POSITIVITY AND CANONICAL BASES IN RANK 2 CLUSTER ALGEBRAS OF FINITE AND AFFINE TYPES

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To Borya Feigin on the occasion of his jubilee

Abstract. The main motivation for the study of cluster algebras initiated in [6], [8], [1] was to design an algebraic framework for understanding total positivity and canonical bases in semisimple algebraic groups. In this paper, we introduce and explicitly construct the canonical basis for a special family of cluster algebras of rank 2.

Key words and phrases. Cluster algebras, affine root systems, Newton polygons.

ju-bi-lee
1 : a year of emancipation and restoration provided by ancient Hebrew law to be kept every 50 years by the emancipation of Hebrew slaves, restoration of alienated lands to their former owners, and omission of all cultivation of the land
2 a : a special anniversary; especially : a 50th anniversary b : a celebration of such an anniversary Merriam-Webster Dictionary

1. Introduction

This paper continues the study of cluster algebras initiated in [6], [8], [1]. The principal motivation for this theory was to design an algebraic framework for understanding total positivity and canonical bases in semisimple algebraic groups; see [17] for more details. Since its inception, the theory of cluster algebras has developed several interesting connections:

• Discrete dynamical systems [7].
• Al. Zamolodchikov’s Y-systems in thermodynamic Bethe Ansatz [9].
• A new family of convex polytopes (generalized associahedra) including as special cases Stasheff’s associahedron, and Bott–Taubes cyclohedron [9], [2].

Received July 9, 2003; in revised form December 14, 2003.
The first named author supported in part by US Department of Education GAANN grant.
The second named author supported in part by NSF (DMS) Grant No. 0200299.

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Amid these exciting developments, virtually no progress has been made (at least, documented) towards the original motivation. This paper presents the first step in this direction by introducing and explicitly constructing the canonical basis $B$ for a special class of rank 2 cluster algebras of finite and affine type. More precisely, in Theorem 2.3 we show that $B$ is uniquely determined by a certain positivity property. Needless to say, we hope to be able one day to extend the results of this paper to a much more general class of cluster algebras.

Restricting our attention to rank 2 cluster algebras allows us to give all the necessary definitions in a quick and painless way. Thus, the treatment in this paper is completely self-contained, and no preliminary knowledge on cluster algebras is assumed from the reader.

2. Main Results

Let $\mathcal{F} = \mathbb{Q}(y_1, y_2)$ be the field of rational functions in two (commuting) independent variables $y_1$ and $y_2$ with rational coefficients. Given positive integers $b$ and $c$, recursively define elements $y_m \in \mathcal{F}$ for $m \in \mathbb{Z}$ by the relations

$$y_{m-1}y_{m+1} = \begin{cases} y_m^b + 1 & \text{for } m \text{ odd;} \\ y_m^c + 1 & \text{for } m \text{ even.} \end{cases} (2.1)$$

The main object of study in this paper is the cluster algebra $\mathcal{A}(b, c)$: by definition, this is a subring of $\mathcal{F}$ generated by the $y_m$ for $m \in \mathbb{Z}$. The elements $y_m$ are called cluster variables and the relations (2.1) are called the exchange relations. The sets \{\{y_m, y_{m+1}\}\} for $m \in \mathbb{Z}$ are called clusters.

In the terminology of [6], $\mathcal{A}(b, c)$ is a (coefficient-free) cluster algebra of rank 2. The most general definition of cluster algebras given in [6] allows terms in the right hand side of (2.1) to have nontrivial coefficients. In Section 6, we will show that for rank 2 cluster algebras, it suffices to restrict to the coefficient-free case since the coefficients may be removed by a rescaling of the $y_m$.

In this paper, we introduce and explicitly construct the canonical basis $B$ in $\mathcal{A}(b, c)$ for $bc \leq 4$. More precisely, in Theorem 2.3 we show that $B$ is uniquely determined by a certain positivity property. We expect this result to hold in much greater generality; in particular, in a sequel to this paper we are planning to investigate whether the assumption $bc \leq 4$ can be lifted.

Before stating the new results, we briefly discuss some known properties of $\mathcal{A}(b, c)$.

2.1. Laurent phenomenon. It is clear from (2.1) that every cluster of $\mathcal{A}(b, c)$ is a transcendence basis of the ambient field $\mathcal{F}$, so every element of $\mathcal{A}(b, c)$ is uniquely expressed as a rational function in $y_m$ and $y_{m+1}$, for every $m \in \mathbb{Z}$. According to the Laurent phenomenon established in [6], [7], all these rational functions are actually Laurent polynomials with integer coefficients. The following stronger result is a
special case of the results in [1]:

\[ A(b, c) = \bigcap_{m \in \mathbb{Z}} \mathbb{Z}[y_m^{\pm 1}, y_{m+1}^{\pm 1}] = \bigcap_{m=0}^2 \mathbb{Z}[y_m^{\pm 1}, y_{m+1}^{\pm 1}], \quad (2.2) \]

where \( \mathbb{Z}[y_m^{\pm 1}, y_{m+1}^{\pm 1}] \) denotes the ring of Laurent polynomials with integer coefficients in \( y_m \) and \( y_{m+1} \). The symmetry of the exchange relations (2.1) allows the second intersection in (2.2) to be taken over any three consecutive clusters.

2.2. Standard monomial basis. Another special case of the results in [1] is the following:

the set \( \{ y_0^{a_0} y_1^{a_1} y_2^{a_2} y_3^{a_3} : a_m \in \mathbb{Z}_{\geq 0}, a_0a_2 = a_1a_3 = 0 \} \) is a \( \mathbb{Z} \)-basis of \( A(b, c) \). (2.3)

Again, in view of the obvious symmetry, the variables \( y_0, \ldots, y_3 \) in (2.3) can be replaced by any four consecutive cluster variables. We refer to the basis in (2.3) as a standard monomial basis.

2.3. Finite generation. As an immediate corollary of (2.3) (cf. [1]), we obtain that the algebra \( A(b, c) \) is finitely generated and has the following presentation:

\[ A(b, c) = \mathbb{Z}[y_0, y_1, y_2, y_3]/\langle y_0 y_2 - y_1^b - 1, y_1 y_3 - y_2^c - 1 \rangle . \quad (2.4) \]

2.4. Finite type classification. The structure of \( A(b, c) \) is closely related to the root system associated with the (generalized) Cartan matrix

\[ A = A(b, c) = \begin{pmatrix} 2 & -b \\ -c & 2 \end{pmatrix} . \quad (2.5) \]

Some properties of this root system will be reviewed in Section 3. Borrowing the terminology from the theory of Kac–Moody algebras (see [13]), we say that \( A(b, c) \) is of finite (resp. affine, indefinite) type if so is \( A(b, c) \), i.e., if \( bc \leq 3 \) (resp. \( bc = 4 \), \( bc > 4 \)). The following result is proved in [6, Theorem 6.1] (it is extended to cluster algebras of an arbitrary rank in [8]).

**Proposition 2.1.** The algebra \( A(b, c) \) is of finite type if and only if it has finitely many distinct cluster variables. More precisely, if \( bc \leq 3 \) then

\[ y_m = y_n \iff m \equiv n \mod (h+2), \]

where \( h \) is the Coxeter number given by Table 1 below; whereas, if \( bc \geq 4 \) then all \( y_m \) are distinct.

<table>
<thead>
<tr>
<th>bc</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>( \geq 4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h )</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>( \infty )</td>
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</table>
2.5. Positivity and canonical basis. We now turn to the statements of the new results. We will introduce the “canonical” \( \mathbb{Z} \)-basis \( B \) in \( A(b, c) \), in the cases when \( A(b, c) \) is of finite or affine type (note that the basis in (2.3) is not canonical, because it involves an arbitrary choice of four consecutive cluster variables). Our construction of \( B \) is based on the following concept of positivity.

Definition 2.2. A non-zero element \( y \in A(b, c) \) is positive if for every \( m \in \mathbb{Z} \), all the coefficients in the expansion of \( y \) as a Laurent polynomial in \( y_m \) and \( y_{m+1} \) are positive.

The set of positive elements is a semiring, i.e., is closed under addition and multiplication. The following theorem which is the main result of the paper gives a complete description of this semiring in the finite and affine type cases.

Theorem 2.3. Suppose that \( bc \leq 4 \), i.e., \( A(b, c) \) is of finite or affine type. Then there exists a unique \( \mathbb{Z} \)-basis \( B \) of \( A(b, c) \) such that the semiring of positive elements of \( A(b, c) \) consists precisely of positive integer linear combinations of elements of \( B \).

The uniqueness of \( B \) is clear: it consists of all indecomposable positive elements (those that cannot be written as a sum of two positive elements). We call \( B \) the canonical basis in \( A(b, c) \).

Our next result provides a parametrization of \( B \). Let \( Q = \mathbb{Z}^2 \) be a lattice of rank 2 with a fixed basis \( \{\alpha_1, \alpha_2\} \). We will sometimes write a point \( \alpha = a_1 \alpha_1 + a_2 \alpha_2 \in Q \) simply as \( \alpha = (a_1, a_2) \).

Theorem 2.4. In the situation of Theorem 2.3, for every \( \alpha = (a_1, a_2) \in Q \), there is a unique basis element \( x[\alpha] \in B \) of the form

\[
x[\alpha] = \frac{N_\alpha(y_{1}, y_{2})}{y_{1}^{a_1} y_{2}^{a_2}},
\]

where \( N_\alpha \) is a polynomial with constant term 1. The correspondence \( \alpha \mapsto x[\alpha] \) is a bijection between \( Q \) and \( B \).

By its definition in Theorem 2.3, the canonical basis \( B \) is invariant under any automorphism of \( A(b, c) \) that preserves the semiring of positive elements; we refer to such an automorphism as positive. In view of the exchange relations (2.1), for every \( p \in \mathbb{Z} \), there is a positive automorphism \( \sigma_p \) of \( A(b, c) \) (for arbitrary \( b \) and \( c \)) acting on cluster variables by an involutive permutation \( \sigma_p(y_m) = y_{2p-m} \). The group of automorphisms of \( A(b, c) \) generated by all \( \sigma_p \) acts transitively on the set of clusters; it is easy to see that this group is generated by any two elements \( \sigma_p \) and \( \sigma_{p+1} \), for instance, by \( \sigma_1 \) and \( \sigma_2 \).

Theorem 2.5. The bijection between \( B \) and \( Q \) given in Theorem 2.4 translates the action of each \( \sigma_p \) on \( B \) into a piecewise linear transformation of \( Q \). In particular, the action of \( \sigma_1 \) and \( \sigma_2 \) on \( Q \) is given by

\[
\sigma_1(a_1, a_2) = (a_1, c \max(a_1, 0) - a_2), \quad \sigma_2(a_1, a_2) = (b \max(a_2, 0) - a_1, a_2).
\]

Remark 2.6. The piecewise linear transformations in (2.7) are a special case of the transformations \( \tau_+ \) and \( \tau_- \) appearing in [9, (1.7)]. Note that they can be viewed as “tropicalizations” of exchange relations (2.1). We expect that a similar interpretation exists for cluster algebras of higher rank.
2.6. Explicit realization of $\mathcal{B}$. We now describe the canonical basis $\mathcal{B}$ explicitly. An element of the form $y_m^p y_{m+1}^q$ for some $m \in \mathbb{Z}$ and $p, q \in \mathbb{Z}_{\geq 0}$ is called a cluster monomial.

