in U_P f v' U_P$, so that $v \in f v' U_P U_{P'}$. We show that

$$P' \cap U_P U_{P'} = U_P.$$  \hfill (a)

Assume that $x \in P' \cap U_P U_{P'}$. We must show that $x \in U_P$. We have $u x \in U_{P'}$ with $u_1 \in U_{P'}$. Let $x' = u x \in P'$. Then $x' \in P' \cap U_P U_{P'} = U_P \cap U_P$ (by Lemma 8.6, which is applicable since $P', \tilde{P}'$ have a common Levi). Thus, $x' \in U_P$, hence $x \in U_P$, as required.

Applying (a) to $v' f^{-1} v \in P' \cap U_P U_{P'}$, we see that $v' f^{-1} v \in U_P$, so that $v' = f^{-1} d v$ for some $d \in U_P$. We have $d = f v' v^{-1} \in U_P \cap P'$. Hence $d \in U_P \cap U_P$. Thus, (ii) holds. The converse is immediate. The lemma is proved.

**8.17.** We consider a new group structure $(d, f) \bullet (d', f') = (f' d f^{-1} d', f' f)$ on $(U_P \cap U_P') \times (U_P \cap P')$ and a new group structure $(u, v) \bullet (u', v') = (u' u, v v')$ on $U_P \times (U_P \cap P')$. Then

$$\theta: (U_P \cap U_P') \times (U_P \cap P') \to U_P \times (U_P \cap P'), \quad (d, f) \mapsto (f, f^{-1} d),$$

is an (injective) group homomorphism for these new group structures. We can reformulate condition (ii) in 8.16 as follows:

$$\theta((P' \cap U_P) \times (U_P \cap P')) \ni (x, \gamma) \quad \theta((U_P \cap U_P') \times (U_P \cap P')) \ni (x, \gamma).$$

We see that $\kappa$ defines a bijection

$$\theta((P' \cap U_P') \times (U_P \cap P')) \ni (x, \gamma) \quad \theta((U_P \cap U_P') \times (U_P \cap P')) \ni (x, \gamma).$$

One can check that this is in fact an isomorphism of algebraic varieties. Since $U_P \times (U_P \cap P')$ is a connected unipotent group and $\theta((P' \cap U_P') \times (U_P \cap P'))$ is a connected closed subgroup of it, we see that

(a) $F$ is isomorphic to an affine space of dimension $\dim(U_P / (U_P \cap U_P'))$.  

8.18. Let $J, J' \subset I$ and $y \in J' W^{\delta(J)}$ be such that $\text{Ad}(y) \delta(J) = J'$. Let
$$Z_{J,y,\delta} = \{(P, P', \gamma); P \in \mathcal{P}_J, P' \in \mathcal{P}_{J'}, \gamma \in U_P \setminus \mathcal{A}_y(P, P')/U_P \}.$$ 

To any $(P, P', \gamma) \in Z_{J,y,\delta}$ we associate a sequence $(J_n, J'_n, u_n)_{n \geq 0}$ with $J_n, J'_n \subset I$, $u_n \in W$, a sequence $(y_n)_{n \geq 0}$ with $y_n \in J'_n W^{\delta(J_n)}$, $\text{Ad}(y_n) \delta(J_n) = J'_n$ and a sequence $(P_n, P'_n, \gamma_n)_{n \geq 0}$ with $P_n \in \mathcal{P}_{J_n}, P'_n \in \mathcal{P}_{J'_n}, \gamma_n \in \mathcal{A}_{y_n}(P_n, P'_n)$. We set
$$P_0 = P, \quad P'_0 = P', \quad \gamma_0 = \gamma, \quad J_0 = J, \quad J'_0 = J', \quad u_0 = \text{pos}(P'_0, P_0), \quad y_0 = y.$$

Assume that $n \geq 1$, that $P_n, P'_n, \gamma_m, J_m, J'_m, u_m, y_m$ are already defined for $m < n$ and that $u_m = \text{pos}(P'_m, P_m), P_m \in \mathcal{P}_{J_m}, P'_m \in \mathcal{P}_{J'_m}$ for $m < n$. Let
$$J_n = J_{n-1} \cap \delta^{-1} \text{Ad}(y_{n-1} u_{n-1}) J_{n-1}, \quad J'_n = J_{n-1} \cap \text{Ad}(u_{n-1} y_{n-1}) \delta(J_{n-1}),$$
$$P_n = y_{n-1}^{-1} (g_{n-1} P_{n-1})^{\nu_{n-1} - \nu_{n-1}'} \gamma_{n-1} P_{n-1} \in \mathcal{P}_{J_n}, \quad P'_n = P_{n-1}'' \in \mathcal{P}_{J'_n}$$
where
$$g_{n-1} = \gamma_{n-1} \cap \delta^{-1} \mathcal{A}_{y_{n-1}}(P_{n-1}, P'_{n-1}),$$
$$u_n = \text{pos}(P'_n, P_n), \quad y_n = u_{n-1}^{-1} y_{n-1}, \quad \gamma_n = U_{P_n} g_{n-1} U_{P_n}.$$ 

This completes the inductive definition; the definition makes sense (it is independent of choices) by 8.9–8.11. We have $(J_n, J'_n, u_n)_{n \geq 0} \in S(J, \text{Ad}(y) \delta)$ (see 2.3). We write $(J_n, J'_n, u_n)_{n \geq 0} = (\beta(P, P', \gamma)$. For $s \in S(J, \text{Ad}(y) \delta)$ let
$$Z^{s}_{J,y,\delta} = \{(P, P', \gamma) \in Z_{J,y,\delta}; \beta(P, P', \gamma) = s \}.$$ 

Clearly, $(Z^{s}_{J,y,\delta})_{s \in S(J, \text{Ad}(y) \delta)}$ is a partition of $Z_{J,y,\delta}$ into locally closed subvarieties. The $G$-action on $Z^{s}_{J,y,\delta}$ given by $g: (P, P', \gamma) \mapsto (g P, g P', g \gamma g^{-1})$ preserves each of the pieces $Z^{s}_{J,y,\delta}$. Now $(P, P', \gamma) \mapsto (P_1, P'_1, \gamma_1)$ is a morphism $f: Z^{s}_{J,y,\delta} \rightarrow Z^{s}_{1,y,\delta}$ where for $s = (J_n, J'_n, u_n)_{n \geq 0} \in S(J, \text{Ad}(y) \delta)$ we set $s^1 = (J_n, J'_n, u_n)_{n \geq 1} \in S(J_1, \text{Ad}(y_1) \delta)$.

**Lemma 8.19.** (a) The morphism $\theta: Z^{s}_{J,y,\delta} \rightarrow Z^{s}_{1,y,\delta}$ is a locally trivial fibration with fibres isomorphic to an affine space of dimension $(l(u_0) + \nu_J - \nu_{J_1})$.

(b) Let $\overline{\theta}$ be the map from the set of $G$-orbits on $Z^{s}_{J,y,\delta}$ to the set of $G$-orbits on $Z^{s}_{1,y,\delta}$ induced by $\theta$. Then $\overline{\theta}$ is a bijection.

We prove (a). Let $(P, P', \gamma) \in Z^{s}_{J,y,\delta}$. From 8.17 and 8.1(a) we see that each fibre of $\theta$ is an affine space of dimension
$$\dim(U_P/(U_P \cap U_{P'})) = l(u_0) + \nu_J - \nu_{J \cap \text{Ad}(u^{-1})_{J'}}, \quad l(u_0) + \nu_J - \nu_{J_1} \quad = l(u_0) + \nu_J - \nu_{J_1}.$$ 

The verification of local triviality is omitted.

We prove (b). From the fact that $\theta$ is surjective (see (a)) and $G$-equivariant, it follows that $\overline{\theta}$ is well defined and surjective. We show that $\overline{\theta}$ is injective. Let $(P, P', \gamma), (\tilde{P}, \tilde{P}', \tilde{\gamma})$ be two triples in $Z^{s}_{J,y,\delta}$ whose images under $\theta$ are in the same $G$-orbit; we must show that these two triples are in the same $G$-orbit. Since $\theta$ is $G$-equivariant, we may assume that $\theta(P, P', \gamma) = \theta(\tilde{P}, \tilde{P}', \tilde{\gamma}) = (Q, Q', \gamma_1) \in Z^{s}_{1,y,\delta}$. Define $F$ in terms of $(Q, Q', \gamma_1)$ as in 8.13. Then $(P, P', \gamma) \in F, (\tilde{P}, \tilde{P}', \tilde{\gamma}) \in F$. Since $P, \tilde{P}$ are parabolics of the same type containing $Q'$ we have
Let $g \in \gamma \cap A'_f(P, P')$. By Lemma 8.15, there exist $u \in U_P, v \in U_P \cap P'$ such that $P' = uP', \tilde{\gamma} = U_{sv}uvgU_P$. We have also $P = uP$ (since $uv \in P$), $P' = uP'$ (since $v \in P'$),
\[
\tilde{\gamma} = uU_PuvgU_P = uvU_PgvU_P = uvU_PgU_Pv^{-1}u^{-1} = uv\gamma v^{-1}u^{-1}
\]
(since $v$ normalizes $U_P$ and $uv \in U_P$). Thus, $(\tilde{P}, \tilde{P}', \tilde{\gamma})$ is obtained by the action of $uv \in G$ on $(P, P', \gamma)$, hence $(\tilde{P}, \tilde{P}', \tilde{\gamma})$ is in the $G$-orbit of $(P, P', \gamma)$. The lemma is proved.

**Lemma 8.20.** Let $s = (J_n, J'_n, u_n)_{n \geq 0} \in S(J, \text{Ad}(y)\delta)$. Then $Z^s_{J,y,\delta}$ is an iterated affine space bundle (with fibre dimension $l(w) + \nu_J - \nu_{J_m}$, $w = u_0u_1 \cdots u_m$, $m \gg 0$) over a fibre bundle over $P_{J_m}$ with fibres isomorphic to $P'/U_P$ with $P \in P_{J_m}$, $m \gg 0$. In particular, $Z^s_{J,y,\delta} \neq \emptyset$.

Assume first that $s$ is such that $J_n = J'_n = J$ and $u_n = 1$ for all $n \geq 0$. (Then $\text{Ad}(y)\delta(J) = J$, $y \in J^W(\delta)$. In this case, $Z^s_{J,y,\delta}$ is the set of all $(P, P', \gamma)$ with $P = P' \in P_J$, $\gamma \in U_P \backslash A_p(P, P)/U_P$. (The associated sequence $(P_n, P'_n, \gamma_n)$ is in this case $P_n = P'_n = P$, $\gamma_n = \gamma$.) Thus, $Z^s_{J,y,\delta}$ is a locally trivial fibre over $P_J$ with fibres isomorphic to $P'/U_P$ for $P \in P_J$ and the lemma holds.

We now consider a general $s$. For any $r \geq 0$ let
\[
s_r = (J_n, J'_n, u_n)_{n \geq r} \in S(J_r, \text{Ad}(y_r)\delta)
\]
($y_r$ as in 8.18). By 8.19(a) we have a sequence of affine space bundles
\[
Z^s_{J,y,\delta} \to Z^s_{J_{r+1}, y_{r+1}, \delta} \to Z^s_{J_{r+2}, y_{r+2}, \delta} \to \cdots
\]
where for $r \gg 0$, $Z^s_{J_{r}, y_{r}, \delta}$ is as in the first part of the proof. By 8.19(a), the sum of dimensions of fibres of the maps in this sequence is
\[
\sum_{n \geq 0} (l(u_n) + \nu_{J_n} - \nu_{J_{n+1}}) = \sum_{n \geq 0} l(u_n) + \nu_{J_0} - \nu_{J_m} = l(w) + \nu_J - \nu_{J_m}
\]
where $m \gg 0$. The lemma follows.

**Remark.** If $\hat{G}$ is defined over the finite field $\mathbf{F}_p$ and $k$ is the algebraic closure of $\mathbf{F}_p$, then the number $N$ of rational points of $Z^s_{J,J',w}$ over a sufficiently large finite subfield $\mathbf{F}_q$ of $k$ equals
\[
\#(G(F_q)q^{l(w) + \nu_J - \nu_{J_m}}.
\]
Indeed, for $m \gg 0$ we have $N = \#(G(F_q)/P_m(F_q))q^{l(w) + \nu_J - \nu_{J_m}}$. Note that $l(w) + \nu_J - \nu_{J_m} \geq 0$.

**8.21.** In the setup of 8.20, the maps in 8.20(a) induce bijections on the sets of $G$-orbits (see 8.19(b)). Thus we obtain a canonical bijection between the set of $G$-orbits on $Z^s_{J,J',w}$ and the set of $G$-orbits on $Z^s_{J_{r}, y_{r}, \delta}$ with $r$ large enough so that $J_r = J'_r = J_{r+1} = J'_{r+1} = \cdots$, $u_r = u_{r+1} = \cdots = 1$. This last set of orbits is canonically the set of $Q$-orbits on $U_Q \backslash A_p(Q, Q)/U_Q$ where $Q \in P_J$. The $Q$-action (conjugation) factors through $Q/U_Q$. Let $L^*$ be a Levi of $Q$. Then
\[
C^* = \{g \in G^1; gL^* = L^*, \text{pos}(Q, gQ) = y_r\}
is a connected component of $N_G(L^s)$. We have an obvious bijection

$$C^s \cong U_Q \setminus A_{y_r}(Q, Q)/U_Q$$

under which the action of $L^s$ on $C^s$ by conjugation corresponds to the action of $Q/U_Q$ on $U_Q \setminus A_{y_r}(Q, Q)/U_Q$ by conjugation. Thus we obtain a canonical bijection between the set of $G$-orbits on $Z_{J,y,\delta}$ and the set of $L^s$-conjugacy classes in $C^s$ (a connected component of an algebraic group with identity component $L^s$). Putting together these bijections we obtain a bijection

$$G \setminus Z_{J,y,\delta} \leftrightarrow \bigcup_{s \in S(J, \text{Ad}(y)\delta)} L^s \setminus C^s$$ (a)

where $G \setminus Z_{J,y,\delta}$ is the set of $G$-orbits on $Z_{J,y,\delta}$ and $L^s \setminus C^s$ is the set of $L^s$-orbits on $C^s$ (for the conjugation action).