**Theorem 2.7.** If $bc \leq 3$, i. e., $\mathcal{A}(b, c)$ is of finite type, then the basis $\mathcal{B}$ in Theorem 2.3 is the set of all cluster monomials.

Turning to the affine types, we notice that there are only two cases to consider: $(b, c) = (2, 2)$, or $(b, c) = (1, 4)$ (the case $(b, c) = (4, 1)$ reduces to the latter one by shifting the indices of all cluster variables by 1). In each of these two cases, we introduce an element $z \in \mathcal{A}(b, c)$ by setting

$$z = \begin{cases} y_0 y_3 - y_1 y_2 & \text{if } (b, c) = (2, 2); \\ y_0 y_3 - (y_1 + 2) y_2^2 & \text{if } (b, c) = (1, 4). \end{cases} \tag{2.8}$$

Let $T_0, T_1, \ldots$ be the sequence of Chebyshev polynomials of the first kind given by $T_0 = 1$, and $T_n(t + t^{-1}) = t^n + t^{-n}$ for $n > 0$. In each of the affine types, we define the sequence $z_1, z_2, \ldots$ of elements of $\mathcal{A}(b, c)$ by setting $z_n = T_n(z)$.

**Theorem 2.8.** If $(b, c)$ is one of the pairs $(2, 2)$ or $(1, 4)$ then the basis $\mathcal{B}$ in Theorem 2.3 is the disjoint union of the set of all cluster monomials and the set $\{z_n : n \geq 1\}$.

**Remark 2.9.** Theorems 2.7 and 2.8 imply in particular that in the finite or affine type, every cluster variable $y_n$ is a positive element of $\mathcal{A}(b, c)$ in the sense of Definition 2.2. Jim Propp informed us that he (resp. Gregg Musiker) found a combinatorial proof of this fact for $\mathcal{A}(2, 2)$ (resp. $\mathcal{A}(1, 4)$). Dylan Thurston told us that he found a topological interpretation of $\mathcal{A}(2, 2)$ leading to yet another proof of positivity for this case. As conjectured in [6], the positivity of cluster variables is expected to hold for cluster algebras of arbitrary rank; cf. also Remark 5.6 below.

Following [6], for $b, c$ arbitrary, we identify $Q$ with the root lattice associated with the Cartan matrix (2.5), so that $\alpha_1$ and $\alpha_2$ become simple roots (see Section 3 for the background on the corresponding root system). For the initial cluster variables $y_1$ and $y_2$, the correspondence (2.6) takes the form

$$y_1 = \frac{1}{y_1^1} = x[\alpha_1], \quad y_2 = \frac{1}{y_2^1} = x[\alpha_2].$$

As shown in [6, Theorem 6.1], the rest of the cluster variables have the following description.

**Proposition 2.10.** In a cluster algebra $\mathcal{A}(b, c)$ with $b, c$ arbitrary (not necessarily of finite or affine type), every cluster variable $y_n$ different from $y_1$ and $y_2$ has the form (2.6) for a positive real root $\alpha$. This correspondence is a bijection between the set of all positive real roots associated with the Cartan matrix $A(b, c)$, and the set of cluster variables different from $y_1$ and $y_2$ in $\mathcal{A}(b, c)$.

In the affine type case $bc = 4$, positive imaginary roots are all positive integer multiples of the root $\delta$ given by

$$\delta = \begin{cases} \alpha_1 + \alpha_2 & \text{if } (b, c) = (2, 2); \\ \alpha_1 + 2 \alpha_2 & \text{if } (b, c) = (1, 4). \end{cases} \tag{2.9}$$
The following result complements Proposition 2.10.

**Proposition 2.11.** Under the correspondence in Theorem 2.4, we have \( x[n\delta] = z_n \) for all \( n > 0 \).

**Remark 2.12.** As a consequence of Theorem 2.5 and Proposition 2.11, each \( z_n \) is invariant under all automorphisms \( \sigma_p \) (see Proposition 5.3). In particular, this means that in the “non-canonical” definition (2.8), the four cluster variables \( y_0, y_1, y_2, \) and \( y_3 \) can be replaced by any four consecutive cluster variables.

**Remark 2.13.** In view of Theorem 2.8, no non-trivial monomial in the \( z_n \) belongs to the canonical basis \( B \). This is in contrast with the behavior of cluster variables, since \( B \) contains all cluster monomials. This contrast between the “real” generators \( y_m \) and the “imaginary” generators \( z_n \) seems to be a special case of a general phenomenon taking place in any cluster algebra of infinite type (cf. [14]).

The rest of the paper is organized as follows. In Section 3, after a necessary background on root systems of rank 2, we state and prove some results about cluster monomials which are valid in an arbitrary algebra \( A(b, c) \) (not assumed to be of finite or affine type). In particular, we show (Corollary 3.3) that the cluster monomials are always linearly independent. Sharpening Proposition 2.10, we compute the Newton polygon of every cluster variable \( y_m \) viewed as a Laurent polynomial in \( y_1 \) and \( y_2 \) (Proposition 3.5): this description turns out to be crucial for the proof of Theorem 2.3.

Section 4 contains the proofs of our main results (Theorems 2.3, 2.4, 2.5, and 2.7) in the finite type case, while Section 5 treats the affine types. To avoid a case-by-case analysis of the algebras \( A(2, 2) \) and \( A(1, 4) \), we introduce in Section 5 an interesting 2-parameter deformation of \( A(2, 2) \), and extend to it most of the results in question.

Finally, in Section 6 we deal with more general cluster algebras of rank 2, in which the two monomials on the right hand side of (2.1) have non-trivial coefficients. We show (Proposition 6.1 and Theorem 6.3) that this seemingly more general case reduces to the coefficient-free case by a rescaling of cluster variables.

## 3. Cluster Monomials and their Newton Polygons

### 3.1. Background on rank 2 root systems.

In this section we work with an arbitrary cluster algebra of rank 2; so the indefinite case \( bc > 4 \) is not excluded.

We start by recalling some basic facts about the root system associated with a 2 × 2 Cartan matrix \( A = A(b, c) \) (more details can be found in [13]).

First of all, the Weyl group \( W = W(A) \) is a group of linear transformations of \( Q \) generated by two simple reflections \( s_1 \) and \( s_2 \) whose action in the basis of simple roots is given by

\[
\begin{bmatrix}
-1 & b \\
0 & 1
\end{bmatrix}, \quad 
\begin{bmatrix}
1 & 0 \\
c & -1
\end{bmatrix}.
\]

(3.1)

Since both \( s_1 \) and \( s_2 \) are involutions, each element of \( W \) is one of the following:

\[
w_1(m) = s_1s_2s_1\cdots s_{(m)}, \quad w_2(m) = s_2s_1s_2\cdots s_{(m+1)};
\]
here $⟨m⟩$ stands for the element of $\{1, 2\}$ congruent to $m$ modulo 2, and both products are of length $m ≥ 0$. It is well known that $W$ is finite if and only if $bc ≤ 3$. The Coxeter number $h$ of $W$ is the order of $s_1s_2$ in $W$; it is given by Table 1. In the finite case, $W$ is the dihedral group of order $2h$, and its elements can be listed as follows: $w_3(0) = w_3(0) = e$ (the identity element), $w_1(h) = w_2(h) = w_0$ (the longest element), and $2h − 2$ distinct elements $w_1(m), w_2(m)$ for $0 < m < h$. In the infinite case, all elements $w_1(m)$ and $w_2(m)$ for $m > 0$ are distinct.

A vector $α ∈ Q$ is a real root for $A$ if it is $W$-conjugate to a simple root. Let $Φ^{re}$ denote the set of real roots for $A$. It is known that $Φ^{re} = Φ^{re}_+ ∪ (−Φ^{re}_+)$, where

$$Φ^{re}_+ = \{α = a_1α_1 + a_2α_2 ∈ Φ^{re} : a_1, a_2 ≥ 0\}$$

is the set of positive real roots.

In the finite case, $Φ^{re}_+$ has cardinality $h$, and we have

$$Φ^{re}_+ = \{w_1(m)α_{m+1} : 0 ≤ m < h\}.$$  

In the infinite case, we have

$$Φ^{re}_+ = \{w_1(m)α_{m+1} : i ∈ \{1, 2\}, m ≥ 0\},$$

with all the roots $w_1(m)α_{m+1}$ distinct.

To introduce imaginary roots, consider the $W$-invariant symmetric scalar product on $Q$ given by

$$(α, α) = ca_1^2 − bcα_1α_2 + ba_2^2 \quad (α = a_1α_1 + a_2α_2 ∈ Q).$$  \(3.2\)

According to [13], the set of imaginary roots can be defined as follows:

$$Φ^{im} = \{α ∈ Q − \{0\} : (α, α) ≤ 0\}. \quad (3.3)$$

Similarly to the case of real roots, we have $Φ^{im} = Φ^{im}_+ ∪ (−Φ^{im}_+)$, where

$$Φ^{im}_+ = \{α = a_1α_1 + a_2α_2 ∈ Φ^{im} : a_1, a_2 ≥ 0\}$$

is the set of positive imaginary roots. In view of (3.3) and (3.2), a vector $a_1α_1 + a_2α_2$ from $Q$ belongs to $Φ^{im}_+$ if and only if $a_1, a_2 > 0$, and

$$\frac{bc − \sqrt{bc(bc − 4)}}{2b} ≤ \frac{a_2}{a_1} ≤ \frac{bc + \sqrt{bc(bc − 4)}}{2b}. \quad (3.4)$$

In particular, in the affine type case $bc = 4$, positive imaginary roots are precisely the positive integer multiples of the minimal such root $δ$ given by (2.9). Note also that in the indefinite case $bc > 4$, the upper and lower bounds for $a_2/a_1$ in (3.4) are irrational numbers (indeed, in this case, $(bc − 3)^2 < bc(bc − 4) < (bc − 2)^2$, hence $bc(bc − 4)$ is not a perfect square); so both inequalities in (3.4) can be replaced by strict ones.

3.2. Parameterizing cluster monomials. Returning to cluster algebras, we start with the following result.

**Proposition 3.1.** Every cluster monomial has a (unique) presentation of the form (2.6), with $α ∈ Q − Φ^{im}_+$. This correspondence is a bijection between the set of all cluster monomials and $Q − Φ^{im}_+$. 