9. Comparison of Two Partitions

9.1. In the case where $y = 1$, Sections 3 and 8 provide two methods to partition $Z_{J,y,\delta}$. In this section we show that the resulting partitions of $Z_{J,y,\delta}$ are the same. Lemmas 9.2, 9.3 hold for any $y$, but in 9.4, 9.5 we assume that $y = 1$.

**Lemma 9.2.** Let $(P, P', \gamma) \in Z_{J,y,\delta}$. Let $n \geq 1$. Let $P'_1, P_n$ be as in 8.18. We have $\text{pos}(P', P_n) = \text{pos}(P', P_1^n) \text{pos}(P_1, P_n)$.

Let $z = \text{pos}(P', P'_1), \tilde{z} = \text{pos}(P', P_n), x = \text{pos}(P', P_n)$. We have $z = \text{pos}(P', P)$ hence $z \in J W^J$. We have also $\tilde{z} \in J W^J$. Using 2.1(b) with $x, x'$ replaced by $z, \tilde{z}$, we see that $\tilde{z} = z v$ with $v \in W_J$. Let $B, B' \in B$ be such that $B \subset P', B' \subset P_n$, $\text{pos}(B, B') = \tilde{z}$. Since $z \in W_J$, we have $l(zv) = l(z) + l(v)$. Hence there is a unique $B'' \in B$ such that $\text{pos}(B, B'') = z$, $\text{pos}(B'', B') = v$. Since $B' \subset P$ and $\text{pos}(B'', B') \subset W_J$, we have $B'' \subset P$. Since $B \subset P', B'' \subset P$ and $\text{pos}(B, B'') = \text{pos}(P', P) = z$, we have $B'' \subset P_P' = P'_1$. Since $B'' \subset P_1, B' \subset P_n$, we have $\text{pos}(B'', B') \geq x$, hence $v \geq x$. We can find $B_1, B_2 \in B$ such that $B_1 \subset P'_1, B_2 \subset P_n, \text{pos}(B_1, B_2) = x$. Since $\text{pos}(P', P) = z$ and $B_1 \subset P_P'$, we can find $B_0 \in B$ such that $B_0 \subset P', \text{pos}(B_0, B_1) = z$. Since $z \in W_J, x \in W_J$, we have $\text{pos}(B_0, B_2) = z x$. We have $B_0 \subset P_P', B_2 \subset P_n$ hence $\text{pos}(B_0, B_2) \geq \text{pos}(P_1, P_n)$, that is, $zx \geq z = z v$. Thus, we have $v \geq x$ and $zx \geq z v$. Since $x \in W^J$ and $x, v \in W_J$ we have $x = v$. Thus, $z = zx$. The lemma is proved.

**Lemma 9.3.** Let $(P, P', \gamma) \in Z_{J,y,\delta}$. Let $P'_n, P_n, u_n, \gamma_n$ be as in 8.18. For any $n \geq 0$ we have $\text{pos}(P', P_n) = u_0 u_1 \ldots u_n$.

We argue by induction on $n = 0$. For $n = 0$ the result is clear. Assume now that $n \geq 1$. We have $\text{pos}(P', P'_n) = \text{pos}(P', P_P') = \text{pos}(P', P) = u_0$. Using the induction hypothesis for $(P'_1, P_1, \gamma_1)$ and $n - 1$ (instead of $(P, P', \gamma)$ and $n$) we see that $\text{pos}(P'_1, P_n) = u_1 \ldots u_n$. By Lemma 9.2 we have $\text{pos}(P', P_n) = \text{pos}(P', P'_1) \text{pos}(P'_1, P_n) = u_0 u_1 \ldots u_n)$. The lemma is proved.

**Lemma 9.4.** Let $J \subset I$. Let $(P, P', g) \in Z_{J,\delta}$. To $(P, P', g)$ we associate $P^u, P^n, u_n$ as in 4.11. To $(P, P', U_P g U_P)$ we associate $P_n, P'_n, \gamma_n, u_n, y_n$ as in 8.18 (with $y = 1$). For any $n \geq 0$, the following hold:
We argue by induction on \( n \). The result is obvious for \( n = 0 \): we have \( P_0 = P^0 = P \); \( s P_0 = P' \) contains a Levi of \( P \cap P' \); we have \( P'(P'^T) = P''P \). We have \( w_0 = w_0 \). We now assume that \( n \geq 1 \) and that \((a_{n-1}), (b_{n-1}), (c_{n-1}), (d_{n-1}), (e_{n-1})\) hold.

We show that \((a_n)\) holds. Using \((c_{n-1}), (a_{n-1}), (d_{n-1})\), we have
\[
P_n = g^{-1}(s P_{n-1}) (P'_{n-1})^{-1} g = g^{-1}(s P_{n-1}) (P'_{n-1})^{-1} g = g^{-1}(P''_{n-1}) (P'_{n-1})^{-1} g = P_n^0;
\]
hence \((a_n)\) holds.

We show that \((b_n)\) holds, using \((a_{n-1}), (b_{n-1}), (c_{n-1}), (a_n)\). It suffices to show that \( Y' = P'_n \) satisfies the hypotheses of Lemma 3.2(a) with \( P, P', a, J, \delta(J) \) replaced by \( P_{n-1}, P''_{n-1}, w_{n-1}, J_{n-1}, \delta(J_{n-1}) \). By definition we have \( Y' \cap Y_{n-1} = P''_{n-1}. \)

The type of \( Y' \) is \( J_{n-1} \cap J \). Since \( (P'_{n-1})^{A_{n-1}}(P'_{n-1})^{-1} \delta(J_{n-1}) \) and that of \( (P''_{n-1})^{A_{n-1}} \)

is \( J_{n-1} \cap J \). Thus, \( Y' \) and \( (P''_{n-1})^{A_{n-1}} \) have the same type. Since \( s P_n = s P_{n-1} = P''_{n-1} = P''_{n-1} \) and \( P'_{n-1} \), \( s P_n \) are in good position, pos\((P'_{n-1})^{A_{n-1}} P_n = (u_0 u_1 \ldots u_{n-1})^{-1} = w_{n-1} \), we see that \( (P''_{n-1})^{A_{n-1}}, P''_{n-1} \) are in good position, pos\((P''_{n-1})^{A_{n-1}}, P''_{n-1} = w_{n-1}. \)

Thus, the hypotheses of Lemma 3.2(a) are satisfied and \((b_n)\) holds.

We show that \((c_n)\) holds, using \((a_n), (b_n), (c_{n-1})\). Since \( g \in \gamma_{n-1} \cap A_{\gamma_{n-1}}(P_{n-1}, P'_{n-1}) \) and \( g \in \gamma_{n-1} \cap A_{\gamma_{n-1}}(P_{n-1}, P''_{n-1}) \) (see 8.10) we have \( g \in \gamma_{n-1} \cap A_{\gamma_{n-1}}(P_{n-1}, P'_{n-1}) \). It remains to show that \( s P_n \) contains a Levi of \( P_n \cap P'_n \) or equivalently, that \( (P''_{n-1})^{A_{n-1}} \) contains a Levi of \( (P''_{n-1})^{A_{n-1}} \cap g^{-1}(P''_{n-1})^{-1} g \). This follows from Lemma 3.2(b) with \( P, P', Z \) replaced by \( P_{n-1}, P''_{n-1}, g^{-1}(P''_{n-1})^{-1} g \).

We show that \((d_n)\) holds, using \((a_n), (b_n)\). This follows from Lemma 3.2(d) with \( P, P', Z \) replaced by \( P_{n-1}, P''_{n-1}, g^{-1}(P''_{n-1})^{-1} g \).

We show that \((e_n)\) holds, using \((a_n)\). Since pos\((P''_{n-1})^{A_{n-1}} = w_n, P''_{n-1} \subset P'_{n-1} \) and \( w_n \in J \) we have pos\((P'_{n-1})^{A_{n-1}} = w_n. \) We also have pos\((P'_{n-1})^{A_{n-1}} = u_0 u_1 \ldots u_n \) (see Lemma 9.3). Since \( P'' = P_n \), we have \( w_n = u_0 u_1 \ldots u_n \).

This completes the inductive proof.

**Proposition 9.5.** Let \( J \subseteq I \). Let \( s = (J_n, J'_n, u_n)_{n \geq 0} \in S(J, \delta) \) and let \( t \in T(J, \delta) \) be the corresponding element under the bijection in 2.4. Then \( t Z_{J,\delta} = t Z_{J,\delta} \).

Let \( P \in P_J, P' \in P_{J'}(J), g \in A_1(P, P') = A_1'(P, P') \). Assume that
\[
(P, P', U_P g U_P) \in Z_{J,\delta}^s, \quad (P, P', U_P g U_P) \in Z_{J,\delta}^s
\]
where \( s = (J_n, J'_n, u_n)_{n \geq 0} \in S(J, \delta) \). By Lemma 9.4(c), we see that \( \tilde{u}_0 \tilde{u}_1 \ldots \tilde{u}_n = u_0 u_1 \ldots u_n \) for all \( n \). Using Proposition 2.5, we have \( t Z_{J,\delta} \subset Z_{J,\delta}^s. \)
Conversely, let \((Q, Q', \gamma) \in Z^*_J, \delta\). We have \((Q, Q', \gamma) \in t' Z_{J, \delta}\) for a unique \(t' \in T(J, \delta)\). By the first part of the proof we have \((Q, Q', \gamma) \in Z^*_J, \delta\) where \(s' \in S(J, \delta)\) corresponds to \(t'\) under the bijection in 2.4. We have \((Q, Q', \gamma) \in Z^*_J, \delta\) and the sets \(Z^*_J, \delta, Z'_{J, \delta}\) are either disjoint or coincide. Thus, \(s = s'\) hence \(t = t'\). We see that \(Z^*_J, \delta \subset t' Z_{J, \delta}\). The proposition is proved.

10. Example

10.1. We consider an example. Let \(V\) be a finite dimensional \(k\)-vector space. Let 
\[ G = \hat{G} = G^1 = \mathrm{GL}(V). \]
Consider two \(n\)-step filtrations \(V_s, V'_s\):
\[ 0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n = V, \quad 0 \subset V'_1 \subset V'_2 \subset \cdots \subset V'_n = V \]
of \(V\). The type of \(V_s\) is defined by the set
\[ J = \{ i \in [1, n - 1]; \dim V_k \neq i \text{ for any } k \in [0, n]\}. \]
To \(V_s, V'_s\) we associate two \(n^2\)-step filtrations \(X_s, X'_s\):
\[ 0 = X_{10} \subset X_{11} \subset X_{12} \subset \cdots \subset X_{1n} = X_{20} \subset X_{21} \subset X_{22} \subset X_{2n} = X_{30} \subset \cdots \subset X_{n,n-1} \subset X_{nn} = V, \]
\[ 0 = X'_{10} \subset X'_{11} \subset X'_{12} \subset \cdots \subset X'_{1n} = X'_{20} \subset X'_{21} \subset X'_{22} \subset X'_{2n} = X'_{30} \subset \cdots \subset X'_{n,n-1} \subset X'_{nn} = V, \]
where
\[ X_{ij} = V_{i-1} + (V_i \cap V_j), \quad (i, j) \in [1, n] \times [0, n], \]
\[ X'_{ij} = V'_{i-1} + (V'_i \cap V'_j), \quad (i, j) \in [1, n] \times [0, n]. \]
Here the indexing set is \([1, n] \times [0, n]\) with the identifications \(1n = 20, \ldots, (n-1)n = n0\). We have \(X_{i0} = V_{i-1}, X'_{i0} = V'_{i-1}\) for \(i \in [1, n]\). Hence \(X_s\) (resp. \(X'_s\)) is a refinement of \(V_s\) (resp. \(V'_s\)). If the stabilizer of \(V_s\) (resp. \(V'_s\)) is the parabolic \(P\) (resp. \(P'\)) then the stabilizer of \(X_s\) (resp. \(X'_s\)) in \(G\) is the parabolic \(P^{裴}\) (resp. \(P'^{裴}\)). By Zassenhaus’ lemma, we have a canonical isomorphism
\[ t: X'_{ij}/X'_{i,j-1} \tilde{\rightarrow} X_{ij}/X_{j,i-1} \quad \text{for all } (i, j) \in [1, n] \times [1, n]. \]
Assume that we are given a permutation \(\sigma: [1, n] \rightarrow [1, n]\) and vector space isomorphisms
\[ a_i: V_i/V_{i-1} \rightarrow V'_{\sigma(i)}/V'_{\sigma(i)-1} \quad \text{for } i \in [1, n]. \]
Define a third \(n^2\)-step filtration \(Y_s\) (refining \(V_s\)):
\[ 0 = Y_{10} \subset Y_{11} \subset Y_{12} \subset \cdots \subset Y_{1n} = Y_{20} \subset Y_{21} \subset Y_{22} \subset Y_{2n} = Y_{30} \subset \cdots \subset Y_{n,n-1} \subset Y_{nn} = V, \]
\[ Y_{i0} = V_{i-1} \quad \text{for } i \in [1, n], \]
\(Y_{ij}\) is the subspace of \(V_{ij}\) containing \(V_{i-1}\) such that \(a_i\) carries the subspace \(Y_{ij}/Y_{i0}\) of \(V_i/V_{i-1}\) onto the subspace \(X'_{\sigma(i),j}/X'_{\sigma(i),j-1}\) of \(V'_{\sigma(i)}/V'_{\sigma(i)-1}\).
The composition
\[ Y_{ij}/Y_{i,j-1} \xrightarrow{a_i} X'_{\sigma(i),j}/X'_{\sigma(i),j-1} \xrightarrow{t} X_{j,i}/X_{j,i-1} \]
is an isomorphism $b_{ij}$ for $(i, j) \in \{1, n\} \times \{1, n\}$. Define a permutation $\tau: [1, n] \times [1, n] \to [1, n] \times [1, n]$ by $\tau(i, j) = (j, \sigma(i))$.

Let $\Sigma$ be the set of all quadruples $(V_*, V'_*, a_i)$ with $V_*, V'_*$ as above (of prescribed types) and $a_i$ are as above ($\sigma$ is fixed). Then $\Sigma$ may be identified with a set $Z_{J,y,1}$ attached to $G = \text{GL}(V)$. Here $J, \text{Ad}(y)J$ are the types of $V_*, V'_*$. Let $(V_*, V'_*, a_i) \in \Sigma$. We define a sequence $(V^m_*, V'^m_*, \sigma^m, a^m_{i, \sigma^m})_{m \geq 0}$ of quadruples of the same kind as $(V_*, V'_*, a_i)$. Assume that $m \geq 1$ and that $(V^{m-1}_*, V'^{m-1}_*, \sigma^{m-1}, a^{m-1}_{i, \sigma^{m-1}})$ is already defined. Then $(V^m_*, V'^m_*, \sigma^m, a^m_{i, \sigma^m})$ is attached to $(V^{m-1}_*, V'^{m-1}_*, \sigma^{m-1}, a^{m-1}_{i, \sigma^{m-1}})$ in the same way as $(Y_*, X_*, \tau, b_{ij})$ was attached to $(V_*, V'_*, \sigma, a_i)$. Then $V^m_*, V'^m_*$ are $n^m$-step filtrations of $V$ and, for $m > 0$, $V^m_*, V'^m_*$ are refinements of $V^m_*$. Let $J_m$ (resp. $J'_m$) be the type of $V^m_*$ (resp. $V'^m_*$). The set of all $(V_*, V'_*, a_i) \in \Sigma$ such that $J_m, J'_m$ and the relative position of $V^m_*, V'^m_*$ is specified for each $m \geq 0$ is a locally closed subvariety of $\Sigma$. Thus we obtain a partition of $\Sigma$ which coincides with the partition $Z_{J,y,1} = \bigcup_k Z_{J,y,1}^k$.

**10.2.** Let $V$ be a $k$-vector space of dimension $d \geq 2$. Let $\Sigma$ be the set of all quadruples $(V_1, V'_1, a, b)$ where $V_1, V'_1$ are lines in $V$, and $a: V_1 \rightleftarrows V'_1$, $b: V/V_1 \rightleftarrows V/V'_1$ are isomorphisms. (This is a special case of the situation in 10.1 where we omit the 0 and $d$ dimensional members of a 2-step filtration.) We describe explicitly in this case the partition of $\Sigma$ given in 10.1. For any $k \in \{1, d\}$ let $\Sigma^k$ be the set of all quadruples

$$(0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k, V'_1, a, b)$$

where $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k$ is a (partial) flag in $V$, $\dim V_j = j$ for all $j$, $V'_1$ is a line in $V$, $a: V_1 \rightleftarrows V'_1$, $b: V/V_1 \rightleftarrows V/V'_1$ are isomorphisms and

$$V'_1 \cap V_{k-1} = 0,$$

$$b(V'_j/V_1) = (V_{j-1} + V'_j)/V'_1 \quad \text{for } j \in [1, k].$$

Define $\pi_k: \Sigma^k \to \Sigma$ by $\pi_k(0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k, V'_1, a, b) = (V_1, V'_1, a, b)$. Then $\pi_k$ is injective and $(\pi_k(\Sigma^k))_{k \in \{1, d\}}$ is a partition of $\Sigma$ into $d$ locally closed subvarieties. This is a special case of the partition of $\Sigma$ in 10.1 and of the partition of $Z_{J,y,\delta}$ in 8.18.

**10.3.** Let $V$ be a $k$-vector space of dimension $d \geq 2$. Let $\Sigma$ be the set of all quadruples $(V_1, H, a, b)$ where $V_1$ is a line in $V$, $H$ is a hyperplane in $V$, and $a: V_1 \rightleftarrows V/H$, $b: V/V_1 \rightleftarrows H$ are isomorphisms. (This is a special case of the situation in 10.1 where we omit the 0 and $d$ dimensional members of a 2-step filtration.) We describe explicitly in this case the partition of $\Sigma$ given in 10.1. For any $k \in \{1, d\}$ let $\Sigma^k$ be the set of all quadruples

$$(0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_k, H, a, b)$$

where $0 = V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_{k-1} \subset V_k$ is a (partial) flag in $V$, $\dim V_j = j$ for all $j$, $H$ is a hyperplane in $V$, $a: V_1 \rightleftarrows V/H$, $b: V/V_1 \rightleftarrows H$ are isomorphisms and

$$V_{k-1} = V_k \cap H,$$

$$b(V_j/V_1) = V_{j-1} \quad \text{for } j \in [1, k].$$
Def\n
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11. PARABOLIC CHARACTER SHEAVES ON Z_{J,y,\delta}

11.1. Assume that we are in the setup of 8.18. Let \(x = (x_1, x_2, \ldots, x_r)\) be a sequence in \(W\) such that

\[ r \geq 2, \quad x_r = y. \]  

(a) As in 4.2 we define

\[ Y_{\alpha} = \{(B_0, B_1, \ldots, B_r, g) \in B^{r+1} \times G^2; \ \text{pos}(B_{i-1}, B_i) = x_i, i \in [1, r], B_r = ^gB_0\}. \]

Let \(Y'_{\alpha}\) be the set of all \((B_0, B_1, \ldots, B_{r-1}, \gamma)\) where \((B_0, B_1, \ldots, B_{r-1}) \in B^r\) satisfies \(\text{pos}(B_{i-1}, B_i) = x_i, i \in [1, r-1]\) and \(\gamma \in U_{P'} \setminus A_{y}(P, P') / U_{P}\) (with \(P \in \mathcal{P}_{J}, P' \in \mathcal{P}_{J'}\) given by \(P_0 \subset P, B_{r-1} \subset P'\)) satisfies \(\text{pos}(B_{r-1}, ^gB_0) = x_r\) for some/any \(g \in \gamma\). (This definition is correct since \(U_{P} \subset B_0, \ U_{P'} \subset B_{r-1}\).) We have an affine space bundle

\[ \omega: Y_{\alpha} \to Y'_{\alpha}, \quad (B_0, B_1, \ldots, B_r, g) \mapsto (B_0, B_1, \ldots, B_{r-1}, U_{P'}gU_{P}) \]

with \(P, P'\) as above). Define

\[ \Pi_{\alpha}: Y'_{\alpha} \to Z_{J,y,\delta}, \quad \Pi_{\alpha}(B_0, B_1, \ldots, B_{r-1}, \gamma) = (P, P', \gamma) \]

where \(P \in \mathcal{P}_{J}, P' \in \mathcal{P}_{J'}\) are given by \(B_0 \subset P, B_{r-1} \subset P'\).

If \(\mathcal{L} \in S(T)\) (see 4.1) is such that \(x_1x_2 \ldots x_r \in W_{2}^A\) (see 4.1), the local system \(\hat{\mathcal{L}}\) on \(Y_{\alpha}\) is defined (see 4.2); it is \(\omega^*\) of a well defined local system on \(Y'_{\alpha}\) again by \(\hat{\mathcal{L}}\). We set

\[ K^\mathcal{L}_{\alpha} = (\Pi_{\alpha})_{*}\hat{\mathcal{L}} \in \mathcal{D}(Z_{J,y,\delta}). \]

Now assume in addition that

\[ x_i \in I \cup \{1\} \quad \text{for} \ i \in [1, r-1]. \]  

(b)

Let \(Y'_{\alpha}^{+}\) be the set of all \((B_0, B_1, \ldots, B_{r-1}, \gamma)\) where \((B_0, B_1, \ldots, B_{r-1}) \in B^r\) satisfies \(\text{pos}(B_{i-1}, B_i) = \{x_i, 1\}, i \in [1, r-1]\), and \(\gamma \in U_{P'} \setminus A_{y}(P, P') / U_{P}\) (with \(P \in \mathcal{P}_{J}, P' \in \mathcal{P}_{J'}\) given by \(B_0 \subset P, B_{r-1} \subset P'\)) satisfies \(\text{pos}(B_{r-1}, ^gB_0) = y\) for some/any \(g \in \gamma\). Define

\[ \Pi^{+}_{\alpha}: Y'_{\alpha}^{+} \to Z_{J,y,\delta}, \quad \Pi^{+}_{\alpha}(B_0, B_1, \ldots, B_{r-1}, \gamma) = (P, P', \gamma) \]

where \(P \in \mathcal{P}_{J}, P' \in \mathcal{P}_{J'}\) are given by \(B_0 \subset P, B_{r-1} \subset P'\). Now \(Y_{\alpha}^{+}\) is an open dense subset of \(Y'_{\alpha}^{+}\) and it carries the local system \(\mathcal{L}\). The intersection cohomology complex \(IC(Y_{\alpha}^{+}, \hat{\mathcal{L}})\) is a constructible sheaf \(\hat{\mathcal{L}}\) on \(Y_{\alpha}^{+}\) (cf. 4.3); we set

\[ K^\mathcal{L}_{\alpha}^{+} = (\Pi^{+}_{\alpha})_{*}\hat{\mathcal{L}} \in \mathcal{D}(Z_{J,y,\delta}). \]

Now \(\Pi^{+}_{\alpha}\) is a proper morphism. (Indeed, the condition \(\text{pos}(B_{r-1}, ^gB_0) = y\) in the definition of \(Y'_{\alpha}^{+}\) can be replaced by the closed condition \(\text{pos}(B_{r-1}, ^gB_0) \leq y\) since \(^gB_0 \subset ^gP, B_{r-1} \subset P'\) and \(\text{pos}(P', ^gP) = y\).) Hence we may apply the decomposition theorem [BBD] and we see that

\[ K^\mathcal{L}_{\alpha}^{+} \text{ is a semisimple complex on } Z_{J,y,\delta}. \]
Proposition 11.2. Let $\mathcal{L} \in S(T)$ and let $A$ be a simple perverse sheaf on $Z_{J,y,\delta}$. The following conditions on $A$ are equivalent:

(i) $A \rightarrow K^L_x$ for some $x$ as in 11.1(a) with $x_1x_2 \ldots x_r \in W^1_L$;
(ii) $A \rightarrow K^L_x$ for some $x \in W$ such that $xy \in W^1_L$;
(iii) $A \rightarrow K^L_x$ for some $x$ as in 11.1(b) with $x_1x_2 \ldots x_r \in W^1_L$.

(Cf. 4.4.)

11.3. Let $C^L_{J,y,\delta}$ be the set of isomorphism classes of simple perverse sheaves on $Z_{J,y,\delta}$ which satisfy the equivalent conditions 11.2(i)–(iii) with respect to $\mathcal{L}$. The simple perverse sheaves on $Z_{J,y,\delta}$ which belong to $C^L_{J,y,\delta}$ for some $\mathcal{L} \in T$ are called parabolic character sheaves; they (or their isomorphism classes) form a set $C_{J,y,\delta}$.

11.4. Let $(P, P', \gamma) \in Z_{J,y,\delta}$. Let $\gamma' = \gamma \cap A'_\nu(P, P')$. Let $z = \text{pos}(P', P)$. Let $(P_1, P'_1) = \alpha(P, P', g)$ where $g \in \gamma'$ (see 8.10, 8.11). We have $\text{pos}(P'_1, \bar{g}P_1) = z^{-1}y$ and $\text{pos}(P'^P, P'^{P'}) = z$ (see 8.10). Let $w \in W$. We can write uniquely $w = ab$, $a \in W^J$, $b \in W^J$. Let

$X' = \{(B, B') \in B \times B; \text{pos}(B, B') = w, \text{pos}(\tilde{B}, \tilde{B}') = y, B \subset P, B' \subset P'\}$. Here $g \in \gamma'$; the choice of $g$ is irrelevant since $U_P \subset B, U_{P'} \subset B'$. Set $b' = y^{-1}by \in W_{\tilde{a}(J)}$. (Recall that $gW_{\tilde{a}(J)} = W_{\tilde{a}(J)}g$.) Let

$X' = \{(\tilde{B}, B, \tilde{B}') \in B^3; \text{pos}(\tilde{B}, B) = \delta^{-1}(b'), \text{pos}(B, \tilde{B}') = az, \text{pos}(\tilde{B}', b) = z^{-1}y, B \subset P_1, \tilde{B}' \subset P'_1\}.$

(We have automatically $B \subset P$.) Here $g \in \gamma'$; the choice of $g$ is irrelevant since $U_{P_1} \subset B'$, $U_{P'_1} \subset \tilde{B}$ (another choice of $g$ is in $U_{P'_1}gU_{P_1}$ by 8.11). Define

$X \rightarrow X', \ (B, B') \mapsto (\tilde{B}, B, \tilde{B}')$

as follows. Define $R$ by $\text{pos}(B, R) = a$, $\text{pos}(R, B') = b$ (we have $l(w) = l(a) + l(b)$). Define $B'$ by $\text{pos}(B, B') = az$, $\text{pos}(\tilde{B}', R) = z^{-1}$ (we have $l(az) + l(z^{-1}) = l(a)$). Define $S_y$ (in terms of $g \in \gamma'$) by $\text{pos}(R, S_y) = y, \text{pos}(S_y, b) = \tilde{b}$; we use $l(b) + l(y) = l(by) = l(yb') = l(y) + l(b')$. Set $\tilde{B} = g^{-1}S_y$. We will show below that

(i) if $(B, B') \in X$ then $(\tilde{B}, B, \tilde{B}') \in X'$;
(ii) $\tilde{B}$ is independent of the choice of $g$ in $\gamma'$.