Proof. According to [6, Theorem 6.1] or Proposition 2.10 above, every cluster variable has the form \( y_m = x(\alpha(m)) \); here \( \alpha(m) = -\alpha_m \) for \( m \in \{1, 2\} \), and the rest of the cluster variables are in a bijection with the set of all positive real roots. Clearly, every cluster monomial \( y_m^n y_{m+1}^p \) also has the form \( x[\alpha] \) with \( \alpha = pa(m) + qa(m+1) \). To complete the proof it is enough to show the following:

For every \( m \in \mathbb{Z} \), the vectors \( \alpha(m) \) and \( \alpha(m+1) \) form a \( \mathbb{Z} \)-basis of \( Q \). \hspace{1cm} (3.5)

For every \( m \in \mathbb{Z} \), the vectors \( \alpha(m) \) and \( \alpha(m+1) \) are the only positive real roots contained in the additive semigroup \( \mathbb{Z}_{\geq 0}\alpha(m) + \mathbb{Z}_{\geq 0}\alpha(m+1) \). \hspace{1cm} (3.6)

The union \( \bigcup_{m \in \mathbb{Z}} (\mathbb{Z}_{\geq 0}\alpha(m) + \mathbb{Z}_{\geq 0}\alpha(m+1)) \) is equal to \( Q - \Phi^m \). \hspace{1cm} (3.7)

In the finite type, the properties (3.5)–(3.7) are seen by an easy inspection (they are in fact special cases of the results in [9, Theorems 1.8,1.10] valid for all cluster algebras of finite type). So let us assume we are in the infinite type case \( bc \geq 4 \). As shown in [6, Theorem 6.1], the positive roots \( \alpha(m) \) are given by

\[
\alpha(m+3) = w_1(m)\alpha(m+1), \quad \alpha(-m) = w_2(m)\alpha(m+2) \quad (m \geq 0).
\]

In particular, we have \( \alpha(3) = \alpha_1 \) and \( \alpha(0) = \alpha_2 \). Thus, properties (3.5) and (3.6) hold for \( m \in \{0, 1, 2\} \), and we also have

\[
\bigcup_{m=0}^{2} (\mathbb{Z}_{\geq 0}\alpha(m) + \mathbb{Z}_{\geq 0}\alpha(m+1)) = Q - (\mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_2).
\]

We now check (3.5) and (3.6) for \( m \geq 3 \) (the case \( m \leq 0 \) follows by the obvious symmetry between \( \alpha_1 \) and \( \alpha_2 \). As shown in [6, (6.7)], the roots \( \alpha(m) \) for \( m \in \mathbb{Z} - \{1, 2\} \) satisfy the recurrence

\[
\alpha(m+1) + \alpha(m-1) = \begin{cases} 
bc\alpha(m) & \text{if } m \text{ is odd;} \\
ca\alpha(m) & \text{if } m \text{ is even.}
\end{cases}
\]

Writing each \( \alpha(m) \) as \( a_{m1}\alpha_1 + a_{m2}\alpha_2 \), we conclude from (3.9) that

\[
\det \begin{pmatrix} a_{m1} & a_{m+1,1} \\ a_{m2} & a_{m+1,2} \end{pmatrix} = \det \begin{pmatrix} a_{m-1,1} & a_{m1} \\ a_{m-1,2} & a_{m2} \end{pmatrix} = \cdots = \det \begin{pmatrix} a_{21} & a_{31} \\ a_{22} & a_{32} \end{pmatrix} = 1 \hspace{1cm} (3.10)
\]

for \( m \geq 2 \). This proves (3.5), as well as the fact that for \( m \geq 3 \), the vectors \( \alpha(m-1) \) and \( \alpha(m+1) \) lie on the opposite sides of the ray through \( \alpha(m) \). It follows that the sequence \( (a_{m2}/a_{m1})_{m \geq 3} \) is strictly increasing. In view of (3.4), to finish the proof of (3.6) and (3.7), it is enough to show that

\[
\lim_{m \to \infty} \frac{a_{m2}}{a_{m1}} = \frac{bc - \sqrt{bc(bc - 4)}}{2b} \hspace{1cm} (3.11)
\]

Let us abbreviate \( u_m = a_{m2}/a_{m1} \). By (3.8), for \( m \geq 3 \) we have \( \alpha(m+2) = s_1 s_2 \alpha(m) \). Remembering (3.1), we conclude that

\[
u_{m+2} = \frac{c - u_m}{bc - 1 - bu_m} \quad (m \geq 3).
\]
An elementary check shows that the transformation
\[ u \mapsto f(u) = \frac{c - u}{bc - 1 - bu} \]
preserves the interval \([0, \frac{bc - \sqrt{bc(bc - 4)}}{2b}]\), and we have \(f(u) > u\) for \(0 \leq u < \frac{bc - \sqrt{bc(bc - 4)}}{2b}\). This implies (3.11) and completes the proof of Proposition 3.1. □

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Root lattice for \(b = c = 2\). Lattice points in the closed shaded sector correspond to cluster monomials \(y_p^q y_j^i\).}
\end{figure}

**Example 3.2.** We illustrate Proposition 3.1 by explicitly computing the family \((\alpha(m))_{m \in \mathbb{Z}}\) for the affine types. First, let \((b, c) = (2, 2)\). Using the recurrence (3.9) and induction on \(|m|\), we obtain that
\[
\alpha(m) = \begin{cases} 
(m - 2)\alpha_1 + (m - 3)\alpha_2 & \text{if } m \geq 2; \\
-\alpha_1 + (1 - m)\alpha_2 & \text{if } m \leq 1.
\end{cases} \tag{3.12}
\]
These roots and the corresponding cluster variables are shown in Figure 1.

Similarly, for \((b, c) = (1, 4)\), we get
\[
\langle m \rangle \alpha(m) = \begin{cases} 
(m - 2)\alpha_1 + 2(m - 3)\alpha_2 & \text{if } m \geq 2; \\
-\alpha_1 + 2(1 - m)\alpha_2 & \text{if } m \leq 1.
\end{cases} \tag{3.13}
\]
Corollary 3.3. In any cluster algebra of rank 2, the cluster monomials are linearly independent.

Proof. For $\gamma = g_1\alpha_1 + g_2\alpha_2 \in Q$, we abbreviate $y^\gamma = y_1^{g_1}y_2^{g_2}$. We will use the product partial order on $Q = \mathbb{Z}^2$:

$$\gamma_1 \geq \gamma_2 \iff \gamma_1 - \gamma_2 \in Q_+ = \mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_2.$$  

By Proposition 3.1, cluster monomials can be parametrized by $Q - \Phi_+^m$ so that the cluster monomial corresponding to $\alpha \in Q - \Phi_+^m$ has the form

$$x[\alpha] = y^{-\alpha} + \sum_{\gamma > -\alpha} c_\gamma y^\gamma. \quad (3.14)$$

Now suppose that a (finite) integer linear combination of the cluster monomials $x[\alpha]$ is equal to 0. Let $S \subset Q - \Phi_+^m$ be the set of all $\alpha$ such that $x[\alpha]$ occurs with a non-zero coefficient in this linear combination. If $S$ is non-empty, pick a maximal element $\alpha \in S$; in view of (3.14), the (Laurent) monomial $y^{-\alpha}$ does not occur in any $x[\beta]$ for $\beta \in S - \{\alpha\}$, which gives a desired contradiction. □

We now turn our attention to positive automorphisms $\sigma_p$ of $A(b, c)$ which appear in Theorem 2.5. By definition, each $\sigma_p$ acts on the set of all cluster monomials; identifying the latter set with $Q - \Phi_+^m$ as in Proposition 3.1, we obtain the action of $\sigma_p$ on $Q - \Phi_+^m$.

Proposition 3.4. The action of $\sigma_1$ and $\sigma_2$ on $Q - \Phi_+^m$ is given by (2.7).

Proof. By symmetry, it is enough to prove the claim for $\sigma_1$. In the finite type case, it is seen by inspection, so we shall deal with the infinite type case $bc \geq 4$. Remembering the definition, we see that the action of $\sigma_1$ on $Q - \Phi_+^m$ is given as follows: $\sigma_1(\alpha(m)) = \alpha(2 - m)$ for $m \in \mathbb{Z}$, and $\sigma_1$ acts linearly in each cone $\mathbb{Z}_{\geq 0}\alpha(m) + \mathbb{Z}_{\geq 0}\alpha(m + 1)$; here $\alpha(1) = -\alpha_1$, $\alpha(2) = -\alpha_2$, and the rest of the $\alpha(m)$ are all positive real roots labeled according to (3.8). This description implies that $\sigma_1(-\alpha_1) = -\alpha_1$, and $\sigma_1(\alpha(m)) = s_2\alpha(m)$ for $m \neq 1$, i.e., when $\alpha(m)$ is either a positive real root, or $-\alpha_2$. It follows that the action of $\sigma_1$ on $\alpha = (a_1, a_2) \in Q - \Phi_+^m$ is given as follows:

$$\sigma_1(\alpha) = \begin{cases} (a_1, -a_2) & \text{if } a_1 \leq 0; \\ s_2(a_1, a_2) = (a_1, ca_1 - a_2) & \text{if } a_1 \geq 0. \end{cases}$$

This is clearly equivalent to the first equality in (2.7), and we are done. □

3.3. Newton polygons of cluster variables. Sharpening (3.14), we now give an explicit description of the Newton polygon of any cluster variable $y_m$. Recall that the Newton polygon $\text{Newt}(x)$ of a Laurent polynomial $x \in \mathbb{Z}[y_1^{\pm 1}, y_2^{\pm 1}]$ is the convex hull in $Q_\mathbb{R} = \mathbb{R}\alpha_1 \oplus \mathbb{R}\alpha_2$ of all lattice points $\gamma$ such that the monomial $y^\gamma$ appears with a non-zero coefficient in the Laurent expansion of $x$. We will say that $x$ is monic if every monomial corresponding to a vertex of $\text{Newt}(x)$ appears in the Laurent expansion of $x$ with coefficient 1. For $\alpha = a_1\alpha_1 + a_2\alpha_2 \in Q_+$, let $\Delta(\alpha)$ denote the triangle (possibly degenerate) in $Q_\mathbb{R}$ with vertices $-\alpha$, $-\alpha + ba_2\alpha_1$, and $-\alpha + ca_1\alpha_2$. 
Proposition 3.5. For every positive real root $\alpha$, the corresponding cluster variable $x[\alpha]$ is a monic Laurent polynomial in $y_1$ and $y_2$, and its Newton polygon $\text{Newt}(x[\alpha])$ is equal to the triangle $\Delta(\alpha)$.