Assume that (i) is already established (for fixed $g \in \gamma'$). Let us replace $g$ by $g_1 \in \gamma'$. Then $g_1 = u'gu$ where $u \in U_P, u' \in U_{P'}$. We have $\text{pos}(R, S_{g_1}) = y, \text{pos}(S_{g_1}, u'guB) = b'$. Hence

$\text{pos}(u'^{-1}R, u'^{-1}S_{g_1}) = y, \text{pos}(u'^{-1}S_{g_1}, b) = b'$. 

Since $\text{pos}(R, B') \in W_{\tilde{a}(J)}$ we have $R \subset P'$ hence $u' \subset R$ and $u'^{-1}R = R$ and $\text{pos}(R, u'^{-1}S_{g_1}) = y, \text{pos}(u'^{-1}S_{g_1}, b) = \tilde{b}$. It follows that $S_y = u'^{-1}S_{g_1}$. Hence

$u'^{-1}g^{-1}u'^{-1}S_{g_1} = u'^{-1}g^{-1}u'^{-1}(u'S_y) = u'^{-1}g^{-1}S_y = u'^{-1} \tilde{B} = \tilde{B}$

(since by (i) we have $\tilde{B} \subset P_1 \subset P$, hence $u \in \tilde{B}$). Thus (ii) is verified.
We now verify (i) (with fixed $g \in \gamma'$). Since $P^{\mu'}, \mu' P, g P_1$ have a common Levi $L_0$, there is a canonical bijection $0B \leftrightarrow 1B \leftrightarrow 2B$ between the sets of Borels of $P^{\mu'}, \mu' P$ respectively, defined by $0B = U_{P^{\mu'}} \beta, 1B = U_{P'} \beta, 2B = U_{P_1} \beta$ where $\beta$ is a Borel of $L_0$ or equivalently by

$$
pos(0B, 1B) = z^{-1}, \quad \pos(1B, 2B) = y, \quad \pos(0B, 2B) = z^{-1}y$$

(any two of these three conditions implies the third).

Assume that $B, B', R, S = S' \subset g B, \tilde{B}, \tilde{B}'$ are as above. We have $R \subset P'\mu$ (since $a = \pos(B, R) = \pos(B, P')$) and $\tilde{B}' \subset P^{\mu'}$ (since $z^{-1} = \pos(\tilde{B}', R) = \pos(\tilde{B}', P')$).

Hence $\tilde{B}' \leftrightarrow R \leftrightarrow Y$ under the bijection above where $Y$ is a Borel of $g P_1$. (We have $\pos(\tilde{B}', R) = z^{-1}, \pos(R, Y) = y, \pos(\tilde{B}', Y) = z^{-1}y$.)

Since $P^\mu, gP$ have a common Levi, there is a canonical bijection $B^0 \leftrightarrow B_1$ between the sets of Borels of $P^\mu, gP$ respectively, defined by $\pos(B^0, B_1) = y$. Since $\pos(R, Y) = y, we have $R \leftrightarrow Y$ under this bijection; since $\pos(R, S) = y$, we have $R \leftrightarrow S$ under this bijection; hence $Y \leftrightarrow S$. We see that $S \subset g P_1$ and $\tilde{B} \subset P_1$. We have $\pos(S, gB) = b'$ hence $\pos(\tilde{B}, B) = \delta'(b')$. This completes the proof of (i).

Define

$$X' \xrightarrow{\mu'} X, \quad (\tilde{B}, B, \tilde{B}') \mapsto (B, B')$$

as follows. Choose $g \in \gamma'$. Define $R, S$ by the condition that $\tilde{B}' \leftrightarrow R \leftrightarrow S$ under the canonical bijection above. Define $B'$ by the condition $\pos(R, B') = b, \pos(B', gB) = y$ (we use $l(b) + l(y) = l(by) = l(yb') = l(y) + l(b')$). We will show that

(iii) if $(\tilde{B}, B, \tilde{B}') \in X'$ then $(B, B') \in X$;

(iv) $B'$ is independent of the choice of $g$ in $\gamma'$.

Now (iv) follows from (iii) in the same way that (ii) follows from (i). We now verify (iii) (with fixed $g \in \gamma'$). Assume that $\tilde{B}, B, \tilde{B}', R, B'$ are as above. We have $R \subset P^{\mu'}$ hence $R \subset P'$. Since $\pos(R, B') \in W_{J'}$, we have $B' \subset P'$. We have $\pos(B, \tilde{B}') = az, \pos(\tilde{B}', R) = z^{-1}, l(az) + l(z^{-1}) = l(a)$ hence $\pos(B, R) = a$. This, together with $\pos(R, B') = b, l(ab) = l(a) + l(b)$ implies $\pos(B, B') = ab = w$.

Thus $(B, B') \in X$ and $\mu'$ is well defined.

From the definitions we see that $\mu'$ is the inverse of $\mu$.

11.5. For $z \in J' W'$ let $Z_{J,y,\delta,z} = \{(P, P', \gamma) \in Z_{J,y,\delta}; \pos(P, P') = z\}$. As in 8.10, 8.11 we have a well defined map

$$Z_{J,y,\delta,z} \to Z_{J_1,z^{-1}y,\delta}, \quad (P, P', \gamma) \mapsto (P^1, P'^1, \gamma_1)$$

where $\gamma_1$ is given by $\gamma_1 \subset \gamma \cap \mathcal{A}_y(P, P')$. Let $^yY_{w,y}^{\mu'}$ be the inverse image of $Z_{J,y,\delta,z}$ under the canonical map $^yY_{w,y}^{\mu'} \to Z_{J,y,\delta}$. For $a, b'$ as in 11.4, let

$$^yY_{\delta^{-1}(b'),az,z^{-1}y}^{\mu'} = z^{-1}^yY_{\delta^{-1}(b'),az,z^{-1}y}^{\mu'} \times Z_{J_1,z^{-1}y,\delta} \to Z_{J,y,\delta,z}$$

where the fibre product is formed using the canonical maps

$$z^{-1}^yY_{\delta^{-1}(b'),az,z^{-1}y} \to Z_{J_1,z^{-1}y,\delta} \to Z_{J,y,\delta,z}$$

The results in 11.4 provide an isomorphism

$$^yY_{w,y}^{\mu'} z \simeq ^yY_{\delta^{-1}(b'),az,z^{-1}y}^{\mu'}$$
compatible with the natural maps of the two sides into \( Z \). Hence in the cartesian diagram

\[
\begin{array}{ccc}
Y''_{\delta^{-1}(b'),az,z^{-1}y} & \longrightarrow & z^{-1}y_{Y''_{\delta^{-1}(b'),az,z^{-1}y}} \\
\uparrow & & \uparrow \\
Z_{J,y,\delta,z} & \longrightarrow & Z_{J',y,\delta,z}
\end{array}
\]

we may substitute \( Y''_{\delta^{-1}(b'),az,z^{-1}y} \) by \( Y'_{w,y} \) and we obtain a cartesian diagram

\[
\begin{array}{ccc}
y_{Y'_{w,y}} & \longrightarrow & z^{-1}y_{Y'_{\delta^{-1}(b'),az,z^{-1}y}} \\
\uparrow & & \uparrow \\
Z_{J,y,\delta,z} & \longrightarrow & Z_{J',z^{-1}y,\delta,z}
\end{array}
\]

11.6. In the setup of 11.5, let \( \mathcal{L}, \mathcal{L}' \in S(T) \) be such that

\[
\mathcal{L}' = \text{Ad}(\delta^{-1}(b'^{-1}))^* \mathcal{L}
\]

(we have \( \delta^{-1}(b'^{-1}) \in W_J \)) and

\[
w_y \in W^1_{\mathcal{L}}, \quad \delta^{-1}(b')az(z^{-1}y) \in W^1_{\mathcal{L}}.
\]

(These two conditions are equivalent. In general, for \( v \in W \) we have \( W^1_{\text{Ad}(v)^* \mathcal{L}} = v^{-1}W^1_{\mathcal{L}} \). In our case we have

\[
\delta^{-1}(b')az(z^{-1}y) = \delta^{-1}(b')(ayb')b'^{-1} = \delta^{-1}(b')wyb'^{-1}.
\]

Let \( \mathcal{L}', \mathcal{L}'' \) the local systems on \( w_{Y''_{w,y}}z^{-1}y_{Y'_{\delta^{-1}(b'),az,z^{-1}y}} \) corresponding as in 11.1 to \( \mathcal{L}, \mathcal{L}' \). From the definitions we see that the inverse image of \( \mathcal{L}' \) under \( w_{Y''_{w,y}}z \rightarrow z^{-1}y_{Y'_{\delta^{-1}(b'),az,z^{-1}y}} \) (in the cartesian diagram above) is the same as the restriction of \( \mathcal{L}'' \) to \( w_{Y''_{w,y}}z \).

11.7. In the setup of 11.4, we assume in addition that \( b \in W_{\text{Ad}(y)}(J_1) \) that is, \( b' \in W_{\text{Ad}(y)}(J_1) \) where \( J_1 = J \cap \delta^{-1} \text{Ad}(y^{-1}z)J \). Assume that \( (B, B') \in \mathcal{X} \). We show that

\[
B \subset P_1.
\]

Let \( R, S, \tilde{B}, \tilde{B}' \) be as in the definition of \( \mu \) in terms of some \( g \in \gamma' \). (Thus, \( \mu(B, B') = (\tilde{B}, B, \tilde{B}') \)). By 11.4, we have \( S \subset \delta P_1 \). Since \( \text{pos}(S, gB) = b' \in W_{\delta(J_1)} \) and \( gP_1 \) has type \( \delta(J_1) \), we have \( gB \subset gP_1 \) and \( B \subset P_1 \).

11.8. In the setup of 11.4 and 11.7, for \( B, B', \tilde{B}, \tilde{B}' \) as in 11.7 we have \( az = \text{pos}(B, B') \in J^1W \) (since \( B \subset P_1 \)) and \( \text{pos}(B, B) = \delta^{-1}(b') \in W_J \), hence

\[
l(\delta^{-1}(b')az) = l(\delta^{-1}(b')) + l(az).
\]

It follows that \( \text{pos}(\tilde{B}, \tilde{B}') = \delta^{-1}(b')az \). We see that \( (\tilde{B}, B, \tilde{B}') \mapsto (\tilde{B}, \tilde{B}') \) defines an isomorphism \( \mathcal{X}' \cong \mathcal{X}' \) where

\[
\mathcal{X}' = \{(\tilde{B}, \tilde{B}') \in B^2; \text{pos}(\tilde{B}, \tilde{B}') = \delta^{-1}(b')az, \text{pos}(\tilde{B}, \tilde{B}') = z^{-1}y, \tilde{B} \subset P_1, \tilde{B}' \subset P_1'\}.
\]
Combining with the isomorphism $\mu \colon \mathcal{X} \xrightarrow{\sim} \mathcal{X}'$ we see that $(B, B') \mapsto (\tilde{B}, \tilde{B}')$ is an isomorphism
\[ \mathcal{X} \xrightarrow{\sim} \mathcal{X}''. \]

Now let
\[ Y''_{\overline{\delta}(b')az,z^{-1}y} = z^{-1}Y'_{\overline{\delta}(b')az,z^{-1}y} \times_{Z_{J,1,z^{-1}y,\delta}} Z_{J,y,\delta,z} \]
where the fibre product is formed using the canonical maps
\[ z^{-1}Y'_{\overline{\delta}(b')az,z^{-1}y} \rightarrow Z_{J,1,z^{-1}y,\delta} \leftarrow Z_{J,y,\delta,z}. \]

Then (a) gives rise to an isomorphism
\[ yY'_{w,y} \rightarrow Y''_{\overline{\delta}(b')az,z^{-1}y} \]
compatible with the natural maps of the two sides into $Z_{J,y,\delta,z}$. As in 11.5 this gives rise to a cartesian diagram
\[ \begin{array}{ccc}
yY'_{w,y} & \xrightarrow{z^{-1}Y'_{\overline{\delta}(b')az,z^{-1}y}} & yY'_{w,y} \\
\downarrow & & \downarrow \\
Z_{J,y,\delta,z} & \rightarrow & Z_{J,1,z^{-1}y,\delta}.
\end{array} \]

Now let $\mathcal{L}, \mathcal{L}'$ be as in 11.6. Let $\tilde{\mathcal{L}}, \tilde{\mathcal{L}}'$ the local systems on $yY'_{w,y}, z^{-1}yY'_{\overline{\delta}(b')az,z^{-1}y}$ corresponding as in 11.1 to $\mathcal{L}, \mathcal{L}'$. From the definitions we see that the inverse image of $\tilde{\mathcal{L}}'$ under $yY'_{w,y} \rightarrow z^{-1}yY'_{\overline{\delta}(b')az,z^{-1}y}$ (in the cartesian diagram above) is the same as the restriction of $\tilde{\mathcal{L}}'$ to $yY'_{w,y}$.

11.9. Let $w = ab, b'$ be as in 11.4. Let $s = (J_n, J'_n, u_n)_{n \geq 0} \in S(J, \text{Ad}(y)\delta)$. For $x$ as in 11.1(a) let $yY'^{x}_{w,y}$ be the inverse image of $Z_{J,y,\delta}^{s}$ under the canonical map $yY'_{w,y} \rightarrow Z_{J,y,\delta}$. In the last (cartesian) diagram in 11.5 (with $z = u_0$), the inverse image of $Z_{J,1,z^{-1}y,\delta}^{s}$
- under $Z_{J,y,\delta,z} \rightarrow Z_{J,1,z^{-1}y,\delta}$ is $Z_{J,y,\delta}^{s}$,
- under $z^{-1}yY'_{\overline{\delta}(b')az,z^{-1}y} \rightarrow Z_{J,1,z^{-1}y,\delta}^{s}$ is $z^{-1}yY'_{\overline{\delta}(b')az,z^{-1}y}^{s}$,
- under $yY'_{w,y} \rightarrow Z_{J,1,z^{-1}y,\delta}$ (the two compositions in the diagram) is $yY'^{x}_{w,y}$.

Therefore that cartesian diagram restricts to a cartesian diagram
\[ \begin{array}{ccc}
yY'^{x}_{w,y} & \xrightarrow{z^{-1}yY'_{\overline{\delta}(b')az,z^{-1}y}^{s}} & yY'^{x}_{w,y} \\
\downarrow & & \downarrow \\
Z_{J,y,\delta}^{s} & \rightarrow & Z_{J,1,z^{-1}y,\delta}^{s}.
\end{array} \]

11.10. In the setup of 11.9 assume in addition that $b, b'$ are as in 11.7. As in 11.9, the cartesian diagram in 11.8 restricts to a cartesian diagram
\[ \begin{array}{ccc}
yY'^{x}_{w,y} & \xrightarrow{z^{-1}yY'_{\overline{\delta}(b')az,z^{-1}y}^{s}} & yY'^{x}_{w,y} \\
\downarrow & & \downarrow \\
Z_{J,y,\delta}^{s} & \rightarrow & Z_{J,1,z^{-1}y,\delta}^{s}.
\end{array} \]
Let $\mathcal{L} \in S(T)$, let $x$ be as in 11.1(a) (with $r = 3$) and let $s \in S(J, \operatorname{Ad}(y)\delta)$.

Let $\mathcal{L} \in S(T)$ be such that $x_1x_2y \in W_2^{1}$ and let $\tilde{\mathcal{L}}$ be the corresponding local system on $\mathcal{Y}_{x_1x_2y}$. Let $A'$ be a simple perverse sheaf on $Z_{y,s}^{*}$ such that $A' \dashv ((\Pi_{x_1x_2y})\tilde{\mathcal{L}})|_{Z_{y,s}^{*}}$. We show that

(a) there exists $x_0 \in W$ such that $x_0y \in W_2^{1}$ and $A' \dashv ((\Pi_{x_0y})\tilde{\mathcal{L}})|_{Z_{y,s}^{*}}$

We denote the local system on $\mathcal{Y}_{x_0y}$ corresponding to $\mathcal{L}$ again by $\tilde{\mathcal{L}}$.

The proof is similar to that of Lemma 4.8. We argue by induction on $l(x_2)$. If $l(x_2) = 0$ then $x_2 = 1$, we may identify $\mathcal{Y}_{x_1x_2y}$ with $\mathcal{Y}_{x_1,y}$ and the result is obvious.

Assume now that $l(x_2) > 0$. We can find $s \in W$ such that $l(s) = 1$, $l(x_2) > l(sx_2)$.

Assume first that $l(x_1s) = l(x_1) + 1$. Then

$\mathcal{Y}_{x_1x_2y} \to \mathcal{Y}_{x_1sx_2y}$,

$(B_0, B_1, B_2, \gamma) \mapsto (B_0, B_1', B_2, \gamma)$

with $B_1'$ given by

$$(\operatorname{pos}(B_1, B_1') = s, \operatorname{pos}(B_1', B_2) = sx_2)$$

is an isomorphism. Under this isomorphism, $\tilde{\mathcal{L}}$ on $\mathcal{Y}_{x_1x_2y}$ corresponds to the analogous local system on $\mathcal{Y}_{x_1sx_2y}$. We have $A' \dashv ((\Pi_{x_1s,x_2y})\tilde{\mathcal{L}})|_{Z_{y,s}^{*}}$. We may apply the induction hypothesis to $x_1s$, $sx_2$, $y$; the desired result follows.

Assume next that $l(x_1s) = l(x_1) - 1$. Then we have a partition $\mathcal{Y}_{x_1x_2y} = Y \cup Y'$ where $Y$ is the open subset defined by $\operatorname{pos}(B_0, B_1') = x_1$ (with $B_1'$ as in (b)) and $\mathcal{Y}'$ is the closed subset defined by $\operatorname{pos}(B_0, B_1') = x_1s$ (with $B_1'$ as in (b)). Let $\Pi: Y \to Z_{y,s}$,

$\Pi: \mathcal{Y} \to \tilde{\mathcal{L}}$, $\Pi: \mathcal{Y} \to \tilde{\mathcal{L}}|_{Z_{y,s}^{*}}$ be the restrictions of $\Pi_{x_1x_2y}$ to $Y$, $\mathcal{Y}'$. By general principles, we have either

(c) $A' \dashv (\Pi_{x_1s}\tilde{\mathcal{L}})|_{Z_{y,s}^{*}}$

(d) $A' \dashv (\Pi_{x_1s}\tilde{\mathcal{L}})|_{Z_{y,s}^{*}}$

where the restriction of $\tilde{\mathcal{L}}$ to $Y$ or $\mathcal{Y}'$ is denoted again by $\tilde{\mathcal{L}}$.

Assume that (d) holds. Then

$\iota: \mathcal{Y} \to \mathcal{Y}_{x_1sx_2y}$,

$(B_0, B_1, B_2, \gamma) \mapsto (B_0, B_1', B_2, \gamma)$

with $B_1'$ as in (b), is a line bundle and $\iota^*\tilde{\mathcal{L}}$ is up to shift the local system on $\mathcal{Y}_{x_1sx_2y}$ attached to $\tilde{\mathcal{L}}$ (we denote it again by $\tilde{\mathcal{L}}$). Since $\Pi = \Pi_{x_1s,x_2y}\iota^*$, it follows that $A' \dashv ((\Pi_{x_1s,x_2y})\tilde{\mathcal{L}})|_{Z_{y,s}^{*}}$. We may apply the induction hypothesis to $x_1s$, $sx_2$, $y$; the desired result follows.

Assume now that (c) holds. Then

$\iota': \mathcal{Y} \to \mathcal{Y}_{x_1sx_2y}$,

$(B_0, B_1, B_2, \gamma) \mapsto (B_0, B_1', B_2, \gamma)$

with $B_1'$ as in (b), makes $\mathcal{Y}'$ into the complement of the zero section of a line bundle over $\mathcal{Y}_{x_1sx_2y}$ and we have $\Pi = \Pi_{x_1s,x_2y}\iota'$. In the case where

(e) the inverse image of the sheaf $\mathcal{L}$ under the coroot $\kappa^* \to T$ corresponding to $y^{-1}x_2^{-1}sx_2y$ is $Q_l$

(so that $x_1sx_2y \in W_2^{1}$), $\tilde{\mathcal{L}}$ (on $\mathcal{Y}'$) is $\iota''$ of the local system on $\mathcal{Y}_{x_1sx_2y}$ (denoted by $\tilde{\mathcal{L}}$) hence we have an exact triangle consisting of $\iota'_*\tilde{\mathcal{L}}$, $\tilde{\mathcal{L}}$ and a shift of $\tilde{\mathcal{L}}$. Hence $A' \dashv ((\Pi_{x_1sx_2y})\tilde{\mathcal{L}})|_{Z_{y,s}^{*}}$. We may apply the induction hypothesis to $x_1$, $sx_2$, $y$;
the desired result follows. In the case where (e) does not hold, we have $i^!\mathcal{L} = 0$ hence $\Pi^!\mathcal{L} = 0$, a contradiction. (a) is proved.

11.12. Let $s = (J_n, J_{n'}_1, u_n)_{n\geq 0} \in S(S, \text{Ad}(y)\delta)$. For $r \gg 0$ we have $J_r = J_{r'} = J_{r+1} = J_{r'+1} = \cdots$, and $u_r = u_{r+1} = \cdots = 1$. Let $P \in \mathcal{P}_J$. Let $L^s$ be a Levi of $P$.

Then

$$C^s = \{g \in G^1; gL^s = L^s, \text{pos}(P, gP) = y_r\}$$

(where $y_r = u_{r-1} \cdots u_0^{-1}y$) is a connected component of $N_G(L^s)$. Let $X$ be a character sheaf on $C^s$ (the definition in 4.5 is applicable since $C^s$ is a connected component of an algebraic group with identity component $L^s$). We regard $X$ as a simple perverse sheaf on $U_P\backslash A_y, (P, P)/U_P$ via the obvious isomorphism $C^s \cong U_P\backslash A_y, (P, P)/U_P$. Now $X$ is $P$-equivariant for the conjugation action of $P$. Hence there is a well defined simple perverse sheaf $X'$ on $G \times_P (U_P\backslash A_y, (P, P)/U_P)$ (with $P$ acting on $G$ by right translation) whose inverse image under

$$G \times (U_P\backslash A_y, (P, P)/U_P) \to G \times_P (U_P\backslash A_y, (P, P)/U_P)$$

is a shift of the inverse image of $X$ under

$$pr_2: G \times (U_P\backslash A_y, (P, P)/U_P) \to U_P\backslash A_y, (P, P)/U_P.$$

We may regard $X'$ as a simple perverse sheaf on $Z^s_{J_{r'-r}, \delta}$ via the isomorphism

$$G \times_P (U_P\backslash A_y, (P, P)/U_P) \cong Z^s_{J_{r'-r}, \delta}, \quad (g, \gamma) \mapsto (gP, g\gamma g^{-1}).$$

Now let $\mathfrak{g}: Z^s_{J_{r'-r}, \delta} \to Z^s_{J_{r'-r'}, \delta}$ be a composition of maps in 8.20(a), a smooth map with connected fibres. Then $\hat{X} = \mathfrak{g}(X')$ is a simple perverse sheaf on $Z^s_{J_{r'-r'}, \delta}$. Let $\bar{X}$ be the simple perverse sheaf on $Z_{J,y,\delta}$ whose support is the closure in $Z_{J,y,\delta}$ of $\hat{X}$ and whose restriction to $Z^s_{J_{r'-r'}, \delta}$ is $\hat{X}$.

Let $C^s_{J_{r'-r'}, \delta}$ be the class of simple perverse sheaves on $Z^s_{J_{r'-r'}, \delta}$ consisting of all $\bar{X}$ as above. Let $C^s_{J_{r'-r'}, \delta}$ be the class of simple perverse sheaves on $Z_{J,y,\delta}$ consisting of all $\bar{X}$ as above (where $s$ varies). The isomorphism classes of objects in $C^s_{J_{r'-r'}, \delta}$ are in bijection with the set of pairs $(s, X)$ where $s \in S(S, \text{Ad}(y)\delta)$ and $X$ is a character sheaf on $C^s$ (as above).

Lemma 11.13. Let $s = (J_n, J_{n'}_1, u_n)_{n\geq 0} \in S(S, \text{Ad}(y)\delta)$. Let $L \in S(T)$ and let $w \in W$ be such that $yw \in W_L^1$; let $\bar{L}$ be the corresponding local system on $wY^w_{w', y}$. Let $A'$ be a simple perverse sheaf on $Z^s_{J', y,\delta}$ such that $A' \sim (\Pi_{w, y})!\bar{L}|Z^s_{J', y,\delta}$. Then $A' \in C^s_{J_{r'-r'}, \delta}$.

More generally, we show that the lemma holds when $J, s$ are replaced by $J_n, s_n$ for any $n \geq 0$. First we show:

(a) if the result is true for $n = 1$ then it is true for $n = 0$.

Let $L', \bar{L}'$ be related to $L, \bar{L}$ as in 11.6. The restrictions of $\bar{L}, \bar{L}'$ to $wY^w_{w', y}^s$, $z^{-1}y Y^s_{\delta^{-1}(b), a, z^{-1}a}$ (where $z = u_0$) are again denoted by $\bar{L}, \bar{L}'$. In the last cartesian diagram in 11.10, $\bar{L}$ is the inverse image of $\bar{L}'$ under

$$wY^w_{w', y} \to z^{-1}y Y^s_{\delta^{-1}(b'), a, z^{-1}a}.$$
(see the last sentence in 11.8). By the change of basis theorem, the direct image with compact support of $\tilde{\mathcal{L}}$ under $\nu Y'_{w,y}^s : Z^*_{\delta,y,\delta} \to \tilde{Z}^*_{\delta,y,\delta}$ is $\vartheta^*$ (as in 8.19) of the direct image with compact support of $\tilde{\mathcal{L}}$ under $z^{-1} Y'_{\delta^{-1}(\nu')az, az^{-1}_{1_{y},s}} : Z^*_{1_{y},z^{-1}_{1_{y},s}} \to \tilde{Z}^*_{1_{y},z^{-1}_{1_{y},s}}$.

In other words,

$$
\left((\Pi_{w,y})_!\tilde{\mathcal{L}}|_{Z^*_{1_{y},z^{-1}_{1_{y},s}}} \right) = \vartheta^* \left(\left((\Pi_{\delta^{-1}(\nu')az, az^{-1}_{1_{y}}})_!\tilde{\mathcal{L}}|_{Z^*_{1_{y},z^{-1}_{1_{y},s}}} \right) \right).
$$

Thus, $A' \vdash \vartheta^* \left(\left((\Pi_{\delta^{-1}(\nu')az, az^{-1}_{1_{y}}})_!\tilde{\mathcal{L}}|_{Z^*_{1_{y},z^{-1}_{1_{y},s}}} \right) \right)$.