Proof. Using the obvious symmetry, it is enough to prove the proposition for the cluster variables $y_m$ with $m \geq 3$. The cluster relations (2.1) imply immediately that

$$y_3 = x[\alpha_1] = y^{-\alpha_1} + y^{-\alpha_1 + c\alpha_2}$$

and

$$y_4 = x[b\alpha_1 + \alpha_2] = y^{-b\alpha_1 - \alpha_2}(y^{c\alpha_2 + 1})^b + y^{-\alpha_2};$$

it follows that both $y_3$ and $y_4$ satisfy the proposition. Now let $m \geq 4$ be such that each of the cluster variables $y_{m-1}, y_m$ and $y_{m+1}$ is different from $y_1$ and $y_2$ (this condition is vacuous for the infinite types). Proceeding by induction on $m$, we assume that the proposition holds for $y_{m-1}$ and $y_m$, and will prove it for $y_{m+1}$. Let $m$ be even (the case of $m$ odd is entirely similar). To find $\text{Newt}(y_{m+1})$, we compute the Newton polygons of both sides of the relation $y_{m-1}y_{m+1} = y_m^2 + 1$. Since the Newton polygon of the product of two Laurent polynomials is the Minkowski sum of their Newton polygons, we obtain

$$\text{Newt}(x[\alpha(m-1)]) + \text{Newt}(x[\alpha(m+1)]) = \text{Newt}(x[\alpha(m)]^n + 1).$$

We will use the following well-known properties of convex polygons in $\mathbb{R}^2$ and their Minkowski addition (see [12, Section 15.3]). With every convex $n$-gon $\Pi$ in $\mathbb{R}^2$ there is associated a set $V(\Pi)$ of $n$ non-zero vectors in $\mathbb{R}^2$ defined as follows: each side $AB$ of $\Pi$ contributes a vector of the same length as $AB$ whose direction is that of the outward normal to $AB$. In the degenerate case $n = 2$ when $\Pi$ is just a line segment, the set $V(\Pi)$ is a pair of opposite vectors, each of length equal to $\Pi$ and perpendicular to $\Pi$. Clearly, the sum of $n$ vectors from $V(\Pi)$ is 0, and no two of them are positively proportional. Conversely, every $n$-set of non-zero vectors in $\mathbb{R}^2$ with these properties has the form $V(\Pi)$, where a convex $n$-gon $\Pi$ is determined uniquely up to translations. Furthermore, for every two convex polygons $\Pi_1$ and $\Pi_2$ in $\mathbb{R}^2$, the set $V(\Pi_1 + \Pi_2)$ associated with their Minkowski sum is obtained by taking the union of $V(\Pi_1)$ and $V(\Pi_2)$ (as a multiset) and replacing any pair of positively proportional vectors by their sum. This implies in particular the cancellation property for the Minkowski addition: a convex polygon $\Pi_2$ is uniquely determined by $\Pi_1$ and $\Pi_1 + \Pi_2$.

Applying the cancellation property to (3.15) we see that the desired equality

$$\text{Newt}(x[\alpha(m+1)]) = \Delta(\alpha(m+1))$$

becomes a consequence of

$$\Delta(\alpha(m-1)) + \Delta(\alpha(m+1)) = \text{Conv}(c\Delta(\alpha(m)) \cup \{0\}),$$

where $\text{Conv}$ stands for the convex hull in $\mathbb{R}^2 = \mathbb{R}^2$.

Our proof of (3.16) is illustrated by Figure 2. We first compute the polygon on the right hand side. Note that if $\alpha = a_1\alpha_1 + a_2\alpha_2 \in Q$ is such that $a_1, a_2 > 0$ then the side of the triangle $\Delta(\alpha)$ opposite to the vertex $-\alpha$ lies on the straight line

$$\{g_1\alpha_1 + g_2\alpha_2: ca_1g_1 + ba_2g_2 + (\alpha, \alpha) = 0\},$$
where \((\alpha, \alpha)\) is given by (3.2). If \(\alpha\) is a positive real root then \((\alpha, \alpha) > 0\), hence \(\Delta(\alpha)\) does not contain the origin. We conclude that \(\text{Conv}(c\Delta(\alpha(m)) \cup \{0\})\) is the quadrilateral with vertices \(-\alpha1(m), -\alpha2(m) + bca(m)\alpha1, 0\), and \(-\alpha2(m) + c^2\alpha1\alpha2\), where as before, we write \(\alpha(m) = a1\alpha1 + a2\alpha2\). It follows easily that the corresponding set \(\text{V}(\text{Conv}(c\Delta(\alpha(m)) \cup \{0\}))\) consists of the following four vectors written in the basis \(\alpha1, \alpha2\):

\[
\{(-c^2a1, 0), (0, -bca2), (ca2, bca2 - ca1), (c^2a1 - ca2, ca1)\}. \tag{3.17}
\]

The position of the quadrilateral itself is uniquely determined by this set and the property that its minimal vertex (with respect to the product order on \(\mathbb{R}^2\)) is \(-\alpha1(m) = (-ca1, -ca2)\).

Turning to the left hand side of (3.16), we use the following easily checked property: if \(\alpha = a1\alpha1 + a2\alpha2 \in Q_+\) then

\[
\text{V}(\Delta(\alpha)) = \{(-ca1, 0), (0, -ba2), (ca1, ba2)\}
\]

(with the convention that if, say, \(a1 > 0\) and \(a2 = 0\) then the zero vector \((0, -ba2)\) does not appear in \(\text{V}(\Delta(\alpha))\)). Using the equality \(\alpha(m - 1) + \alpha(m + 1) = ca(m)\) from (3.9), we see that \(\text{V}(\Delta(\alpha(m - 1)) + \Delta(\alpha(m + 1)))\) is equal to

\[
\{(-c^2a1, 0), (0, -bca2), (ca_{m-1,1}, ba_{m-1,2}), (ca_{m+1,1}, ba_{m+1,2})\}. \tag{3.18}
\]
It follows that $\Delta(\alpha(m - 1)) + \Delta(\alpha(m + 1))$ is a quadrilateral; again, its position is uniquely determined by $V(\Delta(\alpha(m - 1)) + \Delta(\alpha(m + 1)))$ and the property that its minimal vertex is $-(\alpha(m - 1) + \alpha(m + 1)) = -ca(m)$.

To show (3.16), it remains to prove that the sets in (3.17) and (3.18) are equal to each other. Remembering that the vectors in each of the sets must sum to zero, we see it suffices to show that

$$a_{m2} = a_{m-1,1}, \quad ca_{m1} = ba_{m+1,2}.\$$

Once again using (3.9), we can rewrite these equalities as

$$a_{m+1,2} = ca_{m-1,1} - a_{m-1,2}, \quad a_{m-1,1} = ba_{m+1,2} - a_{m+1,1}, \quad (3.19)$$

which is an immediate consequence of $\alpha(m + 1) = s_1s_2\alpha(m - 1)$. This concludes the proof of the equality $\text{Newt}(x[\alpha(m + 1)]) = \Delta(\alpha(m + 1))$.

To complete the proof of Proposition 3.5, it remains to show that if $y_{m-1}$ and $y_m$ are monic then so is $y_{m+1}$. In view of the exchange relations, this is a consequence of the following easily verified general property: a Laurent polynomial equal to the ratio of two monic Laurent polynomials is itself monic. \hfill \Box

We conclude this section with the following corollary of Proposition 3.5.

**Proposition 3.6.** Let $y = y^p_m y^q_{m+1}$ be a cluster monomial containing at least one cluster variable different from $y_1$ and $y_2$. Then there exists a non-zero linear form $\varphi_m$ on $Q_\mathbb{R}$ such that

$$\varphi_m(g_1\alpha_1 + g_2\alpha_2) = c_{m1}g_1 + c_{m2}g_2, \quad c_{m1}, c_{m2} \geq 0,$$

and $\text{Newt}(y)$ lies in the open half-plane $\{\varphi_m < 0\}$. In particular, $\text{Newt}(y)$ has empty intersection with the positive quadrant $Q_+ = \mathbb{Z}_{\geq 0}\alpha_1 + \mathbb{Z}_{\geq 0}\alpha_2$.

**Proof.** It is enough to consider the case $m \geq 2$; by Proposition 2.1, in the finite type case we can also assume without loss of generality that $m \leq \frac{b}{2} + 2$. If $m = 2$ then $y = g^p_2 g^q_1$ for some $p \geq 0$ and $q > 0$; in this case, $\text{Newt}(y)$ is a line segment parallel to $\alpha_2$ and lying in the open half-plane $\{g_1\alpha_1 + g_2\alpha_2; g_1 < 0\}$. Thus, the proposition holds for $m = 2$.

For $m \geq 3$, we define a linear form $\varphi_m$ on $Q_\mathbb{R}$ by setting

$$\varphi_m(g_1\alpha_1 + g_2\alpha_2) = (a_{m+1,2} + ca_{m1} - a_{m2})g_1 + (a_{m,1} + ba_{m+1,2} - a_{m+1,1})g_2.$$

Remembering (3.19), we can rewrite $\varphi_m$ as

$$\varphi_m(g_1\alpha_1 + g_2\alpha_2) = (a_{m,2} + a_{m+1,2})g_1 + (a_{m-1,1} + a_{m,1})g_2, \quad (3.20)$$

so the coefficients of $g_1$ and $g_2$ in $\varphi_m$ are nonnegative. Since

$$\text{Newt}(y) = p\Delta(\alpha(m)) + q\Delta(\alpha(m + 1)),$$

to complete the proof it is enough to show that $\varphi_m$ takes negative values at all vertices of each of the triangles $\Delta(\alpha(m))$ and $\Delta(\alpha(m + 1))$. Clearly, both $\varphi_m(-\alpha(m))$ and $\varphi_m(-\alpha(m + 1))$ are negative since $\varphi_m$ has positive coefficients. As for the
remaining four vertices, a direct calculation using (3.10) yields
\[ \varphi_m(-\alpha(m) + ca_m\alpha_2) = \varphi_m(-\alpha(m + 1) + ba_{m+1,1,2}\alpha_1) = -1, \]
\[ \varphi_m(-\alpha(m) + ba_m\alpha_1) = -\langle\alpha(m), \alpha(m)\rangle - 1, \]
\[ \varphi_m(-\alpha(m + 1) + ca_{m+1,1}\alpha_2) = -\langle\alpha(m + 1), \alpha(m + 1)\rangle - 1. \]
Since \( \alpha(m) \) and \( \alpha(m + 1) \) are real roots, both \( \langle\alpha(m), \alpha(m)\rangle \) and \( \langle\alpha(m + 1), \alpha(m + 1)\rangle \) are positive. Hence all the above values of \( \varphi_m \) are negative, as desired. \[ \square \]

In the affine types, the linear forms \( \varphi_m \) given by (3.20) can be computed explicitly using (3.12) and (3.13). This implies the following corollary which will be used in Section 5.

**Corollary 3.7.** In an affine type, for \( m \geq 3 \), every Laurent monomial \( y_1^{a_1}y_2^{a_2} \) that occurs in the expansion of a non-trivial cluster monomial in \( y_m \) and \( y_{m+1} \) satisfies the following condition:

- \( (2m - 3)g_1 + (2m - 5)g_2 < 0 \) if \( (b, c) = (2, 2) \);
- \( (3m - 3 - \langle m \rangle)g_1 + \frac{3m - 6 - \langle m \rangle}{2}g_2 < 0 \) if \( (b, c) = (1, 4) \).