Since $\vartheta$ is an affine space bundle it follows that there exists a simple perverse sheaf $A''$ on $Z^*_{1_{y},z^{-1}_{1_{y},s}}$ such that $A' = \vartheta(A'')$ and $A'' \vdash (\Pi_{\delta^{-1}(\nu')az, az^{-1}_{1_{y}}})_!\tilde{\mathcal{L}}|_{Z^*_{1_{y},z^{-1}_{1_{y},s}}}$.

Applying 11.11(a) for $\delta^{-1}(\nu')$, $az, az^{-1}_{1}, s_1$ instead of $x_1, x_2, y, s$ we see that there exists $x_0 \in W$ such that $x_0 z^{-1}_{1} y \in W_{P,C}$ and $A'' \vdash (\Pi_{x_0,y})_!\tilde{\mathcal{L}}|_{Z^*_{1_{y},z^{-1}_{1_{y},s}}}$.

By our assumption we have $A'' \in C'_{J,y,\delta}$. From the definitions we have $\vartheta(A'') \in C'_{J,y,\delta}$. Thus, (a) holds.

Similarly, if the result holds for some $n \geq 1$ then it holds for $n - 1$. (The proof is the same as for $n = 1$.) In this way we see that it suffices to prove the result for $n \gg 0$. Thus we may assume that $J_0 = J_0' = J_1 = J_2 = \cdots = J$ and $u_0 = u_1 = \cdots = 1$. Then $W_J = y W_{(J)}$. In our case, $\nu Y'_{w,y}^s \neq \emptyset$ hence there exist $B_0, B_1 \in \mathcal{B}$ such that $\text{pos}(B_0, B_1) = w$ and $B_0, B_1$ are contained in the same parabolic of type $J$. Thus we have $w \in W_J$. Let $P \in \mathcal{P}_J$ with $T \subset P$. Let $L$ be the Levi of $P$ such that $T \subset L$. Then

$$
C = \{ c \in G^1; ^c L = L, \text{pos}(P, ^c P) = y \}
$$

is a connected component of $N_C(L)$. Let $Y'$ be the set of all $(\beta_0, \beta_1, c)$ where $\beta_0, \beta_1$ are Borels of $L$ such that $\text{pos}(\beta_0, \beta_1) = w$ (position with respect to $L$ with Weyl group $W_J$) and $c \in C$ is such that $^c \beta_0 = \beta_1$. Then $P$ acts on $Y'$ by $p: (\beta_0, \beta_1, c) \mapsto (^l \beta_0, ^l \beta_1, ^l c)$ where $l \in L, p \in U_P$. We have a commutative diagram

$$
\begin{array}{c}
G \times P Y' \xymatrix@R=1cm@C=1cm{\ar[r]^-{\nu Y'_{w,y}^s} & } \tilde{\mathcal{L}} \ar[d] & \\
G \times P C \ar[r]^-{\vartheta^*} & Z^*_{\delta,y,\delta} \ar[u]
\end{array}
$$

where the upper horizontal map is

$$
(g, \beta_0, \beta_1, c) \mapsto (^g B_0, ^g B_1, ^g c)
$$

with $B_0 = \beta_0 U_P, B_1 = \beta_1 U_P$, the lower horizontal map is

$$
(g, c) \mapsto (^g P, ^g P, U_P gg_{c^{-1}} U_P),
$$

the left vertical map is $(g, \beta_0, \beta_1, c) \mapsto (g, c)$ and the right vertical map is $\Pi_{w,y}$. This commutative diagram shows that any composition factor of $\bigoplus_p \mathbb{P} H^i((\Pi_{w,y})_!\tilde{\mathcal{L}})$ is of the form $X'$ (notation of 11.12) where $X$ is a character sheaf on $C$; hence it is in $C'_{J,y,\delta}$. The lemma is proved.

**Lemma 11.14.** For $s \in S(J, \text{Ad}(y) \delta)$, $A \in C'_{J,y,\delta}$, we set $A^s = A|_{Z^*_{1_{y},s}}$. Then any composition factor of $\bigoplus_p \mathbb{P} H^i(A^s)$ belongs to $C'_{J,y,\delta}$.
We can find \( \mathcal{L} \in S(T) \) and \( \mathfrak{x} = (x_1, x_2, \ldots, x_r) \) as in 11.1(b) so that \( x_1x_2\ldots x_r \in W_1^1_L \) and \( A = K_{\mathfrak{L}}^L \) (see 11.1, 11.2). Since the complex \( K_{\mathfrak{L}}^L \) is semisimple (see 11.1) we have \( K_{\mathfrak{L}}^L \cong A[m] \oplus K' \) for some \( K' \in D(Z_{J,y, \delta}) \) and some \( m \in \mathbb{Z} \). Hence \( \mathcal{K}_{\mathfrak{L}}^L = Z_{J,y, \delta} \cong A[1] \oplus K' \) for some \( K' \in D(Z_{J,y, \delta}) \). It suffices to show that, if \( A' = K_{\mathfrak{L}}^L | Z_{J,y, \delta} \), then \( A' \in C_{J,y, \delta}^* \). As in \([L,3], 2.11-2.16\) we see that there exists \( \mathcal{L} \in S(T) \), \( w \in W \) such that \( wy \in W_1^1_L \) and \( A' = K_{\mathfrak{L}}^L | Z_{J,y, \delta} \). Using Lemma 11.13 we have \( A' \in C_{J,y, \delta}^* \). The lemma is proved.

**Lemma 11.15.** If \( A \in C_{J,y, \delta} \) then \( A \in C_{J,y, \delta}^* \).

Since \( Z_{J,y, \delta} = \bigcup_{J \subset L} Z_{J,y, \delta}^* \), we can find \( s \in S(J, \text{Ad}(y)\delta) \) such that \( \text{supp}(A) \cap Z_{J,y, \delta}^s \) is open dense in \( \text{supp}(A) \). Then \( A^s = A[Z_{J,y, \delta}^s] \) is a simple perverse sheaf on \( Z_{J,y, \delta}^s \) and \( A^s \in C_{J,y, \delta}^* \) (Lemma 11.14). Now \( A, A^s \) are related just as \( \hat{X}, \hat{X} \) are related in 11.12. Hence \( A \in C_{J,y, \delta}^* \). The lemma is proved.

**Lemma 11.16.** Let \( s = (J_0, J'_n, u_n)_{n \geq 0} \in S(J, \text{Ad}(y)\delta) \), \( C, X, X', \hat{X}, \hat{X}' \) be as in 11.12. For any \( n \geq 0 \) define a simple perverse sheaf \( X_n^s \) on \( Z_{J_0,y_n, \delta}^s \) by \( X_n^s = \mathcal{P}(X_n') \) where \( \mathcal{P} : Z_{J_0,y_n, \delta} \to Z_{J_0,y_n+1, \delta} \) is as in 8.20(a) for \( n \geq 0 \) and \( X_n' = X' \) for \( n \gg 0 \). Define \( a_n \in W_{J_n}^s \) by \( a_n^{-1} = u_nu_{n+1}\ldots u_m \) for \( m \gg 0 \). For any \( n \geq 0 \) there exists \( \mathcal{L}_n \in S(T) \) and \( b_n^s \in W_{\delta(J_n)}^s \) (see 2.6) such that \( a_ny_nb_n^s \in W_1^1_L \) and \( X_n' \to \Pi_{a_ny_n,b_n^{-1}, y}^s | Z_{J_0,y_n, \delta} \).

Assume that the result holds for \( n = 1 \); we show that it holds for \( n = 0 \). By assumption we have \( X_1' \simeq (\Pi_{a_0b_0^{-1}y^{-1}, z, z^{-1}y})| Z_{J_0, y_1, \delta}^s \), where \( L' = L_1, z = u_0, a = a_0 \). (We have \( a_1y_1 = ay_0 \).) We consider the cartesian diagram in 11.10 with \( w = ayb_0y^{-1} \) where \( \delta^{-1}(b')a = ayb_0y^{-1} \). (We have \( b' = \delta(ayb_0y^{-1}a^{-1}) \) in \( \delta(J_n) \) by 2.6.) The inverse image of \( L' \) under \( y\mathcal{Y}_{w,y}^s \to z^{-1}y\mathcal{Y}_{b,y}^s \) is \( \mathcal{L} \) for some \( \mathcal{L} \in S(T) \) (see 11.8). Using the change basis theorem for the cartesian diagram in 11.10 we deduce that \( X_0' \simeq (\Pi_{a_0b_0^{-1}y^{-1}, y})| Z_{J_0, y, \delta}^s \).

The same argument shows that, if the result holds for some \( n \geq 1 \) then it also holds for \( n - 1 \). In this way it suffices to show that the result holds for \( n \gg 0 \). Replacing \( s, n \) by \( s_n, 0 \), we may assume that \( J_0 = J_0' = J_1 = J_1' = \cdots = J_r, u_0 = u_1 = \cdots = 1 \) and \( n = 0 \). Let \( P, L \) be as in 11.13. We can find \( w \in W_J^s \) such that \( X' \to \Pi_{a^{-1}, y}^s | Z_{J,y, \delta}^s \) where \( \mathcal{L} \) is the corresponding local system on \( Y' \). Using the commutative diagram in 11.13 we see that \( X' \to \Pi_{a^{-1}y^{-1}, y}^s | Z_{J,y, \delta}^s \) where \( b' = y^{-1}wy \in W_{\delta(J)} \) and \( \mathcal{L} \) is the local system on \( \mathcal{Y}_{w,y}^s \) corresponding to \( \mathcal{L} \). The lemma is proved.

**Lemma 11.17.** Let \( s = (J_0, J'_n, u_n)_{n \geq 0} \in S(J, \text{Ad}(y)\delta) \). Define \( a \in W_{J'} \) by \( a^{-1} = u_0u_1\ldots u_m \) for \( m \gg 0 \). Let \( b' \in W_{\delta(J_\infty)}^s \). Then the image of

\[
\Pi_{a^{-1}y^{-1}, y}^s : \mathcal{Y}_{w,y}^s \to Z_{J,y, \delta}
\]

is contained in \( Z_{J,y, \delta}^s \).
Let \((B_0, B_1, \gamma) \in Z^\gamma_{\alpha yb, y^{-1}}\). Let \((P, P', \gamma) = \Pi_{\alpha yb, y^{-1}}(B_0, B_1, \gamma)\). We have \(yb^t y^{-1} \in W_{J'}\). Hence
\[
\text{pos}(P', P) = \min(W_{J'} \text{ pos}(B_1, B_0) W_J) = \min(W_{J'} yg^t y^{-1} a^{-1} W_J) = \min(W_{J'} a^{-1} W_J) = u_0.
\]

Thus, \((P, P', \gamma) \in Z_{J, y, \delta, z}\) where \(z = u_0\). Let \((P_1, P'_1, \gamma_1)\) be the image of \((P, P', \gamma)\) under \(Z_{J, y, \delta, z} \rightarrow Z_{J, z^{-1} y, \delta}\) (see 11.5). By the cartesian diagram in 11.8, there exists \((\tilde{B}_0, \tilde{B}_1, \tilde{\gamma}) \in Z_{\gamma_1} y^{-1}(b) az, z^{-1} y\)

such that
\[
\Pi_{\delta^{-1}(b') az, z^{-1} y}(\tilde{B}_0, \tilde{B}_1, \tilde{\gamma}) = (P_1, P'_1, \gamma_1)\).
\]
We have \(\delta^{-1}(b'^{-1}) \in W_{J_m} \subset W_{J_1}\) hence
\[
\text{pos}(P'_1, P_1) = \min(W_{J_1} \text{ pos}(\tilde{B}_1, \tilde{B}_0) W_{J_1}) = \min(W_{J_1} z^{-1} a^{-1} \delta^{-1}(b'^{-1}) W_{J_1}) = \min(W_{J_1} u_0^{-1} a^{-1} W_{J_1}) = u_1.
\]

Now \(\delta^{-1}(b') az = (az)(z^{-1} y)b'_1(y^{-1} z)\) where \(b'_1 = (ay)^{-1} \delta^{-1}(b') ay \in W_{\delta((J_m))}\) (see 2.6). Hence in the previous argument we may replace \(B_0, B_1, \gamma, P, P', a, y, b' \) by \(\tilde{B}_0, \tilde{B}_1, \tilde{\gamma}, P_1, P'_1, a_0, u_0^{-1} y, b'_1 \) and the image \((P_2, P'_2, \gamma_2)\) of \((P_1, P'_1, \gamma_1)\) under \(Z_{J_1, u_0^{-1} y, \delta, u_1} \rightarrow Z_{J_2, u_1^{-1} u_0^{-1} y, \delta, u_1}\) satisfies \(\text{pos}(P'_2, P_2) = u_2\). Continuing this process we find that \((P, P', \gamma) \in Z^*_{J, y, \delta}\). The lemma is proved.

**Lemma 11.18.** If \(A \in C'_{J, y, \delta}\) then \(A \in C_{J, y, \delta}\).

Let \(s, \tilde{X}, \tilde{X}\) be as in the proof of Lemma 11.16. We may assume that \(A = \tilde{X}\). By Lemma 11.16, \(\tilde{X} = (\Pi_{\alpha yb, y^{-1}}) \tilde{L}\) for some \(\tilde{L} \in \mathcal{S}(T)\) with \(\alpha yb' \in W^1_\delta\). By Lemma 11.17, \(\Pi_{\alpha yb, y^{-1}} y^\gamma y^\gamma_{\alpha yb, y^{-1}} \rightarrow Z_{J, y, \delta}\) factors through a map \(H' : y^\gamma y^\gamma_{\alpha yb, y^{-1}} \rightarrow Z_{J, y, \delta}\) and \(\tilde{X} = \Pi' \tilde{L}\). Thus there exists a simple perverse sheaf on \(Z_{J, y, \delta}\) whose support is the closure in \(Z_{J, y, \delta}\) of \(\text{supp} (\tilde{X})\), whose restriction to \(Z^*_{J, y, \delta}\) is \(\tilde{X}\) and which is a composition factor of \(\bigoplus H'(\Pi_{\alpha yb, y^{-1}}) \tilde{L}\); this is necessarily \(\tilde{X}\). We see that \(\tilde{X} \in C_{J, y, \delta}\). The lemma is proved.