### 4. PROOFS FOR FINITE TYPES

Throughout this section, \( \mathcal{A}(b, c) \) is a cluster algebra of finite type. It is enough to consider three cases: type \( A_2 \) with \( (b, c) = (1, 1) \), type \( B_2 \) with \( (b, c) = (1, 2) \), and type \( G_2 \) with \( (b, c) = (1, 3) \). Note that in these cases there are no imaginary roots, so Theorem 2.4 (resp. Theorem 2.5) becomes a consequence of Theorem 2.7 and Proposition 3.1 (resp. Proposition 3.4).

To prove Theorems 2.3 and 2.7, it suffices to show the following:

- Every cluster variable is a positive element of \( \mathcal{A}(b, c) \). \[ (4.1) \]
- Cluster monomials linearly span \( \mathcal{A}(b, c) \). \[ (4.2) \]
- If \( y \in \mathcal{A}(b, c) \) is written as a linear combination of cluster monomials then its every coefficient is equal to some coefficient in the Laurent polynomial expansion of \( y \) with respect to some cluster \( \{y_m, y_{m+1}\} \). \[ (4.3) \]

Recall from Proposition 2.1 that every cluster variable \( y_m \) is equal to one of the variables \( y_1, y_2, \ldots, y_{b+h+2} \). To prove (4.1), it is enough to show that each of the elements \( y_1, \ldots, y_{b+h+2} \) is a positive Laurent polynomial in \( y_1 \) and \( y_2 \) (since all the clusters are interchangeable). This is clear from their explicit calculation (the formulas below are obtained from those in the proof of [6, Theorem 6.1] by specializing all the coefficients to 1):

**Type \( A_2 \):** \( (b, c) = (1, 1) \).

\[ y_3 = \frac{y_2 + 1}{y_1}, \quad y_4 = \frac{y_1 + y_2 + 1}{y_1y_2}, \quad y_5 = \frac{y_1 + 1}{y_2}. \] \[ (4.4) \]

**Type \( B_2 \):** \( (b, c) = (1, 2) \).

\[ y_3 = \frac{y_2^2 + 1}{y_1}, \quad y_4 = \frac{y_1 + y_2^2 + 1}{y_1y_2}, \quad y_5 = \frac{(y_1 + 1)^2 + y_2^2}{y_1y_2^2}, \quad y_6 = \frac{y_1 + 1}{y_2}. \] \[ (4.5) \]
Type $G_2$: $(b, c) = (1, 3)$.

\[
y_3 = \frac{y_2^2 + 1}{y_1}, \quad y_4 = \frac{y_1 + y_2^2 + 1}{y_1 y_2}, \quad y_5 = \frac{(y_1 + 1)^3 + y_2^3 (y_2^3 + 3y_1 + 2)}{y_1 y_2^2},
\]
\[
y_6 = \frac{(y_1 + 1)^2 + y_2^3}{y_1 y_2^2}, \quad y_7 = \frac{(y_1 + 1)^3 + y_2^3}{y_1 y_2^3}, \quad y_8 = \frac{y_1 + 1}{y_2}. \tag{4.6}
\]

To prove (4.2), we notice that cluster monomials are the monomials in the cluster variables $y_m$ that do not contain products of the form $y_m y_{m+n}$ for $n \geq 2$. Our first task is to produce the relations that express every such “forbidden” product as a linear combination of cluster monomials. Without loss of generality we can assume that $n \leq (h + 2)/2$. For $n = 2$, the relations in question are just the exchange relations (2.1). It remains to treat $n = 3$ in the $B_2$ case, and $n = 3, 4$ in the $G_2$ case. A direct calculation using (1.5) and (4.6) yields the following:

Type $B_2$: $(b, c) = (1, 2)$.

\[
y_m y_{m+3} = \begin{cases} y_{m-1} + y_{m+1} & \text{if } m \text{ is odd;} \\ y_{m+2} + y_{m+4} & \text{if } m \text{ is even.} \end{cases} \tag{4.7}
\]

Type $G_2$: $(b, c) = (1, 3)$.

\[
y_m y_{m+3} = \begin{cases} y_{m-1} + y_{m+1}^2 & \text{if } m \text{ is odd;} \\ y_{m+2} + y_{m+4} & \text{if } m \text{ is even.} \end{cases} \tag{4.8}
\]
\[
y_m y_{m+4} = \begin{cases} y_{m-2} + y_{m+2} + 3 & \text{if } m \text{ is odd;} \\ y_{m-2} + y_{m+2} & \text{if } m \text{ is even.} \end{cases} \tag{4.9}
\]

To complete the proof of (4.2), it remains to show that the above relations form a system of straightening relations, i.e., that their repeated application allows one to express every monomial in the cluster variables as a linear combination of cluster monomials.

In type $A_2$, the exchange relations are of the form $y_m y_{m+2} = y_m + 1$. Note that both terms on the right have smaller degree than the product on the left hand side. It follows that if we take any monomial $M$ in the cluster variables which has $y_m y_{m+2}$ as a factor, and replace this factor by $y_m + 1$, then $M$ will be written as a sum of two monomials of smaller degree. The fact that $M$ is a linear combination of cluster monomials follows at once by induction on the degree of $M$.

In types $B_2$ and $G_2$, the argument is similar. The only difference is that we use the degree of monomials in the cluster variables based on the following grading: in type $B_2$ (resp. $G_2$) we set $\deg(y_m) = 3$ (resp. 5) for $m$ odd, and $\deg(y_m) = 2$ (resp. 3) for $m$ even. By inspection, every monomial on the right hand side of (4.7)–(4.9) (and of every exchange relation for $B_2$ and $G_2$) has smaller degree than the “forbidden” product on the left hand side of the same relation. Thus, the proof of (4.2) is completed by the same argument.

To prove (4.3), suppose that an element $y \in A(b, c)$ is written as a linear combination of cluster monomials. Again, since all the clusters are interchangeable, it is enough to show that the coefficient of a cluster monomial $y_1^p y_2^q$ in this linear combination is equal to the coefficient of $y_1^p y_2^q$ in the Laurent polynomial expansion of $y$. 

with respect to \{y_1, y_2\}. This follows at once from Proposition 3.6, which implies that, for \( p, q \geq 0 \), the monomial \( y^p_1 y^q_2 \) does not occur in the Laurent expansion of any other cluster monomial. This concludes the proofs of Theorems 2.3 and 2.7 for the finite type.

5. Proofs for Affine Types

In this section, we prove Theorems 2.3, 2.4, 2.5, 2.8, and Proposition 2.11 for the algebras \( \mathcal{A}(2, 2) \) and \( \mathcal{A}(1, 4) \). To avoid the case-by-case analysis as much as possible, we introduce the following 2-parameter deformation of \( \mathcal{A}(2, 2) \). Let \( \tilde{F} = \mathbb{Q}(q_1, q_2, Y_1, Y_2) \) be the field of rational functions in four (commuting) independent variables. We recursively define elements \( Y_m \in \tilde{F} \) for \( m \in \mathbb{Z} \) by the relations
\[
Y_{m-1}Y_{m+1} = Y_m^2 + q_{(m)}Y_m + 1. \tag{5.1}
\]
Now let \( \tilde{A} \) denote the \( \mathbb{Z}[q_1, q_2] \)-subalgebra of \( \tilde{F} \) generated by the \( Y_m \) for all \( m \in \mathbb{Z} \).

In view of [7, Example 5.3], \( \tilde{A} \subset \mathbb{Z}[q_1, q_2, Y_m^\pm 1, Y_{m+1}^\pm 1] \) for every \( m \in \mathbb{Z} \), i.e., every element of \( \tilde{A} \) is a Laurent polynomial in \( Y_m \) and \( Y_{m+1} \) with coefficients in \( \mathbb{Z}[q_1, q_2] \).

The algebra \( \mathcal{A}(2, 2) \) is obtained from \( \tilde{A} \) by an obvious specialization
\[
Y_m = y_m, \quad q_1 = q_2 = 0. \tag{5.2}
\]
Less obvious is the following observation: the subalgebra of \( \mathcal{A}(1, 4) \) generated by the elements \( y_{m}^{(m)} \) for \( m \in \mathbb{Z} \) (i.e., by the \( y_m \) for \( m \) odd, and the \( y_{m}^2 \) for \( m \) even) is obtained from \( \tilde{A} \) by the specialization
\[
Y_m = y_{m}^{(m)}, \quad q_1 = 2, \quad q_2 = 0. \tag{5.3}
\]
To see this, notice that
\[
Y_{m-1}Y_{m+1} = (y_{m-1}y_{m+1})^2 = (y_m + 1)^2 = Y_m^2 + 2Y_m + 1
\]
for \( m \) odd, and
\[
Y_{m-1}Y_{m+1} = y_{m-1}y_{m+1} = y_m^2 + 1 = Y_m^2 + 1
\]
for \( m \) even.

We introduce an element \( Z \in \tilde{A} \) by setting
\[
Z = Y_0Y_3 - (Y_1 + q_1)(Y_2 + q_2). \tag{5.4}
\]
As before, we define the sequence \( Z_1, Z_2, \ldots \) of elements of \( \tilde{A} \) by setting \( Z_n = T_n(Z) \). In view of (2.8), each element \( z_n \in \mathcal{A}(2, 2) \) (resp. \( z_n \in \mathcal{A}(1, 4) \)) is obtained from \( Z_n \) by a specialization (5.2) (resp. (5.3)).

Returning to the cluster algebras of affine types, we now prove the following result, sharpening Proposition 2.11 in the spirit of Proposition 3.5. Recall that \( \delta \) denotes the minimal positive imaginary root given by (2.9).

**Proposition 5.1.** In the affine type, each \( z_n \) is a monic Laurent polynomial in \( y_1 \) and \( y_2 \), and its Newton polygon \( \text{Newt}(z_n) \) is equal to the triangle \( \Delta(n\delta) \).
Proof. It is enough to show that $Z_n$ is a monic Laurent polynomial in $Y_1$ and $Y_2$, and that its Newton polygon Newt($Z_n$) is a triangle $\Delta_n$ with vertices $(-n, -n)$, $(n, -n)$ and $(-n, n)$; in view of (2.9), the proposition then follows from the specializations (5.2) and (5.3).