**11.19.** For \(P \in P_J\) let \(H_P\) be the inverse image of the connected centre of \(P/U_P\) under \(P \rightarrow P/U_P\). For \(P, \tilde{P} \in P_J\), the groups \(H_P/U_P, H_{\tilde{P}}/U_{\tilde{P}}\) are canonically isomorphic (an element \(h \in G\) that conjugates \(P\) into \(\tilde{P}\) induces an isomorphism \(H_P/U_P \rightarrow H_{\tilde{P}}/U_{\tilde{P}}\) that is independent of the choice of \(h\)). Thus we may identify the groups \(H_P/U_P\) (with \(P \in P_J\)) with a single torus \(\Delta_J\) independent of \(P\). Now \(\Delta_J\) acts (freely) on \(Z_{J, y, \delta}\) by \(\delta : (P, P', \gamma) \mapsto (P, P', \gamma z)\) where \(z \in H_P\) represents \(\delta \in \Delta_J\) and each piece \(Z_{J, y, \delta}^\star\) is \(\Delta_J\)-stable. We set
\[
\mathcal{Z}_{J, y, \delta}^\ast = \Delta_J \backslash Z_{J, y, \delta}, \quad \mathcal{Z}_{J, y, \delta}^\ast = \bigcup \mathcal{Z}_{J, y, \delta}^\ast.
\]

The action of \(G\) on \(Z_{J, y, \delta}\) and on \(Z_{J, y, \delta}^\ast\) commutes with the action of \(\Delta_J\) and induces an action of \(G\) on \(\mathcal{Z}_{J, y, \delta}\) and on \(Z_{J, y, \delta}^\ast\).
For \((P, P', \gamma) \in Z_{J, y, \delta}\) we have naturally
\[
H_P/U_P \subset H_{P_1}/U_{P_1} \subset \cdots \subset H_{P_r}/U_{P_r} \subset \cdots
\]
hence we may identify \(\Delta_J\) with a (closed) subgroup of the centre of \(L^s\) (as above). The bijection 8.21(a) induces a bijection
\[
G'\backslash Z_{J, y, \delta} \leftrightarrow \bigsqcup_{\Phi \in S(J, Ad(y)\delta)} (L^s/\Delta_J) \setminus (C^s/\Delta_J).
\]
(b) Here \(C^s/\Delta_J\) is the orbit space for the free action of \(\Delta_J\) by right translation on \(C^s\) (restriction of the action of the centre of \(L^s\) by right translation) and the action of \(L^s/\Delta_J\) on \(C^s/\Delta_J\) is induced by the conjugation action of \(L^s\) on \(C^s\).

12. Completion

12.1. Assume that \(P, P'\) are two parabolics of \(G\) (as in 0.1) and that \(\Phi: P/U_P \rightarrow P'/U_{P'}\) is an isomorphism of algebraic groups. Let \(H^s_{P, P'}\) be the set of all \((f, f') \in P \times P'\) such that \(\Phi\) carries the image of \(f\) in \(P/U_P\) to the image of \(f'\) in \(P'/U_{P'}\).

Clearly, \(H^s_{P, P'}\) is a closed connected subgroup of \(G \times G\) of dimension \(\dim U_P + \dim U_{P'} + \dim(P/U_P) = \dim G\) and \(P, P', \Phi\) can be reconstructed from \(H^s_{P, P'}\).

Let \(\mathcal{V}_G\) be the (projective) variety whose points are the \(\dim(G)\)-dimensional Lie subalgebras of \(\text{Lie}(G \times G)\). We have \(\text{Lie} H^s_{P, P'} \in \mathcal{V}_G\).

12.2. Assume now that we are in the setup of 3.1. Let \(J, J', y\) be as in 8.8. For \((P, P', \gamma) \in Z_{J, y, \delta}\) we set \(\Phi_\gamma = ba^{-1}\text{Ad}(g): P/U_P \stackrel{\sim}{\rightarrow} P'/U_{P'}\) where \(g \in \gamma\) and \(a, b\) are the obvious isomorphisms in
\[
P/U_P \xrightarrow{\text{Ad}(g)} 9P/U_{P'} \xrightarrow{a} (9P \cap P')/U_{P'\cap P'} \xrightarrow{b} P'/U_{P'}.
\]
Now \(\Phi_\gamma\) is independent of the choice of \(g\) and \((P, P', \gamma) \mapsto \text{Lie} H^s_{P, P'}\) is an embedding
\[
Z_{J, y, \delta} \subset \mathcal{V}_G
\]
(notation of 11.19(a)).

12.3. In the remainder of this section we assume that \(G\) is adjoint. Recall that two parabolics \(Q, Q'\) of \(G\) are said to be opposed if their intersection is a common Levi of \(Q, Q'\). (We then write \(Q \bowtie Q'\).) If \(B \in \mathcal{B}\) and \(Q\) is a parabolic in \(G\), we write \(B \times Q\) if \(B\) is opposed to some Borel of \(Q\).

Let \(J \subset I\). Define \(J^* \subset I\) by \(\{Q; Q \bowtie P\text{ for some } P \in \mathcal{P}_J\} = \mathcal{P}_{J^*}\). Let \(y_J\) be the longest element in \(W(J)\). If \((P, P', g) \in \mathcal{P}_J \times \mathcal{P}_{S(J)^*} \times \mathcal{G}^1\), then \(P' \bowtie gP\) if and only if \(g \in \text{Ad}(g)(P, P')\) (see 3.8). Let
\[
\hat{A}_{y_J}(P, P') = H_P \setminus A_{y_J}(P, P')/U_P = U_{P'} \setminus A_{y_J}(P, P')/H_P,
\]
\[
G^1_J = \{(P, P', \mu); P \in \mathcal{P}_J, P' \in \mathcal{P}_{S(J)^*}, \mu \in \hat{A}_{y_J}(P, P')\} = \hat{Z}_{J, y_J, \delta},
\]
\[
\hat{G}^1 = \bigsqcup_{J \subset I} \hat{G}^1_J.
\]
(notation of 11.19(a)). We define a structure of algebraic variety on \(\hat{G}^1\). We identify \(G^1_J\) with a subvariety of \(\mathcal{V}_G\) by 12.2(a). Since \(G\) is adjoint we have \(G^1_J = G^1\). By
[DP], $G^1$ is the closure of $G^1_j = G^1$ in $V_G$, so that $G^1$ is a (projective) variety. (In [DP] it is assumed that $G = G^1$ but the general case can be easily reduced to this special case; in fact, $G^1$ is isomorphic to the analogous variety in the case $G = G^2$.)

We have a partition

$$G^1 = \bigcup_{J \in I} \bigcup_{s \in S(J, \text{Ad}(y_j) \delta)} \mathcal{Z}^*_J,$$

(see 11.19(a)) refining the partition $G^1 = \bigcup_{J \in I} G^1_J$. Putting together the bijections 11.19(b) we obtain a canonical bijection

$$G^1 \setminus G^1 \sim \bigcup_{J \in I} \bigcup_{s \in S(J, \text{Ad}(y_j) \delta)} (L^s/\Delta_J) \setminus (C^s/\Delta_J).$$

(a)

Here the action of $G$ on $G^1$ is the extension of the conjugation action of $G$ on $G^1$ and $(L^s/\Delta_J) \setminus (C^s/\Delta_J)$ is as in 11.19.

Let $J \subset I$, $s \in S(J, \text{Ad}(y_j) \delta)$ and let $X$ be a character sheaf on $C^s$ that is equivariant for the free action of $\Delta_J$ by right translation. The simple perverse sheaf $\mathcal{X}$ on $Z^*_J$ (see 11.12) is $\Delta_J$-equivariant hence it is a shift of the inverse image of a well defined simple perverse sheaf $\mathcal{X}'$ on $Z^*_J$ under the canonical map $Z^*_J \to Z^*_J$. Let $\mathcal{X}'$ be the simple perverse sheaf on $G^1$ whose support is the closure in $G^1$ of $\text{supp} \mathcal{X}'$ and whose restriction to $Z^*_J$ is $\Delta_J$.

The character sheaves on $G^1$ are by definition the simple perverse sheaves on $G^1$ of the form $\mathcal{X}'$ with $X$ as above. The character sheaves on $G^1$ are in bijection with the set of triples $(J, s, X)$ where $J \subset I$, $s \in S(J, \text{Ad}(y_j) \delta)$ and $X$ is a character sheaf on $C^s$ that is equivariant for the free action of $\Delta_J$ by right translation.

12.4. We now give another definition of a structure of algebraic variety on the set $G^1$ which does not use $V_G$. For $B, \tilde{B} \in \mathcal{B}$ and $J \subset I$, let

$$G^1_{B, \tilde{B}} = \{(P, P', \mu) \in G^1_J, B \times P', \tilde{B} \times P, g(P, P') \sim P^B \}$$

where $g \in \mu$ (an open subset of $G^1_J$). By the the substitution $(P, P', \mu) \mapsto (B_1, B_1', \mu)$ where $B_1 = P^B$, $B_1' = P^{P^B}$, we may identify $G^1_{B, \tilde{B}}$ with the set of all triples $(B_1, B_1', \mu)$ where $B_1, B_1' \in \mathcal{B}$, $\text{pos}(B, B_1') = y_J, \text{pos}(\tilde{B}, B_1) = y_{J^{-1}}, \mu \in \mathcal{A}_J(P, P')$ (with $P \in \mathcal{P}_J, P' \in \mathcal{P}_{B_{-1}}(J)$, defined by $B_1 \subset P, B_1' \subset P'$) and $gB_1 \sim B_1'$ for any $g \in \mu$.

In particular, $G^1_{B, \tilde{B}} = \{g \in G^1, g\tilde{B} \sim B\}$. Moreover, $G^1_{B, \tilde{B}} = \{(B_1, B_1') \in \mathcal{B} \times \mathcal{B}, B_1 \sim B_1', B_1 \sim B_1' \}$ (we omit the $\mu$-component since it is uniquely determined).

Define $\rho_J : G^1_{B, \tilde{B}} \to G^1_{B, \tilde{B}}$ by

$$\rho_J(g) = (g^{-1}P_J, g\tilde{P}_J, gH_{\tilde{P}_J}, g^{-1}U_{P_J}, g)$$

where $P_J \in \mathcal{P}_{B_{-1}}(J), \tilde{P}_J \in \mathcal{P}_{B_{-1}}$. This is well defined: if $B_1, B_2 \in \mathcal{B}$ are opposed and $P_1 \in \mathcal{P}_{B_{-1}}(J), P_2 \in \mathcal{P}_{B_{-1}}(J)$, satisfy $B_1 \subset P_1, B_2 \subset P_2$, then $P_1^{B} \sim P_2^{B}$; apply this to $B_1 = B, B_2 = g\tilde{B}$.

In particular, $g \mapsto (g^{-1}B, g\tilde{B})$ is a morphism $\rho_\mu : G^1_{B, \tilde{B}} \to G^1_{B, \tilde{B}}$. We show that
(a) $\rho_\varphi$ is a principal $\tilde{B}/U_{\tilde{B}}$-bundle.

We fix $(B_1, B'_1) \in G_{\tilde{a}, \tilde{B}}^1$. Let $F = \rho_\varphi^{-1}(B_1, B'_1)$. Since $B_1 \cong \tilde{B}$ and $B'_1 \cong B$, we can find $g_0 \in G$ such that $g_0 B_1 = B, g_0 \tilde{B} = B'_1$. Next we can find $g_1 \in G^1$ such that $g_1 B = B, g_1 B'_1 = B'_1$. We have $g_1^{-1} g_0 B_1 = B, g_1^{-1} g_0 \tilde{B} = B'_1$. Hence $g_1^{-1} g_0 \in F$ and $F \neq \emptyset$.

We show that $F$ is a free homogeneous $\tilde{B}/U_{\tilde{B}}$-space. Since $\tilde{B} \cong B_1$, we have canonically $\tilde{B}/U_{\tilde{B}} = \tilde{B} \cap B_1$. It suffices to show that $F$ is a free homogeneous $\tilde{B} \cap B_1$-space. Now $\tilde{B} \cap B_1$ acts freely on $F$ by $t \cdot g : g \mapsto g t$. Let $g, g' \in F$. Since $g^{-1} B = g'^{-1} B$ and $g g'^{-1} \in G$, we have $g' = bg$ where $b \in B$. Since $g \tilde{B} = g' \tilde{B}$ and $g^{-1} g' \in G$, we have $g' = gb$ where $b \in \tilde{B}$. We have $b = g^{-1} g' = g^{-1} b \in g^{-1} B = B_1$. Thus, $b \in \tilde{B} \cap B_1$. This proves (a).

Similarly,

(b) $\rho_\gamma$ is a principal $H_{\tilde{P}j}/U_{\tilde{P}j}$-bundle. ($\tilde{P}_j$ is as above.)