First of all, combining (5.4) with (5.1), we obtain

$$Z = \frac{Y_1}{Y_2} + \frac{Y_2}{Y_1} + q_1 + q_2 + \frac{1}{Y_1Y_2},$$

(5.5)

implying our claims for $Z_1 = Z$. To deal with $n > 1$, we write $Z$ as $Z = t + t^{-1} + u$, where

$$t = \frac{Y_1}{Y_2}, \quad u = \frac{q_1}{Y_2} + \frac{q_2}{Y_1} + \frac{1}{Y_1Y_2}.$$

Taking the Taylor expansion of the Chebyshev polynomial $T_n$, we get

$$Z_n = T_n(Z) = \sum_{k=0}^{n} \frac{1}{k!} T_n^{(k)}(t + t^{-1})u^k.$$

(5.6)

Since each derivative $T_n^{(k)}$ is a linear combination of the polynomials $T_k$ for $0 \leq k \leq n$, it follows that $Z_n$ is a linear combination of Laurent monomials $t^k u^k$ for $0 \leq k \leq n$ and $-(n-k) \leq k \leq n-k$. The Newton polygon Newt($t^k u^k$) is a triangle (possibly degenerate) with vertices $(\ell - k, -\ell - k)$, $(\ell, -\ell - k)$ and $(\ell - k, -\ell)$, and sides parallel to those of $\Delta_n$. By inspection, all these vertices belong to $\Delta_n$, and so Newt($Z_n$) $\subseteq$ $\Delta_n$. Furthermore, the only term $t^k u^k$ whose Newton polygon may contain the vertex $(-n, -n)$ (resp. $(n, -n)$, $(-n, n)$) of $\Delta_n$ is $u^n$ (resp. $t^n$, $t^{-n}$), and the corresponding Laurent monomial occurs with coefficient 1. But it is clear from (5.6) and the definition of the Chebyshev polynomial $T_n$ that each of the monomials $u^n$ and $t^k u^k$ occurs in $Z_n$ with coefficient 1. This implies the desired equality Newt($Z_n$) = $\Delta_n$, as well as the claim that $Z_n$ is monic. \(\square\)

A more careful analysis of (5.6) implies the following analogue of Proposition 3.6.

Proposition 5.2. (1) Let $n > 0$ and suppose that a point $\gamma \in Q$ belongs to the interior of the side of the triangle $\Delta(n\delta)$ opposite to the vertex $-n\delta$. Then the monomial $y^n$ does not occur in the Laurent expansion of $z_n$ in $y_1$ and $y_2$.

(2) The Laurent expansion of $z_n$ in $y_1$ and $y_2$ contains no monomials $y_1^p y_2^q$ with $p, q \geq 0$.

Proof. As in Proposition 5.1, it is enough to prove the corresponding statements for $Z_n$. Using the notation in the proof of Proposition 5.1, it is easy to see that the only terms $t^k u^k$ whose Newton polygons have non-empty intersection with the side $[(n, -n), (-n, n)]$ of $\Delta_n$ are those with $k = 0$, i.e., those appearing in $T_n(t + t^{-1}) = t^n + t^{-n}$. This implies (1).

Assertion (2) follows at once from (1) by observing that the intersection of $\Delta_n$ with the positive quadrant $\mathbb{R}^2_{>0}$ consists of a single point (namely, the origin), and this point belongs to the interior of the line segment $[(n, -n), (-n, n)]$. \(\square\)

Our next result implies, in particular, that both Propositions 5.1 and 5.2 hold not just for $\{y_1, y_2\}$ but for any choice of “initial” cluster.
Proposition 5.3. For every \( p \in \mathbb{Z} \), the automorphism \( \sigma_p \) preserves each of the elements \( z_n \).

Proof. It suffices to prove the corresponding statement for the elements \( Z_n \in \mathcal{A} \), and the \( \mathbb{Z}[q_1, q_2] \)-automorphisms \( \sigma_p \) acting on the generators by \( \sigma_p(Y_m) = Y_{2p-m} \); the proposition then follows by an appropriate specialization ((5.2) in the case of \( \mathcal{A}(2, 2) \), and (5.3) in the case of \( \mathcal{A}(1, 4) \)). It is enough to show that \( \sigma_1(Z) = Z \).

This is proved by the following calculation using (5.5):

\[
\sigma_1(Z) = \frac{Y_0^2 + q_2Y_0 + (Y_0^2 + q_1Y_1 + 1)}{Y_0Y_1} = \frac{Y_0Y_1^2 + q_2Y_0 + Y_0Y_2}{Y_0Y_1} = \frac{Y_0 + Y_2 + q_2}{Y_1} = \frac{Y_0Y_2 + Y_2^2 + q_2Y_2}{Y_1Y_2} = \frac{Y_2^2 + Y_2 + q_1Y_1 + q_2Y_2 + 1}{Y_1Y_2} = Z, \tag{5.7}
\]

as desired. \( \square \)

The above proof implies, in particular, that \( Z \) can be expressed in any four consecutive variables \( Y_m, \ldots, Y_{m+3} \) as \( Z = Y_mY_{m+3} - (Y_{m+1} + q_{(m+1)})(Y_{m+2} + q_{(m+2)}) \) where as above \( \langle n \rangle \) is 1 for \( n \) odd and 2 for \( n \) even. In the two affine cases this specializes for all even \( m \in \mathbb{Z} \) to

\[
z = \begin{cases} 
  y_m y_{m+2} - y_{m+1} y_{m+3} & \text{if } (b, c) = (2, 2); \\
  y_m^2 y_{m+3} - (y_{m+1} + 2) y_{m+2}^2 & \text{if } (b, c) = (1, 4).
\end{cases}
\]

We are ready for the proofs of main results for the affine types: Theorems 2.3–2.5 and 2.8. We follow the same strategy as in Section 4 with necessary technical modifications.

First of all, Theorem 2.4 for the affine types is a consequence of Theorem 2.8 combined with Propositions 2.11 and 3.1. Second, Propositions 3.1 and 2.11 imply that the cluster monomials together with the elements \( z_n \) are linearly independent: this is proved by the same argument as in Corollary 3.3. Third, Theorem 2.5 follows from Theorem 2.8 combined with Propositions 3.4, 2.11 and 5.3.

As in Section 4, it remains to prove the following counterparts of (4.1)–(4.3):

Every cluster variable and every \( z_n \) is a positive element of \( \mathcal{A}(b, c) \). \( \tag{5.8} \)

Cluster monomials and the elements \( z_n \) linearly span \( \mathcal{A}(b, c) \). \( \tag{5.9} \)

If \( y \in \mathcal{A}(b, c) \) is written as a linear combination of cluster monomials and the \( z_n \)'s then its every coefficient is equal to some coefficient in the Laurent polynomial expansion of \( y \) with respect to some cluster. \( \tag{5.10} \)

Our proofs of (5.8) and (5.9) will use explicit expressions for all minimal “forbidden” products

\[
z_n z_p \ (n, p > 0), \quad z_n y_m \ (n > 0, \ m \in \mathbb{Z}), \quad y_m y_{m+n} \ (m \in \mathbb{Z}, \ n \geq 2), \quad \tag{5.11}
\]

as linear combinations of the cluster monomials and the \( z_n \)’s. In the formulas below we use the convention that \( z_0 = 1 \) and \( z_{-n} = 0 \) for \( n > 0 \). Also \( \langle n \rangle \) has the same meaning as above: it is 1 for \( n \) odd, and 2 for \( n \) even.
Proposition 5.4. (1) For any affine type, the following relation holds for all \( p \geq n \geq 1 \):

\[
z_n z_p = \begin{cases} 
  z_{p-n} + z_{p+n} & \text{if } p > n; \\
  2 + z_{2n} & \text{if } p = n.
\end{cases}
\] (5.12)

(2) In the case \((b, c) = (2, 2)\), for all \( m \in \mathbb{Z} \) and \( n \geq 1 \), we have

\[
z_n y_m = y_{m-n} + y_{m+n},
\] (5.13)

\[
y_m y_{m+n} = y_{\lfloor m + \frac{n}{2} \rfloor} y_{\lceil m + \frac{n}{2} \rceil} + \sum_{k \geq 1} k z_{n-2k}.
\] (5.14)

(3) In the case \((b, c) = (1, 4)\), we have

\[
z_n y_m = \begin{cases} 
  y_{m-2n} + y_{m+2n} & \text{for } m \text{ even}, \\
  y_{m-n} + y_{m+n} + 4 \sum_{k \geq 1} k z_{n-2k} & \text{for } m \text{ odd};
\end{cases}
\] (5.15)

\[
y_m y_{m+2n} = y_{\lfloor m+n \rfloor} + \sum_{k \geq 1} (2k-1) z_{n+1-2k} & \text{for } m \text{ even, } n \geq 0;
\] (5.16)

\[
y_m y_{m+n} = \sum_{1 \leq 2k < n} \min(4k, n-2k) y_{m+4k} + \begin{cases} 
  y_s^2 & \text{if } n \equiv 0 \pmod{3} \\
  y_s y_{\lfloor s \rfloor} & \text{otherwise}
\end{cases}
\] (5.17)

(m even, \( n \geq 1 \) odd), where \( s = m \pm \frac{n}{3} \);

\[
y_m y_{m+2n} = y_{\lfloor m+n \rfloor}^2 + 4 \sum_{k=1}^{n-1} \min(k, n-k) y_{m+2k} + \frac{1}{3} \sum_{k \geq 1} (2k^3 + k) z_{2n-2k}
\] (5.18)

(m odd, \( n \geq 0 \)).

Proof. To prove Proposition 5.4, we first establish the analogs of the relations in question for the elements \( Z_n \) and \( Y_m \) in the algebra \( \tilde{A} \). As above, we use the convention that \( Z_0 = 1 \), and \( Z_{-n} = 0 \) for \( n > 0 \).

Lemma 5.5. The following relations hold for all \( m \in \mathbb{Z} \) and \( p \geq n \geq 1 \):

\[
Z_n Z_p = \begin{cases} 
  Z_{p-n} + Z_{p+n} & \text{if } p > n, \\
  2 + Z_{2n} & \text{if } p = n;
\end{cases}
\] (5.19)

\[
Z_n Y_m = Y_{m-n} + Y_{m+n} + \sum_{k \geq 1} k q_{m+k} Z_{n-k};
\] (5.20)

\[
Y_m Y_{m+n} = Y_{\lfloor m + \frac{n}{2} \rfloor} Y_{\lceil m + \frac{n}{2} \rceil} + \sum_{k=1}^{n-1} \min(k, n-k) q_{m+k} Y_{m+n-k} + \sum_{k \geq 1} c_k Z_{n-1-k}.
\] (5.21)

where the coefficients \( c_k \) are given by

\[
c_{2p} = \frac{p(p+1)(2p+1)}{6} q_1 q_2, \quad c_{2p-1} = \frac{(p-1)p(p+1)}{6}(q_1^2 + q_2^2) + p.
\] (5.22)
Proof. The relation (5.19) follows at once from the definition of Chebyshev polynomials $T_n$.

To prove (5.20), we start with $n = 1$. Proposition 5.3 and the symmetry of indices 1 and 2 in (5.1) reduces the case of general $m$ to a special case $m = 1$; the corresponding relation $Z_1 Y_1 = Y_0 + Y_2 + q_2$ has already appeared in (5.7). The case $n = 2$ now follows since

$$Z_2 Y_m = (Z_1^2 - 2) Y_m = Z_1(Y_{m-1} + Y_{m+1} + q_{(m+1)}) - 2Y_m = Y_{m-2} + Y_{m+2} + q_{(m+1)} Z_1 + 2q_{(m)}.$$ 

Proceeding by induction, we now assume that the products corresponding relation $Y_{n-1} Y_n = Y_{n-1} + 2q_{(n+1)} + q_{(n+1)} (Z_1 Z_{n-1} - Z_{n-1-k})$ satisfy (5.20) for some $n \geq 2$, and we want to show that the same is true for $Y_{n+1} Y_m$. This is done by the following calculation using (5.19):

$$Z_{n+1} Y_m = (Z_1 Z_n - Z_{n-1}) Y_m = Z_1 \left(Y_{m-n} + Y_{m+n} + \sum_{k \geq 1} k q_{(m+k)} Z_{n-k}\right) - \left(Y_{m-n+1} + Y_{m+n-1} + \sum_{k \geq 1} k q_{(m+k)} Z_{n-1-k}\right)$$

$$= Y_{m-n-1} + Y_{m+n+1} + 2q_{(m+n+1)} + \sum_{k \geq 1} k q_{(m+k)} (Z_1 Z_{n-k} - Z_{n-1-k})$$

$$= Y_{m-n-1} + Y_{m+n+1} + (n+1)q_{(m+n+1)} + \sum_{k=1}^n k q_{(m+k)} Z_{n+1-k},$$

as desired.

The proof of (5.21) uses the same strategy but requires a little bit more work. The case $n = 1$ is tautologically true (the two summations in the right hand side being empty), and the case $n = 2$ is the cluster relations (5.1). Proceeding by induction, we now assume that the products $Y_m Y_{m+n-1}$ and $Y_m Y_{m+n}$ satisfy (5.21) for some $n \geq 2$, and we want to show that the same is true for $Y_m Y_{m+n+1}$. Using the relation

$$Y_{m+n+1} = Z_1 Y_m - Y_{m+n+1} - q_{(m+n+1)},$$

which is a special case of (5.20), we obtain

$$Y_m Y_{m+n+1} = Y_m (Z_1 Y_{m+n} - Y_{m+n+1} - q_{(m+n+1)})$$

$$= Z_1 \left(Y_{m+\frac{n-1}{2}} Y_{m+\frac{n+1}{2}} + \sum_{k=1}^{n-1} \min(k, n-k) q_{(m+k)} Y_{m+n-k} + \sum_{k \geq 1} c_k Z_{n-1-k}\right)$$

$$- \left(Y_{m+\frac{n-1}{2}} Y_{m+\frac{n-1}{2}} + \sum_{k=1}^{n-2} \min(k, n-1-k) q_{(m+k)} Y_{m+n-1-k} + \sum_{k \geq 1} c_k Z_{n-2-k}\right) - q_{(m+n+1)} Y_m$$

$$= S_1 + S_2 + S_3,$$
where we abbreviated
\[ S_1 = Z_1 Y_{(m+\frac{z}{2})} Y_{(m+\frac{y}{2})} - Y_{(m+\frac{w}{2})} Y_{(m+\frac{v}{2})}. \]
\[ S_2 = \sum_{k=1}^{n} \min(k, n-k)q_{(m+k)}(Y_{m+n+1-k} + Y_{m+n-1-k} + q_{(m+n+1+k)}) \]
\[ - q_{(m+n+1)} Y_m = \sum_{k=1}^{n-1} \min(k, n-1-k)q_{(m+k)} Y_{m+n-1-k}, \]
\[ S_3 = \sum_{k \geq 1} c_k (Z_1 Z_{n-k} - Z_{n-2-k}). \]

Using (5.19), we simplify
\[ S_3 = \sum_{k \geq 1} c_k Z_{n-k} + c_{n-2} - c_n. \]

To simplify \( S_1 \) and \( S_2 \), we first assume that \( n = 2p \) is even. The routine calculations yield
\[ S_1 = Y_{m+p}(Z_1 Y_{m+p} - Y_{m+p-1}) = Y_{m+p} Y_{m+p+1} + q_{(m+p+1)} Y_{m+p}; \]
\[ S_2 = \sum_{k=1}^{n} \min(k, n+1-k)q_{(m+k)} Y_{m+n+1-k} - q_{(m+p+1)} Y_{m+p} + p^2 q_1 q_2. \]

Plugging in the expressions for \( c_n \) and \( c_{n-2} \) given by the first equality in (5.22), we see that \( S_1 + S_2 + S_3 \) is indeed the right hand side of (5.21) for \( Y_m Y_{m+n+1} \). In the case \( n = 2p - 1 \), similar calculations yield
\[ S_1 = Y_{m+p-1} Y_{m+p+1} + q_{(m+p+1)} Y_{m+p-1} = Y_{m+p} + q_{(m+p)} Y_{m+p} + q_{(m+p+1)} Y_{m+p-1} + 1; \]
\[ S_2 = \sum_{k=1}^{n} \min(k, n+1-k)q_{(m+k)} Y_{m+n+1-k} - q_{(m+p)} Y_{m+p} \]
\[ - q_{(m+p+1)} Y_{m+p-1} + \frac{p(p-1)}{2} (q_1^2 + q_2^2). \]

To conclude that \( S_1 + S_2 + S_3 \) is again the desired expression, it remains to plug in the expressions for \( c_n \) and \( c_{n-2} \) given by the second equality in (5.22). This completes the proof of Lemma 5.5.

Returning to the relations in Proposition 5.4, we notice that all of them in the case \((b, c) = (2, 2)\) are obtained from the corresponding relations in Lemma 5.5 by a specialization (5.2). So we assume throughout the rest of the proof that we are in the case \((b, c) = (1, 4)\).

Using the specialization (5.3), we see that (5.12), the second case in (5.15) and (5.18) are again consequences of the corresponding relations in Lemma 5.5. It remains to prove the first case in (5.15), and also (5.16) and (5.17).

We start with an observation that the specialization (5.3) of (5.20) yields the identity
\[ z_1 y_m^2 = y_{m-1} + y_{m+1} + 2 \ (m \text{ even}). \]
Using the exchange relations (2.1), we obtain the following relation for any even \( m \):
\[
z_1 y_m = \frac{y_{m-1} + y_{m+1} + 2}{y_m} = \frac{(y_{m-2} y_m - 1) + (y_m y_{m+2} - 1) + 2}{y_m} = y_{m-2} + y_{m+2}.
\]
We see that the operator of multiplication by \( z_1 \) preserves the linear span \( V \) of the cluster variables \( y_m \) with even \( m \); and it acts on \( V \) as \( t + t^{-1} \), where \( t \) is the shift operator acting by \( t(y_{2m}) = y_{2m+2} \). It follows that the multiplication by an arbitrary element \( z_n \) acts on \( V \) as \( T_n(t + t^{-1}) = t^n + t^{-n} \), which is precisely the first case in (5.15).

To prove (5.16), we proceed by induction on \( n \). The case \( n = 0 \) is trivial, while \( n = 1 \) is simply the exchange relation. To treat the product \( y_n y_{m+2(n+1)} \) for \( n \geq 1 \), we substitute \( y_{m+2(n+1)} = z_1 y_{m+2n} - y_{m+2(n-1)} \) (see (5.15)) and then use the inductive assumption together with (5.12) and the appropriate specialization of (5.20); the routine details are left to the reader.

The same strategy is used for the proof of (5.17). This completes the proof of Proposition 5.4.

Now we are ready to prove (5.8), i.e., that the cluster variables and the \( z_n \) are positive. By Proposition 5.3, it suffices to check the positivity of the Laurent expansion of each \( z_n \) or \( y_m \) in the variables \( y_1 \) and \( y_2 \). For \( z_n \), the desired positivity is an immediate consequence of (5.6) combined with the following well-known property of Chebyshev polynomials: the derivative of \( T_n \) is a positive linear combination of the polynomials \( T_p \) with \( p < n \). For the sake of completeness, here is the proof: setting \( z = t + t^{-1} \), we obtain
\[
\frac{dT_n(z)}{dz} = \frac{dT_n(t + t^{-1})}{dt} : (dz/dt)^{-1} = n(t^n - t^{-n})/(1 - t^{-2}) = n(t^n - t^{-n})/(t - t^{-1}) = n(t^{n-1} + t^{n-3} + \ldots + t^{-(n-1)}) = n \sum_{0 \leq k \leq \frac{n-1}{2}} T_{n-1-2k}.
\]
To prove that each cluster variable \( y_m \) for \( m \geq 3 \) is a positive Laurent polynomial in \( y_1 \) and \( y_2 \), it suffices to apply one of the formulas (5.14), (5.16) or (5.18) to the product \( y_{(m)} y_m \), and to notice that the right hand side is a positive linear combination of the elements \( z_n \) and cluster monomials containing the variables \( y_p \) with \( \langle m \rangle < p < m \). The desired positivity of \( y_m \) follows by induction on \( m \). The positivity of \( y_m \) for \( m \leq 0 \) follows by Proposition 5.3.

Remark 5.6. Note that the same argument establishes that, in the algebra \( \tilde{A} \) given by (5.1), every element \( Y_m \) is a Laurent polynomial in \( Y_1 \) and \( Y_2 \) whose coefficients are polynomials in \( q_1 \) and \( q_2 \) with positive integer coefficients.

To prove (5.9), it suffices to show that every monomial \( M \) in the variables \( z_n \) and \( y_m \) is a linear combination of the cluster monomials and the \( z_n \)’s. We write \( M \) in the form \( M = z_{m_1}^{a_1} \cdots z_{m_x}^{a_x} y_{m_1}^{b_1} \cdots y_{m_x}^{b_x} \), where \( 0 < n_1 < \cdots < n_x, m_1 < \cdots < m_x \), and the exponents \( a_1, \ldots, a_x \) and \( b_1, \ldots, b_x \) are positive integers. We define the
multi-degree $\mu(M) = (\mu_1(M), \mu_2(M), \mu_3(M)) \in \mathbb{Z}^3_{\geq 0}$ by setting

$$
\begin{align*}
\mu_1(M) &= \sum_{i=1}^s a_i + \sum_{j=1}^t b_j & \text{if } (b, c) = (2, 2); \\
\mu_2(M) &= \sum_{i=1}^s a_i + \sum_{j=1}^t (m_j - 1) b_j & \text{if } (b, c) = (1, 4); \\
\mu_3(M) &= m_4 - m_3; & \mu_3(M) = b_1 + b_8.
\end{align*}
$$

The lexicographic order on $\mathbb{Z}^3_{\geq 0}$ makes it into a well ordered set (i.e., every non-empty subset of $\mathbb{Z}^3_{\geq 0}$ has the smallest element). Therefore, to complete the proof, it suffices to show that every monomial $M$ which has at least one of the “forbidden” products in (5.11) as a factor, can be written as a linear combination of monomials of (lexicographically) smaller multi-degree. We will show that this can be done by replacing some “forbidden” factor of $M$ with its expression given by the appropriate relation in Proposition 5.4. Indeed, if $\sum_{i=1}^s a_i \geq 2$ (resp. $\sum_{i=1}^t a_i = 1$) then one can apply (5.12) (resp. (5.13) or (5.15)), expressing $M$ as a linear combination of monomials with smaller value of $\mu_1$. So we can assume that $M = y_{m_1}^{b_1} \cdots y_{m_s}^{b_s}$ with $m_4 - m_1 \geq 2$. Then we apply (5.14) or one of (5.16)–(5.18) to the product $y_{m_4} y_{m_5}$. By inspection, in the resulting expression for $M$, there is precisely one monomial $M'$ with $\mu_1(M') = \mu_1(M)$, while the rest of the terms have smaller value of $\mu_1$. By further inspection, we have $\mu_2(M') < \mu_2(M)$ (resp. $\mu_3(M') = \mu_3(M)$ and $\mu_3(M') = \mu_3(M) - 2$) if $\min(b_1, b_4) = 1$ (resp. $\min(b_1, b_4) \geq 2$). This concludes the proof of (5.9).