Using simple roots we identify $\tilde{B}/U_{\tilde{B}} = (k^*)^I$ (we use again that $G$ is adjoint). Now $(k^*)^I$ acts on $k^I$ by multiplication on each factor. This may be regarded as an action of $\tilde{B}/U_{\tilde{B}}$ on $k^I$. Using the principal $\tilde{B}/U_{\tilde{B}}$-bundle $\rho_\varphi$ we may form the associated bundle

$$X^{B, \tilde{B}} = G_{I, \tilde{B}} \times_{\tilde{B}/U_{\tilde{B}}} k^I$$

(a $k^I$-bundle over the affine space $G_{\tilde{a}, \tilde{B}}^1$). Now $X^{B, \tilde{B}}$ is an affine space of dimension $\dim(G)$. We have an obvious partition $k^I = \bigsqcup_{J \subset I} (k^*)^J$ and each piece is $\tilde{B}/U_{\tilde{B}}$-stable. Hence there is a partition

$$X^{B, \tilde{B}} = \bigcup_{J \subset I} X_{J, \tilde{B}}$$

where $X_{J, \tilde{B}} = G_{J, \tilde{B}} \times_{\tilde{B}/U_{\tilde{B}}} (k^*)^J$. We may identify $X_{J, \tilde{B}}$ with the orbit space of $G_{J, \tilde{B}}$ by $H_{\tilde{P}j}/U_{\tilde{P}j}$, hence (using (b)) with $G_{J, \tilde{B}}^1$. Thus, we may identify $X_{J, \tilde{B}}$ with the subset of $\bigsqcup_{J \subset I} G_{J}^1$ so that $X^{B, \tilde{B}} \cap G_{J}^1 = G_{J, \tilde{B}}^1$. We show that for any $J$ we have

$$G_{J}^1 = \bigsqcup_{B, \tilde{B} \in B} G_{J, \tilde{B}}^1.$$  

Let $(P, P', \mu) \in G_{J}^1, g \in \mu$. Since $g P \cong P'$, we can find Borels $B_1 \subset P, B'_1 \subset P'$ such that $g B_1 \cong B'_1$. We can find $B \in B$ such that $B \times P'$ and $P' B = B'_1$. We can find $\tilde{B} \in \tilde{B}$ such that $\tilde{B} \times P$ and $\tilde{B} B = B_1$. Then $(P, P', \gamma) \in G_{J, \tilde{B}}^1$. This proves (c).

From (c) we deduce

$$G^1 = \bigsqcup_{B, \tilde{B} \in B} X^{B, \tilde{B}}.$$  

We can now define a structure of algebraic variety on $G^1$ by declaring that $X^{B, \tilde{B}}$ is an open subvariety of $G^1$ for any $B, \tilde{B} \in B$. We see that $G^1$ has a covering by open subsets isomorphic to the affine space of dimension $\dim(G)$. 
12.5. Let $V = \{ (B, B', g) \in B \times B \times G^1; B' \vDash gB \}$. The fibre of $p_{12} : V \to B \times B$ at $(B, B')$ is just $G^1_{B, B'}$. There is a unique free action of the torus $\Delta_{\infty}$ (see 11.19) on $V$ whose restriction to $G^1_{B, B'}$ is the action of $\Delta_{\infty} = B/U_B$ appearing in 12.4(a). By 12.4(a), this makes $V$ into a principal $\Delta_{\infty}$-bundle over

\[(B, B', B_1, B'_1) \in B^4; B_1 \vDash B, B'_1 \vDash B'.\]  

(a)

As in 12.4, we form the associated bundle

\[X = V \times_{\Delta_{\infty}} k^4\]

(a $k^4$-bundle over (a)). For $(B, B') \in B \times B$, the inclusion $G^1_{B, B'} \subset V$ gives rise to an inclusion $G^1_{B, B'} \times_{\Delta_{\infty}} k^4 \subset V \times_{\Delta_{\infty}} k^4$ that is, $X_{B, B'} \subset X$. The subsets $X_{B, B'}$ form a partition of $X$. Let $\pi : X \to G^1$ be the morphism which induces for any $B, B'$ the identity map $X_{B', B'} \to X_{B', B'}$. We have a partition

\[X = \bigsqcup_{J \in I} X_J\]

where $X_J = V \times_{\Delta_{\infty}} (k^*)^J$. We may identify $X_J$ with the orbit space of $X$ by $\Delta_{I-J}$ hence, using 12.4(b), with the set of all quintuples $(B_1, B'_1, B, B', \mu)$ where

\((B, B', B_1, B'_1) \in B^4, \quad \text{pos}(B', B'_1) = y_{J^*}, \quad \text{pos}(B, B_1) = y_{g^{-1}(J)}, \quad \mu \in \tilde{A}_{y_J}(P, P')\)

(with $P \in \mathcal{P}_J, P' \in \mathcal{P}_{\delta(J)}$, defined by $B_1 \subset P, B'_1 \subset P'$ and $gB_1 \vDash B'_1$ for some/any $g \in \mu$). Now $\pi : X \to G^1$ restricts to the map $X_J \to G^1_J$ given by $(B_1, B'_1, B, B', \mu) \mapsto (P, P', \mu)$ (with $P, P'$ as above).

For $w \in W$ we set $V^w = \{ (B, B', g) \in V; \text{pos}(B, B') = w \}$, $X^w = V^w \times_{\Delta_{\infty}} k^4$. The sets $X^w = X^w \cap X_j$ form a partition of $X$. If $(B_1, B'_1, B, B', \mu) \in X^w$, then

\[\text{pos}(B_1, B) = y_{g^{-1}(J)}^{-1}, \quad \text{pos}(B, B') = w, \quad \text{pos}(B'_1, B'_1) = y_{J^*}, \quad \text{pos}(B', gB_1) = y_{\infty}\]

for $g \in \mu$. Hence $(B_1, B'_1, B', gB_1, g) \in Y_\infty$ where $x = (y_{g^{-1}(J)}^{-1}, w, y_{J^*}, y_{\infty})$ (see 4.2).

Let $Y_\infty$ be the set of all $(\beta_0, \beta_1, \beta_2, \beta_3, \gamma)$ where $(\beta_0, \beta_1, \beta_2, \beta_3) \in B^3$ and $\gamma \in U_{P'} \backslash A_{y_J}(P, P'/U_P)$ (with $P \in \mathcal{P}_J, P' \in \mathcal{P}_{\delta(J)}$, defined by $\beta_0 \subset P, \beta_3 \subset P'$) are such that $(\beta_0, \beta_1, \beta_2, \beta_3, \gamma, \mu) \in Y_{\infty}$ for some/any $\gamma, \mu$.

The obvious map $Y_\infty \to Y_\infty$ is an affine space bundle. The obvious map $Y_\infty \to X^w$ is a principal $\Delta_{\delta(J)}$-bundle. Let $\mathcal{L} \in \mathcal{S}(T)$ be such that $y_{g^{-1}(J)}^{-1}wy_{J^*}y_{\infty} \in W_\mathcal{L}$ and such that the associated local system $\hat{L}$ on $Y_\infty$ is the inverse image under the composition $Y_\infty \to Y_\infty \to X^w$ of a local system $\hat{L}_0$ on $X^w$. (In any case, $\hat{L}$ is the inverse image under $Y_\infty \to Y_\infty$ of a well defined local system on $Y_\infty$ and we require that this last local system is $\Delta_{\delta(J)}$-equivariant.) Let $K_{\mu,J}^w$ be the direct image with compact support of $\hat{L}_0$ under $X^w \to G^1$ (restriction of $\pi : X \to G^1$).

We give a second definition of character sheaves on $G^1$ as the simple perverse sheaves on $G^1$ which are composition factors of $\bigoplus J^*H^1(K_{\mu,J}^w)$ for some $w, J, \mathcal{L}$ as above. We expect that this coincides with the definition in 12.3.
12.6. There is a unique simple perverse sheaf $S$ on $G^1$ such that:

(a) $S$ is a direct summand of the perverse sheaf $(\text{pr}_1)_!Q_{[\dim G]}$, $\text{pr}_1 : \{(g, B) \in G^1 \times B; gB = B\} \to G^1$ being the first projection (a small map);

(b) if $J \subsetneq I$, $\delta(J) = J$, then $S$ is not a direct summand of the perverse sheaf $(\text{pr}_1)_!Q_{[\dim G]}$ where $\text{pr}_1 : \{(g, P) \in G^1 \times \mathcal{P}_J; gP = P\} \to G^1$ is the first projection (a small map).

Let $\tilde{S}$ be the simple perverse sheaf on $G^1$ such that $\tilde{S}|_{G^1} = S$. For $x \in \tilde{G}^1$ let $G_x$ be the stabilizer of $x$ in $G$ and let $\mathcal{H}_x^\tau(\tilde{S})$ be the stalk at $x$ of the $\tau$-th cohomology sheaf of $\tilde{S}$.

We conjecture that the following three conditions on $x \in \tilde{G}^1$ are equivalent:

(c) $\mathcal{H}_x^\tau(\tilde{S}) \neq 0$ for some $\tau$;

(d) $\sum_i \dim \mathcal{H}_x^i(\tilde{S}) = 1$;

(e) $G_x$ is a reductive group.

If we assume that $x \in \tilde{G}^1$, then the equivalence of (c), (d), (e) is known.

12.7. A difficulty in proving the conjecture in 12.6 is that the small map in 12.6(a) does not seem to extend to a small map over all of $G^1$. There is one case when such an extension exists (partially). Assume that $G = G^1 = \text{PGL}(V)$ where $V$ is a $k$-vector space of dimension $d \geq 2$. Let

$$Y = \{\tau \in \text{End}(V); \dim \ker(\tau) \leq 1\}/k^*$$

where $k^*$ acts by scalar multiplication. For $\tau$ as above let $\bar{\tau}$ be the image of $\tau$ in $Y$. Let $Y_0 = \{\tau \in \text{End}(V); \dim \ker(\tau) = 1\}/k^*$. We may identify $Y$ with an open subset of $G^1$ so that $Y - Y_0$ corresponds to the open stratum $G^1_{d_0}$. Let $\tilde{Y}$ be the set of all $(\bar{\tau}, V_1 \subset V_2 \subset \cdots \subset V_4)$ where $\bar{\tau} \in Y$ and $V_1 \subset V_2 \subset \cdots \subset V_4$ are subspaces of $V$ (dim $V_i = i$) such that $\tau(V_i) \subset V_i$ for all $i$. Then $\tilde{Y}$ is smooth and $\text{pr}_1 : \tilde{Y} \to Y$ is a small map.

Its restriction to $\text{pr}_1^{-1}(G)$ may be identified with the small map in 12.6(a). Let $K_G = (\text{pr}_1)_!Q_{[\dim G]}$, a perverse sheaf on $Y$. More generally, for any $J \subset I$ we have a perverse sheaf $K_J$ on $Y$ defined like $K_G$ but using flags of type $J$ (see 8.24) in $V$ instead of complete flags. Let $\tilde{S}' = \tilde{S}|_Y$ that is, the simple perverse sheaf on $Y$ such that $\tilde{S}'|_{G^1} = S$. Now $S$ is an alternating sum over $J$ of the perverse sheaves $K_J|_{G^1}$; hence $\tilde{S}'$ is an alternating sum over $J$ of the perverse sheaves $K_J$. Using this, one can compute explicitly the stalks of the cohomology sheaves of $\tilde{S}'$ at any $x \in Y_0$. Thus one can verify that the conjecture in 12.6 holds in our case for $x \in Y_0$.

12.8. Assume that $G = G^1 = \text{PGL}(V)$ where $V$ is a $k$-vector space of dimension 3. We may identify $G^1$ with

$$\{A, B \in \text{End}(V); \{A, B \in \text{End}(V) - \{0\}; AB = BA = \lambda A, \lambda \in C\}/C^* \times C^*;$$

here $C^* \times C^*$ acts by scalar multiplication on $A$ and $B$. The four strata $G^1_J$ are:

$$\{(A, B); AB = BA = \lambda A, \lambda \in C^*\}/C^* \times C^* \cong \{A \in \text{GL}(V)\}/C^* = G,$$

$$\{(A, B); AB = BA = 0, \text{Im}(B) = \text{Ker}(A) = \text{plane}, \text{Im}(A) = \text{Ker}(B) = \text{plane}\}/C^* \times C^*$$

$$\cong \{V_1 = \text{plane}, V_2 = \text{plane}, B : V/V_1 \cong V_2\}/C^*,$$
{(A, B); AB = BA = 0, Im(B) = Ker(A) = line, Im(A) = Ker(B) = plane} / C^* × C^*
≡ \{V_1 = line, V_2 = plane, \tilde{A} : V/V_1 \cong V_2\} / C^*;',

{(A, B); AB = BA = 0, V_1 = Im(B) \subset Ker(A) = V_2, V'_1 = Im(A) \subset Ker(B) = V'_2,
\dim V_i = \dim V'_i = i\} / C^* × C^*
≡ \{V_1 \subset V_2, V'_1 \subset V'_2, \dim V_i = \dim V'_i = i\}.

The closure in \( \bar{G}^1 \) of the variety of unipotent elements in \( G \) is
\[ \bar{G}^1_u = \{(A, B); A = \nu + N \neq 0, B = \nu' + N' \neq 0, N, N' \text{ nilpotent}, \nu, \nu' \in C, NN' = N'N = -\nu N' - \nu' N\} / C^* × C^*. \]
\( \bar{G}^1_u \) intersects the four strata above along the following four sets:
\[ \{A \in GL(V); A = \nu + N, N \text{ nilpotent, } \nu \in C^*\} / CC^* \]
\[ \{V_1, V_2, N': V \to V \text{ regular nilpotent, } \text{Im}(N') = V_2, \text{Ker}(N') = V_1, \dim V_i = i\} / C^* \]
\[ \{V_1, V_2, N: V \to V \text{ regular nilpotent, } \text{Im}(N) = V_2, \text{Ker}(N) = V_1, \dim V_i = i\} / C^* \]
\[ \{V_1 \subset V_2, V'_1 \subset V'_2, V_i = V'_i \text{ or } V_2 = V'_2\}. \]

From this we see that, if \( V \) is defined over \( F_q \), then the number of \( F_q \)-rational points of \( \bar{G}^1_u \) is \( q^6 + 2q^5 + 4q^4 + 5q^3 + 4q^2 + 2q + 1 \). This suggests that \( \bar{G}^1_u \) is a rational homology manifold.

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