To prove (5.10), suppose that an element $y \in \mathcal{A}(b, c)$ is written (uniquely) as a linear combination of cluster monomials and the elements $z_n$. We need to show that every coefficient in this linear combination is equal to some coefficient in the Laurent expansion of $y$ with respect to some cluster. For the coefficients of cluster monomials, this is proved by precisely the same argument as in the proof of (4.3) given in Section 4; the only extra thing to take into account is Part 2 of Proposition 5.2. It remains to deal with the coefficient of each $z_n$. By Proposition 5.3, we can assume without loss of generality that our linear combination involves only cluster variables $y_m$ with $m \geq 3$. It is then enough to show the following: for every $n \geq 1$, there exists a Laurent monomial $y^\gamma = y_1^{n_1} \cdots y_s^{n_s}$ that occurs with coefficient 1 in (the Laurent expansion of) $z_n$ but does not occur in any $y_z$ for $p \neq n$, or in any cluster monomial in the variables $y_m$ for $m \geq 3$. We claim that the following vector $\gamma$ has all the desired properties:

$$
\gamma = (g_1, g_2) = \begin{cases} 
(n, -n) & \text{if } (b, c) = (2, 2); \\
(n, -2n) & \text{if } (b, c) = (1, 4). 
\end{cases}
$$

Indeed, by Proposition 5.1, $\gamma$ is a vertex of Newt$(z_n)$, and so $y^\gamma$ occurs in $z_n$ with coefficient 1. The fact that $y^\gamma$ does not occur in $y_z$ for $p \neq n$ follows from Part 1 of Proposition 5.2. Finally, the fact that $y^\gamma$ does not occur in any cluster monomial $y_m^p y_{m+1}^q$ for $m \geq 3$, follows from Corollary 3.7: an immediate check shows that $\gamma$ does not satisfy the linear inequality given there that must hold on Newt$(y_m^p y_{m+1}^q)$. This concludes the proofs of Theorems 2.3 and 2.8 for the affine type.
Example 5.7. As an illustration of the above proof of (5.10), let \((b, c) = (2, 2)\) and let \(y\) be a positive element that is a \(\mathbb{Z}\)-linear combination of \(y_1^2 y_2\), \(y_4 y_5\) and \(z_4\). We wish to show each of the three coefficients is positive. First we consider \(y\) as a Laurent polynomial in the variables \(\{y_1, y_2\}\). Propositions 3.5 and 5.1 make it an easy exercise to compute the Newton polygons of the three terms; the results are shown in Figure 3. Since the point \(\text{Newt}(y_1^2 y_2)\) does not belong to \(\text{Newt}(y_4 y_5)\) or \(\text{Newt}(z_4)\), the coefficient of \(y_1^2 y_2\) must be positive. Expressing \(y\) as Laurent polynomial in \(\{y_4, y_5\}\) we could generate a similar picture (not shown) implying \(y_4 y_5\) has positive coefficient. Finally, we consider \(y\) as a Laurent polynomial in \(\{y-1, y_0\}\) as shown in Figure 4. Since \(z_4\) is monic and \(\text{Newt}(z_4)\) has a vertex that lies outside \(\text{Newt}(y_1^2 y_2)\) and \(\text{Newt}(y_4 y_5)\), we conclude that \(z_4\) has a positive coefficient.

Remark 5.8. The second equality in (2.2) shows that the Laurentness property is local: if an element of the ambient field \(\mathcal{F}\) of the cluster algebra \(\mathcal{A}(b, c)\) is a Laurent polynomial with respect to three consecutive clusters, then the same is true with respect to any cluster. Unfortunately, there seems to be no hope to extend this to the “positive Laurentness” property, as shown by the following counterexample for the algebra \(\mathcal{A}(2, 2)\). A direct check shows that the element \(y_0 y_1 + y_2 y_3 + y_3 y_4 - z\) has positive Laurent expansion in \(y_1\) and \(y_2\). It follows that, for every \(n \geq 1\), the
Figure 4. Newton polygons of $y_1^2y_2$, $y_4y_5$ and $z_4$ as Laurent polynomials in $\{y_{-1}, y_0\}$ for $b = c = 2$.

6. General Coefficients

According to [6], for a given pair $(b, c)$ of positive integers, the most general form of the corresponding algebra involves the exchange relations

$$\hat{y}_{m-1}\hat{y}_{m+1} = \begin{cases} q_m\hat{y}_m + r_m & \text{for } m \text{ odd;} \\ q_m\hat{y}_m + r_m & \text{for } m \text{ even.} \end{cases} \tag{6.1}$$
Here the coefficients $q_m$ and $r_m$ are the generators of the universal coefficient group $\mathbb{P}$ defined by the relations

$$r_{m-1}r_{m+1} = \begin{cases} q_{m-1}q_{m+1}r_m^c & \text{for } m \text{ odd;} \\ q_{m-1}q_{m+1}r_m^b & \text{for } m \text{ even.} \end{cases}$$  \hfill (6.2)

The group $\mathbb{P}$ is a free abelian group with countably many generators; as a set of free generators, one can choose $\{q_m : m \in \mathbb{Z}\} \cup \{r_1\}$ (cf. [6, Proposition 5.2]). For our current purposes, we prefer to work with the completion $\hat{\mathbb{P}}$ of $\mathbb{P}$ given by

$$\hat{\mathbb{P}} = P \otimes_{\mathbb{Z}} Q;$$  \hfill (6.3)

in plain words, $\hat{\mathbb{P}}$ is obtained from $\mathbb{P}$ by adjoining the roots of all degrees from all the elements of $\mathbb{P}$. Thus, we will work over the ground ring $\mathbb{Z} \hat{\mathbb{P}}$, the integer group ring of $\hat{\mathbb{P}}$. And so we define the completed universal cluster algebra $\hat{\mathcal{A}}(b, c)$ as the $\mathbb{Z} \hat{\mathbb{P}}$-subalgebra generated by the $y_m$ for $m \in \mathbb{Z}$ inside the ambient field of rational functions in $\hat{y}_1$ and $\hat{y}_2$ over the field of fractions of $\mathbb{Z} \hat{\mathbb{P}}$. The coefficient-free cluster algebra $\mathcal{A}(b, c)$ studied above is obtained from $\hat{\mathcal{A}}(b, c)$ by the specialization

$$\hat{y}_m = y_m, \quad q_m = r_m = 1.$$  \hfill (6.4)

The following result shows that this specialization does not lead to any loss of information, since $\hat{\mathcal{A}}(b, c)$ can be obtained from $\mathcal{A}(b, c)$ by an extension of scalars from $\mathbb{Z}$ to $\mathbb{Z} \hat{\mathbb{P}}$.

**Proposition 6.1.** There is a natural $\mathbb{Z} \hat{\mathbb{P}}$-algebra isomorphism $\psi : \mathcal{A}(b, c) \otimes \mathbb{Z} \hat{\mathbb{P}} \rightarrow \hat{\mathcal{A}}(b, c)$ given by

$$\psi(y_m) = \begin{cases} \left(\frac{q_m}{r_m}\right)^{1/b} \hat{y}_m & \text{for } m \text{ odd;} \\ \left(\frac{q_m}{r_m}\right)^{1/c} \hat{y}_m & \text{for } m \text{ even.} \end{cases}$$  \hfill (6.5)

**Proof.** We just need to show that the elements in (6.5) satisfy the coefficient-free exchange relations (2.1). This follows at once from (6.1) and (6.2). \qed

In view of Proposition 6.1, the canonical basis $\mathcal{B}$ in $\mathcal{A}(b, c)$ given by Theorem 2.3 can be viewed as a $\mathbb{Z} \hat{\mathbb{P}}$-basis in $\hat{\mathcal{A}}(b, c)$. Theorem 2.3 extends to $\hat{\mathcal{A}}(b, c)$ in the following way. Let $\mathbb{Z}_{\geq 0}\hat{\mathbb{P}}$ denote the semiring in $\mathbb{Z} \hat{\mathbb{P}}$ consisting of positive integer linear combinations of elements of $\hat{\mathbb{P}}$. The following definition is a counterpart of Definition 2.2.

**Definition 6.2.** A non-zero element $y \in \hat{\mathcal{A}}(b, c)$ is **positive** if for every $m \in \mathbb{Z}$, all the coefficients in the expansion of $y$ as a Laurent polynomial in $\hat{y}_m$ and $\hat{y}_{m+1}$ belong to $\mathbb{Z}_{\geq 0}\hat{\mathbb{P}}$.

A counterpart of Theorem 2.3 can be now stated as follows.

**Theorem 6.3.** Suppose that $bc \leq 4$. Then the semiring of positive elements of $\hat{\mathcal{A}}(b, c)$ consists precisely of $\mathbb{Z}_{\geq 0}\hat{\mathbb{P}}$-linear combinations of elements of $\mathcal{B}$. This property determines $\mathcal{B}$ uniquely up to rescaling by elements of $\hat{\mathbb{P}}$. 

Proof. For the finite (resp. affine) type, the theorem follows at once from the properties (4.1) and (4.3) (resp. (5.8) and (5.10)). □

Acknowledgments. We thank Sergey Fomin for editorial suggestions, and Jim Propp and Dylan Thurston for helpful discussions. Dylan’s remark that Chebyshev polynomials of the first kind appeared naturally in his topological model of cluster algebras was especially stimulating. One of the authors (A. Z.) gratefully acknowledges the hospitality of David Eisenbud and the financial support of MSRI during his visit in May 2003.

Andrei Zelevinsky is especially happy to dedicate this paper to Borya Feigin, an old friend and classmate since 7th grade in the unforgettable Moscow School No. 2. Many years ago, Borya taught me the basics of Kac–Moody algebras and their root systems (he did it in the best possible way, by engaging me in writing a joint survey paper [3] on the subject). I am happy to be able to apply some of his lessons in this work.

References


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